

MUTUAL-VISIBILITY OF THE DISJOINTNESS GRAPH OF SEGMENTS IN $\mathbb{R}^2$

Jesús Leaños, Mario Lomelí-Haro, Christophe Ndjatchi,
and Luis Manuel Ríos-Castro

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Abstract. Let $G = (V(G), E(G))$ be a simple graph, and let $U \subseteq V(G)$. Two distinct vertices $x, y \in U$ are U -mutually visible if G contains a shortest x - y path that is internally disjoint from U . U is called a mutual-visibility set of G if any two vertices of U are U -mutually visible. The mutual-visibility number $\mu(G)$ of G is the size of a largest mutual-visibility set of G . Let P be a set of $n \geq 3$ points in $\mathbb{R}^2$ in general position. The disjointness graph of segments $D(P)$ of P is the graph whose vertices are all the closed straight line segments with endpoints in P , two of which are adjacent in $D(P)$ if and only if they are disjoint. In this paper we establish tight lower and upper bounds for $\mu(D(P))$, and show that $D(P)$ has diameter 2 for every point set with n at least 9.

Keywords: disjointness graph of segments, mutual-visibility, straight-line drawings of complete graphs.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph. We recall that a sequence $H := u_0, u_1, \dots, u_m$ of pairwise distinct vertices is a *path* of G if $u_i u_{i+1} \in E(G)$ for all $0 \leq i < m$. The *length* of a path is its number of edges. If u and v are vertices of $V(G)$, then a u - v path is a path with endpoints u and v . The graph G is said to be *connected* if for every pair of distinct vertices $u, v \in V(G)$, there exists a u - v path in G . If G is not connected, then G is *disconnected* and a maximal connected subgraph of G is a *component* of G . In a connected graph G , the *distance* $d_G(u, v)$ between u and v is the length of a shortest u - v path, and the *diameter* $\text{diam}(G)$ of G is the greatest distance between any pair of vertices of G . If G is disconnected, then $\text{diam}(G) = \infty$.

Let U be a subset of $V(G)$. Two vertices of U are U -mutually visible if G contains a shortest path between them that does not contain any other vertices of U . U is called a *mutual-visibility set* if its points are pairwise mutually visible. The *mutual-visibility number* of G , denoted by $\mu(G)$, is the size of any largest mutual-visibility set of G .

It is not hard to see that if G is disconnected and $G_1, \dots, G_k$ are the components of G , then $\mu(G) = \max\{\mu(G_1), \dots, \mu(G_k)\}$.

Let P be a set of $n \geq 3$ points in the plane in general position, i.e., no three points in P are collinear. The *disjointness graph of segments* $D(P)$ is the graph whose vertices correspond to the closed straight-line segments with endpoints in P . Two vertices are adjacent if and only if the corresponding segments are disjoint. Figure 1(a) illustrates a 7-point set P in general position, its associated set of segments $\mathcal{P}$ in Figure 1(b), and the corresponding graph $D(P)$ together with a maximum mutual-visibility set in Figure 1(c).

Our goal in this paper is to study the mutual-visibility number of the disjointness graphs of segments.

The following two subsections provide a brief historical overview of mutual visibility and disjointness graphs of segments, highlighting key prior results.

1.1. MUTUAL VISIBILITY IN GRAPHS

In 2022, Di Stefano introduced the concept of mutual visibility [9], drawing inspiration from classical Euclidean geometry problems like Sylvester’s and the “no-three-in-line” problems. Di Stefano [9] demonstrated that identifying a mutual-visibility set exceeding a specified size is NP-complete, while verifying the mutual visibility of a given set is solvable in polynomial time. Initial connections to other graph-theoretic parameters were also established.

Building on this foundation, Cicerone, Di Stefano, and Klavžar [6] reduced the computation of $\mu(K_m \square K_n)$ to the well-known Zarankiewicz problem [23] and characterized triangle-free graphs G with $\mu(G) = 3$.

In 2024, Cicerone, Di Stefano, Klavžar, and Yero [7] introduced variations of mutual visibility, including total, outer, and dual mutual visibility, and studied the corresponding parameters in diameter-two graphs. Their analysis focused on these parameters within diameter-two graphs, exploring potential research avenues and examining computational complexity.

Another variant, the “mobile mutual-visibility number” $\text{Mob}_\mu(G)$, was defined in 2025 by Dettlaff *et al.* [8], where robots placed on a mutual-visibility set must maintain the property while traversing the graph. It was shown that $\text{Mob}_\mu(G) \leq \mu(G)$, with subsequent work identifying graphs where equality holds and proving that finding the mobile mutual-visibility number is NP-hard.

Finally, exact values for the mutual-visibility number in Kneser, bipartite Kneser, and Johnson graphs were provided by Boruzanlı Ekinci and Bujtás in 2025 [3]. Their work also established connections between total mutual visibility in Kneser and bipartite Kneser graphs and the minimum size of covering designs. Further results on this topic can be found in the references of [3, 4, 7, 8].

1.2. DISJOINTNESS GRAPHS OF SEGMENTS IN THE PLANE

The concept of disjointness graphs of segments in the plane was introduced by Araujo *et al.* in 2005 [2] as a geometric counterpart to Kneser graphs. Recall that for $k, m \in \mathbb{Z}^+$

with $k \leq m/2$, the Kneser graph $KG(m, k)$ has vertices representing the k -subsets of $\{1, 2, \dots, m\}$, with edges connecting disjoint k -subsets. In 1978 Lovász [19] proved that $\chi(KG(m, k)) = m - 2k + 2$, which was conjectured in 1955 by Kneser [16].

The chromatic number $\chi(D(P))$ of $D(P)$ has been investigated in [2, 12–15]. However, determining the exact value of $\chi(D(P))$ remains an open problem. A general lower bound for $\chi(D(P))$ was established in [2]. The exact value of $\chi(D(P))$ is known for two families of point sets: for sets of n points in convex position C_n ,

$$\chi(D(C_n)) = n - \left\lfloor \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rfloor$$

(see [12, 14]), and for double chains with k points in the upper convex chain and $l \geq k$ points in the lower convex chain $C_{l,k}$,

$$\chi(D(C_{l,k})) = k + l - \left\lfloor \sqrt{2l + \frac{1}{4}} - \frac{1}{2} \right\rfloor$$

(see [13]). It is well known that $\chi(D(P)) \leq |P| - 2$, and recently García-Davila *et al.* [15] proved that this upper bound is attained only if $|P| \in \{3, 4, \dots, 16\}$.

Pach, Tardos, and Tóth investigated the chromatic number and clique number of $D(P)$ in $\mathbb{R}^d$ for $d \geq 2$ in [20]. Specifically, they demonstrated that $\chi(D(P))$ is bounded above by a polynomial function of its clique number $\omega(D(P))$ and that determining either $\chi(D(P))$ or $\omega(D(P))$ is NP-hard.

The connectivity number $\kappa(D(P))$ of $D(P)$ has been studied in [11] and [17], establishing the following upper and lower bounds, respectively:

$$\binom{\lfloor \frac{n-2}{2} \rfloor}{2} + \binom{\lceil \frac{n-2}{2} \rceil}{2} \leq \kappa(D(P)) \leq \frac{7n^2}{18} + \Theta(n).$$

Finally, Aichholzer *et al.* [1] established an asymptotic upper bound on the maximum size of certain independent sets in $D(P)$, and Leaños *et al.* [18] more recently provided a complete characterization of all those segment disjointness graphs that are Hamiltonian.

1.3. BASIC NOTATIONS AND CONVENTIONS

Throughout this paper, P is a set of $n \geq 3$ points in general position in the plane. If x and y are distinct points in P , then we will use xy to denote the closed line segment with endpoints x and y . Similarly, $\mathcal{P}$ will denote the set of segments $\{xy : x, y \in P \text{ and } x \neq y\}$, and its elements will be referred to as the *segments* of $\mathcal{P}$. Thus, $\mathcal{P}$ constitutes the vertex set of $D(P)$. Often, no distinction will be made between an element of $\mathcal{P}$ and its corresponding vertex in $D(P)$.

If $e \in \mathcal{P}$ and x is an endpoint of e , we simply write $x \in e$. We shall use $\ell(xy)$ to denote the directed line spanned by x and y , directed from x towards y , and $\ell^+(xy)$ (respectively, $\ell^-(xy)$) to denote the open half-plane lying on the right (respectively, left) side of $\ell(xy)$. We remark that $\mathcal{P}$ naturally defines a straight-line drawing of the

complete graph K_n in the plane. Let x_1y_1 and x_2y_2 be two distinct elements of $\mathcal{P}$, and assume that $x_1y_1 \cap x_2y_2 \neq \emptyset$. Then, their intersection consists exactly of one point $o \in \mathbb{R}^2$, since P is in general position. If o is an interior point of both x_1y_1 and x_2y_2 , we say that they *cross* at o . If the segment $xy \in \mathcal{P}$ is not crossed by any other segment of $\mathcal{P}$, then we will say that xy is *clean* in $\mathcal{P}$.

If $m \in \{3, 4, \dots\}$ and $q_1, q_2, q_3, \dots, q_m$ are points in general position in the plane, then $Q := q_1q_2 \cdots q_m$ will denote the (not necessarily simple) polygon formed by the segments $q_1q_2, q_2q_3, \dots, q_{m-1}q_m, q_mq_1$. If $m = 3$ (resp. $m = 4$), then Q will be called a *triangle* (resp. *quadrilateral*), and so on.

As usual, we will denote by $CH(P)$ the boundary of the convex hull of P , and by $\hat{P}$ the set $P \cap CH(P)$. If x and y are distinct points of $\hat{P}$ and $xy \notin CH(P)$, then xy is a *diagonal* of $\mathcal{P}$. See Figure 1(b).

If m is a nonnegative integer, then $[m] := \emptyset$ if $m = 0$, and $[m] := \{1, \dots, m\}$ if $m > 0$.

1.4. MAIN RESULTS

Our goal in this section is to present our main results about the mutual-visibility number of the disjointness graphs of segments.

Recall that if G is a disconnected graph and $G_1, \dots, G_k$ are its connected components, then $\mu(G) = \max\{\mu(G_1), \dots, \mu(G_k)\}$. Furthermore, $D(P)$ is connected if and only if $n = |P| \geq 5$ (see for instance [11]). In light of these two facts, for the remainder of this paper, it suffices to consider the case where P is an n -point set in general position in the plane with $n \geq 5$, which ensures that $D(P)$ is always connected.

Our main results are the following.

Theorem 1.1. *Let P be a set of $n \geq 5$ points in general position in the plane. Then $\text{diam}(D(P)) \in \{2, 3, 4\}$ if $n = 5$, $\text{diam}(D(P)) \in \{2, 3\}$ if $n \in \{6, 7, 8\}$, and $\text{diam}(D(P)) = 2$ if $n \geq 9$.*

Theorem 1.2. *Let P be a set of $n \geq 5$ points in general position in the plane. Then $\mu(D(P)) \geq \binom{n}{2} - 9$.*

Proposition 1.3. *Consider the set*

$$P = \{p_1 = (121, 204), p_2 = (175, 196), p_3 = (216, 82), p_4 = (189, 51), \\ p_5 = (44, 96), p_6 = (36, 140), p_7 = (127, 135)\}$$

of 7 points in general position, as depicted in Figure 1(a). For this 7-point set P , we have $\mu(D(P)) = \binom{7}{2} - 9 = 12$.

We remark that, with the 7-point set P in Proposition 1.3, the value of $\mu(D(P))$ attains the bound established in Theorem 1.2.

Theorem 1.4. *Let P be a set of n points in general position in the plane. If n is big enough, then $\binom{n}{2} - 5 \leq \mu(D(P)) \leq \binom{n}{2} - 4$.*

Proposition 1.5. *Let C_n be the set of n points in general and convex position in the plane. Then $\mu(D(C_n)) = \binom{n}{2} - 5$ for each integer $n \geq 10$.*

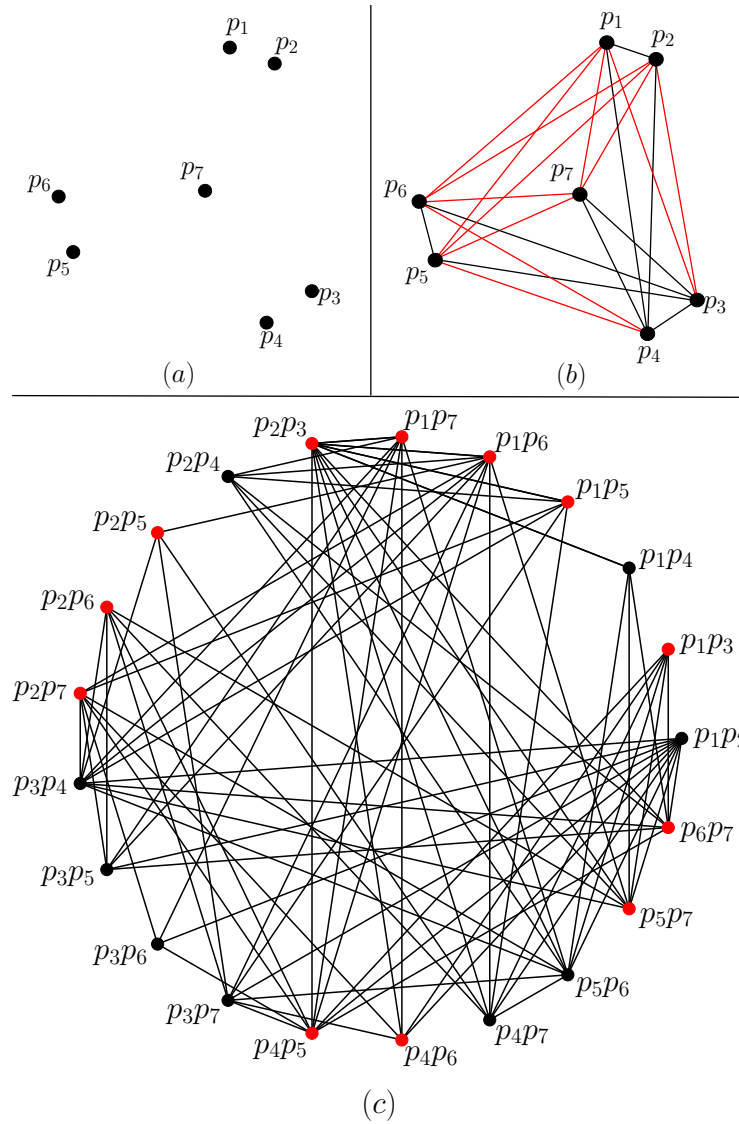


Fig. 1. The 7-point set $P = \{p_1, p_2, \dots, p_7\}$ in general position is shown in (a). In particular, we note that $\hat{P} = P \setminus \{p_7\}$. In (b) we have $\mathcal{P}$, which can be seen as the straight-line drawing of K_7 induced by P . Observe that $CH(P)$ is the convex hexagon formed by the union of segments $p_1p_2, p_2p_3, p_3p_4, p_4p_5, p_5p_6$, and p_6p_1 . The graph in (c) is $D(P)$ and has diameter 3. The red vertex set is a mutual-visibility set of $D(P)$ of size 12. The red vertices in (c) correspond to the red segments in (b)

Let A be the arc of the circumference that goes from $(-5, 2)$ to $(5, 2)$ and pass through the point $(0, 3/2)$. Let B be the reflection of A along the x -axis. Then any point of A (resp. B) is above (resp. below) every straight line spanned by two points of B (resp. A). If $p, q \geq 1$ are integers, then the set of $(p + q)$ points $C_{p,q}$ that results by choosing p points of A and q points of B is a *double chain*.

Proposition 1.6. *Let $p \geq 3$ and $q \geq 6$ be integers. Then $\mu(D(C_{p,q})) = \binom{p+q}{2} - 4$.*

Again, we remark that Propositions 1.5 and 1.6 imply that the asymptotic bounds for $\mu(D(P))$ given by Theorem 1.4 are both best possible.

The remainder of this paper is organized as follows. Section 2 presents Theorem 1.1. The more technical work of this paper is the proof of Theorem 1.2, and it is given in Section 3, which is further subdivided based on the size of $\hat{P}$. Theorem 1.2 is demonstrated for $|\hat{P}| = 3, 4$ in Section 3.3. Section 3.4 then introduces three sufficient conditions that guarantee the validity of Theorem 1.2 for $|\hat{P}| \geq 5$. These conditions will be essential in subsequent sections. The validity of Theorem 1.2 for $|\hat{P}| = 5, 6, 7$ is established in Section 3.5, while the case where $|\hat{P}| \geq 8$ follows directly from results derived in Section 3.4 and is presented in Section 3.6. Section 4 provides a computer-assisted proof of Proposition 1.3. Finally, in Section 5 we show Theorem 1.4 and Propositions 1.5 and 1.6.

2. THE DIAMETER OF $D(P)$: PROOF OF THEOREM 1.1

We recall that $n = |P| \geq 5$ and that the set $\mathcal{P}$ of $\binom{n}{2}$ line segments with endpoints in P is the vertex set of the graph $D(P)$. For brevity, let $G := D(P)$. Since G is not a complete graph, $\text{diam}(G) \geq 2$.

Let a and b be distinct elements of $\mathcal{P}$ such that $d_G(a, b) = \text{diam}(G)$. Then $d_G(a, b) \geq 2$, and so $a \cap b \neq \emptyset$. This inequality and the fact that P has no three collinear points imply that $a \cap b$ consists precisely of one point of $\mathbb{R}^2$, which will be denoted by o . Then a and b cross at o , or o is common endpoint of a and b . An endpoint of a or b that is in exactly one of a or b will be called a *leaf* of $\{a, b\}$. Then, $\{a, b\}$ has exactly two or four leaves.

Let ℓ_a and ℓ_b be the straight lines spanned by a and b , respectively. Clearly, $\mathbb{R}^2 \setminus \{\ell_a, \ell_b\}$ consists of four open connected regions, say R_1, R_2, R_3 , and R_4 . For $i \in \{1, 2, 3, 4\}$, let $P_i := P \cap R_i$.

(i) Suppose that $n \geq 9$. Then $|P_j| \geq 2$ for some $j \in \{1, 2, 3, 4\}$, and so P_j has at least two points, say p_{j_1} and p_{j_2} . Since afb defines a path of length 2 for $f := p_{j_1}p_{j_2}$, we can conclude that $d_G(a, b) = 2$, as required.

(ii) Suppose that $n \in \{6, 7, 8\}$. Since $\text{diam}(G) \geq 2$, it remains to show that $d_G(a, b) \leq 3$. If $|P_i| \geq 2$ for some $i \in \{1, 2, 3, 4\}$, then we can proceed as in previous case and conclude that $d_G(a, b) = 2 \leq 3$, as claimed. Then, we can assume that $|P_i| \leq 1$ for each $i \in \{1, 2, 3, 4\}$. This and $n \geq 6$ imply that there exist $j, k \in \{1, 2, 3, 4\}$ such that $j \neq k$ and $|P_j| = 1 = |P_k|$. Let p_j (respectively, p_k) be the only element of P_j (respectively, P_k). It is not hard to see that, independently of the relative position of P_j and P_k with respect to o , $\mathcal{P}$ always contains at least two disjoint segments f and h

satisfying the following properties: f (respectively, h) is disjoint from b (respectively, a) and joins a point of $\{p_j, p_k\}$ with a leaf of a (respectively, b). These properties of f and h imply that $ahfb$ defines a path of length 3, and hence that $d_G(a, b) \leq 3$, as claimed.

(iii) Suppose that $n = 5$. Since $\text{diam}(G) \geq 2$, it remains to show that $d_G(a, b) \leq 4$. If $|P_i| \geq 2$ for some $i \in \{1, 2, 3, 4\}$, we can proceed as in (i) and conclude that $d_G(a, b) = 2 \leq 4$, as claimed. Similarly, if there exist $j, k \in \{1, 2, 3, 4\}$ such that $j \neq k$ and $|P_j| = 1 = |P_k|$, we can proceed as in (ii) and conclude that $d_G(a, b) = 3 \leq 4$, as claimed. Then, we can assume that $|P_i| = 1$ for some $i \in \{1, 2, 3, 4\}$ and that $|P_j| = 0$ for each $j \in \{1, 2, 3, 4\} \setminus \{i\}$. These facts and $n = 5$ imply that a and b cross at o , and hence $\{a, b\}$ has 4 leaves. Let p_i be the only element of P_i . It is not hard to see that $\mathcal{P}$ always contains at least two segments f and h satisfying the following properties: f (respectively, h) is disjoint from b (respectively, a) and joins p_i with a leaf of a (respectively, b) that is incident with P_i . Let e be the segment of $\mathcal{P}$ that joins the two leaves of $\{a, b\}$ that are disjoint from $f \cup h$. From the choice of f, h , and e it follows that $ahfeb$ defines a path of length 4, and hence that $d_G(a, b) \leq 4$, as claimed.

3. MUTUAL-VISIBILITY OF $D(P)$: PROOF OF THEOREM 1.2

Since the proof of Theorem 1.2 is somewhat intricate, we begin with a brief overview of it to provide the reader with a roadmap for the arguments that follow.

3.1. OVERVIEW OF THE PROOF OF THEOREM 1.2

Our approach relies on Lemma 3.1, which provides the topological and geometric criteria for determining mutual visibility in the disjointness graph $D(P)$. The proof then proceeds through a systematic geometric case analysis according to the cardinality of $\hat{P}$.

A second fundamental ingredient is Theorem 1.1, which establishes that $\text{diam}(D(P)) \leq 3$ almost always. This result makes it possible to develop what we call a *local visibility criterion*. The central idea of such a criterion is to find a subset $\mathcal{S} \subset \mathcal{P}$ satisfying the following two properties: (1) $|\mathcal{S}| \leq 9$, and (2) every pair of distinct segments $a, b \in \mathcal{P}' = \mathcal{P} \setminus \mathcal{S}$ is $\mathcal{P}'$ -mutually visible. Since $\text{diam}(D(P))$ is small, deciding whether a and b are $\mathcal{P}'$ -mutually visible reduces to analyzing the intersections of their neighborhoods with $\mathcal{S}$. A third component of the proof is the identification of three special point configurations. We show that the existence of any one of these configurations in P immediately yields Theorem 1.2.

3.2. MASTER VISIBILITY LEMMA

Our main tool in this paper is the following lemma.

Lemma 3.1. *Let $\mathcal{S}$ be a subset of $\mathcal{P}$ with $s = |\mathcal{S}|$ and let $\mathcal{P}' = \mathcal{P} \setminus \mathcal{S}$. Then $\mu(D(P)) \geq \binom{n}{2} - s$ if each pair of distinct segments a and b of $\mathcal{P}'$ satisfies exactly one of the following conditions.*

- (1) $d_{D(P)}(a, b) = 1$ (or equivalently, $a \cap b = \emptyset$).
- (2) $d_{D(P)}(a, b) = 2$ and there is $f_1 \in \mathcal{S}$ such that $(a \cup b) \cap f_1 = \emptyset$.
- (3) $d_{D(P)}(a, b) = 3$ and there are $f_1, f_2 \in \mathcal{S}$ such that $a \cap f_1 = \emptyset$, $f_1 \cap f_2 = \emptyset$ and $f_2 \cap b = \emptyset$.
- (4) $d_{D(P)}(a, b) = 4$, $n = 5$, and there are $f_1, f_2, f_3 \in \mathcal{S}$ such that $a \cap f_1 = \emptyset = f_3 \cap b$, $a \cap h \neq \emptyset$ for $h \in \{f_2, f_3\}$, $b \cap h \neq \emptyset$ for $h \in \{f_1, f_2\}$, $f_i \cap f_{i+1} = \emptyset$ for $i \in \{1, 2\}$, and $f_1 \cap f_3 \neq \emptyset$.

Proof. We recall that $n = |P| \geq 5$. Let $\mathcal{S}, \mathcal{P}'$ and s be as in the statement of the lemma. Let a and b be distinct segments of $\mathcal{P}'$ and let $t = d_{D(P)}(a, b)$. By Theorem 1.1, we know that t is an element of $\{1, 2, 3, 4\}$.

We now show that a and b are $\mathcal{P}'$ -visible. If (1) holds, then a and b are adjacent in $D(P)$, and so they are $\mathcal{P}'$ -visible. Suppose now that $t \in \{2, 3, 4\}$ and that (t) holds. Then $af_1 \cdots f_{t-1}b$ forms a shortest $a - b$ path in $D(P)$ that is internally disjoint from $\mathcal{P}'$, and so a and b are $\mathcal{P}'$ -visible again. Since $|\mathcal{P}'| = \binom{n}{2} - s$, we are done $\square$

Definition 3.2. If $\mathcal{S}$ is a subset of $\mathcal{P}$ and a, b are distinct segments of $\mathcal{P} \setminus \mathcal{S}$ with $a \cap b \neq \emptyset$, then (a, b) will be called an $\mathcal{S}$ -pair.

3.3. MUTUAL-VISIBILITY OF $D(P)$ WHEN $|\widehat{P}| \in \{3, 4\}$

If x and y are distinct points in the plane, let $d(xy)$ denote the Euclidean distance of the segment xy . The following classical theorem from Euclidean geometry will play a key role in the proof of Proposition 3.7.

Theorem 3.3 (Ceva's Theorem). *Let $v_1v_2v_3$ be a triangle, and let o_1, o_2 , and o_3 be points in the interiors of the sides v_2v_3, v_1v_3 , and v_1v_2 , respectively. Then the lines $\ell(v_1o_1)$, $\ell(v_2o_2)$, and $\ell(v_3o_3)$ are concurrent if and only if*

$$\frac{d(v_1o_3)}{d(o_3v_2)} \cdot \frac{d(v_2o_1)}{d(o_1v_3)} \cdot \frac{d(v_3o_2)}{d(o_2v_1)} = 1.$$

Definition 3.4. Let $v_1v_2v_3$ be a triangle, and let o_1, o_2 , and o_3 be distinct points in the interior of v_2v_3 , listed in counterclockwise order. Let o_4 be a point in the interior of v_1v_3 . See Figure 2. We next construct a set of five points $\{x_1, x_2, x_5, x_6, x_7\}$ and nine lines $\{\ell_1, \ell_2, \dots, \ell_9\}$, which we call the $(5, 9)$ -configuration produced by $(v_1v_2v_3, o_1, o_2, o_3, o_4)$.

Let $\ell_i = \ell(v_1o_i)$ for $i \in \{1, 2, 3\}$, and let $\ell_4 = \ell(v_2o_4)$.

Let $x_1 = \ell_1 \cap \ell_4$ and $x_2 = \ell_2 \cap \ell_4$.

Let $\ell_5 = \ell(v_3x_1)$ and $\ell_6 = \ell(v_3x_2)$.

Let $x_5 = \ell_3 \cap \ell_5$ and $x_6 = \ell_3 \cap \ell_6$.

Let $\ell_7 = \ell(v_2x_5)$ and $\ell_8 = \ell(v_2x_6)$.

Let $x_7 = \ell_7 \cap \ell_2$ and $\ell_9 = \ell(v_3x_7)$.

For $k \in \{5, 6, 7, 8, 9\}$, let o_k be the point of ℓ_k that crosses a side of $v_1v_2v_3$.

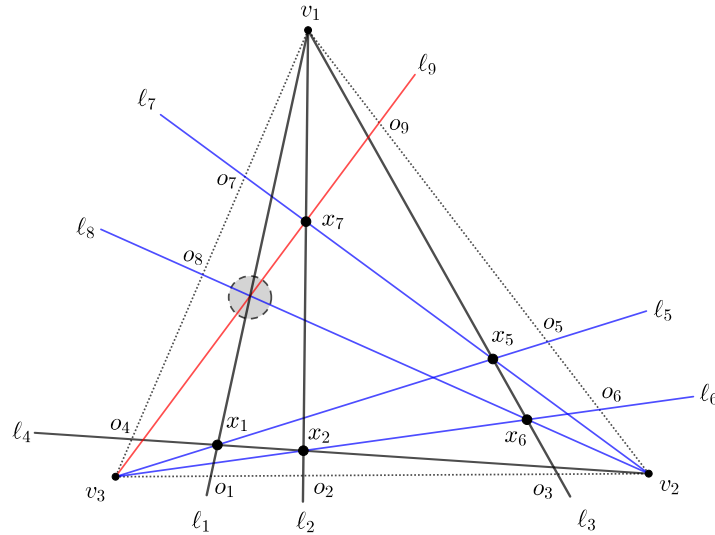


Fig. 2. The $(5, 9)$ -configuration produced by $(v_1v_2v_3, o_1, o_2, o_3, o_4)$

Proposition 3.5. *Let $v_1v_2v_3$ be a triangle, and let o_1, o_2 , and o_3 be distinct points in the interior of v_2v_3 , listed in counterclockwise order. Let o_4 be a point in the interior of v_1v_3 . If $\{x_1, x_2, x_5, x_6, x_7\}$ and $\{\ell_1, \ell_2, \dots, \ell_9\}$ form the $(5, 9)$ -configuration produced by $(v_1v_2v_3, o_1, o_2, o_3, o_4)$, then the lines ℓ_1, ℓ_8 and ℓ_9 are concurrent.*

Proof. Trivially, for each $i \in \{1, 2, 5, 6, 7\}$, the three lines passing through x_i are concurrent. Hence, by Theorem 3.3, the following equalities hold.

$$(1) : \quad \frac{d(v_2o_1)}{d(o_1v_3)} = \frac{d(o_5v_2)}{d(v_1o_5)} \cdot \frac{d(o_4v_1)}{d(v_3o_4)}, \text{ thanks to } x_1.$$

$$(2) : \quad 1 = \frac{d(v_1o_6)}{d(o_6v_2)} \cdot \frac{d(v_2o_2)}{d(o_2v_3)} \cdot \frac{d(v_3o_4)}{d(o_4v_1)}, \text{ thanks to } x_2.$$

$$(3) : \quad 1 = \frac{d(v_1o_5)}{d(o_5v_2)} \cdot \frac{d(v_2o_3)}{d(o_3v_3)} \cdot \frac{d(v_3o_7)}{d(o_7v_1)}, \text{ thanks to } x_5.$$

$$(4) : \quad \frac{d(v_3o_8)}{d(o_8v_1)} = \frac{d(o_6v_2)}{d(v_1o_6)} \cdot \frac{d(o_3v_3)}{d(v_2o_3)}, \text{ thanks to } x_6.$$

$$(5) : \quad \frac{d(v_1o_9)}{d(o_9v_2)} = \frac{d(o_2v_3)}{d(v_2o_2)} \cdot \frac{d(o_7v_1)}{d(v_3o_7)}, \text{ thanks to } x_7.$$

By Theorem 3.3, it suffices to verify that

$$\frac{d(v_2 o_1)}{d(o_1 v_3)} \cdot 1 \cdot 1 \cdot \frac{d(v_3 o_8)}{d(o_8 v_1)} \cdot \frac{d(v_1 o_9)}{d(o_9 v_2)} = 1.$$

This equality follows immediately by replacing each of its five factors with the corresponding expression from the preceding five equations and simplifying the resulting product. $\square$

Definition 3.6. Let $v_1, v_2, \dots, v_m$ be the points in $\widehat{P}$, and assume that they appear in clockwise cyclic order on $\widehat{P}$. For $i \in [m]$, let v_i^- (respectively, v_i^+) be the first (respectively, last) point of $P \setminus \{v_i, v_{i+1}, v_{i+m-1}\}$ that we find when we rotate $\ell(v_i v_{i+1})$ clockwise around v_i , addition taken mod m (i.e., $m + i = i$)

We remark that the points v_i^- and v_i^+ in Definition 3.6 are well defined and distinct, since $n \geq 5$ and P is in general position.

Proposition 3.7. Assume that v_1, v_2, v_3 are the points in $\widehat{P}$ and that they appear in clockwise cyclic order on $\widehat{P}$. Then $v_i^- v_i^+ v_{i+2} v_{i+1}$ form a convex quadrilateral for some $i \in \{1, 2, 3\}$, addition taken mod 3.

Proof. Suppose first that $\{v_1^-, v_1^+\}, \{v_2^-, v_2^+\}, \{v_3^-, v_3^+\}$ are pairwise disjoint. Assume w.l.o.g. that v_1, v_2, v_3 are placed as in Figure 3(a). If $\ell(v_1^+ v_1^-)$ does not cross $v_2 v_3$, then $v_1^- v_1^+ v_3 v_2$ is the required quadrilateral. So, we may assume that $\ell(v_1^+ v_1^-)$ crosses $v_2 v_3$ at o . We only analyze the case in which v_1^+ is closer to o than v_1^- , as depicted in Figure 3(a). The proof for the case in which v_1^- is closer to o than v_1^+ is analogous.

Since v_2^- and v_3^+ are distinct, the segments $v_2 v_2^-$ and $v_3 v_3^+$ must cross each other. Then $v_3^+ \in \ell^-(v_1 v_2^-)$ and each point in $P \setminus \{v_1, v_2, v_3, v_1^-, v_1^+, v_2^-, v_3^+\}$ is in the interior of the convex quadrilateral Q defined by $\ell(v_1 v_1^-)$, $\ell(v_1 v_1^+)$, $\ell(v_2 v_2^-)$ and $\ell(v_3 v_3^+)$. See Figure 3(a).

Since if v_2^+ is on the left of $\ell(v_1 v_2^-)$, then $v_2^- v_2^+ v_1 v_3$ form the required quadrilateral, we assume that v_2^+ is on the right of $\ell(v_1 v_2^-)$. From the definition of v_3^- , it follows that v_3^- must be in the interior of triangle T bounded by $\ell(v_3 v_2^+)$, $\ell(v_1 v_1^+)$ and $\ell(v_2 v_2^+)$. From $v_3^- \in T$ and $v_3^+ \in \ell^-(v_1 v_2^-)$, it follows that $\ell(v_3^- v_3^+)$ crosses $v_1 v_3$.

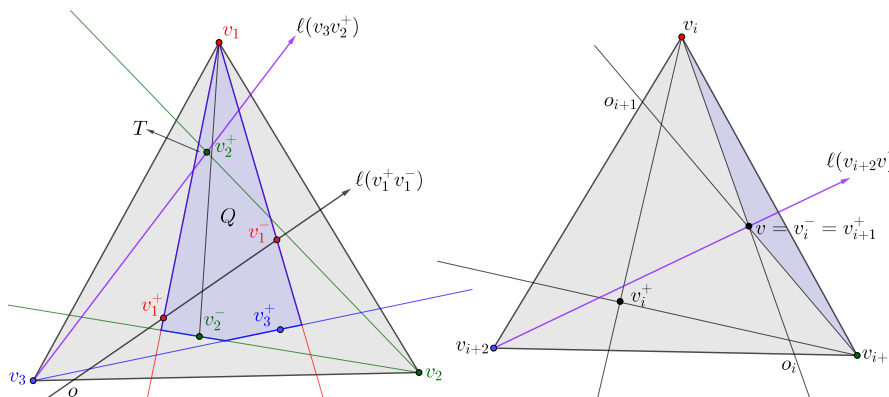
We note that, if we move v_1^+ to $\ell(v_1 v_1^+) \cap \ell(v_2 v_2^-)$, v_1^- to $\ell(v_1 v_1^-) \cap \ell(v_2 v_2^+)$, and v_3^+ to $\ell(v_3 v_3^+) \cap \ell(v_1 v_1^-)$, then we obtain a $(5, 9)$ -configuration with T remaining unchanged. By Proposition 3.5, the interior of T lies on the right of the new $\ell(v_2 v_3^+)$, then the interior of T lies on the right of the original $\ell(v_2 v_3^+)$. Hence, $v_3^- v_3^+ v_2 v_1$ is the required quadrilateral.

Suppose now that $\{v_i^-, v_i^+\}$ intersects either $\{v_{i+1}^-, v_{i+1}^+\}$ or $\{v_{i+2}^-, v_{i+2}^+\}$ for some $i \in \{1, 2, 3\}$. It is easy to see that, up to symmetry, such intersection can occur essentially in two ways: either $v_i^- = v = v_{i+2}^+$ or $v_i^- = v = v_{i+1}^+$.

Suppose that $v_i^- = v = v_{i+2}^+$. Then each point of $P' := P \setminus \{v_1, v_2, v_3, v\}$ is in the interior of the triangle $v_i v v_{i+2}$. If $v_i^+ \in \ell^-(v_{i+1} v_i^-)$ (resp. $v_i^+ \in \ell^+(v_{i+1} v_i^-)$), then $v_i^- v_i^+ v_{i+2} v_{i+1}$ (resp. $v_{i+2} v_{i+2}^+ v_{i+1} v_i$) is the required quadrilateral.

Suppose finally that $v_i^- = v = v_{i+1}^+$. Let $o_i := \ell(v_i v) \cap v_{i+1} v_{i+2}$ and $o_{i+1} := \ell(v_{i+1} v) \cap v_i v_{i+2}$. Then each point of $P \setminus \{v_1, v_2, v_3, v\}$ is in the interior of the convex

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 $i \in \{1, 2, 3\}$, addition taken mod 3

Lemma 3.8. *If $n \geq 5$ and $|\hat{P}| = 3$, then $\mu(D(P)) \geq \binom{n}{2} - 8$.*

u^-u^+wv form a convex quadrilateral. See Figure 4(a).

$P \setminus \{u, v, w, u^-, u^+\}$ are in the interior of the triangle uo^-o^+ . See Figure 4(a).

Let $\mathcal{S} := \{uu^-, uu^+, u^-u^+, vu^-, wu^-, vu^+, wu^+, vw\}$ and let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

Suppose that $(a \cup b) \cap \{u\} \neq \emptyset$. If $a \cap b = \{u\}$, then $(a \cup b) \cap f = \emptyset$ for some $f \in \{wu^+, u^+u^-, u^-v\}$. Suppose w.l.o.g. that $u \in a$ and $u \notin b$. If $a = uw$ (respectively, $a = uv$), then $a \cap b \neq \emptyset$ implies that $w \in b$ (respectively, $v \in b$), and so u^-v (respectively, u^+w) is the required f .

Then, we assume that $w, v \notin a$, and so $a \cap vw = \emptyset$. Since $b \cap vw = \emptyset$ implies that vw is the required f , we can assume $b \cap vw \neq \emptyset$. If $v \in b$ (respectively, $w \in b$), then u^+w (respectively, u^-v) is the required f .

Suppose that $(a \cup b) \cap \{u\} = \emptyset$. If both $v, w \in a \cup b$, then $a \cap b \neq \emptyset$ implies that $(a \cup b) \cap f = \emptyset$ for some $f \in \{uu^-, uu^+, u^-u^+\}$.

Since $v, w \notin a \cup b$ implies that vw is the required f , we can assume w.l.o.g. that $vw \cap a \neq \emptyset$ and $vw \cap b = \emptyset$. If $v \in a$, then $a \cap u^+w = \emptyset$. Since $b \cap u^+w = \emptyset$ implies that u^+w is the required f , we must have that $u^+ \in b$. Then $a \cap b \neq \emptyset$ guarantees that

$(a \cup b) \cap f = \emptyset$ for some $f \in \{wu^-, u^-u\}$. Similarly, if $w \in a$, then $a \cap u^-v = \emptyset$. Since $b \cap u^-v = \emptyset$ implies that u^-v is the required f , we must have that $u^- \in b$. Then $a \cap b \neq \emptyset$ guarantees that $(a \cup b) \cap f = \emptyset$ for some $f \in \{vu^+, u^+u\}$. $\square$

Lemma 3.9. *If $n \geq 5$ and $|\hat{P}| = 4$, then $\mu(D(P)) \geq \binom{n}{2} - 9$.*

Proof. Let v_1, v_2, v_3, v_4 be the points in $\hat{P}$, and assume that they appear in this cyclic order on $\hat{P}$. Trivially, v_1v_2 , v_2v_3 , v_3v_4 and v_4v_1 are clean in $\mathcal{P}$. We assume w.l.o.g. that v_1, v_2, v_3, v_4 are placed as in Figure 4(b).

(a)

(b)

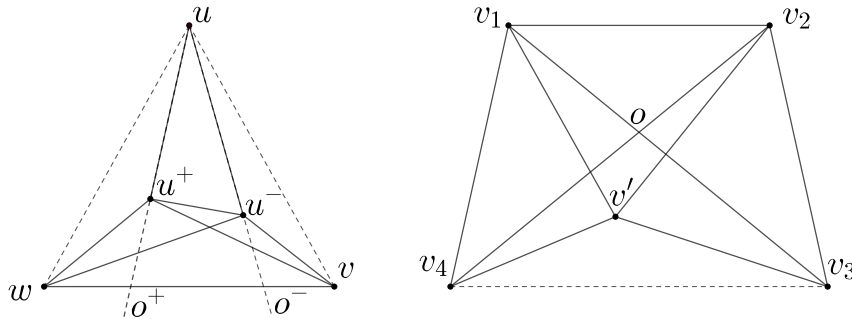


Fig. 4. The set formed by the 8 (respectively, 9) continuous segments of the drawing on the (a) (respectively, (b)) is $\mathcal{S}$

Let $o := v_1v_3 \cap v_2v_4$. Since the points in P are in general position, then $o \notin P$. As $n \geq 5$, at least one of the four triangles formed by $CH(P)$ and its diagonals, must contain points of P in its interior. Assume w.l.o.g. that v_3ov_4 is such a triangle. Let v' be a point of P in the interior of the triangle v_3ov_4 that is closest to v_3v_4 . Then $\triangle := v_3v'v_4$ has no points of P in its interior.

Let $\mathcal{S} := \{v_1v_2, v_1v_3, v_1v', v_1v_4, v_2v_3, v_2v', v_2v_4, v_3v', v_4v'\}$ and let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We note that if $e \in \mathcal{P}'$ is incident with either v_3 or v_4 , then e is disjoint from v_1v_2 . Thus, we can assume that at least one of a or b is disjoint from $\{v_3, v_4\}$, as otherwise v_1v_2 is the required f . Suppose w.l.o.g. that b is disjoint from $\{v_3, v_4\}$. This, together with the choice of v' imply that b intersects at most 2 segments in $\mathcal{Q} := \{v_1v_2, v_2v_3, v_3v', v_4v', v_4v_1\}$.

Since v_3v_4 is clean in $\mathcal{P}$ and $a \cap b \neq \emptyset$, a cannot be v_3v_4 . Similarly, we note that if a intersects at most 2 segments of $\mathcal{Q}$, then $(a \cup b) \cap f = \emptyset$ for some $f \in \mathcal{Q}$, as required. Then we can assume that a intersects at least 3 segments of $\mathcal{Q}$, and so we must have that a crosses v_jv' and $v_i \in a$ for $\{i, j\} = \{3, 4\}$.

We will only analyze the case $i = 3$ and $j = 4$, as the proof for the case $i = 4$ and $j = 3$ is totally analogous. Since if $v_1 \notin b$, then $f = v_1v_4$ is the required segment, we can assume that $v_1 \in b$. From this last assumption, the properties of b , the fact that $\triangle = v_3v'v_4$ is empty, and $a \cap b \neq \emptyset$, it is not hard to see that $f = v_2v'$ is the required segment. $\square$

3.4. SUFFICIENT CONDITIONS

In this section we present three sufficient conditions under which $\mu(D(P)) \geq \binom{n}{2} - 9$ when $m := |\widehat{P}| \geq 5$. These will be often used in the rest of the paper.

Let $v_1, v_2, \dots, v_m$ be the points in $\widehat{P}$, and assume that they appear in clockwise cyclic order on $\widehat{P}$. For $i \in [m]$, let $e_i := v_i v_{i+1}$ where addition is taken mod m .

Proposition 3.10. *If $\mathcal{P}$ has 5 clean segments which are pairwise disjoint, then $\mu(D(P)) \geq \binom{n}{2} - 5$.*

Proof. Let $\mathcal{S} := \{f_1, f_2, f_3, f_4, f_5\}$ be the set of pairwise disjoint segments that are clean in $\mathcal{P}$. Let (a, b) be an $\mathcal{S}$ -pair. Since the segments of $\mathcal{S}$ are clean and pairwise disjoint, then $a \cup b$ is disjoint from at least one $f_i \in \mathcal{S}$, and we are done by Lemma 3.1(2). $\square$

Definition 3.11. Let $x \in P \setminus \widehat{P}$, and let $e_i = v_i v_{i+1}$ and m be as above. We say that a triangle $\Delta := xv_i v_{i+1}$ is a *good-triangle* of P if the following conditions are satisfied:

- (i) x lies in the interior of the quadrilateral $v_i v_{i+1} v_{i+2} v_{i+m-1}$.
- (ii) If $uv \in \mathcal{P} \setminus \{xv_i, xv_{i+1}, e_i\}$ intersects each of the three segments of Δ , then

$$uv \in \{v_i v_{i+2}, v_i v_{i+3}, v_{i+1} v_{i+m-2}, v_{i+1} v_{i+m-1}\}.$$

Evidently, if x satisfies condition (i) and xv_i, xv_{i+1} are clean in $\mathcal{P}$, then $\Delta := xv_i v_{i+1}$ is a good-triangle of P .

Observation 3.12. *If $s \in \mathcal{P} \setminus \{xv_i, xv_{i+1}, e_i\}$ is a segment that intersects the three sides of Δ , then s must have an endpoint at a vertex of Δ . Since e_i is clean, s intersects e_i only if s has an endpoint in $\{v_i, v_{i+1}\}$. Hence, condition (ii) is equivalent to the following: for every $u \in P \setminus \{x, v_i, v_{i+1}\}$, if $v_i u$ is not in the list of (ii) then $v_i u \cap xv_{i+1} = \emptyset$, and if $v_{i+1} u$ is not in that list then $v_{i+1} u \cap xv_i = \emptyset$.*

Proposition 3.13. *If P has a good-triangle and $m = |\widehat{P}| \geq 6$, then $\mu(D(P)) \geq \binom{n}{2} - 9$.*

Proof. By relabelling the points of $\widehat{P}$ if necessary, we may assume that $\Delta := xv_3 v_4$ is a good-triangle of P . By (i), x lies in the interior of the convex quadrilateral $Q := v_2 v_3 v_4 v_5$. Let $\mathcal{L} := \{v_3 v_5, v_3 v_6, v_1 v_4, v_2 v_4\}$ be the list in (ii) for $i = 3$, and let

$$\mathcal{S} := \{e_1, e_3, e_5, xv_3, xv_4\} \cup \mathcal{L}.$$

We remark that e_1, e_3, e_5 are clean in $\mathcal{P}$ and pairwise disjoint. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$. Suppose, to the contrary, that every segment of $\mathcal{S}$ intersects $a \cup b$.

Since P is in general position, two segments of $\mathcal{P}$ with no common endpoint can intersect only if they cross. In particular, a segment of $\mathcal{P}$ intersects an edge $v_j v_{j+1}$ of $CH(P)$ if and only if one of its endpoints is v_j or v_{j+1} . Let $A := \{v_1, v_2\}$, $B := \{v_3, v_4\}$, and $C := \{v_5, v_6\}$. See Figure 5. We will use the following three facts.

- (F1) $a \cup b$ has an endpoint in each of A , B and C , since it intersects e_1 , e_3 , and e_5 .
 (F2) If $s \in \mathcal{P} \setminus \mathcal{S}$ has v_3 (resp. v_4) as an endpoint, then $s \cap xv_4 = \emptyset$ (resp. $s \cap xv_3 = \emptyset$). This follows from Observation 3.12, since $\mathcal{L} \subseteq \mathcal{S}$.
 (F3) Every segment with both endpoints in $A \cup C$ is disjoint from xv_3 and xv_4 . Indeed, the line $\ell(v_2v_5)$ separates the points in $\{x, v_3, v_4\}$ from those in $\{v_1, v_6\}$.

By (F1), at least one endpoint of $a \cup b$ belongs to B . We may assume, without loss of generality, that v_3 is an endpoint of a . The case in which v_4 is an endpoint of $a \cup b$ is symmetric: exchange $v_1 \leftrightarrow v_6$, $v_2 \leftrightarrow v_5$ and $v_3 \leftrightarrow v_4$, which maps Q , $\mathcal{L}$ and $\mathcal{S}$ onto themselves.

Let y be such that $a = v_3y$. Since $xv_3, e_3, v_3v_5, v_3v_6 \in \mathcal{S}$, we have $y \notin \{x, v_4, v_5, v_6\}$. Thus, a has no endpoint in C . By (F2), $a \cap xv_4 = \emptyset$. Since xv_4 intersects $a \cup b$, we get $b \cap xv_4 \neq \emptyset$. Another application of (F2) gives that v_3 is not an endpoint of b .

By (F1), we may write $b = wz$, where $w \in C$. Since $b \cap xv_4 \neq \emptyset$, (F3) implies that $z \notin A$. Then (F1) gives $y \in A$. If $y = v_2$, then $a = e_2$ is clean, and b has no endpoint in $\{v_2, v_3\}$. Hence, $a \cap b = \emptyset$, a contradiction. Therefore, $a = v_1v_3$. We note that $x \in \ell^-(v_1v_3)$ implies that a intersects each of the three segments of Δ . As $a \notin \mathcal{L}$, we have $x \in \ell^+(v_1v_3)$.

Since $a \cap b \neq \emptyset$ and $v_1, v_3 \notin b$, then a and b must cross. Moreover, since $w \in C \subset \ell^+(v_1v_3)$, then $z \in \ell^-(v_1v_3)$. As $z \neq v_2$, the point z lies in the interior of the triangle $v_1v_2v_3$. Thus, $v_4z \notin \mathcal{L}$, and Observation 3.12 gives $v_4z \cap xv_3 = \emptyset$. Consequently, $x \in \ell^+(v_4z)$. Hence, $\ell(v_4z)$ separates x from w , so $b \cap xv_4 = \emptyset$ (note that $v_4 \notin b$). This contradicts $b \cap xv_4 \neq \emptyset$. $\square$

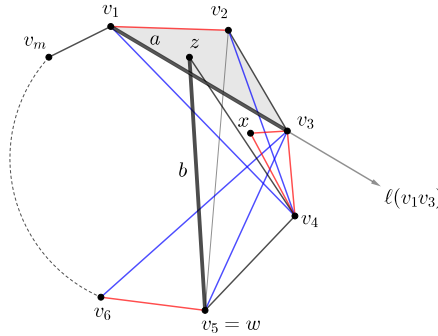


Fig. 5. The red triangle $\Delta = xv_3v_4$ is a good-triangle of P . The segments of $\mathcal{L}$ are shown in blue, and the 9-set $\mathcal{S}$ consists of the blue and red segments. The thick segments are $a = v_1v_3$ and $b = wz$ in the case $w = v_5$. The point z lies in the interior of the triangle $v_1v_2v_3$, and $xv_4 \in \mathcal{S}$ is disjoint from $a \cup b$

Definition 3.14. If uv and xy are disjoint segments of $\mathcal{P}$, we define $\mathcal{D}(uv, xy) := \{uv, ux, uy, vx, vy, xy\}$. Then, $\mathcal{D}(uv, xy)$ defines a straight-line drawing of K_4 . The two segments in $\{ux, uy, vx, vy\}$ that lie in the interior of the convex hull of $\mathcal{D}(uv, xy)$ will be the *diagonals* of $\mathcal{D}(uv, xy)$. We will say that $\mathcal{D}(uv, xy)$ is a *good-2-set* of $\mathcal{P}$ if it satisfies the following conditions:

- uv and xy are clean in $\mathcal{P}$.
- If $e \in \{uv, xy\}$, then e has at least one endpoint in $\widehat{P}$.
- If L and R are the opposite quadrants of the plane defined by the lines spanned by the diagonals of $\mathcal{D}(uv, xy)$ that are internally disjoint from uv and xy , then L (resp. R) has a segment e_l (resp. e_r) of $\mathcal{P}$ in its interior which is not crossed by a segment of $\mathcal{P} \setminus \mathcal{D}(uv, xy)$.

We note that the existence of a good-2-set in $\mathcal{P}$ implies that $n = |P| \geq 8$.

Proposition 3.15. If $\mathcal{P}$ has a good-2-set, then $\mu(D(P)) \geq \binom{n}{2} - 8$.

Proof. Let $\mathcal{D} := \mathcal{D}(uv, xy)$ be a good-2-set of $\mathcal{P}$ and let L, R, e_l , and e_r be as in Definition 3.14. Let $\mathcal{S} := \mathcal{D} \cup \{e_l, e_r\}$ and let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

Let ℓ_1 and ℓ_2 be the straight lines spanned by the diagonals of $\mathcal{D}(uv, xy)$. We assume w.l.o.g. that ℓ_1 and ℓ_2 are directed in such a way that L (resp. R) lies on the left (resp. right) side of ℓ_1 and ℓ_2 . Clearly, the convex hull of $\mathcal{D}$ is either a quadrilateral or a triangle. Then, up to order type isomorphism, there are only two possibilities for $\mathcal{D}$, namely those depicted in Figure 6.

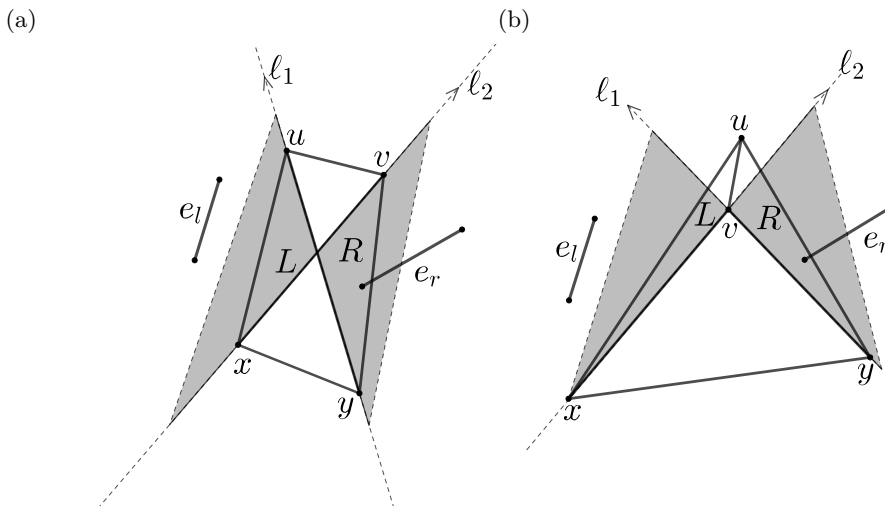


Fig. 6. The convex hull of $\mathcal{D}$ is either a quadrilateral (a) or a triangle (b).

The set formed by the 8 continuous segments is $\mathcal{S}$.

The opposite quadrants L and R are the shaded regions

Since if $(a \cup b) \cap g = \emptyset$ for some $g \in \{e_l, e_r\}$, then $f = g$ is the required segment, we can assume that $a \cup b$ intersects both e_l and e_r . Assume w.l.o.g. that a intersects e_l .

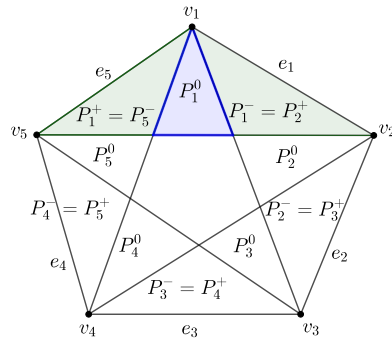
Similarly, we note that if $(a \cup b) \cap g = \emptyset$ for some $g \in \{uv, xy\}$, then $f = g$ is the required segment, and so we can assume that $a \cup b$ intersects both uv and xy . This last, the properties of $\mathcal{D}$, and the fact that uv and xy are clean in $\mathcal{P}$ imply that a (resp. b) shares exactly one common endpoint with exactly one of uv or xy . From these facts and the assumption that a intersects $e_l \in L$, it follows that a is disjoint from the interior of R . See Figure 6. Then b must intersect e_r , as otherwise e_r is the required f . From $b \cap e_r \neq \emptyset$, and the fact that b has exactly one endpoint in $\{u, v, x, y\}$ it is easy to see that b does not intersect the interior of L , and so we have $a \cap b = \emptyset$, a contradiction. $\square$

3.5. MUTUAL-VISIBILITY OF $D(P)$ WHEN $m := |\hat{P}| \in \{5, 6, 7\}$

Let $v_1, v_2, \dots, v_m$ be the points in $\hat{P}$ and assume that they appear in clockwise cyclic order on $\hat{P}$. In this section, addition is taken mod m (i.e., $m + i = i$). Let $\mathcal{D}_m := \{v_i v_j : 1 \leq i < j \leq m\} \subset \mathcal{P}$. The set of segments drawn in Figure 7(a) (respectively, right) is precisely $\mathcal{D}_5$ (respectively, $\mathcal{D}_7$).

For $i \in [m]$, let $e_i := v_i v_{i+1}$ and let P_i be the subset of P that lies in interior of the triangle $v_{i+(m-1)} v_i v_{i+1}$. Let P_i^- (respectively, P_i^+) be the subset of P_i that lies on $\ell^-(v_i v_{i+2})$ (respectively, $\ell^+(v_i v_{i+(m-2)})$) and let $P_i^0 := P_i \setminus (P_i^- \cup P_i^+)$. Then P_i is disjoint union of P_i^-, P_i^0, P_i^+ , and $P_i^- = P_{i+1}^+$ for each $i \in [m]$. See Figure 7(a)

(a)



(b)

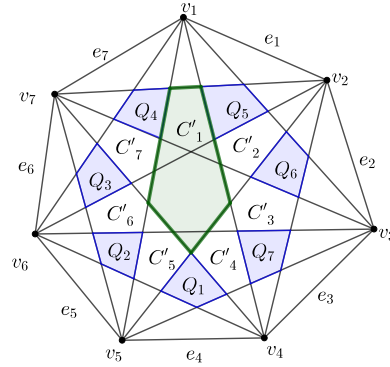


Fig. 7. (a) We have $|\hat{P}| = 5$ showing the labels that we use for the points and the regions of $\mathcal{D}_5$ that are incident with $\hat{P}$. This notation is used in all the proofs given in Section 3.5. (b) We have $|\hat{P}| = 7$, and the labeling depicted is used only in the proof of Lemma 3.20. In particular, the set of points of P that lies in the green pentagon is C_1 , and C'_1 is a subset of C_1

Proposition 3.16. *Let $i \in [m]$ with $P_i \neq \emptyset$, and let u be a point of P_i farthest from the line $\ell(v_{i-1}v_{i+1})$. Then $v_i u$ is clean in $\mathcal{P}$.*

Proof. Let ℓ be the line through u parallel to $\ell(v_{i-1}v_{i+1})$, and let H be the closed half-plane bounded by ℓ that does not contain v_i . By the choice of u , every point of $P \setminus \{v_i\}$ lies in H . Thus, any segment of $\mathcal{P}$ that does not have v_i as an endpoint is contained in H and can intersect $v_i u$ only at u . By general position, this can occur only if u is one of its endpoints. On the other hand, any segment with endpoint v_i intersects $v_i u$ only at v_i . Hence, no segment of $\mathcal{P}$ crosses $v_i u$. $\square$

The following remark follows directly from Proposition 3.16 and the fact that $P_i^0 \cap P_j = \emptyset$ for all $i \neq j$.

Remark 3.17. If $P_i = P_i^0$ and $P_j = P_j^0$ for some $i \neq j$, and $u_i \in P_i$, $u_j \in P_j$ are chosen as in Proposition 3.16, then the segments $v_i u_i$ and $v_j u_j$ are clean and disjoint.

Lemma 3.18. *If $n \geq 5$ and $|\widehat{P}| = 5$, then $\mu(D(P)) \geq \binom{n}{2} - 9$.*

Proof. Note that $m = 5$.

Case 1. Suppose that $P_i = \emptyset$ for some $i \in [5]$. Assume w.l.o.g. that $P_1 = \emptyset$. Let $\mathcal{S} := \{e_1, e_2, e_3, e_4, e_5, v_1 v_3, v_1 v_4, v_2 v_4, v_3 v_5\}$, and let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(1)–(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$, unless $a \cap b = \emptyset$.

We can assume that each segment of $\mathcal{E} := \{e_1, e_2, \dots, e_5\}$ intersects $a \cup b$, as otherwise we are done. Then either a or b intersects at least three elements of $\mathcal{E}$, say a . Since $v_2 v_5$ is the unique segment of $\mathcal{P}'$ with this property, we must have that $a = v_2 v_5$.

Since if $b \cap e_3 = \emptyset$, then $f = e_3$ is the required segment, we can assume either v_3 or v_4 is an endpoint of b . Since $P_1 = \emptyset$, then b cannot cross a , contradicting that $a \cap b \neq \emptyset$.

Case 2. Suppose that P_i^-, P_{i+2}^- are nonempty for some $i \in [5]$. Assume w.l.o.g. that $P_1^- \neq \emptyset$ and $P_3^- \neq \emptyset$. Let $u_1 \in P_1^-$ (resp. $u_3 \in P_3^-$) be a point that is closest to e_1 (resp. e_3). By Case 1 we have $P_5 \neq \emptyset$. Let u_5 be a point of P_5 farthest from $\ell(v_4 v_1)$. By Proposition 3.16, $u_5 v_5$ is clean in $\mathcal{P}$.

Let $\mathcal{S}_1 := \{e_1, u_1 v_1, u_1 v_2\}$, $\mathcal{S}_2 := \{e_3, u_3 v_3, u_3 v_4\}$ and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{u_5 v_5\}$. Then e_1, e_3 and $u_5 v_5$ are clean in $\mathcal{P}$ and pairwise disjoint. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We can assume that $(a \cup b) \cap u_5 v_5 \neq \emptyset$, as otherwise $f = u_5 v_5$ is the required segment. Assume w.l.o.g. that a has an endpoint in $\{u_5, v_5\}$. Then a is disjoint from $\mathcal{S}_i$ for some $i \in \{1, 2\}$. Suppose that a is disjoint from $\mathcal{S}_1$. Then b must have an endpoint in $\{v_1, v_2\}$, as otherwise $f = v_1 v_2$ is the required segment. If $v_1 \in b$ (resp. $v_2 \in b$), then $a \cap b \neq \emptyset$ implies that some $f \in \{u_1 v_2, e_3\}$ (resp. $f \in \{u_1 v_1, e_3\}$) is the required. The case in which a is disjoint from $\mathcal{S}_2$ is totally analogous.

Case 3. Suppose that P_i^-, P_i^+ are nonempty for some $i \in [5]$. Assume w.l.o.g. that $P_1^- \neq \emptyset$ and $P_1^+ \neq \emptyset$. Let $u_1 \in P_1^-$ (resp. $u_5 \in P_1^+$) be a point that is closest to e_1 (resp. e_5). By Case 2, we can assume that P_3^-, P_3^+ and P_4^- are empty. Since $P_4^+ = P_3^-$, these facts and Case 1 imply that $P_3 = P_3^0$ and $P_4 = P_4^0$ are nonempty. Let u_3 (resp. u_4) be

a point of P_3 (resp. P_4) farthest from $\ell(v_2v_4)$ (resp. $\ell(v_3v_5)$). By Proposition 3.16 and Remark 3.17, $u_3 \in P_3^0$, $u_4 \in P_4^0$, and u_3v_3 , u_4v_4 are clean and disjoint.

Let $\mathcal{S}_1 := \{e_1, u_1v_1, u_1v_2\}$, $\mathcal{S}_2 := \{e_5, u_5v_5, u_5v_1\}$, $\mathcal{S}_3 := \{u_3v_3, u_4v_4\}$ and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are triangles whose intersection is v_1 . Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

Suppose first that $v_1 \in a \cup b$. It is easy to see that if $v_1 = a \cap b$, then the required f is some element of $\mathcal{S}_3 \cup \{u_5v_5, u_1v_2\}$. Suppose w.l.o.g. that v_1 is in a but not in b . Then a is incident with at most one segment of $\mathcal{S}_3$. If b is disjoint from $\mathcal{S}_3$, the required f is either u_3v_3 or u_4v_4 . Similarly, if b is disjoint from $\mathcal{S}_1 \cup \mathcal{S}_2$, the required f is either u_1v_2 or u_5v_5 . Then, b must have an endpoint in $\{u_1, v_2, u_5, v_5\}$ and the other in $\{u_3, v_3, u_4, v_4\}$. From these facts it is easy to see that the required f is some element of $\mathcal{S}_3 \cup \{u_1v_2, u_5v_5\}$.

Suppose that $v_1 \notin a \cup b$. We assume that each segment of $\mathcal{S}_1 \cup \mathcal{S}_2$ intersects $a \cup b$, as otherwise we are done. This fact and $v_1 \notin a \cup b$ imply that at least three points of $\{v_2, v_5, u_1, u_5\}$ are endpoints of $a \cup b$, and hence the required f is some element of $\mathcal{S}_3$.

Case 4. Suppose that $P_i^- = P_{i+1}^+$ is nonempty for some $i \in [5]$. Assume w.l.o.g. that $P_1^- \neq \emptyset$. By Cases 2 and 3 we can assume that P_1^+ , P_2^- , P_4^- and P_4^+ are empty. These facts and Case 1 imply that P_3^0 , P_4^0 and P_5^0 are nonempty. Let $u_1 \in P_1^-$ be a point that is closest to e_1 . For $j \in \{3, 4, 5\}$, let u_j be a point of P_j farthest from $\ell(v_{j-1}v_{j+1})$.

Let $\mathcal{S}_1 := \{e_1, u_1v_1, u_1v_2\}$, $\mathcal{S}_2 := \{u_3v_3, u_4v_4, u_5v_5\}$ and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$. Then $\mathcal{S}_1$ forms a triangle and the segments in $\mathcal{S}_2$ are clean in $\mathcal{P}$ and pairwise disjoint. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We assume that each segment of $\mathcal{S}_2$ intersects $a \cup b$, as otherwise we are done. Since the segments in $\mathcal{S}_2$ are clean and pairwise disjoint, then at least three points of $\{u_3, u_4, u_5, v_3, v_4, v_5\}$ are endpoints of $a \cup b$, and hence the required f is some element of $\mathcal{S}_1$.

Case 5. Suppose that $P_i = P_i^0$ for each $i \in [5]$. By this supposition and Case 1, we can assume that $P_i^0 \neq \emptyset$ for each $i \in [5]$. Let u_i be a point of P_i farthest from $\ell(v_{i-1}v_{i+1})$, and let $\mathcal{S} := \{u_1v_1, \dots, u_5v_5\}$. By Proposition 3.16 and Remark 3.17, the segments of $\mathcal{S}$ are clean in $\mathcal{P}$ and pairwise disjoint. Then $\mathcal{S}$ satisfies the hypothesis of Proposition 3.10, and we are done. $\square$

Lemma 3.19. *If $n \geq 6$ and $|\widehat{P}| = 6$, then $\mu(D(P)) \geq \binom{n}{2} - 9$.*

Proof. The strategy of our proof is closely aligned with that utilized in the proof of Lemma 3.18.

Case 1. Suppose that $n = 6$. Let $\mathcal{S} := \{v_1v_4, v_2v_5, v_3v_6, e_1, e_2, e_3, e_4, e_5, e_6\}$. Let (a, b) be an $\mathcal{S}$ -pair. Then $\mathcal{P} \setminus \mathcal{S} = \{v_1v_3, v_3v_5, v_1v_5, v_2v_4, v_4v_6, v_2v_6\}$. By Lemma 3.1(2)–(3), we need to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$ when $d_{D(P)}(a, b) = 2$, and that $\mathcal{S}$ has two segments g_1, g_2 such that $g_1 \cap g_2 = \emptyset$ and $a \cap g_1 = \emptyset = b \cap g_2$, when $d_{D(P)}(a, b) = 3$.

From $a \cap b \neq \emptyset$ it is easy to see that either $a = v_i v_{i+2}$ and $b = v_{i+1} v_{i+5}$ for some $i \in [6]$ or $a = v_j v_{j+2}$ and $b = v_j v_{j+4}$ for some $j \in [6]$. In the former case, $d_{D(P)}(a, b) = 2$ and $f = v_{i+3} v_{i+4}$ is the required segment. In the latter case, $d(a, b) = 3$ and $g_1 = v_{j+4} v_{j+5}$ and $g_2 = v_{j+1} v_{j+2}$ are the required segments.

Case 2. Suppose that $n = 7$. Then $P \setminus \widehat{P}$ contains exactly one point, say x . It is not hard to see that there must exist $i \in [6]$ such that x lies in the interior of the triangle formed by $e_i = v_i v_{i+1}$ and the diagonals $v_i v_{i+3}$ and $v_{i+1} v_{i+4}$. If x lies on $\ell^+(v_{i+2} v_{i+5})$ (resp. $\ell^-(v_{i+2} v_{i+5})$), then it is straightforward to check that $\triangle = x v_i v_{i+1}$ (resp. $\triangle = x v_{i+3} v_{i+4}$) is a good-triangle. The statement follows from the existence of such $\triangle$ and Proposition 3.13.

In view of Cases 1 and 2, from now on we assume $n \geq 8$.

Case 3. Suppose that $P_i, P_{i+3} \neq \emptyset$ for some $i \in [6]$. Assume w.l.o.g. that $P_1, P_4 \neq \emptyset$. Let u_1 (resp. u_4) be a point of P_1 (resp. P_4) farthest from $\ell(v_6 v_2)$ (resp. $\ell(v_3 v_5)$); by Proposition 3.16, $v_1 u_1$ and $v_4 u_4$ are clean. Then $\mathcal{D}(v_2 v_3, v_5 v_6)$ together with $e_l := v_4 u_4$ and $e_r := v_1 u_1$ form a good-2-set of $\mathcal{P}$, and we are done by Proposition 3.15.

Case 4. Suppose that at least two of the sets P_i^-, P_i^0, P_i^+ are nonempty for some $i \in [6]$. Assume w.l.o.g. that $i = 2$. Let u_2 be a point of P_2 farthest from $\ell(v_1 v_3)$. By Proposition 3.16, $u_2 v_2$ is clean in $\mathcal{P}$. We distinguish two cases, according to whether u_2 belongs to P_2^0 or not.

Suppose that $u_2 \in P_2^0$. By hypothesis, $P_2^- \cup P_2^+ \neq \emptyset$. If $P_2^- \neq \emptyset$, then $P_3 \supseteq P_3^+ = P_2^-$ is nonempty; let u_3 be a point of P_3 farthest from $\ell(v_2 v_4)$. Then $\mathcal{D}(u_2 v_2, v_4 v_5)$, $e_l := u_3 v_3$ and $e_r := v_1 v_6$ form a good-2-set of $\mathcal{P}$. If $P_2^+ \neq \emptyset$, then $P_1 \supseteq P_1^- = P_2^+$ is nonempty; let u_1 be a point of P_1 farthest from $\ell(v_6 v_2)$. Then $\mathcal{D}(u_2 v_2, v_5 v_6)$, $e_l := v_3 v_4$ and $e_r := u_1 v_1$ form a good-2-set of $\mathcal{P}$. In both cases we are done by Proposition 3.15.

Suppose that $u_2 \in P_2^- \cup P_2^+$. We consider only the case $u_2 \in P_2^-$, as the case $u_2 \in P_2^+$ is analogous. We recall that $P_2^- \subset P_3$. Let u_3 be a point of P_3 farthest from $\ell(v_2 v_4)$. If $u_2 \neq u_3$, then $\mathcal{D}(u_2 v_2, v_4 v_5)$, $e_l := u_3 v_3$, and $e_r := v_1 v_6$ form a good-2-set, and the result follows from Proposition 3.15. On the other hand, if $u_2 = u_3$, then Proposition 3.16 guarantees that $u_2 v_3$ is also clean in $\mathcal{P}$. Hence, $\triangle := u_2 v_2 v_3$ is a good-triangle of P , and the result follows from Proposition 3.13.

Case 5. Suppose that $P_i^- \neq \emptyset$ and $P_{i+2}^0 \cup P_{i+5}^0 \neq \emptyset$ for some $i \in [6]$. Assume w.l.o.g. that $i = 1$. We only analyze the case $P_3^0 \neq \emptyset$, as the case $P_6^0 \neq \emptyset$ is totally analogous. Then we can assume that $P_4 \cup P_5 \cup P_6 = \emptyset$ by Case 3. Similarly, by Case 4 we can assume that $(P_1 \setminus P_1^-) \cup (P_2 \setminus P_2^+) \cup (P_3 \setminus P_3^0) = \emptyset$.

Let u_2 be a point of $P_2 = P_2^+$ farthest from $\ell(v_1 v_3)$, and let u_3 be a point of $P_3 = P_3^0$ farthest from $\ell(v_2 v_4)$. By Proposition 3.16, $u_2 v_2$ and $u_3 v_3$ are clean. Then $\mathcal{D}(u_2 v_2, v_4 v_5)$, $e_l := u_3 v_3$ and $e_r := v_1 v_6$ form a good-2-set of $\mathcal{P}$, and we are done by Proposition 3.15.

Case 6. Suppose that $P_i^- \neq \emptyset$ for some $i \in [6]$. Assume w.l.o.g. that $i = 1$. Then we can assume that $P_4 \cup P_5 = \emptyset$, $(P_1 \setminus P_1^-) \cup (P_2 \setminus P_2^+) = \emptyset$ and $P_3^0 \cup P_6^0 = \emptyset$ by Cases 3, 4 and 5, respectively.

Suppose that $|P_1^-| = 1$. Let x be the point in P_1^- . Then the segments of the triangle $\triangle := xv_1v_2$ are clean in $\mathcal{P}$, and so $\triangle$ is a good-triangle. The statement follows from the existence of such $\triangle$ and Proposition 3.13.

Now, suppose that $|P_1^-| \geq 2$. Let x and y be distinct points of P_1^- . Let $\mathcal{S}_1 := \{e_1, e_3, e_5\}$, $\mathcal{S}_2 := \{v_1v_4, v_2v_5, v_3v_6, v_2v_6, v_1v_3\}$ and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{xy\}$. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1 (2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We can assume that $(a \cup b) \cap e_i \neq \emptyset$ for $e_i \in \mathcal{S}_1$, as otherwise $f = e_i$ is the required. Then either a or b intersects exactly two segments of $\mathcal{S}_1$. Since $(a \cup b) \cap e_1 \neq \emptyset$. Then a or b has an endpoint in $\{v_1, v_2\}$. We only analyze the case $v_2 \in a \cup b$, as the case $v_1 \in a \cup b$ is analogous. It follows from the choice of $\mathcal{S}_2$ that if $v_2 = a \cap b$, then a and b have their other endpoint in e_3 , and $f = e_5$ is the required. Suppose w.l.o.g. that v_2 is in a but not in b . Then a has its other endpoint in e_3 and so $a \in \{v_2v_3, v_2v_4\}$. Then $a \cap e_5 = \emptyset$ and, as a consequence, b must have an endpoint in e_5 . Since $a \cap b \neq \emptyset$ and $v_2v_5, v_2v_6, v_3v_6 \in \mathcal{S}_2$, then $b \in \{v_4v_5, v_3v_5, v_4v_6\}$. Then $f = xy$ is the required.

Case 7. Suppose that $P_i^0 \neq \emptyset$ for some $i \in [6]$. Assume w.l.o.g. that $i = 1$. Then we can assume that $P_4 = \emptyset$, $P_1 \setminus P_1^0 = \emptyset$ and $P_2^- \cup P_5^- = \emptyset$ by Cases 3, 4 and 5, respectively. Let u_1 be a point of $P_1 = P_1^0$ farthest from $\ell(v_6v_2)$. By Proposition 3.16, v_1u_1 is clean.

Suppose that $P_2^0 \cup P_6^0 \neq \emptyset$. We only analyze the case $P_2^0 \neq \emptyset$, as the case $P_6^0 \neq \emptyset$ is totally analogous. Let u_2 be a point of $P_2 = P_2^0$ farthest from $\ell(v_1v_3)$; by Proposition 3.16, v_2u_2 is clean. Then $\mathcal{D}(v_1u_1, v_3v_4)$, $e_l := v_5v_6$ and $e_r := v_2u_2$ form a good-2-set of $\mathcal{P}$, and we are done by Proposition 3.15.

Suppose that $P_3^0 \cup P_5^0 \neq \emptyset$. We only analyze the case $P_3^0 \neq \emptyset$, as the case $P_5^0 \neq \emptyset$ is totally analogous. By previous case we can assume that $P_2^0 \cup P_6^0 = \emptyset$.

Let u_3 be a point of P_3 farthest from $\ell(v_2v_4)$. Let $\mathcal{S}_1 := \{e_1, e_2, e_5\}$, $\mathcal{S}_2 := \{v_1u_1, v_2v_5, v_2v_6, v_3v_5, v_3v_6, v_4u_3\}$ and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1 (2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We can assume that $(a \cup b) \cap e_i \neq \emptyset$ for $e_i \in \mathcal{S}_1$, as otherwise $f = e_i$ is the required. Since $v_2v_5, v_2v_6, v_3v_5, v_3v_6 \in \mathcal{S}_2$, neither a nor b can go from e_2 to e_5 . Then we can assume w.l.o.g. that a intersects e_2 and that b intersects e_5 . Similarly, we can assume that $(a \cup b) \cap g \neq \emptyset$ for each $g \in \{v_1u_1, v_4u_3\}$, as otherwise $f = g$ is the required.

Since a intersects e_2 , it has an endpoint in $\{v_2, v_3\}$. Assume first that $v_2 \in a$. If $a \cap v_4u_3 \neq \emptyset$, then necessarily $b \cap v_1u_1 \neq \emptyset$, and the line $\ell(v_2v_5)$ separates a from b . Hence, $a \cap b = \emptyset$, a contradiction. Otherwise, $a \cap v_4u_3 = \emptyset$, and therefore $b \cap v_4u_3 \neq \emptyset$ and $a \cap v_1u_1 \neq \emptyset$. The line $\ell(v_3v_6)$ again separates a from b , yielding $a \cap b = \emptyset$, a contradiction.

Now, suppose that $v_3 \in a$. If $a = v_1v_3$, then $b \cap v_4u_3 \neq \emptyset$, and the line $\ell(v_3v_6)$ separates a from b , and so we have $a \cap b = \emptyset$, a contradiction. Thus, we may assume that $v_1 \notin a$. Then $b \in \{v_1v_5, v_1v_6\}$, as otherwise $f = e_1$ is the required. The facts that $P_6 = \emptyset$, $v_1 \notin a$ and $b \in \{v_1v_5, v_1v_6\}$ imply that $a \cap b = \emptyset$, a contradiction.

Suppose that $(P_1^0 \cup P_c) \setminus \{u_1\}$ contains at least one point x , where $P_c := P \setminus ((\bigcup_{i=1}^6 P_i) \cup \hat{P})$. By previous cases we know that $P_1^0 = \bigcup_{i=1}^6 P_i$. Let $\mathcal{S}_1 := \{e_1, e_3, e_5\}$, $\mathcal{S}_2 := \{v_1v_3, v_1v_4, v_2v_5, v_3v_6, v_2v_6\}$ and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{u_1x\}$. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1 (2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We can assume that $(a \cup b) \cap e \neq \emptyset$ for every $e \in \mathcal{S}_1$, as otherwise $f = e$ is the required. Since $(a \cup b) \cap e_1 \neq \emptyset$. Then a or b has an endpoint in $\{v_1, v_2\}$.

First, suppose $v_1 \in a \cup b$. If $v_1 = a \cap b$, then $f = e_3$ is the required. Assume w.l.o.g. that $v_1 \in a$ and $v_1 \notin b$. This implies that b has an endpoint on e_3 . If a (respectively, b) has its other endpoint in e_5 , then $a \in \{v_1v_5, v_1v_6\}$ (respectively, $b \in \{v_3v_5, v_4v_5, v_4v_6\}$). Since $a \cap b \neq \emptyset$, then $f = u_1x$ is the required.

Suppose now that $v_2 \in a \cup b$. If $v_2 = a \cap b$, then $f = e_5$ is the required. Assume w.l.o.g. that $v_2 \in a$ and $v_2 \notin b$. Then a has its other endpoint in e_3 , implying $a \in \{v_2v_3, v_2v_4\}$. Consequently, $a \cap e_5 = \emptyset$, and thus, b must have an endpoint in e_5 . Given that $a \cap b \neq \emptyset$ and $v_2v_5, v_2v_6, v_3v_6 \in \mathcal{S}$, we have $b \in \{v_3v_5, v_4v_5, v_4v_6\}$. Therefore, $f = u_1x$ works.

Case 8. Suppose that $P_c \neq \emptyset$. From Cases 6, 7, $P_i^- = P_{i+1}^+$ and $P_i = P_i^- \cup P_i^0 \cup P_i^+$ it follows that $P_i = \emptyset$ for each $i \in [6]$ and therefore $P = \hat{P} \cup P_c$. By Case 2, we can assume that $|P_c| \geq 2$. Let x and y be distinct points of P_c .

Let $\mathcal{S}_1 := \{e_1, e_3, e_5\}$, $\mathcal{S}_2 := \{v_1v_4, v_2v_5, v_3v_6\}$ and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{xy\}$. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1 (2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We can assume that $(a \cup b) \cap e_i \neq \emptyset$ for $e_i \in \mathcal{S}_1$, as otherwise $f = e_i$ is the required. Then either a or b intersects exactly two segments of $\mathcal{S}_1$. By relabelling the points in $\hat{P}$ if necessary, we may assume that a has an endpoint in e_1 and the other in e_3 . Then $a \in \{v_1v_3, v_2v_3, v_2v_4\}$. Then $a \cap e_5 = \emptyset$ and, as a consequence, b must have an endpoint in e_5 . Since $a \cap b \neq \emptyset$ and $v_2v_5, v_3v_6 \in \mathcal{S}_2$, then $b \in \{v_4v_5, v_3v_5, v_1v_5, v_1v_6, v_2v_6, v_4v_6\}$. Then $f = xy$ is the required. $\square$

Lemma 3.20. *If $n \geq 7$ and $|\hat{P}| = 7$, then $\mu(D(P)) \geq \binom{n}{2} - 9$.*

Proof. Our proof strategy is very similar to that in the proof of Lemma 3.18.

Case 1. Suppose that $\hat{P} = P$. Let $\mathcal{S} := \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and let (a, b) be an $\mathcal{S}$ -pair. If $(a \cup b) \cap e_i = \emptyset$ for some $e_i \in \mathcal{S}$, we are done by Lemma 3.1 (2). Then we can assume that $a \cup b$ intersects each segment of $\mathcal{S}$, and hence $a \cup b$ has four endpoints in $\hat{P}$. Then $d_{D(P)}(a, b) = 3$ and there must exist $j \in [7]$ such that $v_j \in a \cup b$ and $v_{j+1} \in a \cup b$. By relabelling the points of $\hat{P}$ if necessary, we may assume that $j = 1$. Then $a = v_1v_4$, $b = v_2v_6$, and so $f_1 = v_5v_6$, $f_2 = v_3v_4$ satisfy Lemma 3.1 (3).

Case 2. Suppose that $P_i \neq \emptyset$ for some $i \in [7]$. Assume w.l.o.g. that $i = 1$, and let x be a point of P_1 farthest from $\ell(v_7v_2)$; by Proposition 3.16, v_1x is clean. Then $\mathcal{D}(v_2v_3, v_6v_7)$, $e_l := v_1x$ and $e_r := v_4v_5$ form a good-2-set of $\mathcal{P}$, and we are done by Proposition 3.15.

By Cases 1 and 2, for the rest of the proof we assume $P_i = \emptyset$ for each $i \in [7]$ and $n \geq 8$.

We need some additional terminology. For $i \in [7]$, let $Q(e_i)$ be the subset of P that lies in the interior of the quadrilateral $v_{i+6}v_iv_{i+1}v_{i+2}$, and let Δ_i be the subset of P that lies in the interior of the triangle $v_iv_{i+3}v_{i+4}$. We define $C_i := \Delta_i \setminus (Q(e_{i+2}) \cup Q(e_{i+4}) \cup P_i)$, $C'_i := C_i \cap Q(e_i) \cap Q(e_{i+6})$ and $Q_i := (Q(e_{i+2}) \cap Q(e_{i+4})) \setminus (P_{i+3} \cup P_{i+4})$. Then $P = \bigcup_{i=1}^7 (P_i \cup C_i \cup Q_i)$. See Figure 7(b).

Case 3. Suppose that $P \setminus \widehat{P}$ has two distinct points x and y such that xy crosses at most one segment g of $\mathcal{D}_7$. We recall that $\mathcal{D}_7$ is the subset of $\mathcal{P}$ formed by all segments with both endpoints in $\widehat{P}$. If xy does not cross any segment of $\mathcal{D}_7$, we let $g := \emptyset$. Let $\mathcal{S} := \{e_1, e_2, \dots, e_7\} \cup \{xy\} \cup \{g\}$ and let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1 (2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

If there exists a segment $f \in \{e_1, e_2, \dots, e_7\}$ such that $(a \cup b) \cap f = \emptyset$, then f is the required segment. Hence, we may assume that every segment in $\{e_1, e_2, \dots, e_7\}$ intersects $a \cup b$. It follows that $a \cup b$ has four endpoints in $\{v_1, v_2, \dots, v_7\}$, and therefore $f = xy$ is the required segment.

Case 4. Suppose that $|Q_i| = 1 = |Q_{i+2}|$ for some $i \in [7]$. Assume w.l.o.g. that $i = 1$. Let u_1 (respectively, u_3) be the only point of Q_1 (respectively, Q_3). Let $\mathcal{S}_1 := \{e_3, e_4, v_3u_1, v_5u_1\}$, $\mathcal{S}_2 := \{e_6, v_6u_3, v_7u_3\}$, $\mathcal{S}_3 := \{e_1\}$ and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1 (2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We can assume that $(a \cup b) \cap e \neq \emptyset$ for $e \in \mathcal{S}$, as otherwise $f = e$ is the required. Then $a \cup b$ has an endpoint in e_1 . Assume w.l.o.g. that $a \cap e_1 \neq \emptyset$. If a intersects a segment of $\mathcal{S}_2$, then b must intersect each segment of $\mathcal{S}_1$ and so $b \in \{v_3v_5, v_4u_1\}$. Since these imply $a \cap b = \emptyset$ a contradiction, then a must have its other endpoint in $\{v_3, v_4, v_5, u_1\}$, and so $b \cap e_6 \neq \emptyset$. These facts and $a \cap b \neq \emptyset$ imply that the required f is some of $\{e_3, v_3v_6, v_3v_7\}$; recall that $(a \cup b) \cap e \neq \emptyset$ for each $e \in \mathcal{S}$.

Case 5. Suppose that $C_i \neq \emptyset$ and $Q_i = \emptyset$ for some $i \in [7]$. Assume w.l.o.g. that $i = 1$. Let $x \in C_1$. By Case 2 we can assume $\bigcup_{i=1}^7 P_i = \emptyset$.

Case 5.1. Suppose that $C_4 \cap Q(e_3) = \emptyset$. Let $\mathcal{S}_1 := \{e_1, e_2, e_4, e_7\}$, $\mathcal{S}_2 := \{v_1x, v_3x, v_6x\}$, $\mathcal{S}_3 := \{v_1v_4, v_1v_5\}$, and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$. We can assume that $(a \cup b) \cap e \neq \emptyset$ for $e \in \mathcal{S}$, as otherwise we are done.

Suppose that $v_1 \in a \cup b$. Assume w.l.o.g. that $v_1 \in a$. Then $v_j \in b$ for some $j \in \{4, 5\}$. Since if $a = v_1v_3$ (resp. v_1v_6) then $a \cap b \neq \emptyset$ implies that $f = v_6x$ (resp. $f = v_3x$) is the required, then we can assume that $a \notin \{v_1v_3, v_1v_6\}$. This implies that b has its other endpoint in $\{v_2, v_3\}$, and so $a \cap b \neq \emptyset$, $Q_1 \cup (C_4 \cap Q(e_3)) = \emptyset$ and $a \notin \mathcal{S}_3$ together imply that $f = v_6x$ is the required.

Suppose that $v_1 \notin a \cup b$. Then v_2 and v_7 must be endpoints of $a \cup b$. If $v_2v_7 \in \{a, b\}$, then the required f is some element of $\{v_3x, e_4, v_6x\}$. We may then assume w.l.o.g. that $v_2 \in a$ and $v_7 \in b$. Since $e_4 \cap g \neq \emptyset$ for some $g \in \{a, b\}$, then the required f is some element of $\{v_1x, v_3x, v_6x\}$.

Case 5.2. Suppose that $C_4 \cap Q(e_3) \neq \emptyset$. Assume first that $|C_4 \cap Q(e_3)| \geq 2$. Let x and y be two distinct points of $C_4 \cap Q(e_3)$. Then xy crosses at most $v_3v_6 \in \mathcal{D}_7$, and we are done by Case 3.

Assume now that $|C_4 \cap Q(e_3)| = 1$. Let $u \in C_4 \cap Q(e_3)$. Let $\mathcal{S}_1 := \{e_1, e_5, e_6, e_7\}$, $\mathcal{S}_2 := \{v_3u, v_4u, e_3\}$, $\mathcal{S}_3 := \{v_1v_4, v_3v_6\}$, and $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Let (a, b) be an $\mathcal{S}$ -pair. By Lemma 3.1 (2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We can assume that $(a \cup b) \cap e_3 \neq \emptyset$, as otherwise $f = e_3$ is the required. Assume w.l.o.g. that $a \cap e_3 \neq \emptyset$. Suppose first that $v_3 \in a$. If $a \cap \{v_1, v_2, v_5, v_7\} = \emptyset$, then $b = v_1v_6$. This last implies $a \cap b = \emptyset$, a contradiction.

Thus, we must have that $a \cap \{v_1, v_2, v_5, v_7\} \neq \emptyset$. If $a \cap \{v_1, v_2\} \neq \emptyset$ then b must intersect each of $\{e_5, e_6, v_4u\}$. This last implies $a \cap b = \emptyset$, a contradiction. If $a = v_3v_5$, then $a \cap b \neq \emptyset$ implies that some $f \in \{e_1, e_6\}$ is the required. Similarly, if $a = v_3v_7$, then some $f \in \{e_1, e_5, v_4u\}$ is the required. The case in which $v_4 \in a$ can be handled in an analogous way.

Case 6. Suppose that $|Q_i| = 1 = |Q_{i+1}|$ for some $i \in [7]$. Assume w.l.o.g. that $i = 1$. By Case 4, we can assume that $Q_3 \cup Q_4 \cup Q_6 \cup Q_7 = \emptyset$. Then either $C_3 \cup C_4 \cup C_6 \cup C_7 = \emptyset$ or we are done by Case 5. For $j \in \{1, 2\}$, let u_j be the only point of Q_j .

If there is $x \in C'_5$, then xu_1 crosses at most $v_2v_5 \in \mathcal{D}_7$, and we are done by Case 3. Thus, we can assume that $C'_5 = \emptyset$. Then $Q(e_5) = \{u_1, u_2\}$.

Let x be the first point of $Q(e_5)$ that we find when we rotate $\ell(v_4v_6)$ clockwise around v_4 . If $x = u_1$ (resp. $x = u_2$), then v_3v_5 (resp. v_5v_7) is the only segment of $\mathcal{P}$ that crosses v_4x (resp. v_6x). Then $\mathcal{D}(v_5v_6, v_2v_3)$ (resp. $\mathcal{D}(v_1v_7, v_4v_5)$), $e_l := v_4x$ (resp. $e_l := v_6x$) and $e_r := v_1v_7$ (resp. $e_r := v_2v_3$) form a good-2-set of $\mathcal{P}$, and we are done by Proposition 3.15.

Case 7. Suppose that $|Q_i| = 1$ for some $i \in [7]$. Assume w.l.o.g. that $i = 1$. Let x be the only point in Q_1 . By Cases 3, 4, 5, 6, we can assume that $C'_4 \cup C'_5 \cup Q_2 \cup Q_3 \cup Q_6 \cup Q_7 \cup C_2 \cup C_3 \cup C_6 \cup C_7 = \emptyset$. Then $Q(e_3) \cup Q(e_5) = \{x\}$ and v_4v_6 is the only segment of $\mathcal{P}$ that crosses v_5x . Then $\mathcal{D}(v_3v_4, v_6v_7)$, $e_l := v_1v_2$ and $e_r := v_5x$ form a good-2-set of $\mathcal{P}$, and we are done by Proposition 3.15. $\square$

3.6. MUTUAL-VISIBILITY OF $D(P)$ WHEN $|\hat{P}| \geq 8$

Lemma 3.21. *If $m := |\hat{P}| \in \{8, 9\}$, then $\mu(D(P)) \geq \binom{n}{2} - 8$.*

Proof. Let $v_1, v_2, \dots, v_m$ be the points in $\hat{P}$, and assume that they appear in this cyclic order on $\hat{P}$. Then $\mathcal{D}(v_1v_2, v_5v_6)$, $e_l := v_7v_8$ and $e_r := v_3v_4$ form a good-2-set of $\mathcal{P}$, and we are done by Proposition 3.15. $\square$

Lemma 3.22. *If $m := |\hat{P}| \geq 10$, then $\mu(D(P)) \geq \binom{n}{2} - 5$.*

Proof. Let $v_1, v_2, \dots, v_m$ be the points in $\hat{P}$, and assume that they appear in this cyclic order on $\hat{P}$. For $i \in [m-1]$, let $e_i := v_i v_{i+1}$. Since $\{e_1, e_3, e_5, e_7, e_9\}$ is a set of pairwise disjoint segments that are clean in $\mathcal{P}$, we are done by Proposition 3.10. $\square$

Proof of Theorem 1.2. It follows directly from Lemmas 3.8, 3.9, 3.18, 3.19, 3.20, 3.21, and 3.22. $\square$

4. THEOREM 1.2 IS THE BEST POSSIBLE: PROOF OF PROPOSITION 1.3

Proof of Proposition 1.3. Let $P := \{p_1, p_2, \dots, p_7\}$ be the 7-point set given in Proposition 1.3 and depicted in Figure 1(a). Clearly, P is in general position and $|\hat{P}| = 6$. Then $\mu(D(P)) \geq \binom{7}{2} - 9 = 12$ by Case 2 of Lemma 3.19.

Moreover, $\text{diam}(D(P)) \in \{2, 3\}$ by Theorem 1.1. Hence, every $\mathcal{S}$ -pair (a, b) satisfies $d_{D(P)}(a, b) \in \{2, 3\}$.

To prove that $\mu(D(P)) \leq 12$, we implemented an algorithm (described below) in Mathematica (Wolfram Language) [22] that verifies the non-existence of a mutual-visibility set of size 13 in $V(D(P))$. To be more precise, for each 13-subset $\mathcal{P}'$ of $V(D(P))$, the algorithm tests whether every pair of non-adjacent vertices $a, b \in \mathcal{P}'$ admits the shortest a - b path whose interior vertices all belong to $\mathcal{S} = V(D(P)) \setminus \mathcal{P}'$. As soon as a pair fails the test, the candidate $\mathcal{P}'$ is rejected and the algorithm moves on to the next 13-subset. It returns $\mathcal{P}'$ only if all such pairs pass the test, and it returns NIL only after every 13-subset has been examined.

For each of the $\binom{21}{13} = 203490$ 13-subsets $\mathcal{P}'$ of $V(D(P))$, the program found a pair of non-adjacent vertices $a, b \in \mathcal{P}'$ such that every shortest a - b path in $D(P)$ contains at least one interior vertex belonging to $\mathcal{P}'$. Hence, $D(P)$ contains no mutual-visibility set of size 13. Since every subset of a mutual-visibility set is again a mutual-visibility set [5], then $D(P)$ has no mutual-visibility set of size at least 13, and therefore $\mu(D(P)) \leq 12$. $\square$

The interested reader can find and/or execute in [21] the program that we have used to verify that $D(P)$ has no mutual-visibility sets of size 13.

5. TIGHT ASYMPTOTIC BOUNDS FOR $\mu(D(P))$

Our aim in this section is to show Theorem 1.4 and Propositions 1.5 and 1.6.

Proof of Theorem 1.4. We first show that if $n \geq 9$, then $\mu(D(P)) \leq \binom{n}{2} - 4$. Since every subset of a mutual-visibility set is again a mutual-visibility set [5] then, it is enough to show the following: for every set $\mathcal{E} := \{x_1y_1, x_2y_2, x_3y_3\}$ of three distinct segments of $\mathcal{P}$, the set $\mathcal{P}' := \mathcal{P} \setminus \mathcal{E}$ is not a mutual-visibility set. Let $V(\mathcal{E})$ be the set of endpoints of the segments of $\mathcal{E}$.

By Theorem 1.1, $\text{diam}(D(P)) = 2$. It is therefore enough to find two distinct segments $a, b \in \mathcal{P}'$ such that

- (i) $a \cap b \neq \emptyset$, and
- (ii) $(a \cup b) \cap x_iy_i \neq \emptyset$ for each $i \in [3]$.

Indeed, by (i) and the fact that $\text{diam}(D(P)) = 2$, we have $d_{D(P)}(a, b) = 2$. Thus, every shortest a - b path in $D(P)$ has the form $a f b$, where $f \cap (a \cup b) = \emptyset$. By (ii), we have $f \notin \mathcal{E}$, and hence $f \in \mathcal{P}'$. Therefore, any shortest a - b path intersects $\mathcal{P}'$, which implies that $\mathcal{P}'$ is not a mutual-visibility set of $D(P)$, as required.

Algorithm 1: Determining whether $V(D(P))$ contains a mutual-visibility set of size 13.

Input: $D(P) = (V(D(P)), E(D(P)))$.
Output: A mutual-visibility set $\mathcal{P}'$ of 13 elements, or NIL if no such $\mathcal{P}'$ exists.

```

1  foreach 13-subset  $\mathcal{P}'$  of  $V(D(P))$  do
2     $\mathcal{S} = V(D(P)) \setminus \mathcal{P}'$ ;
3    valid = 1;
4    foreach  $\mathcal{S}$ -pair  $(a, b)$  do
5      flag = 0;
6      if  $d_{D(P)}(a, b) == 2$  then
7        foreach  $c \in \mathcal{S}$  do
8          if  $a - c - b$  is a path then
9            flag = 1;
10           break;
11       else if  $d_{D(P)}(a, b) == 3$  then
12         foreach 2-set  $\{c, c'\} \subset \mathcal{S}$  do
13           if  $a - c - c' - b$  is a path or  $a - c' - c - b$  is a path then
14             flag = 1;
15             break;
16       if flag == 0 then
17         print  $\mathcal{P}'$  is not a mutual-visibility set due to the pair of non-adjacent
18           vertices  $a, b$ ;
19         valid = 0;
20         break;
21   if valid == 1 then
22     print  $\mathcal{P}'$  is a mutual-visibility set;
23     return  $\mathcal{P}'$ ;
24 return NIL;

```

We say that $U \subseteq P$ covers $\mathcal{E}$ if every segment of $\mathcal{E}$ has an endpoint in U .

Suppose first that two distinct points $u_1, u_2 \in V(\mathcal{E})$ cover $\mathcal{E}$. Since $|V(\mathcal{E})| \leq 6 < n$, there exists a point $z \in P \setminus V(\mathcal{E})$. Then $a := zu_1$ and $b := zu_2$, which are not in $\mathcal{E}$, satisfy (i) and (ii).

Suppose now that no two points of $V(\mathcal{E})$ cover $\mathcal{E}$. Then no two segments of $\mathcal{E}$ share an endpoint: if u were a common endpoint of two of them, then u and an endpoint $u' \neq u$ of the third segment would cover $\mathcal{E}$. So the points in $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ are pairwise distinct. Then $a := x_1x_2$ and $b := x_1x_3$, which are not in $\mathcal{E}$, satisfy (i) and (ii).

We now show the lower bound. By Erdős–Szekeres theorem [10], there is the smallest integer $f_2^{ES}(10)$ such that if $|P| \geq f_2^{ES}(10)$, then P has a 10-subset, say Q , in convex position.

Let $m := |\widehat{Q}| \geq 10$. Assume that $v_1, v_2, \dots, v_m$ are the points of $\widehat{Q}$, and that they appear in this cyclic order. Let $\mathcal{S} := \{v_{2i-1}v_{2i} : i \in [5]\}$. Then the 5 segments in $\mathcal{S}$ are pairwise disjoint. By Lemma 3.1(2), it is enough to show that if

(a, b) is an $\mathcal{S}$ -pair, then $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

It is easy to see that if $h \in \mathcal{P} \setminus \mathcal{S}$, then h intersects at most 2 segments of $\mathcal{S}$. Thus, if $a, b \in \mathcal{P} \setminus \mathcal{S}$, $a \cup b$ is disjoint from at least one segment f of $\mathcal{S}$, and we are done. $\square$

Proof of Proposition 1.5. By Lemma 3.22, we only need to show that $\mu(D(\mathcal{C}_n)) \leq \binom{n}{2} - 5$. As in the proof of Theorem 1.4, it is enough to show the following. For every set $\mathcal{E} := \{x_1y_1, x_2y_2, x_3y_3, x_4y_4\}$ of four distinct segments of $\mathcal{C}_n$, there are two distinct segments $a, b \in \mathcal{C}' := \mathcal{C}_n \setminus \mathcal{E}$ such that

- (i) $a \cap b \neq \emptyset$, and
- (ii) $(a \cup b) \cap x_iy_i \neq \emptyset$ for each $i \in [4]$.

We use $V(\mathcal{E})$ and “covers” as in the proof of Theorem 1.4. We note that $|V(\mathcal{E})| \leq 8 < n$.

Case 1. There exists a segment $st \in \mathcal{E}$ that shares no endpoint with any segment of $\mathcal{E}' := \mathcal{E} \setminus \{st\}$. If there exist two distinct points $u_1, u_2 \in V(\mathcal{E}')$ that cover $\mathcal{E}'$, then $a := su_1$ and $b := su_2$, which are not in $\mathcal{E}$, satisfy (i) and (ii). Thus, we may assume that no two points of $V(\mathcal{E}')$ cover $\mathcal{E}'$. Then, as in the proof of Theorem 1.4, the three segments of $\mathcal{E}'$ pairwise share no endpoint. Consequently, no two segments of $\mathcal{E}$ share an endpoint, and hence the eight points in $\{x_1, y_1, \dots, x_4, y_4\}$ are pairwise distinct. Recall that x_1, x_2, x_3, x_4 are in convex position. Let a and b be the two diagonals of the convex quadrilateral they determine. Then a and b cross, neither belongs to $\mathcal{E}$, and together they cover $\mathcal{E}$, as required.

Case 2. Every segment of $\mathcal{E}$ shares an endpoint with another segment of $\mathcal{E}$. If there exist two distinct points $u_1, u_2 \in V(\mathcal{E})$ that cover $\mathcal{E}$, then, for any $z \in \mathcal{C}_n \setminus V(\mathcal{E})$, the segments $a := zu_1$ and $b := zu_2$, which are not in $\mathcal{E}$, satisfy (i) and (ii). It remains to show that such a pair $\{u_1, u_2\}$ exists.

If there exists a point $u \in V(\mathcal{E})$ that is an endpoint of at least three segments of $\mathcal{E}$, then u , together with any endpoint $u' \neq u$ of the remaining segment (or any other point $u' \in V(\mathcal{E})$ if there is no remaining segment), covers $\mathcal{E}$. Thus, we may assume that every point of $V(\mathcal{E})$ is an endpoint of at most two segments of $\mathcal{E}$. Consider the graph with vertex set $V(\mathcal{E})$ and edge set $\mathcal{E}$. Each of its components is a path or a cycle, and, by the assumption of Case 2, each component has at least two edges. Since there are four edges, either there are two components, each of which is a path with two edges, and their two middle vertices cover $\mathcal{E}$; or there is a single component, which is either a path $u_1u_2u_3u_4u_5$ or a cycle $u_1u_2u_3u_4u_1$, and so $\{u_2, u_4\}$ covers $\mathcal{E}$. This proves that such a pair exists. $\square$

Proof of Proposition 1.6. By the first part of the proof of Theorem 1.4 we need only to show that $\mu(D(C_{p,q})) \geq \binom{n}{2} - 4$. Let $a_1, a_2, \dots, a_p$ be the points of $C_{p,q} \subset A$ from left to right. Similarly, let $b_1, b_2, \dots, b_q$ be the points of $C_{p,q} \subset B$ from left to right. Let $h_1 := b_1b_2$, $h_2 := b_3b_4$, $h_3 := b_5b_6$ and $\mathcal{B} := \{h_1, h_2, h_3\}$.

Let $\mathcal{S} := \{a_1a_2\} \cup \mathcal{B}$. We remark that the segments in $\mathcal{S}$ are clean in $\mathcal{C}_{p,q}$ and pairwise disjoint. Let $\mathcal{C}' := \mathcal{C}_{p,q} \setminus \mathcal{S}$. Consider two distinct segments $a, b \in \mathcal{C}'$ such that $a \cap b \neq \emptyset$. By Lemma 3.1(2), it is enough to show that $\mathcal{S}$ has a segment f such that $(a \cup b) \cap f = \emptyset$.

We may assume that $a \cup b$ intersects a_1a_2 , as otherwise $f = a_1a_2$ is the required. Assume w.l.o.g. that a intersects a_1a_2 . Then a is disjoint from at least two segments of $\mathcal{B}$, say h_i and h_j . Then b must intersect both h_i and h_j , as otherwise $f \in \{h_i, h_j\}$. Then $b \cap h_k = \emptyset$ for $h_k \in \mathcal{B} \setminus \{h_i, h_j\}$. Since if $a \cap h_k = \emptyset$, then $f = h_k$ is the required, we must have that $a \cap h_k \neq \emptyset$, which implies $a \cap b = \emptyset$, a contradiction. $\square$

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Jesús Leaños
jleanos@uaz.edu.mx

Unidad Académica de Matemáticas
Universidad Autónoma de Zacatecas
Calzada Solidaridad y Paseo La Bufa
Col. Hidráulica, CP. 98060
Zacatecas, Zac., México

Mario Lomelí-Haro

lomeli@mixteco.utm.mx

 <https://orcid.org/0000-0003-3921-1369>

Centro de Modelación Matemática, Vinculación y Consultoría
Universidad Tecnológica de la Mixteca
México

Christophe Ndjatchi (corresponding author)

mndjatchi@ipn.mx

 <https://orcid.org/0000-0003-2702-9981>

Academia de Físico–Matemáticas
Instituto Politécnico Nacional, UPIIZ
P.C. 098160, Zacatecas, México

Luis Manuel Ríos–Castro
lrisc@ipn.mx

Academia de Físico–Matemáticas
Instituto Politécnico Nacional, CECYT18
P.C. 098160, Zacatecas, México

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