

NORMALIZED SOLUTIONS OF A CRITICAL KIRCHHOFF EQUATION IN HIGH DIMENSIONAL SPACES

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Communicated by Vicențiu D. Rădulescu

Abstract. The following well-known critical Kirchhoff equation with prescribed mass,

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + \lambda u = \mu |u|^{q-2} u + |u|^{2^*-2} u & \text{in } \mathbb{R}^N, \ N \geq 4, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, \end{cases}$$

has been extensively studied, where a, b, c, μ , and q are constants with $a, c > 0$ and $2 < q < 2^*$, while λ is a real Lagrange multiplier. When $\mu = 0$, we obtain complete results on the existence, nonexistence, and multiplicity of normalized solutions. When $\mu > 0$ and $N \geq 5$, we investigate the existence of positive normalized solutions, partly filling a gap in the existing literature. Lastly, when $\mu > 0$ and $N = 4$, we obtain a mountain-pass solution by using a new threshold for the energy functional. The main results of this paper broaden and improve the existing results on normalized solutions of the Kirchhoff equation in high-dimensional spaces.

Keywords: Kirchhoff equation, variational methods, high dimensional spaces, Palais–Smale condition.

Mathematics Subject Classification: 35J60, 47J30, 35J20.

1. INTRODUCTION

As is well known, the Kirchhoff equation arises as a model of certain physical phenomena. For example, it is related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where u is the lateral displacement at position x and time t , E is Young's modulus, ρ is the mass density, h is the cross-sectional area, L is the length, and P_0 is the

initial axial tension. For more background, we refer readers to [1, 18]. Compared with the Schrödinger equation, the Kirchhoff equation contains a nonlocal term $(\int_{\mathbb{R}^N} |\nabla u|^2 dx) \Delta u$, so it is no longer a pointwise identity. Thus, the nonlocal nature of the problem causes mathematical difficulties and makes the search for solutions to the Kirchhoff equation more challenging and interesting. Since Lions [24] introduced an abstract functional-analytic framework for the equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(u) \quad \text{in } \mathbb{R}^+ \times \Omega,$$

the Kirchhoff equation has attracted considerable attention from mathematicians. For relevant results, see [8, 9].

The following special form of a Kirchhoff equation,

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + \lambda u = f(u) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

has been widely studied in the past several decades. Here a, b are two positive constants, λ denotes either a prescribed constant or an unknown parameter, the spatial dimension satisfies $N \geq 1$, and the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$. On the one hand, equation (1.1) with a fixed constant λ is referred to as a fixed-frequency problem. In this case, one can obtain solutions to equation (1.1) by seeking unconstrained critical points of the associated energy functional. A large body of work has been established in this setting; see [5, 12, 14, 27, 31, 32, 42] and the references therein.

On the other hand, motivated by some physical phenomena, considerable attention has been paid by physicists to solutions of equation (1.1) with a prescribed mass constraint $\int_{\mathbb{R}^N} |u|^2 dx = c$, which are commonly referred to as normalized solutions. In this case, λ serves as an unknown parameter and acts as the Lagrange multiplier corresponding to the mass constraint,

$$\bar{S}_c := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c \right\}$$

or

$$S_c := \left\{ u \in H_{rad}^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c \right\},$$

where c denotes a positive fixed constant. To obtain normalized solutions of equation (1.1), we seek critical points of the associated energy functional restricted to the constraint manifold $\bar{S}_c$. In what follows, we systematically review the existing literature concerning normalized solutions of the Kirchhoff equation.

Set $a = 1$, $b = 0$ and $f(u) = \mu|u|^{q-2}u + |u|^{p-2}u$. Then equation (1.1) simplifies to the standard nonlinear Schrödinger equation,

$$-\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad (1.2)$$

subject to the mass constraint $\int_{\mathbb{R}^N} |u|^2 dx = c$. A substantial body of literature has focused on normalized solutions to this mass-constrained Schrödinger equation. For example, for $N \geq 3$, Soave [28] pioneered the analysis of equation (1.2) under the exponent ranges $2 < q < 2 + \frac{4}{N}$ and $2 < q \leq 2 + \frac{4}{N} \leq p < 2^*$. Here $2 + \frac{4}{N}$ denotes the L^2 -critical exponent for the Schrödinger equation, which is strictly less than the Sobolev critical exponent 2^* . In that work, Soave established the existence and asymptotic behavior of normalized ground states, along with their orbital stability. In the Sobolev critical case, Soave [29] subsequently constructed normalized ground-state solutions and characterized their asymptotic profiles for $2 < q < 2 + \frac{4}{N}$, $q = 2 + \frac{4}{N}$, and $2 + \frac{4}{N} < q < 2^*$, respectively. Later, Jeanjean and Le [17] resolved an open problem posed in Soave [29] and proved the existence of a second distinct normalized solution to (1.2) in dimension $N \geq 4$ with $2 < q < 2 + \frac{4}{N}$. Moreover, Wei and Wu [30] extended these results and obtained a second normalized solution to equation (1.2) with $2 < q < 2 + \frac{4}{3}$ and $N = 3$.

For $a, b > 0$ and the power-type nonlinearity $f(u) = |u|^{q-2}u$, equation (1.1) reduces to the following Kirchhoff equation,

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + \lambda u = |u|^{q-2}u \quad \text{in } \mathbb{R}^N, \quad (1.3)$$

subject to the mass constraint $\int_{\mathbb{R}^N} |u|^2 dx = c$. The equation (1.3) has been extensively investigated in the past several decades. In particular, Ye [37] pointed out that $2 + \frac{8}{N}$ is the L^2 -critical exponent for the Kirchhoff equation, which plays a dominant role in the study of normalized solutions to equation (1.3). This exponent determines whether the constrained functional I remains bounded from below on $\tilde{S}_c$. Precisely by employing scaling techniques and the concentration-compactness principle, Ye [37–39] proved the existence, nonexistence and concentration behavior of normalized solutions in three subcases $2 < q < 2 + \frac{8}{N}$, $q = 2 + \frac{8}{N}$ and $2 + \frac{8}{N} < q < 2^*$. Furthermore, Zeng and Zhang [40] also established the existence and uniqueness of normalized solutions in this case.

If $f(u) = \mu|u|^{q-2}u + |u|^{p-2}u$, equation (1.1) reduces to the following Kirchhoff equation,

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \quad (1.4)$$

subject to the mass constraint $\int_{\mathbb{R}^N} |u|^2 dx = c$. For $N = 3$ and $\mu > 0$, Li *et al.* [21] investigated the existence and asymptotic behavior of normalized solutions to the above equation. For $2 < q < \frac{10}{3}$ and $\frac{14}{3} < p < 6$, the authors established the existence of a local-minimum solution and a mountain-pass solution via Ekeland's variational principle and the construction of a specialized Palais–Smale sequence. Furthermore, in the Sobolev critical case with $2 < q < \frac{10}{3}$, their work also verified the existence of a local-minimum solution for (1.4). Nevertheless, the associated energy functional has a convex-concave structure in this setting, which suggests the potential existence of a second solution and motivates further exploration. To fill this gap, Feng *et al.* [10] and

Chen *et al.* [7] established the existence of a second mountain-pass solution through distinct analytical strategies. In addition, Hu and Mao [15] derived the existence and nonexistence of normalized solutions by using constrained minimization and the concentration-compactness principle, thereby partially extending the pioneering results in [21] and [28]. The case of the negative coupling parameter $\mu < 0$ was addressed by Carrião *et al.* [4]. For further results, readers may refer to [35, 36].

It is well known that the L^2 -critical exponent $2 + \frac{8}{N}$ and the Sobolev critical exponent $2^* = \frac{2N}{N-2}$ play crucial roles in the qualitative analysis of the Kirchhoff equation. Specifically, the L^2 -critical exponent is a sharp threshold for the boundedness from below of the functional on $\bar{S}_c$, whereas the Sobolev critical exponent governs the well-posedness of the associated variational framework. Notably, $2 + \frac{8}{N} < 2^*$ for $N \leq 3$, and nearly all of the aforementioned literature focuses on the low-dimensional case $N \leq 3$. Compared with the case $N \leq 3$, few results are available on normalized solutions of equation (1.1) for $N \geq 4$. When $1 \leq N \leq 4$ and $f(u) = |u|^{p-2}u$ with $2 < p < 2^*$, Guo *et al.* [13] showed the existence and blow-up behavior of normalized solutions to equation (1.1) with a trapping potential or a constant vanishing potential. When $2^* = 4$ for $N = 4$ and $f(u) = |u|^{p-2}u$ with $2 < p \leq 4$, Li *et al.* [22] proved the existence and nonexistence of normalized minimizers and revealed subtle differences among the potential cases.

This paper is concerned with the following mass-prescribed Kirchhoff equation involving a Sobolev critical exponent in high dimensions:

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + \lambda u = \mu |u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \quad N \geq 4, \quad (1.5)$$

subject to the mass constraint $\int_{\mathbb{R}^N} |u|^2 dx = c$. Here a, c are fixed positive constants, b, μ denote real parameters, λ acts as the Lagrange multiplier and $2 < q < 2^*$. Variationally speaking, normalized solutions to equation (1.5) correspond to critical points of the energy functional $I_{b,\mu} : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$I_{b,\mu}(u) = \frac{a}{2} |\nabla u|_2^2 + \frac{b}{4} |\nabla u|_2^4 - \frac{\mu}{q} |u|_q^q - \frac{1}{2^*} |u|_{2^*}^{2^*}, \quad (1.6)$$

where the functional is restricted to the manifold $\bar{S}_c$ or S_c .

We first focus on equation (1.5) in dimension $N = 4$. A striking feature is the identity $2 + \frac{8}{N} = \frac{2N}{N-2} = 4$, which corresponds to the so-called mass-energy doubly critical phenomenon: the exponent simultaneously coincides with the mass critical exponent and the Sobolev critical exponent. This double criticality introduces substantial new obstacles in the analysis of (1.5). In particular, the competition between the nonlocal Kirchhoff term and the Sobolev critical nonlinearity in the functional (1.6) greatly complicates the analysis of the functional's geometric structure and the boundedness argument for Palais-Smale sequences. Moreover, once $q > 2 + \frac{8}{N} = 4$, equation (1.5) loses its well-defined variational structure. For this reason, the exponent $2 + \frac{4}{N} = 3$, rather than 4, serves as the key critical threshold for our problem. Recall the sharp

Sobolev constant S introduced in (2.1); it yields the fundamental relation linking the nonlocal term and the critical term in $I_{b,\mu}$:

$$S^{-2}|\nabla u|_2^4 \geq |u|_4^4.$$

This inequality inspires us to characterize the geometric properties of (1.6) by drawing a comparison between the parameter b and the constant S^{-2} . For $0 < b < S^{-2}$ and $\mu > 0$, it follows from (1.6), (2.1) and (2.3) that for all $u \in \tilde{S}_c$,

$$I_{b,\mu}(u) \geq \frac{a}{2}|\nabla u|_2^2 - \frac{1-bS^2}{4S^2}|\nabla u|_2^4 - \frac{\mu}{q}C_q^q c^{\frac{4-q}{2}}|\nabla u|_2^{2q-4}.$$

When $2 < q < 3$, $I_{b,\mu}$ exhibits a convex-concave geometry, which heuristically indicates that (1.5) may admit two distinct normalized solutions: a local minimizer and a mountain-pass solution. Recently, Kong and Chen [19, 20] addressed this case and established the existence of the first solution under appropriate assumptions on μ and c , whereas the existence of the second solution was left unresolved. Later, Fang *et al.* [11] derived a mountain-pass solution for the case $q = 3$ with sufficiently small μ via delicate energy estimates. Furthermore, Zhang *et al.* [41] studied equation (1.5) with $\mu = 1$ and $b > S^{-2}$ and systematically analyzed the existence and nonexistence of normalized solutions by dividing the discussion into three separate regimes: $2 < q < 3$, $q = 3$ and $3 < q < 4$.

It is worth mentioning that little work has addressed normalized solutions to equation (1.5) for $N \geq 5$. This is primarily because $2^* < 4$, which gives the nonlocal term a stronger effect on the energy functional than the Sobolev critical term. Consequently, some compactness arguments developed for $N \leq 4$ are not applicable to $N \geq 5$. For $N \geq 5$, Zhang *et al.* [41] established the existence and nonexistence of normalized solutions to equation (1.5) with $\mu = 1$. More specifically, their Theorem 1.5 guarantees the existence of a normalized solution for $q \in (2, 2^*)$, provided that the parameters a , b , and S satisfy the following key condition:

$$\left(\frac{2a}{4-2^*}\right)^{\frac{4-2^*}{2}} \left(\frac{2b}{2^*-2}\right)^{\frac{2^*-2}{2}} > \frac{1}{S^{\frac{2^*}{2}}}. \quad (b^*)$$

In fact, condition (b^*) is equivalent to

$$b > b_0 := \frac{2}{N-2} \left(\frac{N-4}{a(N-2)}\right)^{\frac{N-4}{2}} S^{-\frac{N}{2}}. \quad (1.7)$$

The critical threshold b_0 , first proposed in Xie and Zhou [33], plays a significant role in structure of the energy functional and the limit equation. Based on the aforementioned literature, an intriguing question arises: for $N \geq 5$, $\mu > 0$, $q \in (2, 2^*)$ and $b < b_0$, does equation (1.5) admit a normalized solution? Resolving this open question constitutes one of the primary objectives of the present work. Drawing on the analytical frameworks developed in [10, 21, 26, 29, 34], we further establish partial theoretical results for the aforementioned open problem. To begin our analysis, we first establish the existence,

nonexistence and multiplicity of normalized solutions to equation (1.5) in the case $\mu = 0$. The second major goal of this work is to explore the open problem proposed in Kong and Chen [19] with $N = 4$: whether a second mountain-pass normalized solution exists for $2 < q < 3$, $\mu > 0$ and $0 < b < S^{-2}$. Kong *et al.* [20] partly solved this problem for b small enough (with the order ϵ). For this open problem, we provide a positive answer and establish the existence of two distinct normalized solutions: one is a local minimizer, and the other is a mountain-pass solution for $0 < b < S^{-1}$. It is worth pointing out that we adopt an approach different from theirs to obtain the existence of the local minimizer. Additionally, we derive nonexistence criteria for normalized solutions in the case $\mu < 0$.

2. MAIN RESULTS

Before stating the main theorems, we first introduce some notations and preliminary results. For $N \geq 3$, the best Sobolev embedding constant S is defined by

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{|\nabla u|_2^2}{|u|_{2^*}^2}. \quad (2.1)$$

This constant S is achieved by the following function, for any $\varepsilon > 0$ and $y \in \mathbb{R}^N$,

$$U_{\varepsilon,y}(x) = \left(\frac{\sqrt{N(N-2)}\varepsilon}{\varepsilon + |x-y|^2} \right)^{\frac{N-2}{2}}, \quad x \in \mathbb{R}^N, \quad (2.2)$$

which is the unique positive solution of the following equation,

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \quad (S^*)$$

Moreover, $S^{N/2} = |U|_{2^*}^2 = |\nabla U|_2^2$. We introduce the Pohozaev manifold

$$\mathcal{P}_{c,b,\mu} := \{u \in S_c : P_{b,\mu}(u) = 0\},$$

where

$$P_{b,\mu}(u) := a|\nabla u|_2^2 + b|\nabla u|_2^4 - \mu\delta_q|u|_q^q - |u|_{2^*}^{2^*} \quad \text{with } \delta_q = \frac{N(q-2)}{2q}.$$

By the Pohozaev identity, every critical point of the functional $I_{b,\mu}|_{S_c}$ belongs to $\mathcal{P}_{c,b,\mu}$. Thus, if $\inf_{u \in \mathcal{P}_{c,b,\mu}} I_{b,\mu}(u)$ is attained by a normalized solution of equation (1.5), then that solution is a radial ground-state solution of equation (1.5) on S_c . For any $u \in S_c$ and $s \in \mathbb{R}$, the function $s * u$ is defined by

$$(s * u)(x) := e^{\frac{N}{2}s} u(e^s x) \quad \text{for } x \in \mathbb{R}^N,$$

then $s * u \in S_c$. This map $(s, u) \in \mathbb{R} \times H^1(\mathbb{R}^N) \mapsto (s * u) \in H^1(\mathbb{R}^N)$ is continuous by Lemma 3.5 in [2]. Moreover, we define the fiber map

$$\Psi_u^{b,\mu}(s) := I_{b,\mu}(s * u) = \frac{ae^{2s}}{2}|\nabla u|_2^2 + \frac{be^{4s}}{4}|\nabla u|_2^4 - \mu \frac{e^{q\delta_q s}}{q}|u|_q^q - \frac{e^{2^* s}}{2^*}|u|_{2^*}^{2^*}.$$

It follows from some direct calculations that

$$(\Psi_u^{b,\mu})'(s) = ae^{2s}|\nabla u|_2^2 + be^{4s}|\nabla u|_2^4 - \mu\delta_q e^{q\delta_q s}|u|_q^q - e^{2^*s}|u|_{2^*}^{2^*} = P_{b,\mu}(s * u),$$

and the Pohozaev manifold can be rewritten as

$$\mathcal{P}_{c,b,\mu} = \{u \in S_c : (\Psi_u^{b,\mu})'(0) = 0\}.$$

Obviously, s is a critical point of $\Psi_u^{b,\mu}(s)$ if and only if $s*u \in \mathcal{P}_{c,b,\mu}$. $u \in \mathcal{P}_{c,b,\mu}$ if and only if 0 is a critical point of $\Psi_u^{b,\mu}(s)$. To determine the locations and types of critical points of $I_{b,\mu}|_{S_c}$, we decompose $\mathcal{P}_{c,b,\mu}$ into the disjoint union $\mathcal{P}_{c,b,\mu} = \mathcal{P}_{c,b,\mu}^+ \cup \mathcal{P}_{c,b,\mu}^0 \cup \mathcal{P}_{c,b,\mu}^-$, where

$$\begin{aligned}\mathcal{P}_{c,b,\mu}^+ &:= \{u \in \mathcal{P}_{c,b,\mu} : (\Psi_u^{b,\mu})''(0) > 0\}, \\ \mathcal{P}_{c,b,\mu}^- &:= \{u \in \mathcal{P}_{c,b,\mu} : (\Psi_u^{b,\mu})''(0) < 0\}, \\ \mathcal{P}_{c,b,\mu}^0 &:= \{u \in \mathcal{P}_{c,b,\mu} : (\Psi_u^{b,\mu})''(0) = 0\},\end{aligned}$$

for

$$(\Psi_u^{b,\mu})''(0) = 2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 - \mu q\delta_q^2|u|_q^q - 2^*|u|_{2^*}^{2^*}.$$

We recall the Gagliardo–Nirenberg inequality that exists an optimal constant $C_{N,q}$ depending on N and q such that

$$|u|_q \leq C_{N,q}|\nabla u|_2^{\delta_q}|u|_2^{1-\delta_q}, \quad u \in H^1(\mathbb{R}^N), \quad (2.3)$$

where $C_q := C_{N,q}$ simplifies the notation.

Next, we simplify some notations. When $\mu = 0$, we will replace $I_{b,\mu}$, $P_{b,\mu}$, $\Psi_u^{b,\mu}$, $\mathcal{P}_{c,b,\mu}$, $\mathcal{P}_{c,b,\mu}^\pm$ and $\mathcal{P}_{c,b,\mu}^0$ by I_0 , P_0 , Ψ_u^0 , $\mathcal{P}_0$, $\mathcal{P}_0^\pm$ and $\mathcal{P}_0^0$, respectively. When $\mu \neq 0$, we will replace $I_{b,\mu}$, $P_{b,\mu}$, $\Psi_u^{b,\mu}$, $\mathcal{P}_{c,b,\mu}$, $\mathcal{P}_{c,b,\mu}^\pm$ and $\mathcal{P}_{c,b,\mu}^0$ by I , P , Ψ_u , $\mathcal{P}$, $\mathcal{P}^\pm$ and $\mathcal{P}^0$, respectively. In the case $N \geq 5$, before addressing the perturbation case, it is necessary to study the existence, nonexistence and multiplicity of normalized solutions to equation (1.5) with $\mu = 0$.

Theorem 2.1. *Assume that $N \geq 5$ and $\mu = 0$, equation (1.5) has the following results.*

- (i) *If $0 < b < b_0$, equation (1.5) exists two solutions satisfying $\varphi_\pm \in \mathcal{P}_0^\pm$ such that*

$$\inf_{u \in \mathcal{P}_0^-} I_0(u) = I_0(\varphi_-) = c_{N,-} > 0, \quad \inf_{u \in \mathcal{P}_0^+} I_0(u) = I_0(\varphi_+) = c_{N,+}, \quad (2.4)$$

and $c_{N,-} > c_{N,+}$. Moreover, $c_{N,+} \leq 0$ for $b \in (0, b_1]$ and $c_{N,+} > 0$ for $b \in (b_1, b_0)$, where b_0 is defined in (1.7) and b_1 is defined by

$$b_1 := \frac{4}{N} \left(\frac{N-4}{aN} \right)^{\frac{N-4}{2}} S^{-\frac{N}{2}}.$$

In particular, φ_- is a mountain-pass solution in S_c and

$$\inf_{u \in S_c} I_0(u) = \begin{cases} I_0(\varphi_+), & \text{if } b \in (0, b_1], \\ 0, & \text{if } b \in (b_1, b_0). \end{cases}$$

(ii) If $b < 0$, there exists $\varphi_1 \in \mathcal{P}_0$ such that

$$\inf_{u \in \mathcal{P}_0} I_0(u) = \inf_{u \in \mathcal{P}_0^-} I_0(u) = I_0(\varphi_1) = c_N > 0. \quad (2.5)$$

In particular, φ_1 is a mountain-pass solution in S_c .

- (iii) If $b > b_0$, then $I_0|_{S_c}$ has no critical points. Consequently, equation (1.5) has no solutions in S_c . If $b = b_0$, then equation (1.5) exists a solution satisfying $\varphi_1 \in \mathcal{P}_0$.
- (iv) If the equation (S^*) has n radial solutions in $L^2(\mathbb{R}^N)$ that are pairwise distinct $D^{1,2}(\mathbb{R}^N)$ -norms, $n \in \mathbb{N}$, then there exists $b_n > 0$ with $b_n \leq b_0$ such that equation (1.5), with $b \in (0, b_n)$, has $2n$ solutions $\varphi_{i,\pm} \in \mathcal{P}_0^\pm$, $i \in \{1, 2, \dots, n\}$, satisfying

$$I_0(\varphi_{i,-}) > I_0(\varphi_{i,+}),$$

$$I_0(\varphi_{1,\pm}) < I_0(\varphi_{2,\pm}) < \dots < I_0(\varphi_{n,\pm}) < \frac{a^2}{N(N-4)b},$$

where $\varphi_\pm = \varphi_{1,\pm}$.

Remark 2.2. More details about Theorem 2.1 are as following.

(i) By virtue of norm-preserving scaling, Theorem 2.1 generalizes the results of Xie and Zhou [33] to normalized solutions $u \in S_c$, whereas the original work investigates general solutions $u \in D^{1,2}(\mathbb{R}^N)$. The existence of normalized solutions for equation (1.5) with $b = 0$ has been extensively studied. Compared with the results in Soave [29], the investigation of equation (1.5) with $b > 0$ is more challenging due to the extra contribution of the nonlocal term $(\int_{\mathbb{R}^N} |\nabla u|^2 dx) \Delta u$. Specifically, the energy functional $I(u)$ comprises four distinct terms, whose interactions substantially shape the functional structure and render the classification of critical points on S_c more intricate. Furthermore, the nonlocal term complicates the compactness recovery of Palais–Smale sequences, since the weak convergence $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ fails to ensure strong convergence

$$|\nabla u_n|_2^2 \int_{\mathbb{R}^N} \nabla u_n \nabla \varphi dx \rightarrow |\nabla u|_2^2 \int_{\mathbb{R}^N} \nabla u \nabla \varphi dx \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N).$$

(ii) In the proof of Theorem 2.1, we restrict the functional I_0 on the Pohozaev manifold $\mathcal{P}_0$ and establish that $I_0|_{\mathcal{P}_0}$ is bounded from below. The structural behavior of I_0 depends sensitively on the parameter b . Specifically, I_0 admits a concave-convex structure for $0 < b < b_0$. By decomposing the Pohozaev manifold and adopting a fiber mapping approach, we acquire two distinct normalized solutions to equation (1.5). Further, by decomposing the constraint sphere S_c and constructing a minimax characterization, we prove that one solution is a global minimizer and the other is a mountain-pass solution. Analogous arguments yield a mountain-pass solution for the case $b < 0$, while nonexistence holds for all $b > b_0$. While the existence of two normalized solutions was previously derived in Qi and Zou [26], our work not only recovers such existence results but also further investigates qualitative properties of these solutions.

Now we consider the perturbation case $\mu > 0$.

Theorem 2.3. Assume that $N \geq 5$, $b_1 < b < b_0$, $\mu > 0$ and $2 < q < 2 + \frac{4}{N}$.

- (i) There exists $\mu_2 > 0$ such that for all $\mu, c > 0$ satisfying $\mu c^{\frac{q(1-\delta_q)}{2}} < \mu_2$, equation (1.5) possesses a positive radial normalized solution with negative energy level, i.e., $m(b, \mu) < 0$.
- (ii) If $u_{b,\mu}$ is a solution of equation (1.5) obtained by (i), then

$$m(b, \mu) \rightarrow 0 \quad \text{and} \quad |\nabla u_{b,\mu}|_2^2 \rightarrow 0 \quad \text{as} \quad \mu \rightarrow 0^+.$$

Remark 2.4. There are two explanations about Theorem 2.3.

(i) Theorem 2.3 extends the results in [10, 19], where the authors investigated equation (1.5) for spatial dimensions $N = 3, 4$. Unlike the low-dimensional cases $N = 3, 4$, the nonlocal term $|\nabla u|_2^4$ always dominates the four components of the energy functional $I(u)$ when $N \geq 5$, which renders the problem more challenging and nontrivial. For $2 < q < 2 + \frac{4}{N}$, the compactness recovery of Palais–Smale sequences varies significantly with the dimension. Specifically, Feng *et al.* [10] exploited the continuity and monotonicity of energy levels for $N = 3$, while Kong and Chen [19] imposed extra constraints on the parameters μ and c for $N = 4$. Nevertheless, such strategies fail to work for $N \geq 5$, thereby necessitating a new approach to overcome this obstacle.

(ii) One of the most crucial steps in processing with the critical Sobolev problem is overcoming the lack of compactness. The case $b < b_0$ is totally different from the case $b > b_0$ in recovering the compactness results. According to Theorem 1.5 in Zhang *et al.* [41], compactness can be recovered under the essential condition $b > b_0$. In fact, the limit equation (1.5) with $\mu = 0$ admits no normalized solutions, as stated in Theorem 2.1 (iii). This property serves as an intrinsic ingredient for compactness recovery. More precisely, for $b > b_0$, the function $h(t)$ is strictly monotonically increasing for any $t > 0$, where

$$h(t) := \frac{a}{2}t^2 + \frac{b}{4}t^4 - \frac{S^{-\frac{2^*}{2}}}{2^*}t^{2^*}. \quad (2.6)$$

A direct calculation shows that, for any $u \in S_c$,

$$I(u) = \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{1}{2^*}|u|_{2^*}^{2^*} - \frac{1}{q}|u|_q^q \geq h(|\nabla u|_2) - \frac{1}{q}C_q^q c^{\frac{q(1-\delta_q)}{2}}|\nabla u|_2^{q\delta_q}.$$

Since $2 < q < 2 + \frac{4}{N}$, the constrained functional $I|_{S_c}$ admits a convex structure. Combining the minimization argument with Ekeland's variational principle, we obtain a Palais–Smale sequence $\{u_n\}$ for $I|_{S_c}$ satisfying $P(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Ultimately, the validity of the Palais–Smale condition essentially follows from the nonexistence of normalized solutions to the limit equation. In contrast, $b < b_0$ is much more intricate. The main difficulty arises not only from verifying the Palais–Smale condition, but also from the complex convex-concave-convex structure of the constrained functional. In this paper, we establish the existence of a normalized solution for $b_1 < b < b_0$. It is an interesting question that whether equation (1.5) possesses more normalized solutions for $0 < b < b_0$.

Theorem 2.5. Assume that $N \geq 5$, $0 < b < b_0$, $\mu < 0$ and $2 < q < 2^*$.

- (i) If u is a critical point for $I|_{S_c}$ (not necessarily positive, or even real-valued), then the associated Lagrange multiplier $\lambda < 0$ and $I(u) > c_{N,+}$.
- (ii) The equation

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + \lambda u = \mu |u|^{q-2} u + |u|^{2^*-2} u, \quad u > 0 \quad \text{in } \mathbb{R}^N \quad (2.7)$$

has no solution $u \in H^1(\mathbb{R}^N) \cap u \in L^p(\mathbb{R}^N)$ for $\lambda < 0$, where $p \in (0, \frac{N}{N-2}]$.

Remark 2.6. The proof of Theorem 2.5 is inspired by Theorem 1.2 in [29], which considers the Schrödinger equation with $\mu < 0$.

In the case $N = 4$, we first give some results of equation (1.5) with $\mu = 0$.

Theorem 2.7. Assume that $N = 4$ and $\mu = 0$. Then

$$\inf_{u \in S_c} I_0(u) = \begin{cases} 0, & \text{if } b \geq S^{-2}, \\ -\infty, & \text{if } 0 < b < S^{-2}, \end{cases}$$

where $\inf_{u \in S_c} I_0(u)$ is not attained for $b \geq S^{-2}$. Moreover, for $0 < b < S^{-2}$, we have

$$\inf_{u \in \mathcal{P}_0} I_0(u) = \inf_{u \in \mathcal{P}_0^-} I_0(u) = \Lambda = \frac{a^2 S^2}{4(1 - bS^2)},$$

which cannot be attained.

Before stating the results on $N = 4$ of the perturbation case $\mu > 0$, we define some constants as follows:

$$k_0 := \frac{2aS^2(3-q)}{(1-bS^2)(4-q)}, \quad (2.8)$$

$$c_0 := \left[\frac{2aS^2(3-q)}{(1-bS^2)(4-q)} \right]^2 \left[\frac{q(1-bS^2)}{4\mu C_q^q S^2(3-q)} \right]^{\frac{2}{4-q}}, \quad (2.9)$$

and

$$c_1 := \left[\frac{aqk_0^{3-q}}{2\mu C_q^q(4-q)(1-bS^2)} \right]^{\frac{2}{4-q}}. \quad (2.10)$$

Besides, the set $A_{k_0}(c)$ is defined by

$$A_{k_0}(c) := \{u \in S_c : |\nabla u|_2^2 < k_0\}, \quad (2.11)$$

and the auxiliary functional $J : H^1(\mathbb{R}^4) \rightarrow \mathbb{R}$ is defined by

$$J(u) := \frac{a}{2(1-bS^2)} |\nabla u|_2^2 + \frac{b}{4(1-bS^2)} |\nabla u|_2^4 - \frac{\mu}{q} |u|_q^q - \frac{1}{4} |u|_4^4. \quad (2.12)$$

Theorem 2.8. Assume that $N = 4$, $0 < b < S^{-2}$, $\mu > 0$, $2 < q < 3$ and $c \in (0, c_0)$.

- (i) Equation (1.5) has a positive local-minimizer solution $\tilde{u}$ in S_c for some $\tilde{\lambda}_c > 0$. Moreover, $\tilde{u}$ satisfies

$$\tilde{u} \in A_{k_0}(c), \quad I(\tilde{u}) < 0.$$

- (ii) The auxiliary functional $J(u)$ has a positive local minimizer $\bar{u}$ in S_c for some $\bar{\lambda}_c > 0$. Moreover, $\bar{u}$ satisfies

$$\bar{u} \in A_{k_0}(c), \quad J(\bar{u}) < 0.$$

Remark 2.9. Theorem 2.8(i) is similar to Theorem 1.1 in [19]; however, the proof in [19] relies on a decomposition of the Pohozaev manifold, whereas our method does not. In addition, $J(u) \geq I(u)$ for every $u \in S_c$ when $0 < b < S^{-2}$. Thus, an argument similar to the proof of Theorem 2.8(i) yields Theorem 2.8(ii), which plays an important role in the proof of Theorem 2.10. In particular, for $b > S^{-2}$ and $2 < q < 3$, the functional I has a convex structure and is therefore coercive. Using a minimization argument, the authors of [41] proved that equation (1.5) has a global minimizer for every $c > 0$.

Theorem 2.10. Assume that $N = 4$, $0 < b < S^{-2}$, $\mu > 0$, $2 < q < 3$, and $c \in (0, \min\{c_0, c_1\})$. Then equation (1.5) has a mountain-pass normalized solution $\hat{u}$ in S_c for some $\hat{\lambda}_c > 0$. Moreover, $\hat{u}$ satisfies

$$0 < I(\hat{u}) < J(\bar{u}) + \Lambda.$$

Remark 2.11. When $b = 0$, the authors in [6, 17, 30] built different test functions to obtain the unified mountain-pass bound

$$M(c) < m(c) + \frac{S^2}{4},$$

an essential ingredient for regaining compactness of Palais–Smale sequences in proving mountain-pass solutions with $2 < q < 3$, where $M(c)$ and $m(c)$ stand for the mountain-pass and ground-state energies, respectively. Since small perturbations preserve the main structural features of the functional, we construct a similar mountain-pass threshold for $0 < b < S^{-2}$ to verify Theorem 2.10.

Remark 2.12. For $N = 3$ and $2 < q < \frac{10}{3}$, the authors [10] and [7] derived distinct mountain-pass level thresholds to guarantee the existence of mountain-pass solutions. The former constructed a test function constrained to S_c via superposing the ground-state solution to (1.5) with $b = 0$ and Aubin–Talenti bubbles. Through elaborate computations, they established the strict bound

$$M(c) < m(c) + \frac{(aS)^{\frac{3}{2}}}{3}.$$

In contrast, the latter introduced an auxiliary functional and verified the existence of its local minimizer. Combining this minimizer with Aubin–Talenti bubbles yielded an alternative test functional, which gave rise to the strict inequality

$$M(c) < \bar{m}(c) + \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{\frac{3}{2}}}{24},$$

where $\bar{m}(c)$ denotes the energy associated with this local minimizer. Motivated by the arguments in [7], we adopt a similar strategy to prove Theorem 2.3: we first define an auxiliary functional $J(u)$ as in (2.12) and establish the existence of its local minimizer. We then construct a trial function by superposing this minimizer with Aubin–Talenti bubbles, which finally delivers the following mountain-pass level threshold valid for $N = 4$:

$$M(c) < \bar{m}(c) + \frac{a^2 S^2}{4(1 - bS^2)}.$$

In particular, setting $a = 1$ and $b = 0$ recovers the mountain-pass threshold for the classical Schrödinger equation stated in Remark 2.11. Kong et al. [20] established local minimum and mountain-pass normalized solutions for the $N = 4$ doubly critical Kirchhoff problem via Pohozaev manifold decomposition for b small enough (with the order ϵ). Our work develops an alternative auxiliary functional $J(u)$ argument independent of manifold splitting, and we derive an explicit closed-form mountain-pass energy threshold $\Lambda = \frac{a^2 S^2}{4(1 - bS^2)}$ not present in their work for $0 < b < S^{-1}$.

This paper is organized as follows. In Section 3, we study the pure critical Kirchhoff equation, namely, the case $\mu = 0$ with $N \geq 5$ or $N = 4$, and prove Theorems 2.1, 2.5, and 2.7. In Section 4, we establish a compactness result for Palais–Smale sequences of $I(u)$ and prove Theorem 2.3. Finally, we prove Theorems 2.8 and 2.10 in Section 5. We also introduce notation used throughout the paper.

- $H_{rad}^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u(x) = u(|x|)\}$.
- $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $\|u\|_{D^{1,2}} = |\nabla u|_2$.
- $|\cdot|_q$ is the standard norm in $L^q(\mathbb{R}^N)$ for $q \geq 1$.
- $o(1)$ means a quantity that tends to 0.
- $\rightarrow$ and $\rightharpoonup$ denote the strong convergence and weak convergence, respectively.
- For any $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$.
- C, M are positive constants possibly different in different places.

3. THE PROOF OF THEOREMS 2.1, 2.5 AND THEOREM 2.7

In this section, we mainly prove the results of pure critical Kirchhoff equation, i.e., the case $\mu = 0$, which will be used in the rest of the paper.

The proof of Theorem 2.1. For $N \geq 5$, we assume that

$$E_0(u) = \frac{1}{2}|\nabla u|_2^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}$$

and $\mathcal{P}_c := \{u \in S_c : |\nabla u|_2^2 = |u|_{2^*}^{2^*}\}$. By Proposition 2.2 in [29], we know that $\inf_{u \in \mathcal{P}_c} E_0(u) = \frac{1}{N}S^{\frac{N}{2}}$ is achieved by $U_{\varepsilon_c, 0}$, which is defined in (2.2) for the unique choice of $\varepsilon_c > 0$ such that $|U_{\varepsilon_c, 0}|_2 = c$.

(i) For $0 < b < b_0$, we denote the two roots of $g(t) = 0$ by $t_{\pm}$, $0 < t_- < t_+$, where

$$g(t) := bS^{\frac{N}{2}}t^2 - t^{\frac{4}{N-2}} + a. \quad (3.1)$$

Moreover, $g'(t_-) < 0$ and $g'(t_+) > 0$. Let $s_{\pm}$ satisfy $e^{s_{\pm}} = t_{\pm}$ and

$$\varphi_{\pm}(x) := (s_{\pm} * U_{\varepsilon_c, 0})(x) = e^{\frac{N}{2}s_{\pm}} U_{\varepsilon_c, 0}(e^{s_{\pm}} x) \in S_c.$$

By the above setting and the fact that $\pm g'(t_{\pm}) > 0$, we can easily check that $\varphi_{\pm} \in \mathcal{P}_0^{\pm}$. Thanks to the assumption $0 < b < b_0$, we define the roots of equation $f(t) = 0$ by $\xi_{\pm}$ ($\xi_- < \xi_+$), where

$$f(t) := bt^2 - S^{-\frac{N}{N-2}} t^{\frac{4}{N-2}} + a. \quad (3.2)$$

Moreover, $f'(\xi_-) < 0$ and $f'(\xi_+) > 0$ satisfying

$$\xi_{\pm} = S^{\frac{N}{4}} t_{\pm} = |\nabla \varphi_{\pm}|_2. \quad (3.3)$$

For any $u \in \mathcal{P}_0$, by (2.1), we have

$$a|\nabla u|_2^2 + b|\nabla u|_2^4 = |u|_{2^*}^{2^*} \leq S^{-\frac{2^*}{2}} |\nabla u|_2^{2^*},$$

which implies $\xi_- \leq |\nabla u|_2 \leq \xi_+$. It follows that

$$\begin{aligned} (\Psi_u^0)''(0) &= 2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 - 2^*|u|_{2^*}^{2^*} - 2^*(\Psi_u^0)'(0) \\ &= \frac{2(N-4)}{N-2} b|\nabla u|_2^2 \left(|\nabla u|_2^2 - \frac{2a}{(N-4)b} \right) \\ &= \frac{2(N-4)}{N-2} b|\nabla u|_2^2 (|\nabla u|_2^2 - \eta^2), \end{aligned}$$

where $\eta := (\frac{2a}{(N-4)b})^{1/2}$. Then we know that

$$\begin{aligned} \mathcal{P}_0^- &= \{u \in \mathcal{P}_0 : \xi_- \leq |\nabla u|_2 < \eta\}, \\ \mathcal{P}_0^+ &= \{u \in \mathcal{P}_0 : \eta < |\nabla u|_2 \leq \xi_+\}, \\ \mathcal{P}_0^0 &= \{u \in \mathcal{P}_0 : |\nabla u|_2 = \eta\}. \end{aligned}$$

For any $u \in \mathcal{P}_0$, it follows that

$$\begin{aligned} \inf_{u \in \mathcal{P}_0^-} I_0(u) &= \inf_{u \in \mathcal{P}_0^-} \left\{ \frac{a}{2} |\nabla u|_2^2 + \frac{b}{4} |\nabla u|_2^4 - \frac{1}{2^*} |u|_{2^*}^{2^*} - \frac{1}{2^*} (\Psi_u^0)'(0) \right\} \\ &= \inf_{u \in \mathcal{P}_0^-} \left\{ \frac{1}{N} a |\nabla u|_2^2 - \frac{N-4}{4N} b |\nabla u|_2^4 \right\} \\ &= \frac{1}{N} a \xi_-^2 - \frac{N-4}{4N} b \xi_-^4 = c_{N,-} = I_0(\varphi_-). \end{aligned} \quad (3.4)$$

Similarly, it follows that

$$\inf_{u \in \mathcal{P}_0^+} I_0(u) = \frac{1}{N} a \xi_+^2 - \frac{N-4}{4N} b \xi_+^4 = c_{N,+} = I_0(\varphi_+). \quad (3.5)$$

Moreover, by Theorem 1.2 in Xie and Zhou [33], we know that $c_{N,-} > 0$, $c_{N,-} > c_{N,+}$ for any $0 < b < b_0$, $c_{N,+} \leq 0$ for $0 < b \leq b_1$ and $c_{N,+} > 0$ for $b_1 < b < b_0$.

We claim that

$$\inf_{u \in A_{\xi_-}} I_0(u) = 0 \quad \text{and} \quad \inf_{u \in A_{\xi_-}^c} I_0(u) = I_0(\varphi_+),$$

where $A_{\xi_-} := \{u \in S_c : |\nabla u|_2 < \xi_-\}$ and ξ_- is a constant defined by (3.3). Moreover,

$$\inf_{u \in S_c} I_0(u) = \begin{cases} I_0(\varphi_+), & \text{if } b \in (0, b_1], \\ 0, & \text{if } b \in (b_1, b_0). \end{cases}$$

Then, φ_+ is a global minimum solution in S_c for $0 < b \leq b_1$, and a local minimum solution in S_c for $b_1 < b < b_0$. In fact, by (2.1) and direct computations, we obtain that

$$I_0(u) \geq \frac{a}{2} |\nabla u|_2^2 + \frac{b}{4} |\nabla u|_2^4 - \frac{S^{-\frac{2^*}{2}}}{2^*} |\nabla u|_2^{2^*} = h(|\nabla u|_2) > 0 \quad \text{for any } u \in A_{\xi_-},$$

where $h(t)$ is defined in (2.6). Hence, $\inf_{u \in A_{\xi_-}} I_0(u) \geq 0$. For any $s \in \mathbb{R}$, set

$$u_s(x) := (s * U_{\varepsilon_c, 0})(x) = e^{\frac{N}{2}s} U_{\varepsilon_c, 0}(e^s x) \in S_c. \quad (3.6)$$

Since $|\nabla u_s|_2 = e^s S^{\frac{N}{4}} < \xi_-$ for $s < -M$, it follows that $u_s \in A_{\xi_-}$ and

$$I_0(u_s) = e^{2s} \frac{a}{2} S^{\frac{N}{2}} + e^{4s} \frac{b}{4} S^N - e^{2^*s} \frac{1}{2^*} S^{\frac{N}{2}} \rightarrow 0 \quad \text{as } s \rightarrow -\infty, \quad (3.7)$$

which implies $\inf_{u \in A_{\xi_-}} I_0(u) = 0$. Furthermore, since $I_0(u) \geq h(|\nabla u|_2) > 0$ for any $u \in A_{\xi_-}$, $\inf_{u \in A_{\xi_-}} I_0(u)$ has no minimizer.

On the one hand, for any $u \in \mathcal{P}_0^+$, we have $|\nabla u|_2 > \eta > \xi_-$, which implies $\mathcal{P}_0^+ \subset A_{\xi_-}^c$. Therefore, $\inf_{u \in \mathcal{P}_0^+} I_0(u) \geq \inf_{u \in A_{\xi_-}^c} I_0(u)$. On the other hand, Remark 1.5 in [33] gives

$$\inf_{u \in A_{\xi_-}^c} I_0(u) \geq \inf_{t \geq \xi_-} h(t) = h(\xi_+) = c_{N,+} = \inf_{u \in \mathcal{P}_0^+} I_0(u). \quad (3.8)$$

Thus, $\inf_{u \in A_{\xi_-}^c} I_0(u) = I_0(\varphi_+) = c_{N,+}$ holds. Moreover, by the fact that $c_{N,+} \leq 0$ for $0 < b \leq b_1$ and $c_{N,+} > 0$ for $b_1 < b < b_0$, we have

$$\inf_{u \in S_c} I_0(u) = \begin{cases} I_0(\varphi_+), & \text{if } b \in (0, b_1], \\ 0, & \text{if } b \in (b_1, b_0). \end{cases}$$

We also claim that φ_- is a mountain-pass solution in S_c and that $c_{N,-}$ is the mountain-pass level. Indeed, by (3.6), (3.7), and $c_{N,-} > 0$, one can find $s_0 < -M$ such that $u_{s_0} \in A_{\xi_-}$ and $I_0(u_{s_0}) < c_{N,-} = I_0(\varphi_-)$. Combining this with $c_{N,+} < c_{N,-}$ and $|\nabla \varphi_-|_2 = \xi_- < \xi_+ = |\nabla \varphi_+|_2$, we obtain

$$\max\{I_0(u_{s_0}), I_0(\varphi_+)\} < I_0(\varphi_-) \leq \inf_{u \in \partial A_{\xi_-}} I_0(u).$$

Thus, we can define a mountain pass level as following

$$m_- = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_0(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], S_c) : \gamma(0) = u_{s_0} \text{ and } \gamma(1) = \varphi_+\}$. We claim that

$$m_- = c_{N,-}.$$

Since $|\nabla \gamma(0)|_2 < \xi_- < |\nabla \gamma(1)|_2$ for every $\gamma \in \Gamma$, there exists $t_\gamma \in (0, 1)$ such that $\gamma(t_\gamma) \in \partial A_{\xi_-}$. Therefore,

$$\sup_{t \in [0,1]} I_0(\gamma(t)) \geq I_0(\gamma(t_\gamma)) \geq \inf_{u \in \partial A_{\xi_-}} I_0(u) \geq I_0(\varphi_-),$$

which implies $m_- \geq c_{N,-}$. Set $r(t) = ts_+ + (1-t)s_0$ and $\gamma_1(t) = u_{r(t)} \in C([0, 1], S_c)$, where, by (3.6),

$$\gamma_1(0) = u_{r(0)} = s_0 * U_{\varepsilon_c, 0} = u_{s_0} \quad \text{and} \quad \gamma_1(1) = u_{r(1)} = s_+ * U_{\varepsilon_c, 0} = \varphi_+.$$

Then $\gamma_1 \in \Gamma$ and

$$m_- \leq \sup_{t \in [0,1]} I_0(\gamma_1(t)) = \max_{t \in [0,1]} I_0(u_{r(t)}) = I_0(\varphi_-),$$

which implies $m_- \leq c_{N,-}$. Therefore, $m_- = c_{N,-}$.

(ii) For $b < 0$, it is easy to check that the equation $g(t) = 0$ has a positive root t_1 . Moreover, $g'(t_1) < 0$. Let s_1 satisfy $e^{s_1} = t_1$ and set

$$\varphi_1(x) := (s_1 * U_{\varepsilon_c, 0})(x) = e^{\frac{N}{2}s_1} U_{\varepsilon_c, 0}(e^{s_1}x) \in S_c,$$

Combining this with $g'(t_1) < 0$ yields $\varphi_1 \in \mathcal{P}_0^-$. Set

$$\xi_1 = |\nabla \varphi_1|_2, \tag{3.9}$$

Then ξ_1 is a positive root of $f(t) = 0$. Moreover, $f(t) > 0$ for $t < \xi_1$ and $f(t) < 0$ for $t > \xi_1$. It follows from $u \in \mathcal{P}_0$ and (2.1) that

$$a|\nabla u|_2^2 + b|\nabla u|_2^4 = |u|_{2^*}^{2^*} \leq S^{-\frac{2^*}{2}} |\nabla u|_2^{2^*},$$

which implies $|\nabla u|_2 \geq \xi_1$. For any $u \in \mathcal{P}_0$, since $a > 0$ and $b < 0$, we have

$$(\Psi_u^0)''(0) = \frac{2(N-4)}{N-2} b |\nabla u|_2^4 - \frac{4a}{(N-4)} |\nabla u|_2^2 < 0,$$

which implies $u \in \mathcal{P}_0^-$; hence $\mathcal{P}_0 = \mathcal{P}_0^-$. Finally, an argument similar to that for (3.4) gives

$$\inf_{u \in \mathcal{P}_0} I_0(u) = \inf_{u \in \mathcal{P}_0^-} I_0(u) = \frac{1}{N} a \xi_1^2 - \frac{N-4}{4N} b \xi_1^4 = c_N = I_0(\varphi_1) > 0.$$

We claim that φ_1 is a mountain-pass solution in S_c and that c_N is the mountain-pass level. Indeed, (3.6) gives

$$\lim_{s \rightarrow -\infty} |\nabla u_s|_2 = 0, \quad \lim_{s \rightarrow -\infty} I_0(u_s) = 0, \quad \lim_{s \rightarrow +\infty} |\nabla u_s|_2 = +\infty, \quad \lim_{s \rightarrow +\infty} I_0(u_s) = -\infty.$$

Then, we can take $s_2 < -M$ such that $|\nabla u_{s_2}| < \xi_1$ and $I_0(u_{s_2}) < c_N$, and take $s_3 > M$ such that $|\nabla u_{s_3}| > \xi_1$ and $I_0(u_{s_3}) < 0$, where ξ_1 is a constant defined by (3.9). Hence, it follows that

$$\max\{I_0(u_{s_2}), I_0(u_{s_3})\} < I_0(\varphi_1) \leq \inf_{u \in \partial A_{\xi_1}} I_0(u).$$

Thus, we can define a mountain pass level as following

$$m_1 = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_0(\gamma(t)),$$

where $\Gamma = \{\gamma(t) \in C([0,1]) \cap S_c : \gamma(0) = u_{s_2} \text{ and } \gamma(1) = u_{s_3}\}$. Finally, it is easy to prove that

$$c_N = m_1.$$

(iii) For $b > b_0$, it is easy to check that $f(t) > 0$ by some direct computations. Assume that u is a critical point of I_0 on S_c . Then $u \in \mathcal{P}_0$, and

$$\begin{aligned} 0 &= a|\nabla u|_2^2 + b|\nabla u|_2^4 - |u|_2^{2^*} \\ &\geq a|\nabla u|_2^2 + b|\nabla u|_2^4 - S^{-\frac{2^*}{2}} |\nabla u|_2^{2^*} = |\nabla u|_2^2 f(|\nabla u|_2) \geq 0, \end{aligned}$$

which implies $|\nabla u|_2 = 0$ and $u \equiv 0$ contradicts to $u \in S_c$. For $b = b_0$, it is suffice to prove that $g(t) = 0$ admits one root t_0 . Moreover, $g'(t_0) = 0$. Let s_0 satisfy $e^{s_0} = t_0$ and

$$\varphi_1(x) := (s_0 * U_{\varepsilon_c, 0})(x) = e^{\frac{N}{2}s_0 \pm} U_{\varepsilon_c, 0}(e^{s_0 \pm} x) \in S_c.$$

By some similar computations, we can prove that φ_1 is a solution.

(iv) For any $n \in \mathbb{N}$, let $\{u_i\}_{i=1}^n$ be n solutions of equation (S^*) satisfying

$$|\nabla U|_2 = |\nabla u_1|_2 < |\nabla u_2|_2 < \dots < |\nabla u_n|_2. \quad (3.10)$$

Set

$$v_i^\varepsilon(x) := \varepsilon^{\frac{N-2}{2}} u_i(\varepsilon x), \quad 1 \leq i \leq n, \quad \varepsilon > 0,$$

which are solutions of equation (S^*) as well. Moreover, we can choose a unique ε_i such that $|v_i^{\varepsilon_i}|_2^2 = c$, i.e., $v_i^{\varepsilon_i} \in S_c$. Let

$$b_n = \frac{2}{N-2} \left(\frac{N-4}{a(N-2)} \right)^{\frac{N-4}{2}} |\nabla u_n|_2^{-2},$$

it is easy to check that the existence of two positive roots $t_{i,\pm}(t_{i,-} < t_{i,+})$ for equation $g_i(t) = 0$ defined by

$$g_i(t) = b|\nabla v_i^{\varepsilon_i}|_2^2 t^2 - t^{\frac{4}{N-2}} + a, \quad t \geq 0,$$

where $i \in \{1, 2, \dots, n\}$ and $b \in (0, b_n)$. Let $s_{i,\pm}$ satisfy $e^{s_{i,\pm}} = t_{i,\pm}$ and

$$\varphi_{i,\pm}(x) := (s_{i,\pm} * v_i^{\varepsilon_i})(x) \in S_c,$$

then we can check that $\varphi_{i,\pm} \in \mathcal{P}_0$. Set $\xi_{i,\pm} := |\nabla \varphi_{i,\pm}|_2$ for $i \in \{1, 2, \dots, n\}$, then $\xi_{i,\pm}$ are the roots of equations $f_i(t) = 0$ defined by

$$f_i(t) = bt^2 - (|\nabla v_i^{\varepsilon_i}|_2^{-1}t)^{\frac{4}{N-2}} + a, \quad t \geq 0,$$

where $i \in \{1, 2, \dots, n\}$ and $b \in (0, b_n)$. With a similar argument of Lemma 3.3 in [33], one obtains

$$\xi_{i,-} < \eta < \xi_{i,+} \quad \text{and} \quad \xi_{i,+}^2 - \eta^2 > \eta^2 - \xi_{i,-}^2,$$

which implies $\varphi_{i,\pm} \in \mathcal{P}_0^\pm$ and $I_0(\varphi_{i,-}) > I_0(\varphi_{i,+})$. By the above setting and (3.10), we know that $|\nabla v_i^{\varepsilon_i}|_2 < |\nabla v_{i+1}^{\varepsilon_{i+1}}|_2$ for $i \in \{1, 2, \dots, n-1\}$, which implies that $f_i(t) < f_{i+1}(t)$ and

$$0 < \xi_{1,-} < \xi_{2,-} < \dots < \xi_{n,-} < \eta < \xi_{n,+} < \dots < \xi_{2,+} < \xi_{1,+}. \quad (3.11)$$

For any $u \in \mathcal{P}_0$, we have

$$I_0(u) = I_0(u) - \frac{1}{2^*}(\Psi_u^0)'(0) = \frac{1}{N}a|\nabla u|_2^2 - \frac{N-4}{4N}b|\nabla u|_2^4 =: \mathcal{I}_0(|\nabla u|_2^2),$$

where

$$\mathcal{I}_0(t) = \frac{1}{N}at - \frac{N-4}{4N}bt^2, \quad t \geq 0.$$

It is easy to check that $\mathcal{I}_0(t)$ is increasing in $(0, \eta^2)$ and decreasing in $(\eta^2, +\infty)$. Combining with (i) and (3.11), one obtains $\varphi_\pm = \varphi_{1,\pm}$ and

$$I_0(\varphi_{1,\pm}) < I_0(\varphi_{2,\pm}) < \dots < I_0(\varphi_{n,\pm}) < \frac{a^2}{N(N-4)b}.$$

The proof is complete. $\square$

The proof of Theorem 2.5. (i) Let u be a critical point of $I(u)$ on S_c . Then u solves (1.5) for some $\lambda \in \mathbb{R}$, testing (1.5) by u and integrating over $\mathbb{R}^N$, we have

$$a|\nabla u|_2^2 + b|\nabla u|_2^4 + \lambda|u|_2^2 = \mu|u|_q^q + |u|_{2^*}^{2^*}. \quad (3.12)$$

Recalling that $P(u) = 0$, i.e.,

$$P(u) = a|\nabla u|_2^2 + b|\nabla u|_2^4 - \mu\delta_q|u|_q^q - |u|_{2^*}^{2^*} = 0, \quad (3.13)$$

then we infer that $\lambda|u|_2^2 = \mu(1 - \delta_q)|u|_q^q$ by (3.12) and (3.13). Since $S_c \ni u \neq 0$, $\mu < 0$ and $0 < \delta_q < 1$, we conclude that $\lambda < 0$. It follows from $P(u) = 0$ and (2.1) that

$$a|\nabla u|_2^2 + b|\nabla u|_2^4 = \mu\delta_q|u|_q^q + |u|_{2^*}^{2^*} \leq |u|_{2^*}^{2^*} \leq S^{-\frac{2^*}{2}}|\nabla u|_2^{2^*}.$$

Thus, by (3.2), we have

$$\xi_- < |\nabla u|_2 < \xi_+. \quad (3.14)$$

Moreover,

$$\begin{aligned} I(u) &= I(u) - \frac{1}{2^*} P(u) = \frac{1}{N} a |\nabla u|_2^2 - \frac{N-4}{4N} b |\nabla u|_2^4 - \frac{\mu}{q} \left(1 - \frac{\delta_q q}{2^*}\right) |u|_q^q \\ &\geq \frac{1}{N} a |\nabla u|_2^2 - \frac{N-4}{4N} b |\nabla u|_2^4 \geq \frac{1}{N} a \xi_+^2 - \frac{N-4}{4N} b \xi_+^4 \\ &= c_{N,+}, \end{aligned}$$

where $0 < \delta_q q < 2^*$, $\mu < 0$, (3.5) and (3.14) are used.

(ii) Assume that $u \in H^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ is a solution of equation (2.7), where $p \in (0, \frac{N}{N-2}]$. By the Brézis–Kato argument [3], we know that $u, |\Delta u| \in L^\infty(\mathbb{R}^N)$, and standard gradient estimates for the Poisson equation also imply that $|\nabla u| \in L^\infty(\mathbb{R}^N)$. It follows from $u \in L^2(\mathbb{R}^N)$ that

$$u(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (3.15)$$

Combining with (3.15) and the fact that $\lambda < 0$ from (i), we have

$$-\Delta u = \frac{1}{a + b |\nabla u|_2^2} (-\lambda u + \mu |u|^{q-2} u + |u|^{2^*-2} u) \geq \frac{-\lambda}{2(a + b |\nabla u|_2^2)} u(x) > 0,$$

for $|x| > R_0$. Thus, u is superharmonic at infinity. Now we define

$$m(r) := \min_{|x|=r} u(x).$$

Since $u > 0$ in $\mathbb{R}^N$, we know that $m(r) > 0$. The Hadamard three spheres theorem in [25] implies that $m(r)$ satisfies

$$m(r) \geq \frac{m(r_1)(r_1^{2-N} - r_2^{2-N}) + m(r_2)(r_2^{2-N} - r_1^{2-N})}{r_1^{2-N} - r_2^{2-N}} \quad \text{for } R_0 < r_1 < r < r_2. \quad (3.16)$$

By (3.15), $m(r_2) \rightarrow 0$ as $r_2 \rightarrow \infty$. Letting $r_2 \rightarrow \infty$ in (3.16), we find that the function $r \mapsto r^{N-2} m(r)$ is nondecreasing for $r > R_0$. Therefore,

$$m(r) \geq \frac{m(R_0) R_0^{N-2}}{r^{N-2}} \quad \text{for } r > R_0,$$

which implies that

$$|u|_p^p \geq C \int_{R_0}^{+\infty} |m(r)|^p r^{N-1} dr \geq C \int_{R_0}^{+\infty} r^{p(2-N)+N-1} dr = \infty$$

for any $p \in (0, \frac{N}{N-2}]$, which contradicts the assumption $u \in L^p(\mathbb{R}^N)$ for $p \in (0, \frac{N}{N-2}]$.

Thus, equation (2.7) has no solution. The proof is complete. $\square$

The proof of Theorem 2.7. If $b \geq S^{-2}$, then $\mathcal{P}_0 = \emptyset$. Indeed, if $u \in \mathcal{P}_0$, we have

$$0 \leq a|\nabla u|_2^2 = |u|_4^4 - b|\nabla u|_2^4 \leq |u|_4^4 - S^{-2}|\nabla u|_2^4 \leq 0,$$

which implies $u \equiv 0$, contradicting to $u \in S_c$. Thus, $\mathcal{P}_0 = \emptyset$. For any $u \in S_c$, one obtains

$$I_0(u) = \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}|\nabla u|_2^4 - \frac{1}{4}|u|_4^4 \geq \frac{a}{2}|\nabla u|_2^2 + \frac{1}{4}(b - S^{-2})|\nabla u|_2^4 > 0,$$

which implies $\inf_{u \in S_c} I_0(u) \geq 0$. Since $s * u \in S_c$ for every $s \in \mathbb{R}$,

$$I_0(s * u) = \frac{ae^{2s}}{2}|\nabla u|_2^2 + \frac{be^{4s}}{4}|\nabla u|_2^4 - \frac{e^{4s}}{4}|u|_4^4 \rightarrow 0 \quad \text{as } s \rightarrow -\infty.$$

Thus, $\inf_{u \in S_c} I_0(u) = 0$. Since $I_0(u) > 0$ for every $u \in S_c$, this infimum cannot be attained.

If $0 < b < S^{-2}$, we know that $\inf_{u \in S_c} I_0(u) = -\infty$ by Theorem 1.1(a) in Li *et al.* [22]. In this case, set $S_c^+ := \{u \in S_c : |u|_4^4 - b|\nabla u|_2^4 > 0\}$. Similarly, we can prove that $\mathcal{P}_0 = \mathcal{P}_0^-$ and

$$\inf_{u \in \mathcal{P}_0^-} I_0(u) = \inf_{u \in \mathcal{P}_0} I_0(u) = \inf_{u \in S_c^+} \max_{s \in \mathbb{R}} I_0(s * u).$$

Moreover,

$$\begin{aligned} \inf_{u \in S_c^+} \max_{s \in \mathbb{R}} I_0(s * u) &= \inf_{u \in S_c^+} \frac{a^2}{4} \frac{|\nabla u|_2^4}{|u|_4^4 - b|\nabla u|_2^4} \\ &\geq \inf_{u \in H^1(\mathbb{R}^4) \setminus \{0\}} \frac{a^2}{4} \frac{|\nabla u|_2^4}{|u|_4^4 - b|\nabla u|_2^4} = \frac{a^2}{4} \frac{S^2}{1 - bS^2} = \Lambda. \end{aligned}$$

It remains to prove that

$$\inf_{u \in S_c^+} \frac{a^2}{4} \frac{|\nabla u|_2^4}{|u|_4^4 - b|\nabla u|_2^4} \leq \Lambda.$$

Choose a radially symmetric function $\psi \in C_0^\infty(\mathbb{R}^4)$ with $0 \leq \psi \leq 1$, $\psi = 1$ for $x \in B_1(0)$, and $\psi = 0$ for $x \in B_2^c(0)$. Moreover, we may choose ψ to be nonincreasing in $|x|$. Define

$$u_\varepsilon(x) := \psi(x)U_{\varepsilon,0}(x) \quad \text{and} \quad v_\varepsilon(x) := \frac{\sqrt{c}u_\varepsilon(x)}{|u_\varepsilon|_2} \in S_c,$$

where $U_{\varepsilon,0}(x)$ is given in (2.2). Recall that $S^2 = |\nabla U_{\varepsilon,0}|_2^2$ and $S = |U_{\varepsilon,0}|_4^2$. Appendix A of [29] gives

$$|\nabla u_\varepsilon|_2^2 = S^2 + O(\varepsilon), \quad |u_\varepsilon|_4^2 = S + O(\varepsilon^2).$$

For sufficiently small ε , we have

$$\frac{|v_\varepsilon|_4^4}{|\nabla v_\varepsilon|_2^4} = S^{-2} + O(\varepsilon) > b,$$

which implies $v_\varepsilon \in S_c^+$. Thus,

$$\begin{aligned} \inf_{u \in S_c^+} \frac{a^2}{4} \frac{|\nabla u|_2^4}{|u|_4^4 - b|\nabla u|_2^4} &\leq \frac{a^2}{4} \frac{|\nabla v_\varepsilon|_2^4}{|v_\varepsilon|_4^4 - b|\nabla v_\varepsilon|_2^4} \\ &= \frac{a^2}{4} \left(\frac{S^2}{1 - bS^2} + O(\varepsilon) \right) = \Lambda + O(\varepsilon). \end{aligned}$$

The proof is complete. $\square$

4. THE PROOF OF THEOREM 2.3

First, we present the compactness result of the Palais–Smale sequences.

Proposition 4.1. *Assume that $N \geq 5$, $0 < b < b_0$, $\mu > 0$ and $2 < q < 2^*$. If $\{u_n\} \subset S_c$ is a Palais–Smale sequence for $I|_{S_c}$ at energy level $m \neq 0$, with $m < c_{N,+}$ and*

$$P(u_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

where $c_{N,+}$ is defined by (2.4). Then, up to subsequence, one of the following alternatives holds:

- (i) either $u_n \rightharpoonup u \neq 0$ weakly in $H_{rad}^1(\mathbb{R}^N)$ but not strongly, where u solves

$$-(a + bA^2)\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,$$

for some $\lambda > 0$ and $m - c_{N,+} \geq K(u) := \left(\frac{a}{2} + \frac{bA^2}{4}\right)|\nabla u|_2^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} - \frac{\mu}{q}|u|_q^q$,
where $A := \lim_{n \rightarrow \infty} |\nabla u_n|_2 > 0$.

- (ii) or $u_n \rightarrow u \neq 0$ strongly in $H_{rad}^1(\mathbb{R}^N)$ for some $u \in H_{rad}^1(\mathbb{R}^N)$. Moreover, $u \in S_c$, $I(u) = m$ and u solves equation (1.5).

Proof. It follows from $I(u_n) \rightarrow m$ that

$$\begin{aligned} \frac{a}{2}|\nabla u_n|_2^2 + \frac{b}{4}|\nabla u_n|_2^4 &\leq \frac{1}{2^*}|u_n|_{2^*}^{2^*} + \frac{\mu}{q}|u_n|_q^q + m + 1 \\ &\leq \frac{1}{2^*}S^{-\frac{2^*}{2}}|\nabla u_n|_2^{2^*} + \frac{\mu}{q}C_q^q|\nabla u_n|_2^{q\delta_q}c^{\frac{q(1-\delta_q)}{2}} + m + 1, \end{aligned}$$

combining with $q\delta_q < 2^* < 4$, which implies that $|\nabla u_n|_2 \leq C$ for some C . Then $\{u_n\}$ is bounded in $H_{rad}^1(\mathbb{R}^N)$ and for some $u \in H_{rad}^1(\mathbb{R}^N)$,

$$u_n \rightharpoonup u \quad \text{in } H_{rad}^1(\mathbb{R}^N), \quad u_n \rightarrow u \quad \text{in } L^q(\mathbb{R}^N), \quad u_n(x) \rightarrow u(x) \quad \text{a.e. on } \mathbb{R}^N.$$

Since $\{u_n\}$ is a Palais–Smale sequence of $I|_{S_c}$, the Lagrange multiplier rule yields a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that, for every $\phi \in H_{rad}^1(\mathbb{R}^N)$,

$$\begin{aligned} (a + b|\nabla u_n|_2^2) \int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx - \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n \phi dx \\ - \mu \int_{\mathbb{R}^N} |u_n|^{q-2} u_n \phi dx + \lambda_n \int_{\mathbb{R}^N} u_n \phi dx = o(1)\|\phi\|. \end{aligned} \tag{4.1}$$

In particular, taking $\phi = u_n$ in (4.1), then,

$$-\lambda_n c = (a + b|\nabla u_n|_2^2)|\nabla u_n|_2^2 - |u_n|_{2^*}^{2^*} - \mu|u_n|_q^q + o(1).$$

The boundedness of $\{u_n\}$ in $H_{rad}^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$ implies that $\lambda_n \rightarrow \lambda$ up to subsequence. Combining with $P(u_n) = o(1)$, we have

$$\lambda_n c = \mu(1 - \delta_q)|u_n|_q^q + o(1).$$

Letting $n \rightarrow \infty$, we obtain $\lambda c = \mu(1 - \delta_q)|u|_q^q$. Since $\mu > 0$ and $0 < \delta_q < 1$, we have $\lambda \geq 0$, with $\lambda = 0$ if and only if $u = 0$. If $\lambda = 0$ and $u = 0$, set $\lim_{n \rightarrow \infty} |\nabla u_n|_2 = A \geq 0$. Then

$$\begin{aligned} aA^2 + bA^4 &= \lim_{n \rightarrow \infty} (a|\nabla u_n|_2^2 + b|\nabla u_n|_2^4) = \lim_{n \rightarrow \infty} |u_n|_{2^*}^{2^*} \\ &\leq S^{-\frac{2^*}{2}} \lim_{n \rightarrow \infty} |\nabla u_n|_2^{2^*} = S^{-\frac{2^*}{2}} A^{2^*}, \end{aligned}$$

which implies that $A = 0$ or $\xi_- \leq A \leq \xi_+$, where $\xi_{\pm}$ are the roots of $f(t) = 0$ by (3.2). If $\xi_- \leq A \leq \xi_+$, it follows that

$$\begin{aligned} c_{N,+} > m &= \lim_{n \rightarrow \infty} I(u_n) = \lim_{n \rightarrow \infty} (I(u_n) - \frac{1}{2^*} P(u_n)) \\ &= \frac{1}{N} aA^2 - \frac{N-4}{4N} bA^4 \geq \frac{1}{N} a\xi_+^2 - \frac{N-4}{4N} b\xi_+^4 = c_{N,+}, \end{aligned}$$

which is impossible. The case $A = 0$ is also impossible for $m \neq 0$. So $\lambda > 0$ and $u \neq 0$ hold. Since $u_n \rightharpoonup u \neq 0$ in $H_{rad}^1(\mathbb{R}^N)$, we set $\lim_{n \rightarrow \infty} |\nabla u_n|_2 = A > 0$. Then, by (4.1), we obtain

$$\begin{aligned} (a + bA^2) \int_{\mathbb{R}^N} \nabla u \nabla \phi dx - \int_{\mathbb{R}^N} |u|^{2^*-2} u \phi dx - \mu \int_{\mathbb{R}^N} |u|^{q-2} u \phi dx \\ + \lambda \int_{\mathbb{R}^N} u \phi dx = 0, \end{aligned} \quad (4.2)$$

for any $\phi \in H_{rad}^1(\mathbb{R}^N)$, which means u is a weak solution of

$$-(a + bA^2)\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N.$$

It follows from the Pohozaev identity that

$$Q_A(u) := (a + bA^2)|\nabla u|_2^2 - \mu\delta_q|u|_q^q - |u|_{2^*}^{2^*} = 0. \quad (4.3)$$

Set $v_n := u_n - u$. Then $v_n \rightharpoonup 0$ in $H_{rad}^1(\mathbb{R}^N)$ and $v_n \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$. By the Brézis–Lieb lemma, we have $|\nabla u_n|_2^2 = |\nabla v_n|_2^2 + |\nabla u|_2^2 + o(1)$ and

$$|u_n|_q^q = |v_n|_q^q + |u|_q^q + o(1) = |u|_q^q + o(1), \quad |u_n|_{2^*}^{2^*} = |v_n|_{2^*}^{2^*} + |u|_{2^*}^{2^*} + o(1).$$

We can rewrite $P(u_n) = o(1)$ as

$$P(u_n) = (a + bA^2)|\nabla u_n|_2^2 - \mu\delta_q|u_n|_q^q - |u_n|_{2^*}^{2^*} + o(1) = o(1). \quad (4.4)$$

Set $\lim_{n \rightarrow \infty} |\nabla v_n|_2 = l$. Then (4.3) and (4.4) imply that

$$\begin{aligned} al^2 + bl^4 &:= \lim_{n \rightarrow \infty} (a|\nabla v_n|_2^2 + b|\nabla v_n|_2^4) \leq \lim_{n \rightarrow \infty} (a + bA^2)|\nabla v_n|_2^2 \\ &= \lim_{n \rightarrow \infty} |v_n|_{2^*}^{2^*} \leq S^{-\frac{2^*}{2}} \lim_{n \rightarrow \infty} |\nabla v_n|_2^{2^*} = S^{-\frac{2^*}{2}} l^{2^*}, \end{aligned}$$

which implies that $\xi_- \leq l \leq \xi_+$ or $l = 0$.

(i) If $\xi_- \leq l \leq \xi_+$, we define the function

$$k(t) := \frac{1}{4}at^2 - \left(\frac{1}{2^*} - \frac{1}{4}\right)S^{-\frac{2^*}{2}}t^{2^*} \quad \text{for } t \in (0, \infty).$$

With some careful analysis and some basic computations, we can obtain that

$$\min_{t \in [\xi_-, \xi_+]} k(t) = k(\xi_+) = c_{N,+}.$$

Using the above analysis, we have

$$\begin{aligned} m &= \lim_{n \rightarrow \infty} I(u_n) = K(u) + \lim_{n \rightarrow \infty} \left\{ \left(\frac{a}{2} + \frac{bA^2}{4} \right) |\nabla v_n|_2^2 - \frac{1}{2^*} |v_n|_{2^*}^{2^*} \right\} \\ &= K(u) + \lim_{n \rightarrow \infty} \left\{ \frac{1}{4}a|\nabla v_n|_2^2 - \left(\frac{1}{2^*} - \frac{1}{4} \right) |v_n|_{2^*}^{2^*} \right\} \\ &\geq K(u) + \frac{1}{4}al^2 - \left(\frac{1}{2^*} - \frac{1}{4} \right) S^{-\frac{2^*}{2}} l^{2^*} \\ &\geq K(u) + \min_{t \in [\xi_-, \xi_+]} k(t) \\ &= K(u) + c_{N,+}, \end{aligned}$$

where $K(u) := \left(\frac{a}{2} + \frac{bA^2}{4} \right) |\nabla u|_2^2 - \frac{1}{2^*} |u|_{2^*}^{2^*} - \frac{\mu}{q} |u|_q^q$. In this case, (i) holds.

(ii) If $l = 0$, we know that $u_n \rightarrow u$ in $D^{1,2}(\mathbb{R}^N)$ and $L^{2^*}(\mathbb{R}^N)$. Testing (4.1) and (4.2) with $\phi = u_n - u$ and subtracting them, one obtains

$$(a + bA^2)|\nabla(u_n - u)|_2^2 + \lambda|u_n - u|_2^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies that $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$ and hence in $H_{rad}^1(\mathbb{R}^N)$. Thus, (ii) holds. $\square$

Lemma 4.2. *If $b_1 < b < b_0$ and the case (i) in Proposition 4.1 holds, there exists $\mu_1 > 0$ such that $K(u) \geq -c_{N,+}$ for $\mu c^{q(1-\delta_q)/2} < \mu_1$.*

Proof. For any $t > 0$, we define

$$f_1(t^2) := f(t)t^2 - \mu c^{\frac{q(1-\delta_q)}{2}} C_q^q \delta_q t^{q\delta_q} = bt^4 - S^{-\frac{2^*}{2}} t^{2^*} + at^2 - \mu c^{\frac{q(1-\delta_q)}{2}} C_q^q \delta_q t^{q\delta_q}. \quad (4.5)$$

Since $N \geq 5$ and $2 < q < 2 + \frac{4}{N}$, we have $0 < q\delta_q < 2$ and $2^* < 4$, let $\xi_0^\mu, \xi_\pm^\mu$ be the positive roots of $f_1(t) = 0$ and satisfy that

$$0 < \xi_0^\mu < \xi_-^\mu < \xi_+^\mu.$$

Those three roots are bounded for $\mu c^{\frac{q(1-\delta_q)}{2}}$ small. Recalling that $\xi_\pm$ are the roots of $f(t) = 0$ in (3.2), it is evident to check that for $\mu c^{\frac{q(1-\delta_q)}{2}}$ small

$$\xi_0^\mu < \xi_-^\mu < \xi_- < \xi_+ < \xi_+^\mu \leq C(q).$$

By the definition of $f_1(t)$ and the boundedness of ξ_i^μ , we have

$$|f(\xi_i^\mu)(\xi_i^\mu)^2| = |f_1(\xi_i^\mu) + \mu c^{\frac{q(1-\delta_q)}{2}} C_q^q \delta_q (\xi_i^\mu)^{q\delta_q}| \leq \mu c^{\frac{q(1-\delta_q)}{2}} C_q^q \delta_q C(q)^{q\delta_q} \rightarrow 0,$$

as $\mu c^{\frac{q(1-\delta_q)}{2}} \rightarrow 0^+$, where $i = 0, -$ or $+$. Thus,

$$\xi_0^\mu \rightarrow 0^+ \quad \text{and} \quad \xi_\pm^\mu \rightarrow \xi_\pm \quad \text{as} \quad \mu c^{\frac{q(1-\delta_q)}{2}} \rightarrow 0^+. \quad (4.6)$$

It follows from (4.3) that

$$\begin{aligned} a|\nabla u|_2^2 + b|\nabla u|_2^4 &< (a + bA^2)|\nabla u|_2^2 = |u|_{2^*}^{2^*} + \mu\delta_q|u|_q^q \\ &\leq S^{-\frac{2^*}{2}}|\nabla u|_2^{2^*} + \mu c^{\frac{q(1-\delta_q)}{2}} C_q^q \delta_q |\nabla u|_2^{q\delta_q}, \end{aligned}$$

which implies $f_1(|\nabla u|_2) < 0$. By the above setting, we know that

$$0 < |\nabla u|_2 < \xi_0^\mu \quad \text{or} \quad \xi_-^\mu < |\nabla u|_2 < \xi_+^\mu. \quad (4.7)$$

Then $b_1 < b < b_0$ implies $c_{N,+} > 0$. By (4.6) and (4.7), one obtains

$$\begin{aligned} K(u) &= \left(\frac{a}{2} + \frac{bA^2}{4}\right)|\nabla u|_2^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} - \frac{\mu}{q}|u|_q^q - \frac{1}{4}Q_A(u) \\ &= \frac{a}{4}|\nabla u|_2^2 - \left(\frac{1}{2^*} - \frac{1}{4}\right)|u|_{2^*}^{2^*} - \left(\frac{1}{q\delta_q} - \frac{1}{4}\right)\delta_q\mu|u|_q^q \\ &\geq \frac{a}{4}|\nabla u|_2^2 - \left(\frac{1}{2^*} - \frac{1}{4}\right)S^{-\frac{2^*}{2}}|\nabla u|_2^{2^*} - \left(\frac{1}{q\delta_q} - \frac{1}{4}\right)\mu\delta_q C_q^q c^{\frac{q(1-\delta_q)}{2}}|\nabla u|_2^{q\delta_q} \quad (4.8) \\ &\geq \min_{0 < t < \xi_+^\mu} \left\{ \frac{a}{4}t^2 - \left(\frac{1}{2^*} - \frac{1}{4}\right)S^{-\frac{2^*}{2}}t^{2^*} \right\} - \left(\frac{1}{q\delta_q} - \frac{1}{4}\right)\mu\delta_q C_q^q c^{\frac{q(1-\delta_q)}{2}}(\xi_+^\mu)^{q\delta_q} \\ &\geq 0 - c_{N,+} = -c_{N,+} \end{aligned}$$

for some $\mu c^{\frac{q(1-\delta_q)}{2}} < \mu_1$ small enough. $\square$

Consider the constrained functional $I|_{S_c}$, by (2.1) and (2.3), we have

$$\begin{aligned} I(u) &= \frac{a}{2} |\nabla u|_2^2 + \frac{b}{4} |\nabla u|_2^4 - \frac{\mu}{q} |u|_q^q - \frac{1}{2^*} |u|_{2^*}^{2^*} \\ &\geq \frac{a}{2} |\nabla u|_2^2 + \frac{b}{4} |\nabla u|_2^4 - \frac{\mu}{q} C_q^q c^{\frac{q(1-\delta_q)}{2}} |\nabla u|_2^{q\delta_q} - \frac{S^{-\frac{2^*}{2}}}{2^*} |\nabla u|_2^{2^*} =: h_1(|\nabla u|_2). \end{aligned}$$

Hence, $f_1(t) = th_1'(t)$, where $f_1(t)$ is defined by (4.5). Clearly, ξ_-^μ is the maximum point of $h_1(t)$, while ξ_0^μ and ξ_+^μ are the minimum points of $h_1(t)$. Set

$$m(b, \mu) = m(c, b, \mu) := \inf_{u \in S_c} I(u).$$

Lemma 4.3. *There exists $\mu_2 > 0$ such that, if $\mu c^{\frac{q(1-\delta_q)}{2}} < \mu_2$ and $b_1 < b < b_0$, then*

- (i) $h_1(\xi_+^\mu) > 0$,
- (ii) $h_1(t)$ has a unique zero point $\xi_0^{\mu,1}$ and $h_1(t) \geq 0$ for $t \geq \xi_0^{\mu,1}$.

Proof. (i) Since $h'(t) = tf(t)$, ξ_+ is the minimum point of $h(t)$ and $h(t) > h(\xi_+)$ for $t > \xi_+$. Combining this with $c_{N,+} > 0$ for $b_1 < b < b_0$, $\xi_+^\mu > \xi_+$, and (3.8), we obtain

$$\begin{aligned} h_1(\xi_+^\mu) &= \frac{a}{2} (\xi_+^\mu)^2 + \frac{b}{4} (\xi_+^\mu)^4 - \frac{\mu}{q} C_q^q c^{\frac{q(1-\delta_q)}{2}} (\xi_+^\mu)^{q\delta_q} - \frac{S^{-\frac{2^*}{2}}}{2^*} (\xi_+^\mu)^{2^*} \\ &= h(\xi_+^\mu) - \frac{\mu}{q} C_q^q c^{\frac{q(1-\delta_q)}{2}} (\xi_+^\mu)^{q\delta_q} > h(\xi_+) - \frac{\mu}{q} C_q^q c^{\frac{q(1-\delta_q)}{2}} (\xi_+^\mu)^{q\delta_q} \quad (4.9) \\ &= c_{N,+} - \frac{\mu}{q} C_q^q c^{\frac{q(1-\delta_q)}{2}} (\xi_+^\mu)^{q\delta_q} > 0 \end{aligned}$$

for some $\mu c^{\frac{q(1-\delta_q)}{2}} < \mu_2$ small enough. By (4.8), we also know that $\mu_2 < \mu_1$.

(ii) Since $h_1(0^+) = 0^-$, $h_1(+\infty) = +\infty$, and (4.9) holds, the function $h_1(t)$ has a unique zero $\xi_0^{\mu,1}$ and satisfies $h_1(t) \geq 0$ for $t \geq \xi_0^{\mu,1}$. $\square$

Lemma 4.4. *If $\mu c^{\frac{q(1-\delta_q)}{2}} < \mu_2$ and $b_1 < b < b_0$, then*

$$\inf_{u \in A_{\xi_0^{\mu,1}}} I(u) = m(b, \mu) < 0,$$

where $A_{\xi_0^{\mu,1}} := \{u \in S_c : |\nabla u|_2 < \xi_0^{\mu,1}\}$.

Proof. For any $u \in S_c$, since $\lim_{s \rightarrow -\infty} I(s * u) = 0^-$ and $\lim_{s \rightarrow +\infty} I(s * u) = +\infty$, we can choose a small $s_0 \in \mathbb{R}$ such that $I(s_0 * u) < 0$. Therefore, $m(b, \mu) < 0$. By Lemma 4.3(ii), for any $u \in A_{\xi_0^{\mu,1}}^c$, we have $I(u) \geq h_1(|\nabla u|_2) \geq 0$, which implies $\inf_{u \in A_{\xi_0^{\mu,1}}} I(u) = m(b, \mu) < 0$. $\square$

The proof of Theorem 2.3. (i) Let $\{v_n\}$ be a minimizing sequence for $m(b, \mu)$. Since $I(|v_n|) = I(v_n)$, we may assume that every $v_n \in S_c$ is nonnegative and radially decreasing. Choose $s_n \in \mathbb{R}$ such that $s_n * v_n \in \mathcal{P}$; the scaling parameter s_n are selected at a global minimum of the fibre map, then

$$I(s_n * v_n) = \min\{I(s * v_n) : s \in \mathbb{R}\} \leq I(v_n).$$

Consequently, we obtain a new minimizing sequence $\{w_n = s_n * v_n\}$ for $m(b, \mu)$, with

$$w_n \in S_c \cap \mathcal{P} \quad \text{and} \quad P(w_n) = 0$$

for every n . Hence, the Ekeland's variational principle guarantees the existence of a new minimizing sequence $\{u_n\} \subset S_c$ for $m(b, \mu)$, with the property that $|u_n - w_n|_{H^1} \rightarrow 0$ as $n \rightarrow +\infty$, which is also a Palais–Smale sequence for I on S_c with $P(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus, $\{u_n\}$ satisfies all the assumptions of Proposition 4.1. We shall show that the alternative (ii) in Proposition 4.1 occurs. Otherwise, by Lemmas 4.2 and 4.4, we have

$$0 > m(b, \mu) \geq K(u) + c_{N,+} \geq 0.$$

Consequently, after passing to a subsequence, $u_n \rightarrow u$ in $H_{rad}^1(\mathbb{R}^N)$, and u solves equation (1.5) for some $\lambda > 0$. Moreover, u is nonnegative and radially decreasing, and the strong maximum principle implies that $u > 0$.

(ii) Let $u_{b,\mu} \in S_c$ be a global minimizer of $I(u)$ given by (i). By Lemma 4.4, we also have $u_{b,\mu} \in A_{\xi_0^{\mu,1}}$. It is easy to check that $\xi_0^{\mu,1} \rightarrow 0$ as $\mu \rightarrow 0^+$, and then $|\nabla u_{b,\mu}|_2 \rightarrow 0$ as well. From (2.1) and (2.3), we have

$$\begin{aligned} 0 > m(b, \mu) &= I(u_{b,\mu}) = \frac{a}{2} |\nabla u_{b,\mu}|_2^2 + \frac{b}{4} |\nabla u_{b,\mu}|_2^4 - \frac{\mu}{q} |u_{b,\mu}|_q^q - \frac{1}{2^*} |u_{b,\mu}|_{2^*}^{2^*} \\ &\geq \frac{a}{2} |\nabla u_{b,\mu}|_2^2 + \frac{b}{4} |\nabla u_{b,\mu}|_2^4 - \frac{\mu}{q} C_q^q c^{\frac{q(1-\delta_q)}{2}} |\nabla u_{b,\mu}|_2^{q\delta_q} \\ &\quad - \frac{S^{-\frac{2^*}{2}}}{2^*} |\nabla u_{b,\mu}|_2^{2^*} \rightarrow 0 \quad \text{as } \mu \rightarrow 0^+, \end{aligned}$$

which implies that $m(b, \mu) \rightarrow 0$ as $\mu \rightarrow 0^+$. The proof is complete. $\square$

5. THE PROOF OF THEOREM 2.8 AND THEOREM 2.10

5.1. THE PROOF OF THEOREM 2.8

In the subsection, we study the existence of the local minimizer for equation (1.5) with $N = 4$ and hence complete the proof of Theorem 2.8. For any $u \in S_c$, by $0 < b < S^{-2}$, (2.1) and (2.3), we have

$$\begin{aligned} J(u) &\geq I(u) = \frac{a}{2} |\nabla u|_2^2 + \frac{b}{4} |\nabla u|_2^4 - \frac{\mu}{q} |u|_q^q - \frac{1}{4} |u|_4^4 \\ &\geq \frac{a}{2} |\nabla u|_2^2 - \frac{\mu C_q^q}{q} c^{\frac{4-q}{2}} |\nabla u|_2^{2q-4} - \frac{1-bS^2}{4S^2} |\nabla u|_2^4 = |\nabla u|_2^2 f_c(|\nabla u|_2^2), \end{aligned} \tag{5.1}$$

where the function $f_c(k)$ is defined by

$$f_c(k) := \frac{a}{2} - \frac{\mu C_q^q}{q} c^{\frac{4-q}{2}} k^{q-3} - \frac{1-bS^2}{4S^2} k, \quad \forall c, k \in (0, +\infty). \quad (5.2)$$

In the following, similar to Lemma 2.1 in [16], we give some results of $f_c(k)$.

Lemma 5.1. *Assume that $0 < b < S^{-2}$, $2 < q < 3$ and $\mu > 0$. For any $c > 0$ fixed, then the function $f_c(k)$ has a unique global maximum and the maximum value satisfies*

$$\max_{k \in (0, +\infty)} f_c(k) = f_c(k_c) = \begin{cases} > 0, & \text{if } c < c_0, \\ = 0, & \text{if } c = c_0, \\ < 0, & \text{if } c > c_0, \end{cases} \quad (5.3)$$

where c_0 is defined in (2.9) and

$$k_c := \left[\frac{4\mu C_q^q S^2 (3-q)}{q(1-bS^2)} \right]^{\frac{1}{4-q}} c^{\frac{1}{2}}. \quad (5.4)$$

In particular, we have $k_{c_0} = k_0$.

Lemma 5.2. *Assume that $0 < b < S^{-2}$, $2 < q < 3$ and $\mu > 0$. Let $c_1 > 0$ and $k_1 > 0$ be such that $f_{c_1}(k_1) \geq 0$. Then for any $c_2 \in (0, c_1]$, it holds that*

$$f_{c_2}(k_2) \geq 0, \quad \forall k_2 \in \left[\frac{c_2}{c_1} k_1, k_1 \right]. \quad (5.5)$$

Proof. It is easy to see that $f_c(k)$ is non-increasing with respect to c . Then, for any $c_2 \in (0, c_1]$, we have

$$f_{c_2}(k_1) \geq f_{c_1}(k_1) \geq 0. \quad (5.6)$$

A direct calculation gives

$$f_{c_2} \left(\frac{c_2}{c_1} k_1 \right) \geq f_{c_1}(k_1) \geq 0. \quad (5.7)$$

If $f_{c_2}(k) < 0$ for some $k \in (\frac{c_2}{c_1} k_1, k_1)$, then $f_{c_2}(k)$ has a local minimum in $(\frac{c_2}{c_1} k_1, k_1)$, contradicting Lemma 5.1, which states that f_{c_2} has a unique global maximum. The proof is complete. $\square$

Lemma 5.3. *Assume that $0 < b < S^{-2}$, $2 < q < 3$ and $\mu > 0$. For any $c \in (0, c_0)$, it holds that*

$$m(c) := m(c, b, \mu) = \inf_{u \in A_{k_0}(c)} I(u) < 0 < \inf_{u \in \partial(A_{k_0}(c))} I(u) \quad (5.8)$$

and

$$\bar{m}(c) := \bar{m}(c, b, \mu) = \inf_{u \in A_{k_0}(c)} J(u) < 0 < \inf_{u \in \partial(A_{k_0}(c))} J(u). \quad (5.9)$$

Proof. For any $u \in S_c$, set $u_t(x) := t^2 u(tx)$. Then $u_t \in S_c$ and $|\nabla u_t|_2^2 = t^2 |\nabla u|_2^2$. Moreover, we have

$$I(u_t) \leq J(u_t) = \frac{at^2}{2(1-bS^2)} |\nabla u|_2^2 + \frac{bt^4}{4(1-bS^2)} |\nabla u|_2^4 - \frac{\mu t^{2q-4}}{q} |u|_q^q - \frac{t^4}{4} |u|_4^4.$$

Since $2 < q < 3$, there exists $t_0 > 0$ small enough such that $|\nabla u_{t_0}|_2^2 < k_0$ and $I(u_{t_0}) \leq J(u_{t_0}) < 0$, which implies that $u_{t_0} \in A_{k_0}(c)$ and $m(c) \leq \bar{m}(c) < 0$. Moreover, by (5.1) and the fact that $f_c(k_0) > f_{c_0}(k_0) = 0$ for any $c \in (0, c_0)$, we have $J(u) \geq I(u) \geq |\nabla u|_2^2 f_c(|\nabla u|_2^2) = k_0 f_c(k_0) > 0$ for any $u \in \partial(A_{k_0}(c))$. Thus, (5.8) and (5.9) hold. The proof is complete. $\square$

Lemma 5.4. Assume that $0 < b < S^{-2}$, $2 < q < 3$ and $\mu > 0$. Then the following assertions hold.

- (i) Let $c \in (0, c_0)$. For any $\alpha \in (0, c)$, we have $m(c) \leq m(\alpha) + m(c - \alpha)$. Moreover, if $m(\alpha)$ or $m(c - \alpha)$ is achieved, then the inequality is strict.
- (ii) The function $c \mapsto m(c)$ is continuous on $(0, c_0)$.

The proof of Lemma 5.4 is similar to the following lemma, so we omit it.

Lemma 5.5. Assume that $0 < b < S^{-2}$, $2 < q < 3$ and $\mu > 0$. Then the following assertions hold.

- (i) Let $c \in (0, c_0)$. For any $\alpha \in (0, c)$, we have $\bar{m}(c) \leq \bar{m}(\alpha) + \bar{m}(c - \alpha)$. Moreover, if $\bar{m}(\alpha)$ or $\bar{m}(c - \alpha)$ is achieved, then the inequality is strict.
- (ii) The function $c \mapsto \bar{m}(c)$ is continuous on $(0, c_0)$.

Proof. (i) Let $\{u_n\} \subset A_{k_0}(\alpha)$ be a minimizing sequence with respect to $\bar{m}(\alpha)$, i.e., $\lim_{n \rightarrow \infty} J(u_n) = \bar{m}(\alpha)$. By Lemma 5.3, we have $\bar{m}(\alpha) < 0$ and $0 > J(u_n) \geq |\nabla u_n|_2^2 f_\alpha(|\nabla u_n|_2^2)$ for $n \in \mathbb{N}$ large enough. Then, in view of Lemma 5.2 and the fact that $f_c(k_0) > f_{c_0}(k_0) = 0$ for any $c \in (0, c_0)$, we have $f_\alpha(k) \geq 0$ for any $k \in [\frac{\alpha}{c} k_0, k_0]$, which implies that

$$|\nabla u_n|_2^2 < \frac{\alpha}{c} k_0$$

for all sufficiently large $n \in \mathbb{N}$. For any $\theta \in (1, \frac{c}{\alpha}]$, set $v_n(x) := u_n(\theta^{-\frac{1}{4}} x)$. Then $|v_n|_2^2 = \theta |u_n|_2^2 = \theta \alpha$ and

$$|\nabla v_n|_2^2 = \theta^{\frac{1}{2}} |\nabla u_n|_2^2 \leq (\frac{c}{\alpha})^{\frac{1}{2}} \frac{\alpha}{c} k_0 < k_0,$$

which means that $v_n \in A_{k_0}(\theta \alpha)$. Thus, it follows from (5.9) that

$$\begin{aligned} \bar{m}(\theta \alpha) &\leq J(v_n) \\ &= \frac{a\theta^{\frac{1}{2}}}{2(1-bS^2)} |\nabla u_n|_2^2 + \frac{b\theta}{4(1-bS^2)} |\nabla u_n|_2^4 - \frac{\mu\theta}{q} |u_n|_q^q - \frac{\theta}{4} |u_n|_4^4 \\ &\leq \frac{a\theta}{2(1-bS^2)} |\nabla u_n|_2^2 + \frac{b\theta}{4(1-bS^2)} |\nabla u_n|_2^4 - \frac{\mu\theta}{q} |u_n|_q^q - \frac{\theta}{4} |u_n|_4^4 \\ &= \theta J(u_n) = \theta \bar{m}(\alpha) + o(1), \end{aligned} \tag{5.10}$$

which implies that

$$\bar{m}(\theta\alpha) \leq \theta\bar{m}(\alpha), \quad \forall \theta \in \left(1, \frac{c}{\alpha}\right].$$

If $\bar{m}(\alpha)$ is achieved by $u \in A_{k_0}(\alpha)$, then we can choose $u_n \equiv u$ in (5.10) and hence the strict inequality holds. Moreover, it follows that

$$\bar{m}(c) = \frac{c-\alpha}{c}\bar{m}(c) + \frac{\alpha}{c}\bar{m}(c) \leq \bar{m}(c-\alpha) + \bar{m}(\alpha)$$

with a strict inequality if $\bar{m}(c-\alpha)$ and $\bar{m}(\alpha)$ is achieved.

(ii) Let $\{\bar{c}_n\} \subset (0, c_0)$ be such that $\lim_{n \rightarrow \infty} \bar{c}_n = c \in (0, c_0)$. For any $\alpha \in (0, c_0)$, Lemma 5.3 implies that $\bar{m}(\alpha) < 0$. If $\bar{c}_n < c$, it follows from (i) that

$$\bar{m}(c) \leq \bar{m}(\bar{c}_n) + \bar{m}(c - \bar{c}_n) < \bar{m}(\bar{c}_n). \quad (5.11)$$

If $\bar{c}_n \geq c$, let $u_n \in A_{k_0}(\bar{c}_n)$ be such that $J(u_n) \leq \bar{m}(\bar{c}_n) + \frac{1}{n}$. Set $v_n = \sqrt{\frac{c}{\bar{c}_n}}u_n$; then $v_n \in A_{k_0}(c)$. Moreover, we have

$$\begin{aligned} \bar{m}(c) &\leq J(v_n) = J(u_n) + [J(v_n) - J(u_n)] \\ &= J(u_n) + \frac{a}{2(1-bS^2)} \frac{c-\bar{c}_n}{\bar{c}_n} |\nabla u|_2^2 + \frac{b}{4(1-bS^2)} \frac{c^2 - \bar{c}_n^2}{\bar{c}_n^2} |\nabla u|_2^4 \\ &\quad - \frac{\mu(c^{\frac{q}{2}} - \bar{c}_n^{\frac{q}{2}})}{q\bar{c}_n^{\frac{q}{2}}} |u|_q^q - \frac{c^2 - \bar{c}_n^2}{4\bar{c}_n^2} |u|_4^4 \\ &= J(u_n) + o(1) \\ &\leq \bar{m}(\bar{c}_n) + o(1). \end{aligned}$$

Combining this with (5.11), we have

$$\bar{m}(c) \leq \bar{m}(\bar{c}_n) + o(1). \quad (5.12)$$

For $\varepsilon > 0$ small, there exists $u \in A_{k_0}(c)$ such that $J(u) < \bar{m}(c) + \varepsilon$. Set $w_n = \sqrt{\frac{\bar{c}_n}{c}}u$. Then $w_n \in A_{k_0}(\bar{c}_n)$ for all sufficiently large $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} J(w_n) = J(u)$, it follows that

$$\bar{m}(\bar{c}_n) \leq J(w_n) = J(u) + [J(w_n) - J(u)] = J(u) + o(1) < \bar{m}(c) + \varepsilon + o(1).$$

Thus, since $\varepsilon > 0$ is arbitrary, we have

$$\bar{m}(\bar{c}_n) < \bar{m}(c) + o(1). \quad (5.13)$$

Both (5.12) and (5.13) conclude that $\lim_{n \rightarrow \infty} \bar{m}(\bar{c}_n) = \bar{m}(c)$. The proof is complete. $\square$

The proof of Theorem 2.8. (i) Let $\{u_n\} \subset A_{k_0}(c)$ be a minimizing sequence for $m(c)$. Since $\{|u_n|\} \subset A_{k_0}(c)$ is also a minimizing sequence for $m(c)$, we may assume that $\tilde{u}_n \geq 0$. Lemma 5.3 gives

$$0 < |\tilde{u}|_2^2 \leq |\tilde{u}_n|_2^2 = c, \quad |\nabla \tilde{u}_n|_2^2 < k_0 < +\infty, \quad I(\tilde{u}_n) = m(c) + o(1). \quad (5.14)$$

Hence, after passing to a subsequence, there exists $\tilde{u} \in H^1(\mathbb{R}^4)$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u} & \text{in } H_{rad}^1(\mathbb{R}^4), \\ \tilde{u}_n \rightarrow \tilde{u} & \text{in } L^s(\mathbb{R}^4), \quad \forall s \in (2, 4), \\ \tilde{u}_n(x) \rightarrow \tilde{u}(x) & \text{a.e. on } \mathbb{R}^4. \end{cases} \quad (5.15)$$

We know that $\tilde{u} \neq 0$. If $\tilde{u} = 0$, then, together with (2.1), (5.2), (5.3) and (5.14), we have

$$\begin{aligned} m(c) + o(1) &= I(\tilde{u}_n) = \frac{a}{2} |\nabla \tilde{u}_n|_2^2 + \frac{b}{4} |\nabla \tilde{u}_n|_2^4 - \frac{1}{4} |\tilde{u}_n|_4^4 - \frac{\mu}{q} |\tilde{u}_n|_q^q \\ &= \frac{a}{2} |\nabla \tilde{u}_n|_2^2 + \frac{b}{4} |\nabla \tilde{u}_n|_2^4 - \frac{1}{4} |\tilde{u}_n|_4^4 + o(1) \\ &\geq \frac{a}{2} |\nabla \tilde{u}_n|_2^2 - \frac{1 - bS^2}{4S^2} |\nabla \tilde{u}_n|_2^4 + o(1) \\ &\geq |\nabla \tilde{u}_n|_2^2 \left(\frac{a}{2} - \frac{1 - bS^2}{4S^2} k_0 \right) + o(1) \\ &= |\nabla \tilde{u}_n|_2^2 \left[f_c(k_0) + \frac{\mu C_q^q}{q} c^{\frac{4-q}{2}} k_0^{q-3} \right] + o(1) \\ &\geq o(1), \end{aligned}$$

which contradicts the fact that $m(c) < 0$. Thus, $\tilde{u} \neq 0$.

Set $v_n := \tilde{u}_n - \tilde{u}$. By (5.15) and the Brézis–Lieb lemma, we obtain

$$\begin{aligned} |\nabla \tilde{u}_n|_2^2 &= |\nabla \tilde{u}|_2^2 + |\nabla v_n|_2^2 + o(1), \\ |\nabla \tilde{u}_n|_4^4 &= |\nabla \tilde{u}|_4^4 + |\nabla v_n|_4^4 + 2|\nabla \tilde{u}|_2^2 |\nabla v_n|_2^2 + o(1) \end{aligned} \quad (5.16)$$

and

$$c = |\tilde{u}_n|_2^2 = |\tilde{u}|_2^2 + |v_n|_2^2 + o(1), \quad |\tilde{u}_n|_4^4 = |\tilde{u}|_4^4 + |v_n|_4^4 + o(1). \quad (5.17)$$

It follows that

$$m(c) = I(\tilde{u}_n) + o(1) = I(\tilde{u}) + I(v_n) + \frac{b}{2} |\nabla \tilde{u}|_2^2 |\nabla v_n|_2^2 + o(1). \quad (5.18)$$

We now prove that $v_n \rightarrow 0$ in $H^1(\mathbb{R}^4)$ and hence that $\tilde{u}_n \rightarrow \tilde{u}$ in $H^1(\mathbb{R}^4)$. The proof is divided into two steps.

First, we claim that $|v_n|_2^2 \rightarrow 0$ as $n \rightarrow \infty$, i.e., $|\tilde{u}|_2^2 = c$. Suppose, by contradiction, that $\tilde{c} := |\tilde{u}|_2^2 < c$. By (5.16) and (5.17), for all sufficiently large $n \in \mathbb{N}$, we have

$$\alpha_n := |v_n|_2^2 \leq c, \quad |\nabla v_n|_2^2 \leq |\nabla \tilde{u}_n|_2^2 < k_0,$$

which implies that $v_n \in A_{k_0}(\alpha_n)$ and $I(v_n) \geq m(\alpha_n)$. Then, together with (5.18), one obtains that

$$m(c) = I(\tilde{u}) + I(v_n) + \frac{b}{2} |\nabla \tilde{u}|_2^2 |\nabla v_n|_2^2 + o(1) \geq I(\tilde{u}) + m(\alpha_n) + o(1). \quad (5.19)$$

By Lemma 5.4(ii) and (5.17), we have

$$m(c) \geq I(\tilde{u}) + m(c - \tilde{c}). \quad (5.20)$$

We also have that $\tilde{u} \in \overline{A_{k_0}(\tilde{c})}$ by the weak limit, which implies that $I(\tilde{u}) \geq m(\tilde{c})$. If $I(\tilde{u}) > m(\tilde{c})$, then it follows from (5.20) and Lemma 5.4(i) that

$$m(c) > m(\tilde{c}) + m(c - \tilde{c}) \geq m(c),$$

which is a contradiction. If $I(\tilde{u}) = m(\tilde{c})$, then the strict inequality in Lemma 5.4(i) holds and

$$m(c) \geq m(\tilde{c}) + m(c - \tilde{c}) > m(c),$$

which is also a contradiction. Thus, $|\tilde{u}|_2^2 = c$ and $\tilde{u} \in \overline{A_{k_0}(c)}$ by the weak limit.

Second, we claim that $|\nabla v_n|_2^2 \rightarrow 0$ as $n \rightarrow \infty$. It follows from (5.2), (5.3), (5.18), and $I(\tilde{u}) \geq m(c)$ that

$$\begin{aligned} o(1) &\geq \frac{a}{2} |\nabla v_n|_2^2 + \frac{b}{4} |\nabla v_n|_2^4 - \frac{1}{4} |v_n|_4^4 + \frac{b}{2} |\nabla \tilde{u}|_2^2 |v_n|_2^2 + o(1) \\ &\geq \frac{a}{2} |\nabla v_n|_2^2 - \frac{1 - bS^2}{4S^2} |\nabla v_n|_2^4 + o(1) \geq |\nabla v_n|_2^2 \left(\frac{a}{2} - \frac{1 - bS^2}{4S^2} k_0 \right) + o(1) \\ &= |\nabla v_n|_2^2 \left[f_c(k_0) + \frac{\mu C_q^q}{q} c^{\frac{4-q}{2}} k_0^{q-3} \right] + o(1), \end{aligned}$$

which implies that $|\nabla v_n|_2^2 \rightarrow 0$ as $n \rightarrow \infty$. Together with the fact that $|v_n|_2^2 \rightarrow 0$ as $n \rightarrow \infty$, we have $v_n \rightarrow 0$ in $H^1(\mathbb{R}^4)$, i.e., $\tilde{u}_n \rightarrow \tilde{u}$ in $H^1(\mathbb{R}^4)$. Thus,

$$|\tilde{u}|_2^2 = c, \quad |\nabla \tilde{u}|_2^2 \leq k_0, \quad I(\tilde{u}) = m(c).$$

Combining this with Lemma 5.3, we know that $|\nabla \tilde{u}|_2^2 < k_0$ and henceforth $\tilde{u} \in A_{k_0}(c)$. By Corollary 2.4 in [7], one obtains that $I'_{|S_c}(\tilde{u}) = 0$ and henceforth there exists a lagrange multiplier $\tilde{\lambda}_c \in \mathbb{R}$ such that

$$-(a + b|\nabla \tilde{u}|_2^2) \Delta \tilde{u} + \tilde{\lambda}_c \tilde{u} = \tilde{u}^3 + \mu |\tilde{u}|^{q-2} \tilde{u}.$$

It is easy to check that $\tilde{\lambda}_c > 0$. Since $\tilde{u} \geq 0$ and $\tilde{u} \neq 0$, the strong maximum principle implies that $\tilde{u} > 0$.

(ii) Similarly to (i), Lemmas 5.3 and 5.4 yield a positive $\bar{u} \in A_{k_0}(c)$ such that $J(\bar{u}) = \bar{m}(c) < 0$ and, for some $\bar{\lambda}_c > 0$,

$$-\left[\frac{a}{(1 - bS^2)} + \frac{b}{(1 - bS^2)} |\nabla \bar{u}_c|_2^2 \right] \Delta \bar{u}_c + \bar{\lambda}_c \bar{u}_c = \bar{u}_c^3 + \mu |\bar{u}_c|^{q-2} \bar{u}_c. \quad (5.21)$$

The proof is complete. $\square$

5.2. THE PROOF OF THEOREM 2.10

In the subsection, we establish the threshold of the mountain pass level to prove the existence of the mountain-pass solution, thereby proving Theorem 2.10. By a standard argument, we obtain the Pohozaev-type functional related to (5.21) as follows.

Lemma 5.6. *Assume that $0 < b < S^{-2}$, $2 < q < 3$, $\mu > 0$ and $c \in (0, c_0)$. It holds that*

$$\frac{a}{(1-bS^2)}|\nabla \bar{u}_c|_2^2 + \frac{b}{(1-bS^2)}|\nabla \bar{u}_c|_2^4 - |\bar{u}_c|_4^4 - \frac{\mu(2q-4)}{q}|\bar{u}_c|_q^q = 0. \quad (5.22)$$

Lemma 5.7. *Assume that $0 < b < S^{-2}$, $2 < q < 3$, $\mu > 0$ and $c \in (0, c_0)$. Then there exists $d > 0$ such that*

$$M(c) := \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} I(\gamma(t)) \geq d > \sup_{\gamma \in \Gamma_c} \max\{I(\gamma(0)), I(\gamma(1))\}, \quad (5.23)$$

where

$$\Gamma_c := \{\gamma \in \mathcal{C}([0,1], S_c) : \gamma(0) = \bar{u}_c, I(\gamma(1)) < 2m(c)\}. \quad (5.24)$$

Proof. Set $d := \inf_{u \in \partial(A_{k_0}(c))} I(u)$. By (5.8), $d > 0$. Let $\gamma \in \Gamma_c$ be arbitrary. Lemma 5.3 gives $\gamma(0) = \bar{u}_c \in A_{k_0}(c) \setminus \partial A_{k_0}(c)$ and $I(\gamma(1)) < 2m(c) < m(c) < 0$. By (5.8), $\gamma(1) \notin A_{k_0}(c)$. Since γ is continuous on $[0,1]$, there exists $t_0 \in (0,1)$ such that $\gamma(t_0) \in \partial A_{k_0}(c)$. Hence, $\max_{t \in [0,1]} I(\gamma(t)) \geq d > 0$. Since $I(\gamma(0)) = I(\bar{u}_c) < J(\bar{u}_c) = \bar{m}(c) < 0$, (5.23) follows. The proof is complete. $\square$

Similar to Lemma 3.11 in [7], we obtain the following lemma.

Lemma 5.8. *Assume that $0 < b < S^{-2}$, $2 < q < 3$, $\mu > 0$ and $c \in (0, c_0)$. Then there exists a sequence $\{u_n\} \subset S_c$ such that*

$$I(u_n) \rightarrow M(c) > 0, \quad I'|_{S_c}(u_n) \rightarrow 0 \quad \text{and} \quad P(u_n) \rightarrow 0. \quad (5.25)$$

Set the functions $U_n(x) := T_n(|x|)$, $n > 1$, where

$$T_n(r) = 2\sqrt{2} \begin{cases} \frac{n}{1+n^2r^2}, & 0 \leq r < 1, \\ \frac{n}{1+n^2}(2-r), & 1 \leq r < 2, \\ 0, & r \geq 2. \end{cases} \quad (5.26)$$

It follows from results in [6] that

$$|U_n|_2^2 = O\left(\frac{\log n}{n^2}\right), \quad n \rightarrow \infty, \quad (5.27)$$

$$|\nabla U_n|_2^2 = S^2 + O\left(\frac{1}{n^2}\right), \quad n \rightarrow \infty, \quad (5.28)$$

$$|U_n|_4^4 = S^2 + O\left(\frac{1}{n^4}\right), \quad n \rightarrow \infty. \quad (5.29)$$

In the following, we consider the superposition of $\bar{u}$ and U_n to construct a test function in S_c . By some delicate calculations, we obtain the threshold of mountain pass level $M(c)$ as follows, which is a crucial step in proving Theorem 2.10.

Lemma 5.9. Assume that $0 < b < S^{-2}$, $2 < q < 3$, $\mu > 0$ and $c \in (0, c_0)$. It holds that

$$M(c) < \bar{m}(c) + \Lambda. \quad (5.30)$$

Proof. Let $\bar{u}_c \in S_c$ be given by Theorem 2.8(ii). Then

$$J(\bar{u}_c) = \bar{m}(c), \quad \bar{\lambda}_c |\bar{u}_c|_2^2 = \frac{\mu(4-q)}{q} |\bar{u}_c|_q^q, \quad \bar{u}_c(x) > 0. \quad (5.31)$$

Set

$$V_{n,t}(x) := \bar{u}_c(x) + tU_n(x), \quad W_{n,t}(x) := \tau V_{n,t}(\tau x). \quad (5.32)$$

Direct computation gives

$$|\nabla W_{n,t}|_2^2 = |\nabla V_{n,t}|_2^2, \quad |W_{n,t}|_4^4 = |V_{n,t}|_4^4 \quad (5.33)$$

and

$$|W_{n,t}|_2^2 = \tau^{-2} |V_{n,t}|_2^2, \quad |W_{n,t}|_q^q = \tau^{q-4} |V_{n,t}|_q^q. \quad (5.34)$$

If we set $\tau := \frac{|V_{n,t}|_2}{\sqrt{c}}$, then $W_{n,t} \in S_c$ and

$$\tau^2 = \frac{|V_{n,t}|_2^2}{c} = 1 + \frac{2t}{c} \int_{\mathbb{R}^4} \bar{u}_c U_n dx + \frac{t^2}{c} |U_n|_2^2. \quad (5.35)$$

Since $\bar{u}_c \in S_c$ is a local minimizer of J , Lemma 5.6 and (5.31) yield

$$\left[\frac{a}{(1-bS^2)} + \frac{b}{(1-bS^2)} |\nabla \bar{u}_c|_2^2 \right] \int_{\mathbb{R}^4} \nabla \bar{u}_c \nabla U_n dx = \int_{\mathbb{R}^4} (\bar{u}_c^3 + \mu \bar{u}_c^{q-1} - \bar{\lambda}_c \bar{u}_c) U_n dx. \quad (5.36)$$

Let $P(t) = S^2 \left[\frac{a+b|\nabla \bar{u}_c|_2^2}{2} t^2 - \frac{1-bS^2}{4} t^4 \right]$. By direct calculations, for any $t > 0$ and $t \neq t_*$, we can check that

$$\begin{aligned} P(t) &< P(t_*) = S^2 \left[\frac{(a+b|\nabla \bar{u}_c|_2^2)^2}{4(1-bS^2)} \right] \\ &= \frac{a^2 S^2}{4(1-bS^2)} + \frac{abS^2 |\nabla \bar{u}_c|_2^2}{2(1-bS^2)} + \frac{b^2 S^2 |\nabla \bar{u}_c|_2^4}{4(1-bS^2)} \\ &= \Lambda + \frac{abS^2 |\nabla \bar{u}_c|_2^2}{2(1-bS^2)} + \frac{b^2 S^2 |\nabla \bar{u}_c|_2^4}{4(1-bS^2)}, \end{aligned} \quad (5.37)$$

where

$$t_*^2 := \frac{a+b|\nabla \bar{u}_c|_2^2}{1-bS^2}. \quad (5.38)$$

Furthermore, (5.36) can be rewritten as

$$(a+b|\nabla \bar{u}_c|_2^2 + bS^2 t_*^2) \int_{\mathbb{R}^4} \nabla \bar{u}_c \nabla U_n dx = \int_{\mathbb{R}^4} (\bar{u}_c^3 + \mu \bar{u}_c^{q-1} - \bar{\lambda}_c \bar{u}_c) U_n dx. \quad (5.39)$$

Based on (5.26) and (5.27), we estimate the terms in (5.39) as follows

$$\int_{\mathbb{R}^4} \bar{u}_c U_n dx = O\left(\frac{\sqrt{\log n}}{n}\right), \quad n \rightarrow \infty, \quad (5.40)$$

$$\left| \int_{\mathbb{R}^4} \nabla \bar{u}_c \nabla U_n dx \right| = \left| \int_{\mathbb{R}^4} U_n \Delta \bar{u}_c dx \right| = O\left(\frac{\sqrt{\log n}}{n}\right), \quad n \rightarrow \infty, \quad (5.41)$$

$$\int_{\mathbb{R}^4} \bar{u}_c^{q-1} U_n dx \leq \left[\int_{\mathbb{R}^4} \bar{u}_c^{2(q-1)} dx \int_{|x| \leq 2} U_n^2 dx \right]^{\frac{1}{2}} = O\left(\frac{\sqrt{\log n}}{n}\right), \quad n \rightarrow \infty, \quad (5.42)$$

and

$$\begin{aligned} \int_{\mathbb{R}^4} \bar{u}_c U_n^3 dx &\geq 2\pi^2 B \int_0^1 r^3 |T_n(r)|^3 dr \geq 32\sqrt{2}\pi^2 B \int_0^1 \frac{n^3 r^3}{(1+n^2 r^2)^3} dr \\ &\geq \frac{32\sqrt{2}\pi^2 B}{n} \int_0^n \frac{s^3}{(1+s^2)^3} ds =: \frac{B_0}{n}, \end{aligned} \quad (5.43)$$

where $B := \inf_{|x| \leq 1} \bar{u}_c(x)$. From elliptic bootstrap, radial positive weak solutions are locally Hölder continuous, hence bounded away from zero on compact ball $B_1(0)$ and $B > 0$. If $p \geq 3$, one has,

$$(1+t)^p \geq 1 + pt + pt^{p-1} + t^p, \quad t \geq 0. \quad (5.44)$$

Moreover, if $p \geq 2$, then

$$(1+t)^p \geq 1 + pt^{p-1} + t^p, \quad t \geq 0. \quad (5.45)$$

It follows from (5.27)–(5.29), (5.31)–(5.35) and (5.37)–(5.45) that for any $t > 0$,

$$\begin{aligned} I(W_{n,t}) &= \frac{a}{2} |\nabla W_{n,t}|_2^2 + \frac{b}{4} |\nabla W_{n,t}|_4^4 - \frac{1}{4} |W_{n,t}|_4^4 - \frac{\mu}{q} |W_{n,t}|_q^q \\ &= \frac{a}{2} |\nabla(\bar{u}_c + tU_n)|_2^2 + \frac{b}{4} |\nabla(\bar{u}_c + tU_n)|_4^4 - \frac{1}{4} |\bar{u}_c + tU_n|_4^4 - \frac{\mu\tau^{q-4}}{q} |\bar{u}_c + tU_n|_q^q \\ &\leq \frac{a}{2} |\nabla \bar{u}_c|_2^2 + \frac{b}{4} |\nabla \bar{u}_c|_4^4 - \frac{1}{4} |\bar{u}_c|_4^4 - \frac{\mu\tau^{q-4}}{q} |\bar{u}_c|_q^q + \frac{at^2}{2} |\nabla U_n|_2^2 \\ &\quad + \frac{bt^4}{4} |\nabla U_n|_4^4 - \frac{t^4}{4} |U_n|_4^4 - t \int_{\mathbb{R}^4} \bar{u}_c^3 U_n dx - t^3 \int_{\mathbb{R}^4} \bar{u}_c U_n^3 dx \\ &\quad - \mu\tau^{q-4} t \int_{\mathbb{R}^4} \bar{u}_c^{q-1} U_n dx + \frac{bt^2}{2} |\nabla \bar{u}_c|_2^2 |\nabla U_n|_2^2 \\ &\quad + (a + b|\nabla \bar{u}_c|_2^2 + bt^2 |\nabla U_n|_2^2) t \int_{\mathbb{R}^4} \nabla \bar{u}_c \nabla U_n dx + bt^2 \left(\int_{\mathbb{R}^4} \nabla \bar{u}_c \nabla U_n dx \right)^2 \end{aligned} \quad (5.46)$$

$$\begin{aligned}
& I(W_{n,t}) \\
& \leq \frac{a}{2} |\nabla \bar{u}_c|_2^2 + \frac{b}{4} |\nabla \bar{u}_c|_2^4 - \frac{1}{4} |\bar{u}_c|_4^4 - \frac{\mu}{q} |\bar{u}_c|_q^q + \frac{at^2}{2} |\nabla U_n|_2^2 + \frac{bt^4}{4} |\nabla U_n|_2^4 \\
& \quad - \frac{t^4}{4} |U_n|_4^4 + \frac{\mu(1-\tau^{q-4})}{q} |\bar{u}_c|_q^q + \mu(1-\tau^{q-4})t \int_{\mathbb{R}^4} \bar{u}_c^{q-1} U_n dx - \bar{\lambda}_c t \int_{\mathbb{R}^4} \bar{u}_c U_n dx \\
& \quad + \frac{bt^2}{2} |\nabla \bar{u}_c|_2^2 |\nabla U_n|_2^2 + b(t^2 |\nabla U_n|_2^2 - t_*^2 S^2) t \int_{\mathbb{R}^4} \nabla \bar{u}_c \nabla U_n dx \\
& \quad - t^3 \int_{\mathbb{R}^4} \bar{u}_c U_n^3 dx + t^2 \left[O\left(\frac{\log n}{n^2}\right) \right] \\
& = \frac{a}{2} |\nabla \bar{u}_c|_2^2 + \frac{b}{4} |\nabla \bar{u}_c|_2^4 - \frac{1}{4} |\bar{u}_c|_4^4 - \frac{\mu}{q} |\bar{u}_c|_q^q + S^2 \left[\frac{a+b|\nabla \bar{u}_c|_2^2}{2} t^2 - \frac{1-bS^2}{4} t^4 \right] \\
& \quad + \frac{\mu |\bar{u}_c|_q^q}{q} \left\{ 1 - \left[1 + \frac{2t}{c} \int_{\mathbb{R}^4} \bar{u}_c U_n dx + t^2 \left(O\left(\frac{\log n}{n^2}\right) \right) \right]^{\frac{q-4}{2}} \right\} \\
& \quad - \bar{\lambda}_c t \int_{\mathbb{R}^4} \bar{u}_c U_n dx \tag{5.47} \\
& \quad + \mu \left\{ 1 - \left[1 + \frac{2t}{c} \int_{\mathbb{R}^4} \bar{u}_c U_n dx + t^2 O\left(\frac{\log n}{n^2}\right) \right]^{\frac{q-4}{2}} \right\} t \int_{\mathbb{R}^4} \bar{u}_c^{q-1} U_n dx \\
& \quad + bS^2(t^2 - t_*^2)t \int_{\mathbb{R}^4} \nabla \bar{u}_c \nabla U_n dx - t^3 \int_{\mathbb{R}^4} \bar{u}_c U_n^3 dx + (t^2 + t^4) \left[O\left(\frac{\log n}{n^2}\right) \right] \\
& \leq \frac{a}{2} |\nabla \bar{u}_c|_2^2 + \frac{b}{4} |\nabla \bar{u}_c|_2^4 - \frac{1}{4} |\bar{u}_c|_4^4 - \frac{\mu}{q} |\bar{u}_c|_q^q + S^2 \left[\frac{a+b|\nabla \bar{u}_c|_2^2}{2} t^2 - \frac{1-bS^2}{4} t^4 \right] \\
& \quad - \frac{B_0 t^3}{n} + bS^2(t^2 - t_*^2)t \left[O\left(\frac{\sqrt{\log n}}{n}\right) \right] + (t^2 + t^4) \left[O\left(\frac{\log n}{n^2}\right) \right] \\
& = I(\bar{u}_c) + P(t) + K_n(t) \\
& = I(\bar{u}_c) + P(t_*) + (P(t) - P(t_*)) + K_n(t) \\
& = I(\bar{u}_c) + \Lambda + \frac{abS^2 |\nabla \bar{u}_c|_2^2}{2(1-bS^2)} + \frac{b^2 S^2 |\nabla \bar{u}_c|_2^4}{4(1-bS^2)} + P_1(t) + K_n(t) \\
& = J(\bar{u}_c) + \Lambda + K(n, t)
\end{aligned}$$

where

$$P_1(t) := P(t) - P(t_*) := S^2 \left[\frac{a+b|\nabla \bar{u}_c|_2^2}{2} - \frac{1-bS^2}{4} (t^2 + t_*^2) \right] (t^2 - t_*^2),$$

and

$$K_n(t) := -\frac{B_0 t^3}{n} + bS^2(t^2 - t_*^2)tO\left(\frac{\sqrt{\log n}}{n}\right) + (t^2 + t^4)O\left(\frac{\log n}{n^2}\right),$$

and $K(n, t) := P_1(t) + K_n(t)$. It suffices to prove that $\sup_{t>0} K(n, t) < 0$ for n large enough. In fact, for $t^2 \geq T_0^2$ or $0 < t^2 \leq t_*^2$, there exists n_0 large enough such that for $n \geq n_0$,

$$K(n, t) < 0 \quad (5.48)$$

since the sign of t^4 or $t^2 - t_*^2$ is negative. For $t > 0$ such that $t_*^2 \leq t^2 \leq t_*^2 + \frac{\log n}{n}$, there holds,

$$\begin{aligned} K(n, t) &= P_1(t) + K_n(t) \leq K_n(t) \\ &= -\frac{B_0 t^3}{n} + bS^2(t^2 - t_*^2)tO\left(\frac{\sqrt{\log n}}{n}\right) + (t^2 + t^4)O\left(\frac{\log n}{n^2}\right) \\ &\leq -\frac{B_0 t^3}{n} + bS^2 tO\left(\frac{(\log n)^{\frac{3}{2}}}{n^2}\right) + (t^2 + t^4)O\left(\frac{\log n}{n^2}\right) < 0, \end{aligned} \quad (5.49)$$

for large n . For $t > 0$ such that $t_*^2 + \frac{\log n}{n} \leq t^2 \leq T_0$, combining this condition with (5.37) gives

$$\begin{aligned} K(n, t) &= P_1(t) + K_n(t) \\ &< -\frac{1 - bS^2 \log n}{4} \frac{t^2 - t_*^2}{n} - \frac{B_0 t^3}{n} + bS^2(t^2 - t_*^2)tO\left(\frac{\sqrt{\log n}}{n}\right) \\ &\quad + (t^2 + t^4)O\left(\frac{\log n}{n^2}\right) < 0, \end{aligned} \quad (5.50)$$

for large n . Thus, combining with (5.48), (5.49) and (5.50), there exists $n_0 \in \mathbb{N}$ such that for $\bar{n} > n_0$, $\sup_{t>0} K(n, t) < 0$ and

$$\sup_{t>0} I(W_{\bar{n}, t}) < \bar{m}(c) + \Lambda. \quad (5.51)$$

By (5.32)–(5.35), we have $W_{\bar{n}, t} = \bar{\tau} V_{\bar{n}, t}(\bar{\tau} x)$, $|W_{\bar{n}, t}|_2^2 = c$ and

$$|\nabla W_{\bar{n}, t}|_2^2 = |\nabla(\bar{u}_c + tU_{\bar{n}})|_2^2 = |\nabla \bar{u}_c|_2^2 + t^2 |\nabla U_{\bar{n}}|_2^2 + 2t \int_{\mathbb{R}^4} \nabla \bar{u}_c \nabla U_{\bar{n}} dx,$$

where $\bar{\tau}^2 = 1 + \frac{2t}{c} \int_{\mathbb{R}^4} \bar{u}_c U_{\bar{n}} dx + \frac{t^2}{c} |U_{\bar{n}}|_2^2$, which implies that $W_{\bar{n}, t} \in S_c$ for all $t > 0$, $W_{\bar{n}, 0} = \bar{u}_c$, and there exists a $\bar{t} > 0$ large enough such that $I(W_{\bar{n}, \bar{t}}) < 2m(c)$. Let $\gamma_{\bar{n}}(t) := W_{\bar{n}, t\bar{t}}$, then $\gamma_{\bar{n}}(0) = \bar{u}_c$ and $I(\gamma_{\bar{n}}(1)) = I(W_{\bar{n}, \bar{t}}) < 2m(c)$. Thus, $\gamma_{\bar{n}} \in \Gamma$ defined in (5.7). Finally, it follows from (5.23) and (5.51) that

$$M(c) \leq \max_{t \in [0, 1]} I(\gamma_{\bar{n}}(t)) \leq \sup_{t>0} I(W_{\bar{n}, t}) < \bar{m}(c) + \Lambda.$$

The proof is complete. $\square$

The proof of Theorem 2.10. By Lemma 5.8, there exists a sequence $\{u_n\} \subset S_c$ such that

$$I(u_n) \rightarrow M(c), \quad I|'_{S_c}(u_n) \rightarrow 0 \quad \text{and} \quad P(u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.52)$$

Using (2.3) and (5.52), one obtains that

$$\begin{aligned} M(c) + o(1) &= I(u_n) - \frac{1}{4}P(u_n) = \frac{a}{4}|\nabla u_n|_2^2 - \frac{\mu(4-q)}{2q}|u_n|_q^q \\ &\geq \frac{a}{4}|\nabla u_n|_2^2 - \frac{\mu(4-q)}{2q}C_q^q c^{\frac{4-q}{2}}|\nabla u_n|_2^{2q-4}, \end{aligned}$$

which implies, since $2 < q < 3$, that $\{u_n\}$ is bounded in $H_{rad}^1(\mathbb{R}^4)$. Thus, after passing to a subsequence, there exists $\hat{u} \in H_{rad}^1(\mathbb{R}^4)$ such that

$$\begin{cases} u_n \rightharpoonup \hat{u} & \text{in } H_{rad}^1(\mathbb{R}^4), \\ u_n \rightarrow \hat{u} & \text{in } L^s(\mathbb{R}^4), \quad \forall s \in (2, 4), \\ u_n(x) \rightarrow \hat{u}(x), & \text{a.e. on } \mathbb{R}^4. \end{cases}$$

By (5.52) and the Lagrange multiplier rule, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\begin{aligned} &a \int_{\mathbb{R}^4} \nabla u_n \nabla \varphi dx + b |\nabla u_n|_2^2 \int_{\mathbb{R}^4} \nabla u_n \nabla \varphi dx - \int_{\mathbb{R}^4} |u_n|^2 u_n \varphi dx \\ &- \mu \int_{\mathbb{R}^4} |u_n|^{q-2} u_n \varphi dx + \int_{\mathbb{R}^4} \lambda_n u_n \varphi dx = o(1) \|\varphi\|, \quad \forall \varphi \in H_{rad}^1(\mathbb{R}^4). \end{aligned} \quad (5.53)$$

In particular, taking $\varphi = u_n$ in (5.53), then we have

$$-\lambda_n c = a |\nabla u_n|_2^2 + b |\nabla u_n|_2^4 - |u_n|_4^4 - \mu |u_n|_q^q + o(1). \quad (5.54)$$

Since $\{u_n\}$ is bounded in $H_{rad}^1(\mathbb{R}^4)$, the sequence $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Thus, after passing to a subsequence, there exists $\hat{\lambda}_c \in \mathbb{R}$ such that $\lambda_n \rightarrow \hat{\lambda}_c$ as $n \rightarrow \infty$.

We now claim that $\hat{u} \neq 0$. Indeed, if $\hat{u} = 0$, then $u_n \rightarrow 0$ in $L^q(\mathbb{R}^4)$. It follows from (2.3) and (5.52) that

$$o(1) = a |\nabla u_n|_2^2 + b |\nabla u_n|_2^4 - |u_n|_4^4 \geq a |\nabla u_n|_2^2 + (b - S^{-2}) |\nabla u_n|_2^4.$$

Set $A := \lim_{n \rightarrow \infty} |\nabla u_n|_2 \geq 0$. Then $aA^2 + (b - S^{-2})A^4 \leq 0$, which implies that either $A^2 \geq \frac{aS^2}{1-bS^2}$ or $A = 0$. If $A^2 \geq \frac{aS^2}{1-bS^2}$, combining this with (5.52) gives

$$M(c) + o(1) = I(u_n) - \frac{1}{4}P(u_n) = \frac{a}{4}|\nabla u_n|_2^2 \geq \frac{a^2 S^2}{4(1-bS^2)} = \Lambda,$$

which contradicts (5.30). If $A = 0$, it follows from (5.52) that $M(c) = 0$, which is impossible. Therefore, $\hat{u} \neq 0$. Moreover, in view of (5.52), (5.54), and $2 < q < 3$, we have

$$\begin{aligned}\hat{\lambda}_c c &= \lim_{n \rightarrow \infty} \lambda_n c = \lim_{n \rightarrow \infty} -(a|\nabla u_n|_2^2 + b|\nabla u_n|_2^4 - |u_n|_4^4 - \mu|u_n|_q^q) \\ &= \lim_{n \rightarrow \infty} \mu\left(\frac{4}{q} - 1\right)|u_n|_q^q = \mu\left(\frac{4}{q} - 1\right)|\hat{u}|_q^q > 0,\end{aligned}$$

which implies that $\hat{\lambda}_c > 0$.

Letting $n \rightarrow \infty$ in (5.53), we have

$$\begin{aligned}&a \int_{\mathbb{R}^4} \nabla \hat{u} \nabla \varphi dx + bA^2 \int_{\mathbb{R}^4} \nabla \hat{u} \nabla \varphi dx + \int_{\mathbb{R}^4} \hat{\lambda}_c \hat{u} \varphi dx \\ &= \int_{\mathbb{R}^4} |\hat{u}|^2 \hat{u} \varphi dx + \mu \int_{\mathbb{R}^4} |\hat{u}|^{q-2} \hat{u} \varphi dx,\end{aligned}\tag{5.55}$$

which means that $\hat{u}$ is a weak solution of

$$-(a + bA^2)\Delta u + \hat{\lambda}_c u = \mu|u|^{q-2}u + |u|^2u \quad \text{in } \mathbb{R}^4.$$

Then, $\hat{u}$ satisfies the following Pohozaev identity

$$(a + bA^2)|\nabla \hat{u}|_2^2 + 2\hat{\lambda}_c |\hat{u}|_2^2 - \frac{4\mu}{q} |\hat{u}|_q^q - |\hat{u}|_4^4 = 0.\tag{5.56}$$

Let $\varphi = \hat{u}$ in (5.55), together with (5.56), we can deduce that

$$\hat{P}(\hat{u}) := (a + bA^2)|\nabla \hat{u}|_2^2 - |\hat{u}|_4^4 - \frac{2\mu(q-2)}{q} |\hat{u}|_q^q = 0.\tag{5.57}$$

Set $v_n := u_n - \hat{u}$. By the Brézis–Lieb lemma, we have

$$A^2 = |\nabla u_n|_2^2 + o(1) = |\nabla v_n|_2^2 + |\nabla \hat{u}|_2^2 + o(1) \quad \text{and} \quad |u_n|_4^4 = |v_n|_4^4 + |\hat{u}|_4^4 + o(1).\tag{5.58}$$

It follows from (5.52), (5.57) and (5.58) that

$$\begin{aligned}o(1) &= P(u_n) = a|\nabla u_n|_2^2 + b|\nabla u_n|_2^4 - |u_n|_4^4 - \frac{2\mu(q-2)}{q} |u_n|_q^q \\ &= \hat{P}(\hat{u}) + (a + bA^2)|\nabla v_n|_2^2 - |v_n|_4^4 + o(1) \\ &= (a + bA^2)|\nabla v_n|_2^2 - |v_n|_4^4 + o(1) \\ &= (a + b|\nabla \hat{u}|_2^2)|\nabla v_n|_2^2 + b|\nabla v_n|_2^4 - |v_n|_4^4 + o(1),\end{aligned}\tag{5.59}$$

together with (5.52) and (5.58), one obtains that

$$\begin{aligned}M(c) + o(1) &= \frac{a}{2} |\nabla u_n|_2^2 + \frac{b}{4} |\nabla u_n|_2^4 - \frac{1}{4} |u_n|_4^4 - \frac{\mu}{q} |u_n|_q^q \\ &= \frac{a}{2} |\nabla v_n|_2^2 + \frac{b}{4} |\nabla v_n|_2^4 - \frac{1}{4} |v_n|_4^4 + \frac{b}{2} |\nabla \hat{u}|_2^2 |\nabla v_n|_2^2 + I(\hat{u}) + o(1) \\ &= \frac{a}{4} |\nabla v_n|_2^2 + \frac{b}{4} |\nabla \hat{u}|_2^2 |\nabla v_n|_2^2 + I(\hat{u}) + o(1).\end{aligned}\tag{5.60}$$

Set $l := \lim_{n \rightarrow \infty} |\nabla v_n|_2^2 \geq 0$. It follows from (2.3) and (5.59) that

$$\begin{aligned} (a + b|\nabla \hat{u}|_2^2)l + bl^2 &= \lim_{n \rightarrow \infty} (a + b|\nabla \hat{u}|_2^2)|\nabla v_n|_2^2 + b|\nabla v_n|_2^4 = \lim_{n \rightarrow \infty} |v_n|_4^4 \\ &\leq \lim_{n \rightarrow \infty} S^{-2}|\nabla v_n|_2^4 = S^{-2}l^2, \end{aligned}$$

which implies that either $l \geq \frac{(a+b|\nabla \hat{u}|_2^2)S^2}{1-bS^2}$ or $l = 0$. If $l \geq \frac{(a+b|\nabla \hat{u}|_2^2)S^2}{1-bS^2}$, we consider two cases: $|\nabla \hat{u}|_2^2 < k_0$ and $|\nabla \hat{u}|_2^2 \geq k_0$.

(i) If $|\nabla \hat{u}|_2^2 < k_0$, Lemmas 5.3 and 5.5 give

$$J(\hat{u}) \geq \bar{m}(|\hat{u}|_2^2) \geq \bar{m}(c).$$

Combining this with (5.60), we obtain

$$\begin{aligned} M(c) + o(1) &= \frac{a}{4}|\nabla v_n|_2^2 + \frac{b}{4}|\nabla \hat{u}|_2^2|\nabla v_n|_2^2 + I(\hat{u}) + o(1) \\ &= \frac{a + b|\nabla \hat{u}|_2^2}{4}l + I(\hat{u}) + o(1) \\ &\geq \frac{(a + b|\nabla \hat{u}|_2^2)^2 S^2}{4(1 - bS^2)} + I(\hat{u}) + o(1) \\ &= \frac{a^2 S^2}{4(1 - bS^2)} + \frac{abS^2|\nabla \hat{u}|_2^2}{2(1 - bS^2)} + \frac{b^2 S^2|\nabla \hat{u}|_2^4}{4(1 - bS^2)} + I(\hat{u}) + o(1) \\ &= \Lambda + J(\hat{u}) + o(1) \\ &\geq \Lambda + \bar{m}(c) + o(1), \end{aligned}$$

which contradicts (5.30).

(ii) If $|\nabla \hat{u}|_2^2 \geq k_0$, equations (2.3), (2.10), (5.57), (5.58), and (5.60) yield

$$\begin{aligned} M(c) + o(1) &= \frac{a}{4}|\nabla v_n|_2^2 + \frac{b}{4}|\nabla \hat{u}|_2^2|\nabla v_n|_2^2 + I(\hat{u}) - \frac{1}{4}\hat{P}(\hat{u}) + o(1) \\ &= \frac{a}{4}|\nabla \hat{u}|_2^2 + \frac{a}{4}|\nabla v_n|_2^2 - \frac{\mu(4-q)}{2q}|\hat{u}|_q^q + o(1) \\ &= \frac{a}{4}|\nabla \hat{u}|_2^2 + \frac{a}{4}l - \frac{\mu(4-q)}{2q}|\hat{u}|_q^q + o(1) \\ &\geq \frac{a^2 S^2}{4(1 - bS^2)} + \frac{a}{4(1 - bS^2)}|\nabla \hat{u}|_2^2 - \frac{\mu(4-q)}{2q}|\hat{u}|_q^q + o(1) \\ &\geq \frac{a^2 S^2}{4(1 - bS^2)} + \frac{a}{4(1 - bS^2)}|\nabla \hat{u}|_2^2 - \frac{\mu(4-q)}{2q}C_q^q c^{\frac{4-q}{2}}|\nabla \hat{u}|_2^{2q-4} + o(1) \\ &= \frac{a^2 S^2}{4(1 - bS^2)} + \left[\frac{a}{4(1 - bS^2)}|\nabla \hat{u}|_2^{6-2q} - \frac{\mu(4-q)}{2q}C_q^q c^{\frac{4-q}{2}} \right] |\nabla \hat{u}|_2^{2q-4} + o(1) \\ &\geq \frac{a^2 S^2}{4(1 - bS^2)} + \left[\frac{a}{4(1 - bS^2)}k_0^{3-q} - \frac{\mu(4-q)}{2q}C_q^q c^{\frac{4-q}{2}} \right] |\nabla \hat{u}|_2^{2q-4} + o(1) \\ &\geq \Lambda + o(1), \end{aligned}$$

which also contradicts (5.30). Therefore, $l = 0$. Hence, $|\nabla u_n|_2 \rightarrow |\nabla \hat{u}|_2$ and $|u_n|_4 \rightarrow |\hat{u}|_4$ as $n \rightarrow \infty$. Moreover, taking $\varphi = u_n - \hat{u}$ in (5.53) and (5.55), respectively, and subtracting the resulting identities, we obtain

$$(a + bA^2)|\nabla(u_n - \hat{u})|_2^2 + \hat{\lambda}_c|u_n - \hat{u}|_2^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies that $|u_n|_2 \rightarrow |\hat{u}|_2$ as $n \rightarrow \infty$. Thus, $u_n \rightarrow \hat{u}$ in $H_{rad}^1(\mathbb{R}^4)$. The proof is complete. $\square$

Acknowledgements

This work was supported by Guangdong Basic and Applied Basic Research Foundation (No: 2026A1515012458), National Natural Science Foundation of China (No: 12331017).

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Received: August 5, 2026.

Revised: October 7, 2026.

Accepted: October 8, 2026.

Published online: October 28, 2026.