

QUASICONTRACTIVE n -STEP LIFTINGS FOR OPERATORS SIMILAR TO CONTRACTIONS

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Abstract. The paper is devoted to operators T acting on a Hilbert space $\mathcal{H}$, that are similar to contractions. In this setting, we investigate quasi-isometric liftings induced by certain intermediate quasicontractive liftings Q for T , which satisfy the condition $Q^*Q\mathcal{H} = \mathcal{H}$. Such operators Q are generated by T and an operator A , which gives the similarity of T to a contraction, this meaning that A is invertible and $A \geq T^*AT \geq T^*T$. In this setting, we introduce the n -step liftings of T with respect to A , which lead to a natural quasi-isometric lifting for T . The aim of this paper is to show that the step-by-step construction of the minimal isometric lifting of a contraction can be adapted to this framework, in order to obtain a minimal quasi-isometric lifting of T .

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1. INTRODUCTION AND PRELIMINARIES

It is well known that the Sz.-Nagy–Foiaş dilation theory of contractions plays a fundamental role in operator theory on Hilbert spaces, with isometric liftings and unitary dilations (or extensions) serving as its central objects of investigation (see [6, 17]).

In the last period of time, these concepts were developed in the context of m -isometries, where many difficult problems appear (see [1–5, 7, 8, 10, 11, 13, 14]). In this setting, it was proved in [5, 9] that the operators T on $\mathcal{H}$ similar to contractions admit 2-isometric liftings S , meaning that $S^{*2}S^2 - 2S^*S + I = 0$, and satisfying the condition $S^*S\mathcal{H} \subset \mathcal{H}$. However, unless S is an isometry, it is not power bounded, which suggests that it may not be a particularly natural lifting for T .

Recently, we proved in [12, 15, 16] that more appropriate liftings for an operator similar to a contraction are the quasi-isometries, that is operators T satisfying $T^*T = T^{*2}T^2$.

In this paper, we continue the study of quasi-isometric liftings. We aim to demonstrate the role of intermediate quasicontractions in obtaining (minimal or non-minimal) such liftings. The basic idea is to construct a minimal lifting by the step-by-step method of successive quasicontractive liftings. Recall that an operator Q is a quasicontraction if $Q^*Q \leq I$.

We use the following terminology throughout the paper. For a complex Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the C^* -algebra of all bounded linear operators on $\mathcal{H}$ having as its unit element the identity operator I ($= I_{\mathcal{H}}$, if it is necessary). The adjoint of an operator $T \in \mathcal{B}(\mathcal{H})$ is T^* , while $\mathcal{R}(T)$ and $\mathcal{N}(T)$ stand for the range and the kernel of T . The operator T is left-invertible if T^*T is invertible in $\mathcal{B}(\mathcal{H})$, that is, T is injective with closed range.

For a subspace $\mathcal{M} \subset \mathcal{H}$, $\overline{\mathcal{M}}$ denotes the closure of $\mathcal{M}$ in the norm topology of $\mathcal{H}$. If $\mathcal{M}$ is closed then $P_{\mathcal{M}}$ stands for the orthogonal projection in $\mathcal{B}(\mathcal{H})$ with $\mathcal{R}(P_{\mathcal{M}}) = \mathcal{M}$. Also, in this case $P_{\mathcal{H},\mathcal{M}}$ means the projection from $\mathcal{H}$ onto $\mathcal{M}$, while $J_{\mathcal{M},\mathcal{H}} = P_{\mathcal{H},\mathcal{M}}^*$ is the natural injection of $\mathcal{M}$ into $\mathcal{H}$.

Let $\mathcal{K}$ be another (complex) Hilbert space which contains $\mathcal{H}$ as a closed subspace (in notation, $\mathcal{H} \subset \mathcal{K}$). An operator $S \in \mathcal{B}(\mathcal{K})$ is a lifting for $T \in \mathcal{B}(\mathcal{H})$ if $P_{\mathcal{K},\mathcal{H}}S = TP_{\mathcal{K},\mathcal{H}}$. But this means $S^*J_{\mathcal{H},\mathcal{K}} = J_{\mathcal{H},\mathcal{K}}T^*$, which says that S is a lifting for T if and only if S^* is an extension for T^* . A lifting S on $\mathcal{K} \supset \mathcal{H}$ for T is called to be *minimal* lifting if $\mathcal{K} = \bigvee_{n \geq 0} S^n \mathcal{H}$, meaning that $\mathcal{K}$ is the closed linear span generated by $S^n h$ for all $h \in \mathcal{H}$ and integers $n \geq 0$. Two minimal liftings S_0 on $\mathcal{K}_0$ and S_1 on $\mathcal{K}_1$ for T are *isomorphic* if there exists a unitary operator U from $\mathcal{K}_0$ onto $\mathcal{K}_1$ such that $US_0 = S_1U$ and $U|_{\mathcal{H}} = I$. Here $U^*U = I_{\mathcal{K}_0}$ and $UU^* = I_{\mathcal{K}_1}$, that is, U and U^* are isometries on $\mathcal{K}_0$ and $\mathcal{K}_1$, respectively.

It is known that every contraction T on $\mathcal{H}$ (i.e., $T^*T \leq I$) has a minimal isometric lifting V on the space $\mathcal{K} = \ell_+^2(\mathcal{D}_T) \oplus \mathcal{H}$, where $\mathcal{D}_T = \mathcal{R}(D_T)$ and $D_T = (I - T^*T)^{1/2}$ is the defect operator of T . In addition, the lifting V is uniquely determined up to an isomorphism. Later V has a unique minimal unitary extension on a larger space $\tilde{\mathcal{K}} \supset \mathcal{K}$, which will be a unitary dilation for T . This classical dilation theory of contractions is the basic tool in the study of liftings and extensions for non-contractive operators, as, for instance, operators similar to contractions.

In this sense, we proved in [12, 15, 16] that every operator T similar to a contraction admits quasi-isometric liftings, which are not isomorphic, in general.

Additionally, there are such liftings with closed range and even left-invertible, with a simple or more complex matrix structure. But all these liftings are induced by T and a similarity $A \geq T^*AT$ of T (that is A invertible in $\mathcal{B}(\mathcal{H})$). It is clear that in this case A can be chosen satisfying also the condition $A \geq T^*T$, or even to be expansive, i.e., $A \geq I$. In the latter case, one has $A \geq T^*AT \geq T^*T$, which will be assumed in the sequel.

Recall that the construction of quasi-isometric liftings S for T induced by A begun in [16], where we showed that S can be obtained either as a lifting of a quasicontractive lifting of T (in [16, Theorem 2.1]), or as a lifting for an operator similar to an isometry which dilates T (in [16, Theorem 2.4]). In this paper, we develop the construction of such liftings S of T by means of more general intermediate quasicontractive operators. In this sense, we use the fact that for every quasicontraction Q there exists an operator B

satisfying $Q^*Q = Q^*BQ \leq B$ (see [16, Theorem 2.3]). This is an essential tool for constructing quasicontractive n -step liftings and the associated quasi-isometric lifting.

The paper is organized as follows. In Section 2, we characterize the operators T similar to contractions in terms of quasicontractive liftings Q satisfying the condition $Q^*QH = \mathcal{H}$. Later, we describe two quasi-isometric liftings of Q , which are not minimal as liftings of T . We show that, however, the minimal liftings for T induced by these two liftings of Q are isomorphic. Among other properties, we refer to the ranges and the kernels of those liftings or their adjoints, as well as to the kernel condition $Q^*Q\mathcal{N}(Q^*) \subset \mathcal{N}(Q^*)$.

The results of this section constitute further contributions to the study of quasi-isometric liftings with different structures.

Section 3 is devoted to the description of quasi-isometric liftings for T generated by the quasicontractive one-step lifting T_1 of T , which has a block matrix on $\mathcal{H} \oplus \mathcal{H}$ with the entries T (on $\mathcal{H} = \mathcal{H} \oplus \{0\}$) and $(A - T^*T)^{1/2}$ (from $\mathcal{H} \oplus \{0\}$ into $\{0\} \oplus \mathcal{H}$). Thus, T_1 is induced by T with respect to the similarity A . We show that T_1 generates a quasi-isometric lifting S for T , satisfying the condition $S^*S\mathcal{H} = \mathcal{H}$, as well as other useful properties.

Next, we construct the sequence $\{T_n\}_{n \geq 1}$ of successive quasicontractive n -step liftings of T , such that T_{n+1} is the quasicontractive one step lifting of T_n , with respect to some operator A_n satisfying $T_n^*T_n = T_n^*A_nT_n \leq A_n$, for $n \geq 1$. We construct the sequence $\{A_n\}_{n \geq 1}$ to be bounded, which ensures that $\{T_n\}_{n \geq 1}$ is strongly convergent. Also, its strong limit defines a quasi-isometric lifting S for all T_n ($n \geq 1$), and so for T , with $S^*S\mathcal{H} = \mathcal{H}$, S being a minimal lifting for T . In addition, the minimal lifting S of T is isomorphic to another natural quasi-isometric lifting of T . Finally, we show that the existence of a suitable lifting A_1 of A implies that T is a quasicontraction.

Obviously, the ideas developed in this section are inspired by the Sz.-Nagy–Foiaş model for the construction of the minimal isometric lifting, where the operator naturally associated with a contraction is $A = I$. In our setting, the construction is carried out at each n -th step using two operators, T_n and A_n , which are closely related to each other. The structure of A_n ensures the boundedness of the sequence $\{T_n\}_{n \geq 1}$, and together these operators lead to the construction of the lifting S .

2. INTERMEDIATE QUASICONTRACTIVE LIFTINGS

In the papers [15, 16], we studied quasi-isometric liftings for operators similar to contractions. In fact, these operators were characterized in terms of certain quasi-isometric liftings, which include even those that are left-invertible.

In the sequel, we refer to the construction of quasi-isometric liftings by using some intermediate quasicontractive liftings. Thus, we adapt, in this more general context, the construction by n -step dilations of the minimal isometric lifting of a contraction (see, for instance, [6, Chap. VII]). We begin with a result which is related to Theorem 2.4 and Theorem 2.8 in [16], which offers a characterization of the operators similar to contractions in the terms of quasicontractive liftings.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the following statements are true.*

- (i) *The operator T is similar to a contraction if and only if T admit a quasicontractive lifting Q on a Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$ such that $Q^*Q\mathcal{H} = \mathcal{H}$. In this case $T^*AT \leq A$ where $A = Q^*Q|_{\mathcal{H}}$, and Q can be chosen with closed range.*
- (ii) *If $Q \in \mathcal{B}(\tilde{\mathcal{H}})$ is a lifting for T as in (i) with $\mathcal{R}(Q)$ closed, then $\mathcal{R}(Q|_{\tilde{\mathcal{H}} \ominus \mathcal{H}})$ is closed and there exists an operator $B \in \mathcal{B}(\tilde{\mathcal{H}})$ satisfying*

$$Q^*Q = Q^*BQ \leq B, \quad \mathcal{N}(B) = \mathcal{N}(Q) \cap \mathcal{N}(B - Q^*Q). \quad (2.1)$$

In fact, $B = E^*E + Q^*Q$, where $E = \begin{bmatrix} E_0 & 0 \end{bmatrix} : \mathcal{R}(Q) \oplus \mathcal{N}(Q^*) \rightarrow \mathcal{R}(Q^*)$, and

$$E_0Q = (Q^*Q - Q^{*2}Q^2)^{1/2}.$$

Proof. (i) Let $T \in \mathcal{B}(\mathcal{H})$ be similar to a contraction. So there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that $A \geq T^*AT$ and $A \geq T^*T$. Let $\mathcal{H}_1 = \mathcal{H} \oplus \mathcal{H}$ and $T_1 \in \mathcal{B}(\mathcal{H}_1)$ be the operator defined as

$$T_1 = \begin{pmatrix} T & 0 \\ (A - T^*T)^{1/2} & 0 \end{pmatrix}. \quad (2.2)$$

It is immediate that

$$T_1^*T_1 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T_1^{*2}T_1^2 = \begin{pmatrix} T^*AT & 0 \\ 0 & 0 \end{pmatrix} \leq T_1^*T_1. \quad (2.3)$$

Hence, T_1 is a quasicontractive lifting for T , which satisfies $T_1^*T_1\mathcal{H} = A\mathcal{H} = \mathcal{H}$. This later implies

$$\overline{\mathcal{R}(T_1^*)} = \overline{T_1^*T_1\mathcal{H}_1} = T_1^*T_1\mathcal{H} = \mathcal{H} = \mathcal{R}(T_1^*),$$

so $\mathcal{R}(T_1)$ is closed. The direct implication and the last assertion in (i) are provided.

For the converse implication of (i), we assume that $Q \in \mathcal{B}(\tilde{\mathcal{H}})$ is a quasicontractive lifting for T with $Q^*Q\mathcal{H} = \mathcal{H}$. Therefore, $\mathcal{N}(Q) \subset \tilde{\mathcal{H}} \ominus \mathcal{H}$, which ensures that $A = Q^*Q|_{\mathcal{H}}$ is invertible in $\mathcal{B}(\mathcal{H})$. Since $\mathcal{H}$ is invariant for the self-adjoint operator Q^*Q , it follows that $\mathcal{H}$ is a reducing subspace for Q^*Q . Thus, we have $Q^*QP_{\mathcal{H}} = P_{\mathcal{H}}Q^*Q$ and so $|Q|P_{\mathcal{H}} = P_{\mathcal{H}}|Q|$. Then, for every $h \in \mathcal{H}$, we obtain

$$\begin{aligned} (T^*ATh, h) &= (Q^{*2}QTh, h) = (Q^*QP_{\mathcal{H}}Qh, P_{\mathcal{H}}Qh) \\ &= (P_{\mathcal{H}}|Q|Qh, P_{\mathcal{H}}|Q|Qh) \leq (Q^{*2}Q^2h, h) \leq (Q^*Qh, h) = (Ah, h). \end{aligned}$$

This means that $T^*AT \leq A$ and because A is invertible we conclude that T is similar to a contraction. The statement (i) is proved.

(ii) Let $Q \in \mathcal{B}(\tilde{\mathcal{H}})$ be a quasicontractive lifting for T with $\mathcal{R}(Q)$ closed and $Q^*Q\mathcal{H} = \mathcal{H}$. We have

$$\mathcal{R}(Q^*) = Q^*Q\tilde{\mathcal{H}} = Q^*Q\mathcal{H} \oplus Q^*Q(\tilde{\mathcal{H}} \ominus \mathcal{H}) = \mathcal{H} \oplus Q^*Q(\tilde{\mathcal{H}} \ominus \mathcal{H}).$$

But this shows that

$$\tilde{\mathcal{H}} \ominus \mathcal{H} = Q^*Q(\tilde{\mathcal{H}} \ominus \mathcal{H}) \oplus \mathcal{N}(Q) = Q^*Q(\tilde{\mathcal{H}} \ominus \mathcal{H}) \oplus \mathcal{N}(Q|_{\tilde{\mathcal{H}} \ominus \mathcal{H}}),$$

whence we firstly infer that $\mathcal{R}(Q^*Q|_{\tilde{\mathcal{H}} \ominus \mathcal{H}})$ is closed into $\tilde{\mathcal{H}} \ominus \mathcal{H}$, and finally that $\mathcal{R}(Q|_{\tilde{\mathcal{H}} \ominus \mathcal{H}})$ is closed, taking account that $\tilde{\mathcal{H}} \ominus \mathcal{H}$ is invariant for Q (as a lifting for T). In passing, we also notice that the relation before $\mathcal{N}(Q) = \mathcal{N}(Q|_{\tilde{\mathcal{H}} \ominus \mathcal{H}})$ follows from the above expression for $\mathcal{R}(Q^*)$.

Next, because $0 \leq Q^*Q - Q^{*2}Q^2 \leq Q^*Q$, the Douglas range inclusion theorem ensures that there exists a contraction $E_0 : \mathcal{R}(Q) \rightarrow \mathcal{R} := \overline{\mathcal{R}(Q^*Q - Q^{*2}Q^2)}$ satisfying

$$E_0Q = (Q^*Q - Q^{*2}Q^2)^{1/2}. \quad (2.4)$$

Then $E = \begin{bmatrix} E_0 & 0 \end{bmatrix}$ is a contraction from $\tilde{\mathcal{H}} = \mathcal{R}(Q) \oplus \mathcal{N}(Q^*)$ into $\mathcal{R}$.

Let $B \in \mathcal{B}(\tilde{\mathcal{H}})$ be the operator defined as

$$B = E^*E + Q^*Q. \quad (2.5)$$

Since $E^*E = E_0^*E_0 \oplus 0$ on $\mathcal{R}(Q) \oplus \mathcal{N}(Q^*)$, for every $h \in \tilde{\mathcal{H}}$, we have

$$(Q^*BQh, h) = \|EQh\|^2 + \|Q^2h\|^2 = \|E_0Qh\|^2 + \|Q^2h\|^2 = (Q^*Qh, h).$$

So we obtain that $Q^*BQ = Q^*Q \leq B$, while from the definition of B we get $\mathcal{N}(B) = \mathcal{N}(Q) \cap \mathcal{N}(B - Q^*Q)$, where

$$\mathcal{N}(B - Q^*Q) = \mathcal{N}(E) = \mathcal{N}(E_0) \oplus \mathcal{N}(Q^*).$$

This completes the statement (ii) and the proof of theorem. $\square$

The next result concerns an interesting lifting of the operators T , Q from Theorem 2.1.

Theorem 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and $Q, E \in \mathcal{B}(\tilde{\mathcal{H}})$ be as in Theorem 2.1 (ii). Then T and Q admit a quasi-isometric lifting S on $\mathcal{K} = \ell_+^2(\overline{\mathcal{R}(E)}) \oplus \tilde{\mathcal{H}}$, having the representation*

$$S = \begin{pmatrix} S_+ & JE \\ 0 & Q \end{pmatrix}, \quad (2.6)$$

where S_+ is the forward shift on $\mathcal{K} \ominus \tilde{\mathcal{H}}$ and $J : \overline{\mathcal{R}(E)} \rightarrow \mathcal{K} \ominus \tilde{\mathcal{H}}$ is the embedding mapping. In addition, $S^*S\mathcal{H} \subset \mathcal{H}$ if and only if $E^*E\mathcal{R}(T) \subset \overline{\mathcal{R}(T)}$. Also, $\mathcal{N}(S) = \mathcal{N}(Q) \cap \mathcal{N}(E)$ and S is an isometry on the subspace

$$(\mathcal{K} \ominus \tilde{\mathcal{H}}) \oplus \mathcal{R}(Q) = \overline{\mathcal{R}(S)} \oplus (\overline{\mathcal{R}(E)} \oplus \mathcal{R}(Q)) \cap \mathcal{N}(\begin{bmatrix} E^* & Q^* \end{bmatrix}). \quad (2.7)$$

Moreover, the following conditions are equivalent:

- (i) $(\mathcal{K} \ominus \tilde{\mathcal{H}}) \oplus \mathcal{R}(Q) \subset \mathcal{N}(S^*S - I)$,
- (ii) $Q^*Q\mathcal{R}(Q) \subset \mathcal{R}(Q)$,
- (iii) $\mathcal{R}(S) \subset \mathcal{N}(S^*S - I)$,
- (iv) $S^*S\mathcal{N}(S^*) \subset \mathcal{N}(S^*)$.

Proof. Preserve the notation from the above proof with respect to T and Q . Since the quasicontraction Q is assumed to be with closed range into $\tilde{\mathcal{H}}$, we can consider, in what follows, the decomposition $\tilde{\mathcal{H}} = \mathcal{R}(Q) \oplus \mathcal{N}(Q^*)$.

Let $\mathcal{L} = \ell_+^2(\overline{\mathcal{R}(E)})$, $\mathcal{K} = \mathcal{L} \oplus \tilde{\mathcal{H}}$ and $S \in \mathcal{B}(\mathcal{K})$ be the lifting of Q and T with representations

$$S = \begin{pmatrix} S_+ & JE \\ 0 & Q \end{pmatrix} \begin{bmatrix} \mathcal{L} \\ \tilde{\mathcal{H}} \end{bmatrix} = \begin{pmatrix} S_+ & JE_0 & 0 \\ 0 & Q_0 & Q_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \mathcal{L} \\ \mathcal{R}(Q) \\ \mathcal{N}(Q^*) \end{bmatrix}. \quad (2.8)$$

Here S_+ is the forward shift on $\mathcal{L}$, E_0 is from (2.4), $Q_0 = Q|_{\mathcal{R}(Q)}$, $Q_1 = P_{\mathcal{R}(Q)}Q|_{\mathcal{N}(Q^*)}$, while $J : \overline{\mathcal{R}(E)} = \overline{\mathcal{R}(E_0)} \rightarrow \mathcal{L}$ is the embedding mapping.

Since $\mathcal{N}(S_+) = \overline{\mathcal{R}(E)}$, it follows that

$$S^*S = \begin{pmatrix} I & 0 \\ 0 & E^*E + Q^*Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}. \quad (2.9)$$

On the other hand, by using the first equality from (2.1), we get

$$\begin{aligned} S^{*2}S^2 &= \begin{pmatrix} S_+^* & 0 \\ E^*J^* & Q^* \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} S_+ & JE \\ 0 & Q \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & E^*E + Q^*BQ \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = S^*S. \end{aligned}$$

Therefore, S is a quasi-isometry, which, as a lifting of Q , satisfies the condition $S^*S\tilde{\mathcal{H}} \subset \tilde{\mathcal{H}}$. However, in general, we do not have $S^*S\mathcal{H} \subset \mathcal{H}$.

In fact, for $h \in \mathcal{H}$, we know that $Q^*Qh \in \mathcal{H}$ and $E^*EP_{\mathcal{N}(T^*)}h = 0$, because $\mathcal{N}(T^*) \subset \mathcal{N}(S^*)$, S^* being an extension for T^* . So, from the matrix representation (2.9) of S^*S , we derive that $S^*S\mathcal{H} \subset \mathcal{H}$ if and only if $E^*E\overline{\mathcal{R}(T)} \subset \mathcal{H}$. If this happens, then for $h \in \overline{\mathcal{R}(T)}$ there exists a sequence $\{h_n\}_{n \geq 1} \subset \mathcal{H}$ such that

$$h = \lim_{n \rightarrow \infty} Th_n = \lim_{n \rightarrow \infty} P_{\mathcal{H}}Qh_n,$$

having in view that Q is a lifting for T . Thus, as $E^*EP_{\mathcal{H}} = P_{\mathcal{H}}E^*E$ by our assumption, one obtains that

$$E^*Eh = \lim_{n \rightarrow \infty} E^*EP_{\mathcal{H}}Qh_n = \lim_{n \rightarrow \infty} P_{\mathcal{H}}E^*E_0Qh_n.$$

Since $\mathcal{R}(E_0^*) \subset \mathcal{R}(Q)$, we later infer $E^*E\overline{\mathcal{R}(T)} \subset \overline{P_{\mathcal{H}}Q\tilde{\mathcal{H}}} = \overline{\mathcal{R}(T)}$.

In conclusion, we have $S^*S\mathcal{H} \subset \mathcal{H}$ if and only if $E^*E\overline{\mathcal{R}(\mathcal{T})} \subset \overline{\mathcal{R}(\mathcal{T})}$.

Next, by (2.6), we have $\mathcal{N}(S) = \mathcal{N}(B) = \mathcal{N}(Q) \cap \mathcal{N}(E)$. On the other hand, it is clear from (2.6) that $\mathcal{R}(S) \subset \mathcal{L} \oplus \mathcal{R}(Q)$. Later, if $\ell \oplus Qh \in [\mathcal{L} \oplus \mathcal{R}(Q)] \ominus \mathcal{R}(S)$, then $\ell \oplus Qh \in \mathcal{N}(S^*)$, so by (2.6) it follows that

$$\ell \in \mathcal{N}(S_+^*) = \overline{\mathcal{R}(E)} \quad \text{and} \quad E^*\ell + Q^*Qh = 0.$$

Therefore, $\ell \oplus Qh$ is orthogonal to $\mathcal{R}(S)$ if and only if $\ell \oplus Qh \in \mathcal{N}([E^* \quad Q^*])$ where $[E^* \quad Q^*] : \mathcal{L} \oplus \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$, which provides the relation (2.7). To finish this part of the proof, it remains to show that S is an isometry on its invariant subspace $\mathcal{L} \oplus \mathcal{R}(Q)$ from (2.8).

Indeed, for $k = \ell \oplus Qh \in \mathcal{L} \oplus \mathcal{R}(Q)$, by (2.4), we get

$$\begin{aligned} \|Sk\|^2 &= \|(S_+\ell \oplus E_0Qh) \oplus Q_0Qh\|^2 \\ &= \|\ell\|^2 + \|E_0Qh\|^2 + \|Q^2h\|^2 \\ &= \|\ell\|^2 + ((Q^*Q - Q^{*2}Q^2)h, h) + \|Q^2h\|^2 \\ &= \|\ell\|^2 + \|Qh\|^2 = \|k\|^2. \end{aligned}$$

Hence, $S|_{\mathcal{L} \oplus \mathcal{R}(Q)}$ is an isometry. But, using the second representation of S from (2.8), this means that $E_0^*E_0 + Q_0^*Q_0 = I$. This later leads to a matrix representation of S^*S , namely

$$S^*S = \begin{pmatrix} I & 0 & 0 \\ 0 & I & Q_0^*Q_1 \\ 0 & Q_1^*Q_0 & Q_1^*Q_1 \end{pmatrix} \text{ on } \mathcal{K} = \mathcal{L} \oplus \mathcal{R}(Q) \oplus \mathcal{N}(Q^*), \quad (2.10)$$

where Q_0, Q_1 are the operators from (2.8).

Concerning the conditions (i)–(iv) we infer from (2.10) that $\mathcal{L} \oplus \mathcal{R}(Q) \subset \mathcal{N}(S^*S - I)$ if and only if $Q_0^*Q_1 = 0$, this meaning the range condition for Q , that is, $Q^*Q\mathcal{R}(Q) \subset \mathcal{R}(Q)$. But in this case we also have $\mathcal{R}(S) \subset \mathcal{N}(S^*S - I)$, which is equivalent to the kernel condition for S , i.e. $S^*S\mathcal{N}(S^*) \subset \mathcal{N}(S^*)$, having in view that $S|_{\overline{\mathcal{R}(S)}}$ is an isometry. These show that (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv).

Finally, let us assume that $S^*S\mathcal{N}(S^*) \subset \mathcal{N}(S^*)$ be as in (iv). Then we have $S^*S\mathcal{N}(Q^*) \subset \mathcal{N}(S^*)$, because $\mathcal{N}(Q^*) \subset \mathcal{N}(S^*)$, S being a lifting for Q . Let $\ell \in \mathcal{L}$, $h \in \tilde{\mathcal{H}}$ and $h' \in \mathcal{N}(Q^*)$ be arbitrarily chosen. Then $S^*Sh' \in \tilde{\mathcal{H}}$, and we obtain

$$(S^*Sh', \ell \oplus Qh) = (S^*Sh', Qh) = (S^*Sh', P_{\tilde{\mathcal{H}}}Sh) = (S^*Sh', Sh) = 0,$$

taking into account that $S^*Sh' \in \mathcal{N}(S^*)$ (by our assumption). Having in view that the choice of ℓ, h, h' was arbitrary, we infer that $S^*S\mathcal{N}(Q^*)$ is orthogonal to $\mathcal{L} \oplus \mathcal{R}(Q)$ in $\mathcal{K}$, hence $S^*S\mathcal{N}(Q^*) \subset \mathcal{N}(Q^*)$. Then, from the representation (2.10), it follows $Q_0^*Q_1 = 0$ this meaning the assertion (i), as we remarked before. Thus, (iv) implies (i), and we proved that all conditions (i), (ii), (iii) and (iv) are equivalent, which completes the proof. $\square$

Remark 2.3. Recall from [16, Theorem 3.1] that two minimal quasi-isometric liftings S and S' for T with $S^*S\mathcal{H} \subset \mathcal{H}$ and $S'^*S'\mathcal{H} \subset \mathcal{H}$ are isomorphic if and only if $S^*S|_{\mathcal{H}} = S'^*S'|_{\mathcal{H}}$. This fact will be used in the sequel, because the structure (2.6) of the lifting S for T and Q suggests that one can also define another isomorphic lifting with S for these operators, as follows.

Proposition 2.4. *Let the operators T , Q and E be as in Theorem 2.2. Let $S' \in \mathcal{B}(\mathcal{K}')$ be the operator with the matrix representation on $\mathcal{K}' = \ell_+^2(\overline{\mathcal{R}(E^*)}) \oplus \widetilde{\mathcal{H}}$,*

$$S' = \begin{pmatrix} V_+ & J'|E| \\ 0 & Q \end{pmatrix}, \quad (2.11)$$

where V_+ is the forward shift on $\mathcal{K}' \ominus \widetilde{\mathcal{H}}$ and $J' : \overline{\mathcal{R}(E^*)} \rightarrow \mathcal{K}' \ominus \widetilde{\mathcal{H}}$ is the embedding mapping. Then S' is a quasi-isometric lifting for T and Q . Moreover, S from (2.6) and S' from (2.11) are minimal isomorphic liftings for Q .

Proof. Since $\mathcal{N}(V_+^*) = \overline{\mathcal{R}(E^*)}$, it follows from (2.11) that

$$S'^*S' = I \oplus (E^*E + Q^*Q) = S'^{*2}S'^2 \text{ on } \ell_+^2(\overline{\mathcal{R}(E^*)}) \oplus \widetilde{\mathcal{H}}, \quad (2.12)$$

hence S' is a quasi-isometry on $\mathcal{K}'$.

Now we show that S' on $\mathcal{K}'$ is a minimal lifting for Q , that is,

$$\mathcal{K}' = \bigvee_{n \geq 0} S'^n \widetilde{\mathcal{H}} =: \mathcal{K}_0. \quad (2.13)$$

Indeed, let $k \in \mathcal{K}' \ominus \mathcal{K}_0$. Then k is orthogonal to $\widetilde{\mathcal{H}}$, so $k \in \mathcal{L}' := \mathcal{K}' \ominus \widetilde{\mathcal{H}}$. Since k is orthogonal to $S'\mathcal{H}$, and as $S'h = J'|E|h \oplus Qh$ for every $h \in \widetilde{\mathcal{H}}$, we get $(k, J'|E|h) = 0$, i.e., $|E|J'^*k = 0$. So $k \in \mathcal{N}(E^*EJ'^*) = \mathcal{R}(V_+)$, taking into account that $\mathcal{N}(V_+^*) = \overline{\mathcal{R}(E^*)} = \overline{\mathcal{R}(J'E^*E)}$ into $\mathcal{L}'$. Hence, there exists $\ell_1 \in \mathcal{L}'$ such that $k = V_+\ell_1$.

Next, we use the fact that k is orthogonal to $S'^2\widetilde{\mathcal{H}}$. Since

$$S'^2h = V_+J'|E|h \oplus J'|E|Qh \oplus Q^2h, \text{ for } h \in \widetilde{\mathcal{H}},$$

we obtain $(V_+\ell_1, V_+J'|E|h) = (\ell_1, J'|E|h) = 0$ for all $h \in \widetilde{\mathcal{H}}$. Thus, $\ell_1 \in \mathcal{R}(V_+)$ and $\ell_1 = V_+\ell_2$ for an element $\ell_2 \in \mathcal{L}'$.

Therefore, we have $k = V_+\ell_1 = V_+^2\ell_2$, and by an inductive argument one obtains that $k = V_+^n\ell_n$, with $\ell_n \in \mathcal{L}'$, for every integer $n \geq 1$. Hence, $\ell_n = V_+^{*n}k \rightarrow 0$ ($n \rightarrow \infty$), which later implies $k = 0$. Having in view the choice (arbitrary) of $k \in \mathcal{K}' \ominus \mathcal{K}_0$, it follows that $\mathcal{K}' = \mathcal{K}_0$, this meaning that S' is a minimal lifting for Q , that is, the equality (2.13) holds true.

In a similar way, one can see that S from (2.6) is also a minimal lifting for Q . But from (2.9) and (2.12) we infer that $S^*S|_{\widetilde{\mathcal{H}}} = S'^*S'|_{\widetilde{\mathcal{H}}}$, and this ensures by Remark 2.3, that the liftings S and S' are isomorphic, meaning that there exists a unitary operator $U : \mathcal{K} \rightarrow \mathcal{K}'$ satisfying $US = S'U$ and $U|_{\widetilde{\mathcal{H}}} = I$. $\square$

Notice that the liftings S from (2.6) and S' from (2.11) for T are not necessarily minimal (as these are for Q), but these induce such liftings for T , as we see in the following corollary.

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, $S' \in \mathcal{B}(\mathcal{K}')$ be the quasi-isometric liftings for T , with the matrix representations (2.6) and (2.11), respectively. Then $S^*S\mathcal{H} \subset \mathcal{H}$ if and only if $S'^*S'\mathcal{H} \subset \mathcal{H}$. If this is the case, then $S_0 = S|_{\mathcal{K}_0}$, where $\mathcal{K}_0 = \bigvee_{n \geq 0} S^n \mathcal{H}$, and $S'_0 = S'|_{\mathcal{K}'_0}$, where $\mathcal{K}'_0 = \bigvee_{n \geq 0} S'^n \mathcal{H}$, are minimal and isomorphic quasi-isometric liftings for T .*

Proof. Preserving the notation before, we have from (2.6) and (2.11) that

$$S^*S\mathcal{H} = (E^*E + Q^*Q)\mathcal{H} = S'^*S'\mathcal{H}.$$

Therefore, $S^*S\mathcal{H}$ and $S'^*S'\mathcal{H}$ are simultaneously included in $\mathcal{H}$. When this happens, the quasi-isometric liftings S_0 and S'_0 for the above T are minimal and if $B = E^*E + Q^*Q$ then

$$S_0^*S_0|_{\mathcal{H}} = P_{\mathcal{H}}S^*S|_{\mathcal{H}} = B|_{\mathcal{H}} = P_{\mathcal{K}'_0}S'^*S'|_{\mathcal{H}} = S_0'^*S_0'|_{\mathcal{H}}.$$

But this equality ensures, by Remark 2.3, that S_0 and S'_0 are isomorphic as minimal liftings for T . $\square$

Additional information concerning the lifting S from (2.6) can certainly be obtained as a consequence of Theorem 2.2.

Corollary 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ and $Q, E \in \mathcal{B}(\widetilde{\mathcal{H}})$, $S \in \mathcal{B}(\mathcal{K})$ be as in Theorem 2.2. If $Q_0 = Q|_{\mathcal{R}(Q)}$ and $Q_1 = Q|_{\mathcal{N}(Q^*)}$, then*

$$\mathcal{N}(E) = \mathcal{N}(I - Q_0^*Q_0) \oplus \mathcal{N}(Q^*), \quad \overline{\mathcal{R}(E^*)} = \overline{\mathcal{R}(I - Q_0^*Q_0)}. \quad (2.14)$$

Furthermore, if $Q^*Q\mathcal{R}(Q) \subset \mathcal{R}(Q)$ as in Theorem 2.2 (ii), then

$$\mathcal{N}(E) = \mathcal{N}(I - Q^*Q) \oplus \overline{\mathcal{R}(I - Q_1^*Q_1)}, \quad \mathcal{N}(S) = \mathcal{N}(Q_1). \quad (2.15)$$

Proof. From Theorem 2.2 we have that S is an isometry on the subspace $(\mathcal{K} \ominus \widetilde{\mathcal{H}}) \oplus \mathcal{R}(Q)$. But using the second block matrix of S in (2.8), this means that $E_0^*E_0 + Q_0^*Q_0 = I$, where $E_0 = E|_{\mathcal{R}(Q)}$. So $\mathcal{N}(E_0) = \mathcal{N}(I - Q_0^*Q_0)$, that later implies the relations (2.14), having in view that $E = \begin{bmatrix} E_0 & 0 \end{bmatrix}$ on $\mathcal{R}(Q) \oplus \mathcal{N}(Q^*)$ and $\mathcal{R}(E^*) = \mathcal{R}(E_0^*) \subset \mathcal{R}(Q)$. Here we also notice that Q_0 is a contraction on $\mathcal{R}(Q)$, Q being a quasi-contraction, hence $\overline{\mathcal{R}(E^*)}$ in (2.14) is even the defect space of Q_0 .

Now, we assume that $Q^*Q\mathcal{R}(Q) \subset \mathcal{R}(Q)$. Then, from the block matrix of Q into that of S in (2.8), we infer that

$$Q^*Q = Q_0^*Q_0 \oplus Q_1^*Q_1 \text{ on } \widetilde{\mathcal{H}} = \mathcal{R}(Q) \oplus \mathcal{N}(Q^*).$$

Therefore, $\mathcal{N}(I - Q^*Q) = \mathcal{N}(E_0) \oplus \mathcal{N}(I - Q_1^*Q_1)$, which later gives

$$\mathcal{N}(E) = \mathcal{N}(E_0) \oplus \mathcal{N}(Q^*) = \mathcal{N}(I - Q^*Q) \oplus \overline{\mathcal{R}(I - Q_1^*Q_1)},$$

meaning the first relation in (2.15).

Finally, since $\mathcal{N}(S) = \mathcal{N}(E) \cap \mathcal{N}(Q)$ and as $\mathcal{N}(Q) \subset \mathcal{R}(I - Q^*Q)$, by the above relation for $\mathcal{N}(E)$, we get

$$\mathcal{N}(S) = \overline{\mathcal{R}(I - Q_1^*Q_1)} \cap \mathcal{N}(Q) = \mathcal{N}(Q_1).$$

This means the second relation in (2.15), which ends the proof. $\square$

Remark that Theorems 2.1 and 2.2 refer to an intermediate quasicontractive lifting Q between T and a quasi-isometric lifting for T , Q having the property $Q^*QH = \mathcal{H}$. A simple example in this context is even the lifting T_1 from (2.2). But one can get other liftings Q of this type, as in the following.

Example 2.7. Let $T, A \in \mathcal{B}(\mathcal{H})$ be such that $A \geq I$, $A \geq T^*AT$ and T be left-invertible. Define the operator Q on $\overline{\mathcal{R}(T^*AT - T^*T)} \oplus \mathcal{H}$, Q having the matrix representation

$$Q = \begin{pmatrix} 0 & (T^*AT - T^*T)^{1/2} \\ 0 & T \end{pmatrix}. \quad (2.16)$$

Clearly, $T^*AT \geq T^*T$ (as $A \geq I$) and immediately follows that

$$Q^*Q = 0 \oplus T^*AT, \quad Q^{*2}Q^2 = 0 \oplus T^{*2}AT^2.$$

Since $T^{*2}AT^2 \leq T^*AT$, we have that Q is a quasicontractive lifting for T . In addition, because $T^*AT \geq T^*T$ and T^*T is invertible, it follows that T^*AT is invertible. Hence, $Q^*QH = T^*AT\mathcal{H} = \mathcal{H}$.

But having in view that $T^{*2}AT^2 \geq T^{*2}T^2$, as well as the fact that T^2 is also left-invertible, it follows that $T^{*2}AT^2$ is invertible. So, for this lifting Q , we have also that $Q^{*2}Q^2\mathcal{H} = \mathcal{H}$.

3. QUASICONTRACTIVE n -STEP LIFTINGS

The natural quasicontractive lifting of an operator T similar to a contraction by an operator A is the operator T_1 defined in (2.2) on the space $\mathcal{H} \oplus \mathcal{H}$, A being expansive satisfying the conditions $A \geq T^*AT \geq T^*T$. This operator T_1 has the closed range, namely, $\mathcal{R}(T_1) = T_1\mathcal{H}$, because $\mathcal{R}(T_1^*) = T_1^*T_1\mathcal{H} = A\mathcal{H} = \mathcal{H}$.

The quasicontraction T_1 induces a quasi-isometric lifting for T , which has the properties from Theorem 2.2, but whose structure can be given more precisely.

Theorem 3.1. *Let $T, A \in \mathcal{B}(\mathcal{H})$ be such that $A \geq I$ and $A \geq T^*AT$. Then T admits a quasi-isometric lifting S on the space $\mathcal{K} = \mathcal{L} \oplus \mathcal{H}' \oplus \mathcal{H}$, where $\mathcal{L} = \ell_+^2(\overline{\mathcal{R}(A - T^*AT)})$, S having the following representations*

$$S = \begin{pmatrix} S_+ & JE \\ 0 & T_1 \end{pmatrix} \begin{bmatrix} \mathcal{L} \\ \mathcal{H}' \oplus \mathcal{H} \end{bmatrix} = \begin{pmatrix} S_+ & E_1 & E_2 \\ 0 & 0 & (A - T^*T)^{1/2} \\ 0 & 0 & T \end{pmatrix} \begin{bmatrix} \mathcal{L} \\ \mathcal{H}' \\ \mathcal{H} \end{bmatrix}. \quad (3.1)$$

Here $\mathcal{H}' = \{0\} \oplus \mathcal{H} \oplus \{0\} = \mathcal{K} \ominus (\mathcal{L} \oplus \mathcal{H})$, S_+ is the forward shift on $\mathcal{L}$, T_1 is the lifting of T from (2.2), J is the embedding mapping of $\mathcal{R}(E)$ into $\mathcal{L}$, and $E \in \mathcal{B}(\mathcal{H}' \oplus \mathcal{H})$ is the operator

$$E = \begin{bmatrix} E_1 & E_2 \end{bmatrix} = \begin{bmatrix} E_0 & 0 \end{bmatrix} \text{ on } \mathcal{H}' \oplus \mathcal{H} = \mathcal{R}(T_1) \oplus \mathcal{N}(T_1^*),$$

with $E_0 : \mathcal{R}(T_1) \rightarrow \mathcal{R}(T_1^*) = \mathcal{H}$ defined as in (2.4) relative to T_1 . Then the lifting S for T is not minimal if $\mathcal{N}(A - T^*T) \neq \{0\}$, and furthermore, $S^*S\mathcal{H} \subset \mathcal{H}$ if and only if for every $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$ there exist $h_0, h_1 \in \mathcal{H}$ with $T_1h_0 = P_{\mathcal{R}(T_1)}h$, $T_1h_1 = P_{\mathcal{R}(T_1)}h'$ such that

$$((A - T^*AT)h_0, h_1) = 0.$$

Additionally, the following relations hold:

$$\overline{\mathcal{R}(S^*)} = \mathcal{L} \oplus \overline{\mathcal{R}(E_1^*)} \oplus \mathcal{H}, \quad \mathcal{N}(E_0) = T_1\mathcal{N}(A - T^*AT). \quad (3.2)$$

Proof. The first block matrix of S in (3.1) is that from (2.6) with T_1 instead of Q , which is developed in the second matrix. Clearly, $E_1 = E|_{\mathcal{H}'}$, $E_2 = E|_{\mathcal{H}}$, while $E_0 : \mathcal{R}(T_1) \rightarrow \mathcal{H}$ is given by

$$E_0T_1h = (T_1^*T_1 - T_1^{*2}T_1^2)^{1/2}h = (A - T^*AT)^{1/2}h, \quad h \in \mathcal{H}.$$

So for $E = \begin{bmatrix} E_0 & 0 \end{bmatrix} : \mathcal{R}(T_1) \oplus \mathcal{N}(T_1^*) \rightarrow \mathcal{H}$ one has $\overline{\mathcal{R}(E)} = \overline{\mathcal{R}(E_0)} = \overline{\mathcal{R}(A - T^*AT)}$. This later shows us that $\mathcal{N}(S_+^*) = \overline{\mathcal{R}(E)}$ into $\mathcal{L}$.

From Theorem 2.2 we have that S is a quasi-isometric lifting for T (and T_1), which can be non-minimal as a lifting of T , this meaning that

$$\mathcal{K}_0 := \bigvee_{n \geq 0} S^n \mathcal{H} \subsetneq \mathcal{K}.$$

Indeed, there exist the vectors $k \in \mathcal{K} \ominus \mathcal{K}_0$, $k \neq 0$, which need to be of the form $k = \ell \oplus h'$ with $\ell \in \mathcal{L}$, $h' \in \mathcal{H}'$ (since $k \perp \mathcal{H}$). In fact, if we get $k \in \mathcal{N}(S^*)$, then $k \perp S^n \mathcal{H}$ for $n \geq 0$, so $k \perp \mathcal{K}_0$. In this case, the condition $S^*(\ell \oplus h') = 0$ means the relations

$$S_+^* \ell = 0, \quad E_1^* \ell = 0, \quad E_2^* \ell + (A - T^*T)^{1/2} J_0^* h' = 0,$$

where $J_0 : \mathcal{H} = \{0\} \oplus \mathcal{H} \rightarrow \mathcal{H}' = \mathcal{H} \oplus \{0\}$ is the embedding mapping. Therefore, $\ell \in \overline{\mathcal{R}(E)} \cap \mathcal{N}(E_1^*)$, while the last relations yield $E^* \ell + T_1^* h' = 0$, i.e., $\ell \oplus h' \in$

$\mathcal{N}([E^* \ T_1^*])$. With these choices of ℓ and h' we conclude that if $k = \ell \oplus h' \neq 0$, then $k \perp \mathcal{K}_0$.

In particular, if $0 \neq h \in \mathcal{N}(A - T^*T)$, then $k = h' = J_0 h \in \mathcal{H}'$ and we have

$$S^*k = T_1^*k = (A - T^*T)^{1/2}J_0^*k = (A - T^*T)^{1/2}h = 0.$$

So, in this case (mentioned in the theorem), one has $0 \neq k \perp \mathcal{K}_0$, which justifies that $\mathcal{K} \neq \mathcal{K}_0$. The first assertion of theorem is proved.

Next, we analyze the inclusion $S^*S\mathcal{H} \subset \mathcal{H}$. Clearly, from (3.1) one obtains

$$S^*S = \begin{pmatrix} I & 0 & 0 \\ 0 & E_1^*E_1 & E_1^*E_2 \\ 0 & E_2^*E_1 & E_2^*E_2 + A \end{pmatrix} \text{ on } \mathcal{K} = \mathcal{L} \oplus \mathcal{H}' \oplus \mathcal{H}.$$

This shows that $S^*S\mathcal{H} \subset \mathcal{H}$ if and only if $E_1^*E_2 = 0$. But for every $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$ we have

$$(E_1h', E_2h) = (Eh', Eh) = (E_0P_{\mathcal{R}(T_1)}h', E_0P_{\mathcal{R}(T_1)}h) = (E_0T_1h_1, E_0T_1h_0)$$

where $h_0, h_1 \in \mathcal{H}$ with $P_{\mathcal{R}(T_1)}h' = T_1h_1$, $P_{\mathcal{R}(T_1)}h = T_1h_0$, having in view that $\mathcal{R}(T_1)$ is closed. Thus, we obtain

$$(E_1h', E_2h) = (T_1^*E_0^*E_0T_1h_1, h_0) = ((A - T^*AT)h_1, h_0).$$

But this shows that $E_1^*E_2 = 0$ if and only if for every $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$ there exists $h_0, h_1 \in \mathcal{H}$ such that $T_1h_0 = P_{\mathcal{R}(T_1)}h$, $T_1h_1 = P_{\mathcal{R}(T_1)}h'$ and satisfying $((A - T^*AT)h_0, h_1) = 0$. It is the second assertion of the theorem.

Finally, we infer from Theorem 2.2 that

$$\mathcal{N}(S) = \mathcal{N}(E) \cap \mathcal{N}(T_1) = \mathcal{N}(E) \cap \mathcal{H}' = \mathcal{N}(E_1),$$

which implies the first relation in (3.2). For the second one, we have for $h \in \mathcal{H}$ that $T_1h \in \mathcal{N}(E_0)$ if and only if $(A - T^*AT)h = 0$, which proves that $\mathcal{N}(E_0) = T_1\mathcal{N}(A - T^*AT)$, this being the second relation in (3.2). $\square$

Remark 3.2. The lifting S from (3.1) induced by T_1 is more useful than that from (2.6), but it has two major inconveniences: it can be non-minimal (as a lifting of T) and $S^*S\mathcal{H} \not\subset \mathcal{H}$ in general. But T_1 generates another quasi-isometric lifting of T , outside these inconveniences, slightly modified S in (3.1).

Theorem 3.3. *Let $T, A \in \mathcal{B}(\mathcal{H})$ be such that $A \geq I$ and $A \geq T^*AT$. Then T admits a minimal quasi-isometric lifting S_1 on $\mathcal{K}_1 = \mathcal{L} \oplus \mathcal{L}' \oplus \mathcal{H}$ where $\mathcal{L} = \ell_+^2(\overline{\mathcal{R}(A - T^*AT)})$, $\mathcal{L}' = \overline{\mathcal{R}(A - T^*T)}$. More exactly, S_1 has under the above decomposition of $\mathcal{K}_1$, the representation*

$$S_1 = \begin{pmatrix} S_+ & C & 0 \\ 0 & 0 & (A - T^*T)^{1/2} \\ 0 & 0 & T \end{pmatrix}, \text{ where } C(A - T^*T)^{1/2} = (A - T^*AT)^{1/2}, \quad (3.3)$$

C being a contraction from $\mathcal{L}'$ into $\mathcal{L}$ with $\overline{\mathcal{R}(C)} = \overline{\mathcal{R}(A - T^*AT)} = \mathcal{N}(S_+^*)$, while S_+ is the forward shift on $\mathcal{L}$. Moreover, S_1 satisfies the relations

$$S_1^* S_1 \mathcal{H} = \mathcal{H}, \quad \overline{\mathcal{R}(S_1)} = \mathcal{L} \oplus \mathcal{R}(T_1), \quad \overline{\mathcal{R}(S_1^*)} = \mathcal{L} \oplus \overline{\mathcal{R}(C^*)} \oplus \mathcal{H}, \quad (3.4)$$

where $T_1 = P_{\mathcal{L}' \oplus \mathcal{H}} S_1|_{\mathcal{L}' \oplus \mathcal{H}}$.

Proof. It is obvious that the operator S_1 from (3.3) is a lifting for T which, having in view the definition of C , satisfies the condition

$$S_1^* S_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & C^* C & 0 \\ 0 & 0 & A \end{pmatrix} = S_1^{*2} S_1^2. \quad (3.5)$$

So S_1 is a quasi-isometry on $\mathcal{K}_1$, while C is a contraction from $\mathcal{L}'$ into $\mathcal{N}(S_+^*) = \overline{\mathcal{R}(A - T^*AT)}$, C being well defined because $0 \leq A - T^*AT \leq A - T^*T$.

Let us show that S_1 is a minimal lifting, i.e.,

$$\mathcal{K}_1 = \bigvee_{n \geq 0} S_1^n \mathcal{H}.$$

Indeed, let $k \in \mathcal{K}_1 \ominus \bigvee_{n \geq 0} S_1^n \mathcal{H}$. Then $k \in \mathcal{K}_1 \ominus \mathcal{H}$, so $k = \ell \oplus \ell'$ with $\ell \in \mathcal{L}$, $\ell' \in \mathcal{L}'$. As $k \perp S_1 \mathcal{H}$, for every $h \in \mathcal{H}$, we have by (3.3) the relations

$$0 = (k, S_1 h) = (\ell \oplus \ell' \oplus 0, 0 \oplus (A - T^*T)^{1/2} h \oplus Th) = (\ell', (A - T^*T)^{1/2} h).$$

Therefore, $\ell' \perp \mathcal{L}' = \overline{\mathcal{R}(A - T^*T)}$, i.e., $\ell' = 0$. Thus, $k = \ell \in \mathcal{L}$ and $k \perp S_1^2 \mathcal{H}$. For every $h \in \mathcal{H}$, this yields

$$\begin{aligned} 0 &= (k, C(A - T^*T)^{1/2} h \oplus (A - T^*T)^{1/2} Th \oplus T^2 h) \\ &= (k, C(A - T^*T)^{1/2} h) = (k, (A - T^*AT)^{1/2} h), \end{aligned}$$

which means, $k \perp \overline{\mathcal{R}(C)} = \mathcal{N}(S_+^*)$. Hence, $k = S_+ \ell_0$ with $\ell_0 \in \mathcal{L}$.

Next, using that $k \perp S_1^3$, we obtain from (3.3), for $h \in \mathcal{H}$,

$$\begin{aligned} 0 &= (k, S_1^3 h) = (S_+ \ell_0, S_+ C(A - T^*T)^{1/2} h \oplus C(A - T^*T)^{1/2} Th) \\ &= (\ell_0, C(A - T^*T)^{1/2} h) + (S_+ \ell_0, C(A - T^*T)^{1/2} Th) \\ &= (\ell_0, C(A - T^*T)^{1/2} h). \end{aligned}$$

It follows that $\ell_0 \perp \overline{\mathcal{R}(C)}$ meaning $\ell \in \mathcal{R}(S_+)$, that is there exists $\ell_1 \in \mathcal{L}$ such that $\ell_0 = S_+ \ell_1$, and later $k = S_+ \ell_0 = S_+^2 \ell_1$. By an induction argument, one obtains $k \in \mathcal{R}(S_+^n)$ for every integer $n \geq 1$, that is, $k = 0$, S_+ being a unilateral shift. In conclusion, S_1 is a minimal lifting of T .

Now, it is clear from (3.5) that $S_1^* S_1 \mathcal{H} = A\mathcal{H} = \mathcal{H}$, as we quoted in (3.4). Also, it is immediate from (3.3) that $\mathcal{R}(S_1) \subset \mathcal{L} \oplus \mathcal{R}(T_1)$, $\mathcal{R}(T_1)$ being closed because $T_1^* T_1 = 0 \oplus A$ on $\mathcal{L}' \oplus \mathcal{H}$, which gives

$$\mathcal{R}(T_1^*) = \mathcal{R}(T_1^* T_1) = T_1^* T_1 \mathcal{H} = A\mathcal{H} = \mathcal{H}.$$

In fact, we have $\overline{\mathcal{R}(S_1)} = \mathcal{L} \oplus \mathcal{R}(T_1)$, since if $k = \ell_0 \oplus T_1 h_0 \perp \mathcal{R}(S_1)$ with $\ell_0 \in \mathcal{L}, h_0 \in \mathcal{H}$, then for every $\ell \in \mathcal{L}, \ell' \in \mathcal{L}'$ and $h \in \mathcal{H}$ we obtain

$$\begin{aligned} 0 &= (\ell_0 \oplus T_1 h_0, (S_+ \ell \oplus C h') \oplus T_1 h) \\ &= (\ell_0, S_+ \ell \oplus C \ell') + (T_1 h_0, T_1 h) \\ &= (S_+^* \ell_0, \ell) + (C^* \ell_0, \ell') + (A h_0, h) \\ &= (S_+^* \ell_0 \oplus C^* \ell_0 \oplus A h_0, \ell \oplus \ell' \oplus h). \end{aligned}$$

Hence, $S_+^* \ell_0 \oplus C^* \ell_0 \oplus A h_0 = 0$, what obliges $\ell_0 = 0$ and $h_0 = 0$, so $k = 0$, taking into account that $\mathcal{N}(S_+^*) = \overline{\mathcal{R}(C)}$, while A is injective. Thus, $\overline{\mathcal{R}(S_1)} = \mathcal{L} \oplus \mathcal{R}(T_1)$, which is the second relation in (3.4).

For the last relation, we see that if $S_1(\ell \oplus \ell' \oplus h) = 0$ for $\ell \in \mathcal{L}, \ell' \in \mathcal{L}'$ and $h \in \mathcal{H}$, then $S_+ \ell = 0, C \ell' = 0$ and $T_1 h = 0$. So $\ell = 0$, and since $\mathcal{N}(T_1) = \mathcal{L}'$, it follows $h = 0$. So $\mathcal{N}(S_1) = \mathcal{N}(C)$, that is, $\overline{\mathcal{R}(S_1^*)} = \mathcal{L} \oplus \overline{\mathcal{R}(C^*)} \oplus \mathcal{H}$, as desired. This completes the proof. $\square$

Corollary 3.4. *For $T, A \in \mathcal{B}(\mathcal{H})$ and $S_1 \in \mathcal{B}(\mathcal{K}_1)$ as in Theorem 3.3, the following statements are equivalent:*

- (i) S_1 is left-invertible in $\mathcal{B}(\mathcal{K}_1)$,
- (ii) $\mathcal{R}[(A - T^* T)^{1/2}] = \mathcal{R}[(A - T^* A T)^{1/2}]$,
- (iii) The contraction C from (3.3) is left-invertible.

Proof. From (3.5) we infer that $S_1^* S_1$ is invertible if and only if $C^* C$ is invertible, which shows that the statements (i) and (iii) are equivalent.

Next, if C is left-invertible and C' is a left inverse of C , then as in the proof of Theorem 2.1 in [15] one can get a partial isometry P and a unitary operator U satisfying $U(A - T^* T)^{1/2} = C' P (A - T^* A T)^{1/2}$. But this implies $A - T^* T \leq \|C'\|^2 (A - T^* A T)$, which by Douglas's range inclusion theorem means that $\mathcal{R}[(A - T^* T)^{1/2}] \subset \mathcal{R}[(A - T^* A T)^{1/2}]$. Because the converse inclusion is also true (as $0 \leq A - T^* A T \leq A - T^* T$), it follows the equality from (ii). Hence, (iii) implies (ii).

Finally, if the relation from (ii) holds, then by Douglas's theorem there exists a constant $\rho > 0$ such that $A - T^* T \leq \rho(A - T^* A T)$. In this case, the operator $C' \in \mathcal{B}(\mathcal{L}'), \mathcal{L}' = \overline{\mathcal{R}(A - T^* A T)} = \overline{\mathcal{R}(A - T^* T)}$, where $C'(A - T^* A T)^{1/2} h = (A - T^* T)^{1/2} h$ for $h \in \mathcal{H}$, satisfies $C' C = I$, that is, C' is a left inverse of C . Consequently (ii) implies (iii). $\square$

Notice that in [15] we proved that every operator T similar to a contraction has a left-invertible quasi-isometric lifting S , but which does not necessarily verify the condition $S^*S\mathcal{H} \subset \mathcal{H}$ (as the above lifting S_1). This last condition is essential from many points of view in the operator dilation theory.

In the sequel, we refer to the step-by-step method for the construction of such a lifting S in relationship with the lifting S_1 from Theorem 3.3. In fact, we adapt to our setting the construction of n -step partial isometric dilations of the minimal isometric dilation of a contraction (see [6]).

For two operators $T, A \in \mathcal{B}(\mathcal{H})$ with $A \geq I$ and $A \geq T^*AT$ we concentrate on the quasicontractive lifting T_1 of T , using a notation different from that in Theorem 3.3. Let $D_{A,T} = (A - T^*T)^{1/2}$ and $\mathcal{D}_{A,T} = \overline{\mathcal{R}(D_{A,T})}$ to be the *defect operator* and *defect space* of T relative to A , respectively. Then the operator T_1 from (3.3) has the representation

$$T_1 = \begin{pmatrix} T & 0 \\ D_{A,T} & 0 \end{pmatrix} \text{ on } \mathcal{H}_1 = \mathcal{H} \oplus \mathcal{D}_{A,T}. \quad (3.6)$$

We call this operator T_1 the *quasicontractive one-step lifting* of T with respect to A .

Now we consider the operator $A_1 \in \mathcal{B}(\mathcal{H}_1)$, $A_1 = A \oplus C^*C$ on $\mathcal{H} \oplus \mathcal{D}_{A,T}$, where C is the contraction from (3.3). It is easy to show that

$$T_1^*T_1 = T_1^*A_1T_1 = A \oplus 0 \leq A_1, \quad A_1 - T_1^*T_1 = 0 \oplus C^*C. \quad (3.7)$$

Thus, one can define the quasicontractive one step lifting T_2 of T_1 with respect to A_1 , on the space $\mathcal{H}_2 = \mathcal{H}_1 \oplus \mathcal{D}_{A_1,T_1}$. Let $A_2 \in \mathcal{B}(\mathcal{H}_2)$ be the operator with the block matrix

$$A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix} \text{ on } \mathcal{H}_2 = \mathcal{H}_1 \oplus \mathcal{D}_{A_1,T_1}, \quad \mathcal{D}_{A_1,T_1} = \overline{\mathcal{R}(C^*)}. \quad (3.8)$$

Using the equalities from (3.7), we get

$$T_2^*T_2 = T_2^*A_2T_2 = A_1 \oplus 0 \leq A_2. \quad (3.9)$$

By an induction argument, one can continue this construction to obtain two sequences $\{T_n\}_{n \geq 1}$ and $\{A_n\}_{n \geq 1}$, where $T_n, A_n \in \mathcal{B}(\mathcal{H}_n)$, satisfying for $n \geq 2$ the conditions

$$A_n = A_{n-1} \oplus I, \quad T_n^*T_n = T_n^*A_nT_n = A_{n-1} \oplus 0 \leq A_n, \quad (3.10)$$

on the space

$$\mathcal{H}_n = \mathcal{H}_{n-1} \oplus \mathcal{D}_{A_{n-1},T_{n-1}}. \quad (3.11)$$

In this context, T_n is called the *quasicontractive n -step lifting* of T with respect to A_n , for $n \geq 1$.

Next, we use these operators T_n and A_n to obtain an interesting quasi-isometric lifting for T , closely related to S_1 from (3.3).

Theorem 3.5. *Let $T, A \in \mathcal{B}(\mathcal{H})$ be such that $A \geq I$ and $A \geq T^*AT$. For each integer $n \geq 1$, let T_n be the quasicontractive n -step lifting of T with respect to A_n on the space $\mathcal{H}_n$, which satisfy the relations (3.7), (3.8), (3.10) and (3.11), where $A_1 = A \oplus C^*C$ with C from (3.3). If*

$$\mathcal{K} := \bigvee_{n \geq 0} \mathcal{H}_n = \overline{\bigcup_{n=0}^{\infty} \mathcal{H}_n}, \text{ where } \mathcal{H}_0 = \mathcal{H}, \quad (3.12)$$

then the sequence $\{T_n P_{\mathcal{H}_n}\}_{n \geq 1}$ strongly converges in $\mathcal{B}(\mathcal{K})$, and the operator S on $\mathcal{K}$ given by

$$Sk = \lim_{n \rightarrow \infty} T_n P_{\mathcal{H}_n} k, \quad k \in \mathcal{K}, \quad (3.13)$$

is a quasi-isometric lifting of T and T_n . Moreover, $S^*S\mathcal{H} = \mathcal{H}, \mathcal{K} \ominus \mathcal{H}_n \subset \mathcal{N}(S^*S - I)$ for $n \geq 1$ and S is a minimal lifting of T . Additionally, the minimal quasi-isometric lifting S of T , is isomorphic with the lifting S_1 of T , from (3.3).

Proof. Since $\mathcal{H}_n \subset \mathcal{H}_{n+1} \subset \dots \subset \mathcal{H}_{n+j}$ for $n \geq 0$ and $j \geq 1$, we have that T_{n+j} is a lifting for T_n , that is, $T_n = P_{\mathcal{H}_n} T_{n+j}|_{\mathcal{H}_n}$.

Let $k \in \bigcup_{n=0}^{\infty} \mathcal{H}_n$ and $n \geq 1$ be such that $k \in \mathcal{H}_n$. Then

$$\|T_n k\|^2 \leq \|T_{n+1} k\|^2 \leq \|T_{n+2} k\|^2 \leq \dots$$

On the other hand, by the relations (3.10) for every $m \geq 2$, we have

$$\|T_m\|^2 = \|T_m^* T_m\| \leq \|A_m\| = \|A_1\| \leq \|A\|, \quad (3.14)$$

because $A_m = A_1 \oplus I$ on $\mathcal{H}_m = \mathcal{H}_1 \oplus (\mathcal{H}_m \ominus \mathcal{H}_1)$. Here we also have in view that $A_1 \geq T_1^* T_1 = A \oplus 0$ (by (3.6) and (3.7)), and that $A \geq I$. Thus, we infer that

$$\|T_m k\|^2 \leq \|A\| \|k\|^2,$$

for k as above and every integer $m \geq n$.

Hence, the sequence $\{\|T_m k\|^2\}_{m \geq n}$ is increasing and bounded, so it is convergent. Since for such n, m we also have

$$\|T_m k - T_n k\|^2 = \|(I - P_{\mathcal{H}_n}) T_m k\|^2 = \|T_m k\|^2 - \|T_n k\|^2,$$

it follows that the sequence $\{T_m k\}_{m \geq n}$ converges in $\mathcal{K}$, for $k \in \mathcal{H}_n$. Having in view that $\{T_n P_{\mathcal{H}_n}\}_{n \geq 1}$ is bounded (by (3.14)), it readily follows that this sequence strongly converges in $\mathcal{B}(\mathcal{K})$. So the operator $S \in \mathcal{B}(\mathcal{K})$ can be defined as in (3.13).

Now it is clear that, for each integer $m \geq 1$, and $k \in \mathcal{K}$, one has

$$P_{\mathcal{H}_m} S k = \lim_{\substack{n \rightarrow \infty \\ (n > m)}} P_{\mathcal{H}_m} T_n P_{\mathcal{H}_n} k = T_m P_{\mathcal{H}_m} k,$$

this meaning that S is a lifting for T_m , and so a lifting of T .

On the other hand, because $T_n^{*2}T_n^2 \leq T_n^*T_n$ for $n \geq 1$, and as

$$\|T_n P_{\mathcal{H}_n} k\|_{n \rightarrow \infty}^2 \rightarrow \|Sk\|^2, \quad \|T_n^2 P_{\mathcal{H}_n} k\|_{n \rightarrow \infty}^2 \rightarrow \|S^2 k\|^2$$

for every $k \in \mathcal{K}$, it follows that $S^{*2}S^2 \leq S^*S$, i.e., S is a quasicontraction. In fact, S is even a quasi-isometry.

Indeed, we first assume by contrary that there exists a nonzero $k_0 \in \bigcup_{n=0}^{\infty} \mathcal{H}_n$ such that

$$\delta := \|Sk_0\|^2 - \|S^2 k_0\|^2 > 0.$$

Then there exists an integer $n_0 \geq 1$ such that $k_0 \in \mathcal{H}_{n_0}$. Having in view the limits quoted before, for $\varepsilon \in (0, \frac{\delta}{2})$ there exists $n > n_0$ such that

$$\|Sk_0\|^2 \leq \|T_{n-1}k_0\|^2 + \varepsilon, \quad \|S^2 k_0\|^2 \geq \|T_n^2 k_0\|^2 - \varepsilon.$$

But these, together with the fact that

$$T_n^{*2}T_n^2 = T_{n-1}^*T_{n-1} \oplus 0 \text{ on } \mathcal{H}_n = \mathcal{H}_{n-1} \oplus \mathcal{D}_{A_{n-1}, T_{n-1}},$$

lead to the relations

$$\delta = \|Sk_0\|^2 - \|S^2 k_0\|^2 \leq \|Sk_0\|^2 - \|T_n^2 k_0\|^2 + \varepsilon = \|Sk_0\|^2 - \|T_{n-1}k_0\|^2 + \varepsilon \leq 2\varepsilon.$$

Since this contradicts the choice of ε with $2\varepsilon < \delta$, we conclude that S is a quasi-isometry on $\bigcup_{n=0}^{\infty} \mathcal{H}_n$. As this subspace is dense in $\mathcal{K}$ (by (3.12)), it follows (by a continuity argument) that S is a quasi-isometry on $\mathcal{K}$.

Next, we analyze the action of S^*S on $\mathcal{H}$. We observe that

$$S^*Sh = S^* \lim_{n \rightarrow \infty} T_n h = \lim_{n \rightarrow \infty} T_n^* T_n h = Ah, \quad h \in \mathcal{H},$$

taking into account that S^* is an extension of T_n^* for $n \geq 1$. Here we also have in view the fact that $T_n^*T_n = A_{n-1} \oplus 0$ on $\mathcal{H}_{n-1} \oplus \mathcal{D}_{A_{n-1}, T_{n-1}}$, together with $A_{n-1} = A_1 \oplus I$ on $\mathcal{H}_1 \oplus (\mathcal{H}_{n-1} \ominus \mathcal{H}_1)$, as well as that $A_1|_{\mathcal{H}} = A$. Therefore, $S^*S\mathcal{H} = A\mathcal{H} = \mathcal{H}$.

To prove that S is a minimal lifting for T , we first remark that

$$D_{A_n, T_n}^2 = A_n - T_n^*T_n = \begin{pmatrix} A_{n-1} & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} A_{n-1} & 0 \\ 0 & 0 \end{pmatrix} = 0 \oplus I$$

on $\mathcal{H}_n = \mathcal{H}_{n-1} \oplus \mathcal{D}_{A_{n-1}, T_{n-1}}$, for all integers $n \geq 2$. This gives, for $n \geq 2$,

$$\mathcal{D}_{A_n, T_n} = \mathcal{D}_{A_{n-1}, T_{n-1}} = \cdots = \mathcal{D}_{A_1, T_1} = \overline{\mathcal{R}(C^*)} \subset \mathcal{D}_{A, T}.$$

Thus, it follows that T_n can be represented on $\mathcal{H}_n = \mathcal{H}_1 \oplus \bigoplus_1^{n-1} \mathcal{D}_{A_1, T_1}$, for $n \geq 2$, in the form

$$T_n = \begin{pmatrix} T_1 & 0 & 0 & \cdots & 0 & 0 \\ D_{A_1, T_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \end{pmatrix}. \quad (3.15)$$

This shows that $T_n|_{\mathcal{H}_j} = T_{n+m}|_{\mathcal{H}_j}$, for $n > j \geq 0$ and every $m \geq 1$, where $\mathcal{H}_0 = \mathcal{H}$. So, by (3.13), we have $T_n|_{\mathcal{H}_j} = S|_{\mathcal{H}_j}$, which later gives by (3.6) and (3.11) that

$$\mathcal{H}_n = \mathcal{H}_{n-1} \vee T_n \mathcal{H}_{n-1} = \mathcal{H}_{n-1} \vee S \mathcal{H}_{n-1}, \quad n \geq 1.$$

So, for $n \geq 1$, one obtains by recurrence that

$$\mathcal{H}_n = \bigvee_{j=0}^n S^j \mathcal{H}.$$

From this and (3.12) we infer that $\mathcal{K} = \bigvee_{j=0}^{\infty} S^j \mathcal{H}$, meaning that S is a minimal lifting for T (and also for all T_n , $n \geq 1$).

Next, remark (using (3.15)) that $S^*S = A \oplus C^*C \oplus I$ on $\mathcal{H} \oplus \mathcal{D}_{A, T} \oplus (\mathcal{K} \ominus \mathcal{H}_1)$, so $\mathcal{K} \ominus \mathcal{H}_1 \subset \mathcal{N}(S^*S - I)$. Hence, $\mathcal{K} \ominus \mathcal{H}_n \subset \mathcal{N}(S^*S - I)$ for $n \geq 1$.

Notice finally that S as a minimal quasi-isometric lifting of T on $\mathcal{K} = \bigvee_{n \geq 0} S^n \mathcal{H}$ has also the property $S^*Sh = Ah$ for $h \in \mathcal{H}$. So, we have

$$S^*S|_{\mathcal{H}} = A = S_1^*S_1|_{\mathcal{H}},$$

where S_1 is the lifting of T from (3.3). Hence, S and S_1 are isomorphic as minimal liftings for T (by Remark 2.3). $\square$

Remark 3.6. In the construction of the lifting S , the operator A_1 , has an essential role, by its diagonal structure on $\mathcal{H}_1 = \mathcal{H} \oplus \mathcal{D}_{A, T}$, which leads to the properties (3.7) and (3.9). We mention that by the above Theorem 2.1 (ii), or by [16, Theorem 2.3], for the quasicontraction T_1 there exist other operators $B \in \mathcal{B}(\mathcal{H}_1)$, not necessarily invertible, such that $T_1^*T_1 = T_1^*BT_1 \leq B$, but $\mathcal{H}$ is not reducing for B , in general. However, such an operator B can be used to obtain a quasi-isometric lifting for T (as in the previous proof). But such a lifting does not have the same properties as S above.

On the other hand, in [15, Theorem 2.5], we proved that for each quasicontraction Q (in particular for T_1) there exists even an invertible operator B satisfying $Q^*Q = Q^*BQ \leq B$.

Now we derive the following consequence from above results.

Corollary 3.7. *For $T, A \in \mathcal{B}(\mathcal{H})$ as in Theorem 3.3 the following statements hold:*

- (i) *If the lifting S_1 of T from (3.3) is left-invertible, then the corresponding operator A_1 for T_1 in (3.7) is invertible, therefore $\mathcal{D}_{A_1, T_1} = \mathcal{D}_{A, T} = \overline{\mathcal{R}(A - T^*T)}$.*
- (ii) *The operator A_1 has the form $A_1 = A \oplus I$ on $\mathcal{H} \oplus \mathcal{D}_{A, T}$ if and only if $T^*AT = T^*T$. In this case T is a quasicontraction.*

Proof. (i) The assumption on T ensures (by Corollary 3.4 of above) that the operator C from (3.3) is left-invertible. So C^*C and later $A_1 = A \oplus C^*C$ are invertible operators. As $0 \oplus C^*C = A_1 - T_1^*T_1$, we obtain that $\mathcal{D}_{A_1, T_1} = \mathcal{R}(C^*) = \overline{\mathcal{R}(A - T^*T)} = \mathcal{D}_{A, T}$.

(ii) Assume that $A_1 = A \oplus I$. Then the condition $T_1^*A_1T_1 = T_1^*T_1$ implies $T^*AT = T^*T$. Conversely, the latter ensures that the operator C in (3.3) is an isometry, hence $A_1 = A \oplus C^*C = A \oplus I$. In this case, the relations $T^*T = T^*AT \leq A$ obviously imply that T is a quasicontraction. $\square$

Remark 3.8. Note that the lifting S_1 from (3.3) is not left-invertible, in general. In fact, $\mathcal{R}(S_1)$ and $\mathcal{R}(C)$ are simultaneously closed and $\mathcal{N}(S_1) = \mathcal{N}(C)$. But we know from [15, Corollary 2.8], that, if the spectral radius of T is strictly less than 1, then there exists A invertible satisfying $T^*T \leq T^*AT \leq A$ such that $\mathcal{R}(A - T^*AT) = \mathcal{R}(A - T^*T)$ and this subspace is closed. In this case, the corresponding lifting S_1 of T (induced by T, A and C as in (3.3)) is left-invertible (by [15, Corollary 3.2]), as we assumed in Corollary 3.7.

Finally, we note that the expansivity of A is not essential for constructing quasi-isometric liftings (from Theorem 3.3 and Theorem 3.5), although it ensures additional properties of these liftings. The basic assumptions on T and A are simply

$$T^*T \leq T^*AT \leq A.$$

Under these assumptions, one can also construct quasicontractive n -step liftings of T with respect to A_n (for $n \geq 1$), whose strong operator limit yields a quasi-isometric lifting of T . We will present this construction in a subsequent paper.

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