

## SOLUTIONS WITH PRESCRIBED MASS FOR THE SOBOLEV-CRITICAL KIRCHHOFF EQUATION WITH TRAPPING POTENTIAL

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**Abstract.** We study prescribed-mass solutions of a nonautonomous Kirchhoff equation with a purely Sobolev-critical nonlinearity and a nonnegative trapping potential. The trapping potential breaks the scaling invariance of the autonomous problem, while the critical nonlinearity causes loss of compactness. We establish a global compactness alternative for constrained Palais–Smale sequences and combine it with sharp energy estimates adapted to the nonlocal Kirchhoff term. Under suitable structural assumptions and an explicit energy-separation condition, we obtain a positive normalized ground state as a local minimizer of the constrained energy. For sufficiently small Kirchhoff parameter  $b > 0$ , we further prove the existence of a distinct positive normalized solution at a mountain-pass level.

**Keywords:** Kirchhoff equation, Sobolev-critical exponent, normalized ground state, mountain-pass solution.

**Mathematics Subject Classification:** 35J15, 35B33, 35A15.

### 1. INTRODUCTION

In this paper, we study the following Kirchhoff equation with a purely Sobolev-critical nonlinearity

$$\begin{cases} - \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = \lambda u + u^5, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 = c^2, & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

where  $c > 0$  is the prescribed mass and  $\lambda$  serves as a Lagrange multiplier. Throughout the paper,  $a > 0$  and  $b > 0$  are standing assumptions on the Kirchhoff parameters, and  $V(x) \geq 0$  is a trapping potential.

The Kirchhoff coefficient in (1.1) depends on the global Dirichlet energy  $\int_{\mathbb{R}^3} |\nabla u|^2 dx$ , so the problem is nonlocal even though, for a sufficiently regular solution, the differential equation is still satisfied pointwise. This nonlocal dependence introduces significant mathematical challenges and makes the problem particularly interesting to study. Equation (1.1) may be viewed as the stationary counterpart of the hyperbolic Kirchhoff-type equation

$$\begin{cases} u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(x, u), & (x, t) \in \Omega \times (0, +\infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times [0, +\infty). \end{cases} \quad (1.2)$$

The equation (1.2) was introduced by Kirchhoff [16] as an extension of the D'Alembert wave equation for free vibrations of elastic strings:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = f(x, u),$$

where  $\rho$  is the mass density,  $u$  the lateral displacement,  $\rho_0$  the initial axial tension,  $h$  the cross-sectional area,  $E$  the Young's modulus,  $L$  the length of the string, and  $f$  the external force. Kirchhoff's model accounts for the length variation of the string caused by transverse vibrations. Early classical studies of Kirchhoff equations include [2] and [21]. The study of (1.2) received substantial attention after Lions [18] developed an abstract framework for such problems. Subsequent contributions include those of Arosio *et al.* [1], Cavalcanti *et al.* [6], and D'Ancona *et al.* [10]. Arosio *et al.* [1] studied a Cauchy–Dirichlet problem related to (1.2) in the Hadamard sense, viewing it as a special case of an abstract second-order Cauchy problem in a Hilbert space. Cavalcanti *et al.* [6] established the existence and uniqueness of regular global solutions for the Kirchhoff–Carrier equation with nonlinear boundary dissipation and obtained uniform decay rates without restrictions on the initial data. D'Ancona *et al.* [10] proved the existence of global classical periodic solutions for degenerate Kirchhoff equations and interpreted Kirchhoff's equation as a quasilinear hyperbolic Cauchy problem describing transverse oscillations of a stretched string. In particular, (1.2) with  $a = 0$  describes a string with zero initial tension; this case is referred to as the degenerate Kirchhoff equation. We refer to [13, 22] for related developments.

### 1.1. PREVIOUS DEVELOPMENTS

Normalized solutions, namely solutions with a prescribed  $L^2$ -norm  $\|u\|_2 = c > 0$ , have attracted considerable attention in recent years. We briefly review the developments most relevant to normalized Kirchhoff problems. We begin with the Gagliardo–Nirenberg inequality, which provides a convenient framework for distinguishing the  $L^2$ -subcritical,  $L^2$ -critical, and  $L^2$ -supercritical regimes. Throughout the paper,  $\|\cdot\|_p$  denotes the usual norm in  $L^p(\mathbb{R}^3)$  for  $1 \leq p < \infty$ .

**Lemma 1.1** ([30]). *Let  $p \in (2, 6)$ . Then there exists a constant*

$$C_p = \left( \frac{p}{2\|U_p\|_2^{p-2}} \right)^{\frac{1}{p}} > 0$$

such that

$$\|u\|_p \leq C_p \|\nabla u\|_2^{\gamma_p} \|u\|_2^{1-\gamma_p}, \quad \forall u \in H^1(\mathbb{R}^3),$$

where

$$\gamma_p := 3 \left( \frac{1}{2} - \frac{1}{p} \right),$$

and  $U_p$  denotes the unique positive radial solution of

$$-\Delta U + \left( \frac{1}{\gamma_p} - 1 \right) U = \frac{2}{p\gamma_p} |U|^{p-2} U, \quad x \in \mathbb{R}^3.$$

In the full class of positive solutions, uniqueness is understood up to translations.

At the Sobolev-critical endpoint  $p = 6$ , the preceding  $L^2$ -based representation of the sharp constant is no longer applicable. Instead, we use the sharp Sobolev inequality

$$\|u\|_6^2 \leq \mathcal{S}^{-1} \|\nabla u\|_2^2, \quad u \in D^{1,2}(\mathbb{R}^3),$$

where

$$\mathcal{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_6^2}.$$

Recently, Ye [32] investigated the following Kirchhoff equation

$$- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u = \lambda u + |u|^{p-2} u, \quad x \in \mathbb{R}^3, \quad (1.3)$$

where  $p \in (2, 6)$ . Ye proved that for any  $c > 0$ ,

$$\begin{cases} \sigma(c) \in (-\infty, 0], & \text{if } p \in (2, \bar{p}), \\ \sigma(c) = -\infty, & \text{if } p \in (\bar{p}, 6), \end{cases} \quad (1.4)$$

where

$$\bar{p} := \frac{14}{3}, \quad \sigma(c) := \inf_{u \in \tilde{S}_c} \tilde{I}(u), \quad (1.5)$$

$$\tilde{I}(u) := \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx$$

and

$$\tilde{S}_c := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = c^2 \right\}.$$

In the  $L^2$ -subcritical case, namely  $p \in (2, \bar{p})$ , Ye obtained global minimizers on  $\tilde{S}_c$  by solving (1.5). For  $p \in (\tilde{p}, \bar{p})$ , where  $\tilde{p} := \frac{10}{3}$ , he also constructed a local minimizer, which is a critical point of  $\tilde{I}|_{\tilde{S}_c}$ . In the  $L^2$ -supercritical case  $p \in (\bar{p}, 6)$ , the direct minimization problem (1.5) is no longer applicable because of (1.4). Ye therefore sought critical points of  $\tilde{I}|_{\tilde{S}_c}$  by minimizing  $\tilde{I}$  on the natural constraint

$$\tilde{\mathcal{M}}_c := \left\{ u \in \tilde{S}_c : \tilde{J}(u) := \frac{\partial}{\partial t} \tilde{I}(u_t) \Big|_{t=1} = 0 \right\},$$

where

$$u_t(x) := t^{\frac{3}{2}} u(tx), \quad \forall t > 0, u \in H^1(\mathbb{R}^3),$$

which preserves the  $L^2$ -norm, i.e.,  $u_t \in \tilde{S}_c$  whenever  $u \in \tilde{S}_c$ . Using the Lagrange multiplier method, Ye showed that  $\tilde{\mathcal{M}}_c$  is a natural constraint for  $\tilde{I}|_{\tilde{S}_c}$ . In the  $L^2$ -critical case  $p = \bar{p}$ , Ye [33] proved that (1.3) admits a solution  $(u_c, \lambda_c)$  satisfying  $\|u_c\|_2 = c > c^*$  for some  $c^* > 0$ . Ye [34] also analyzed the concentration behavior of normalized solutions. Using direct energy estimates and avoiding the concentration–compactness principle, Zeng–Zhang [36] established the existence and uniqueness of normalized solutions to (1.3) when  $p \in (2, \bar{p})$ . Later, Qi–Zou [24] determined the exact number of positive normalized solutions and obtained explicit formulas for them. For combined nonlinearities, in which  $|u|^{p-2}u$  is replaced by  $|u|^{p-2}u + \mu|u|^{q-2}u$ , local and global minimizers together with a mountain-pass solution were obtained in [8, 17] for  $2 < q < \bar{p} < p \leq 6$  or  $\bar{p} < q < p \leq 6$ , using Pohozaev-manifold methods and sharp energy estimates developed in [7, 15, 25, 26, 29]. More recently, Cai–Rădulescu [4] established normalized solutions for Kirchhoff equations with a general nonlinearity by minimizing the energy over a suitable combination of the Nehari and Pohozaev constraints, intersected with the closed  $L^2(\mathbb{R}^3)$ -ball of radius  $c$ . Their assumptions on the nonlinearity  $f$  include the following two regimes:

- (i)  $f$  is  $L^2$ -subcritical at the origin and  $L^2$ -critical at infinity,
- (ii)  $f$  is  $L^2$ -critical at the origin and  $L^2$ -supercritical at infinity.

Several works have also addressed normalized Kirchhoff equations in the presence of a potential  $V$ , including the problem

$$\begin{cases} - \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = \lambda u + |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 = c^2, & \text{in } \mathbb{R}^3, \end{cases} \quad (1.6)$$

where  $\bar{p} < p < 6$ . Cai–Zhang [5] first proved the existence of normalized solutions to (1.6) under suitable assumptions on a nonnegative potential  $V$ . They also obtained positive-energy mountain-pass solutions and proved nonexistence of negative-energy solutions when  $V \leq 0$  is bounded, as well as a local minimizer when the negative potential is sufficiently controlled on a suitable interval. When  $p = \bar{p}$  and  $V \geq 0$  is a trapping potential, Hu–Tang [12] studied the limiting behavior and local uniqueness

of normalized solutions to (1.6) by means of energy estimates and blow-up analysis. More generally, when  $|u|^{p-2}u$  is replaced by  $Q(x)f(u)$ , existence and multiplicity results for normalized solutions were obtained in [9, 31].

The preceding literature mainly concerns Sobolev-subcritical nonlinearities or combinations of subcritical and critical terms, including in the nonautonomous setting. This motivates the present study of normalized solutions for a nonautonomous Kirchhoff equation with a purely Sobolev-critical nonlinearity.

## 1.2. MAIN RESULTS

Throughout the paper, we assume that  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a nonnegative, locally Lipschitz trapping potential. More precisely, for every  $x_0 \in \mathbb{R}^3$ , there exist a neighborhood  $B(x_0)$  of  $x_0$  and a constant  $L > 0$ , depending on  $B(x_0)$ , such that

$$|V(x) - V(y)| \leq L|x - y|, \quad \text{for all } x, y \in B(x_0), \quad (1.7)$$

$$V \geq 0 \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} V(x) = +\infty, \quad (1.8)$$

and there exists a constant  $C_V > 0$  such that

$$-2V(x) \leq \nabla V(x) \cdot x \leq C_V V(x) \quad \text{for a.e. } x \in \mathbb{R}^3. \quad (1.9)$$

Since  $V$  is locally Lipschitz,  $\nabla V$  exists almost everywhere by Rademacher's theorem. Throughout the paper, condition (1.9) and every expression involving  $x \cdot \nabla V(x)$  are understood in this almost-everywhere sense.

We introduce the energy space

$$\mathcal{H} := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (a|\nabla u|^2 + (V+1)u^2) dx < +\infty \right\},$$

endowed with the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^3} (a|\nabla u|^2 + (V+1)u^2) dx.$$

Then solutions of (1.1) correspond to critical points of the energy functional  $I : \mathcal{H} \mapsto \mathbb{R}$ , defined by

$$I(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx$$

constrained to the  $L^2$ -sphere

$$S_c = \left\{ u \in \mathcal{H} : \int_{\mathbb{R}^3} u^2 dx = c^2 \right\}.$$

We first introduce the notion of a normalized ground state.

**Definition 1.2.** We say that  $u_0$  is a normalized ground state of (1.1) if  $u_0$  is a critical point of  $I$  constrained to  $S_c$ , i.e., it solves (1.1), and

$$I(u_0) = \inf \{I(u) : u \in S_c, \nabla_{S_c} I(u) = 0\}.$$

By [19, Section 1] and assumptions (1.7)–(1.8), the operator  $-a\Delta + V$  has a first eigenvalue  $\lambda_1 > 0$  with a positive first eigenfunction  $\psi_1 \in \mathcal{H}$ . We normalize  $\psi_1$  by  $\|\psi_1\|_2^2 = c^2$ . Moreover,  $\psi_1 \in C_{\text{loc}}^{2,\beta}(\mathbb{R}^3)$  for every  $0 < \beta < 1$ , and it solves

$$-a\Delta\psi_1 + V(x)\psi_1 = \lambda_1\psi_1. \quad (1.10)$$

**Theorem 1.3.** Assume that (1.7)–(1.9) hold and that

$$\frac{a}{3}K^2\mathcal{S}^{\frac{3}{2}} + \frac{b}{12}K^4\mathcal{S}^3 > \frac{1}{2}\lambda_1c^2 + \frac{b\lambda_1^2}{4a^2}c^4, \quad (1.11)$$

where

$$K = \left[ \frac{1}{2} \left( b\mathcal{S}^{\frac{3}{2}} + \sqrt{b^2\mathcal{S}^3 + 4a} \right) \right]^{\frac{1}{2}}.$$

Then problem (1.1) admits a positive normalized ground state  $u_b \in S_c$ . Moreover,  $u_b$  is a local minimizer of  $I$  on  $S_c$ .

**Theorem 1.4.** Assume that (1.7)–(1.9) hold. Then there exists  $b_0 > 0$  such that, for every  $0 < b < b_0$  satisfying (1.11), problem (1.1) admits a second positive normalized solution  $v_b \in S_c$ , distinct from the ground state  $u_b$ . Moreover,  $v_b$  is obtained at a mountain-pass level of  $I$  on  $S_c$ .

**Remark 1.5.** Since  $V$  is locally Lipschitz and nonnegative, the strong maximum principle implies that every nontrivial nonnegative solution obtained in Theorems 1.3 and 1.4 is in fact strictly positive in  $\mathbb{R}^3$ . We also note that the assumptions (1.7)–(1.9) are satisfied by the standard polynomial trapping potentials

$$V(x) = |x|^\nu, \quad \nu > 1.$$

Thus, the class of admissible potentials includes the usual confining potentials arising in normalized Schrödinger- and Kirchhoff-type problems.

**Remark 1.6.** The smallness assumption on  $b$  is used only in the construction of the second solution. More precisely, in Lemma 5.2 it allows us to obtain a sharp upper bound for the mountain-pass level below the energy threshold at which critical Sobolev bubbling may occur. This energy separation is the key ingredient in ruling out the noncompact alternative for the mountain-pass Palais–Smale sequence. It remains an interesting open problem to determine whether the second normalized solution persists for arbitrary  $b > 0$ , or whether a different compactness mechanism is required in that regime.

### 1.3. NOVELTY AND CONTRIBUTIONS

The main novelties and contributions of this work can be summarized as follows.

- (i) Compared with the settings considered in [4, 5], we study normalized solutions for a nonautonomous Kirchhoff equation with a purely Sobolev-critical nonlinearity. To the best of our knowledge, the normalized problem in this purely critical and nonautonomous Kirchhoff setting has not been treated in the existing literature. A fundamental difficulty is that the embedding

$$\mathcal{H} \hookrightarrow L^6(\mathbb{R}^3)$$

is continuous but not compact, so bounded Palais–Smale sequences may lose compactness through critical Sobolev bubbles. To capture this phenomenon, we establish a global compactness lemma adapted to both the nonlocal Kirchhoff structure and the trapping potential, which provides a quantitative description of the energy carried by a possible loss of compactness.

- (ii) Our compactness analysis is inspired by the ideas developed for Sobolev-critical Schrödinger equations in [20, 28, 35]. In the present Kirchhoff setting, however, the nonlocal term creates an additional difficulty: under weak convergence, the limiting equation involves the effective coefficient

$$a + bA, \quad A = \lim_{n \rightarrow \infty} \|\nabla u_n\|_2^2,$$

rather than  $a + b\|\nabla u_0\|_2^2$  unless strong convergence has already been established. Consequently, a global compactness alternative alone is not sufficient to recover compactness. We therefore combine the global compactness lemma with sharp energy-level estimates that separate the variational levels from the critical bubbling threshold; in particular, we derive a sharp upper estimate for the mountain-pass level (see Lemma 5.2) and use it to exclude the noncompact alternative in the proof of Theorem 1.4. This strategy yields both a positive normalized ground state and, for sufficiently small  $b > 0$ , a distinct positive normalized solution of mountain-pass type.

- (iii) The closest precedent for our variational strategy is Yu [35, Theorems 1.2–1.3 and Lemmas 2.6, 5.2], who obtained a local-minimum ground state and a second mountain-pass solution for the purely Sobolev-critical Schrödinger equation with a trapping potential. In particular, Yu used an energy-defect alternative and a path obtained by adding a truncated Sobolev bubble to the ground state and then normalizing the sum. We adapt these two ingredients to the Kirchhoff problem, but the nonlocal term requires genuinely additional estimates: the weak-limit equation contains the effective coefficient  $a + bA$ , the bubbling threshold becomes

$$\frac{a}{3}K^2\mathcal{S}^{\frac{3}{2}} + \frac{b}{12}K^4\mathcal{S}^3,$$

and the normalized bubble path produces extra  $b$ -dependent interaction terms (see  $H_4$  in Lemma 5.2). Controlling these effects is essential both for excluding loss of compactness and for proving the strict mountain-pass upper bound.

The paper is organized as follows. Section 2 develops the preliminary compactness results, including the global compactness lemma for constrained Palais–Smale sequences. Section 3 establishes the variational geometry of the energy functional on the mass constraint. Section 4 proves the existence of a positive normalized ground state, while Section 5 establishes the existence of a second positive normalized solution at the mountain-pass level.

## 2. PRELIMINARY RESULTS

We first recall the compactness induced by the trapping potential. By (1.8), the embedding

$$\mathcal{H} \hookrightarrow L^p(\mathbb{R}^3) \quad \text{for } 2 \leq p < 6 \quad (2.1)$$

is compact. Indeed, let  $\{u_n\}$  be bounded in  $\mathcal{H}$ . For every  $R > 0$  sufficiently large,

$$\int_{\mathbb{R}^3 \setminus B_R} |u_n|^2 dx \leq \frac{1}{\inf_{|x| \geq R} V(x)} \int_{\mathbb{R}^3} V(x) u_n^2 dx,$$

and the right-hand side tends to 0 uniformly in  $n$  as  $R \rightarrow +\infty$ , since  $\inf_{|x| \geq R} V(x) \rightarrow +\infty$ . On the other hand, the Rellich–Kondrachov theorem gives compactness in  $L^2(B_R)$  for every fixed  $R > 0$ . Hence,

$$\mathcal{H} \hookrightarrow L^2(\mathbb{R}^3)$$

compactly. Combining this with the continuous Sobolev embedding  $\mathcal{H} \hookrightarrow H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$  and interpolation yields (2.1).

In view of [23], we recall the following Pohozaev identity.

**Lemma 2.1.** *Let  $(u, \lambda) \in \mathcal{H} \times \mathbb{R}$  be a solution of (1.1). Then*

$$a \int_{\mathbb{R}^3} |\nabla u|^2 dx + b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u^2 dx = \int_{\mathbb{R}^3} u^6 dx.$$

We next record a useful sign property of the Lagrange multiplier.

**Lemma 2.2.** *Let  $(u, \lambda) \in \mathcal{H} \times \mathbb{R}$  be a solution of (1.1). Then  $\lambda \geq 0$ .*

*Proof.* Since  $(u, \lambda) \in \mathcal{H} \times \mathbb{R}$  solves (1.1), testing the equation with  $u$  and applying Lemma 2.1 give

$$a \int_{\mathbb{R}^3} |\nabla u|^2 dx + b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \int_{\mathbb{R}^3} V(x) u^2 dx = \lambda \int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |u|^6 dx$$

and

$$a \int_{\mathbb{R}^3} |\nabla u|^2 dx + b \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u^2 dx = \int_{\mathbb{R}^3} u^6 dx.$$

Subtracting the second identity from the first yields

$$\lambda \int_{\mathbb{R}^3} |u|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(x) \cdot xu^2 dx + \int_{\mathbb{R}^3} V(x)u^2 dx.$$

Using (1.9), we conclude that  $\lambda \geq 0$ .  $\square$

As a consequence, we obtain the following lower bound for the energy of constrained critical points.

**Lemma 2.3.** *Let  $(u, \lambda) \in \mathcal{H} \times \mathbb{R}$  be a solution of (1.1). Then*

$$I(u) \geq \frac{a}{3} \|\nabla u\|_2^2 + \frac{b}{12} \|\nabla u\|_2^4.$$

*Proof.* By Lemma 2.2,  $\lambda \geq 0$ . Since  $u$  solves (1.1),

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|_{\mathcal{H}}^2 + \frac{b}{4} \|\nabla u\|_2^4 - \frac{1}{2} c^2 - \frac{1}{6} \|u\|_6^6 \\ &= \frac{1}{2} \|u\|_{\mathcal{H}}^2 + \frac{b}{4} \|\nabla u\|_2^4 - \frac{1}{2} c^2 - \frac{1}{6} (\|u\|_{\mathcal{H}}^2 + b \|\nabla u\|_2^4 - c^2) + \frac{1}{6} \lambda c^2 \\ &= \frac{1}{3} (\|u\|_{\mathcal{H}}^2 - c^2) + \frac{b}{12} \|\nabla u\|_2^4 + \frac{1}{6} \lambda c^2 \\ &\geq \frac{a}{3} \|\nabla u\|_2^2 + \frac{b}{12} \|\nabla u\|_2^4. \end{aligned}$$

This completes the proof.  $\square$

The next estimate describes the behavior of the potential term under dilations and will be used repeatedly below.

**Lemma 2.4.** *For every  $s > 0$ , there exists a constant  $C(s) > 0$  such that, for every  $u \in \mathcal{H}$ ,*

$$\int_{\mathbb{R}^3} V(sx)u^2 dx \leq C(s) \int_{\mathbb{R}^3} V(x)u^2 dx.$$

*Proof.* Fix  $x \in \mathbb{R}^3$ . If  $s \geq 1$ , then by the upper bound in (1.9),

$$\frac{d}{ds} V(sx) = \nabla V(sx) \cdot x = \frac{1}{s} \nabla V(sx) \cdot (sx) \leq \frac{C_V}{s} V(sx)$$

for a.e.  $s > 0$ . Hence,

$$\frac{d}{ds} (s^{-C_V} V(sx)) \leq 0,$$

and therefore

$$V(sx) \leq s^{C_V} V(x), \quad s \geq 1.$$

If  $0 < s < 1$ , the lower bound in (1.9) gives

$$\frac{d}{ds} (s^2 V(sx)) = 2s V(sx) + s \nabla V(sx) \cdot (sx) \geq 0.$$

Consequently,

$$V(sx) \leq s^{-2}V(x), \quad 0 < s < 1.$$

Thus, the conclusion follows with

$$C(s) = \max\{s^{C_V}, s^{-2}\}. \quad \square$$

We now establish the compactness alternative that will be used in the proofs of the main results. The argument is adapted to the present nonlocal setting from the global compactness framework in [27, Ch. III, Thm. 3.1] and the energy-defect alternative used for the corresponding critical Schrödinger problem in [35, Lemma 2.6]. The additional point here is that weak convergence leaves the effective Kirchhoff coefficient  $a + bA$ , so the defect estimate must keep track of  $A = \lim_n \|\nabla u_n\|_2^2$ .

**Lemma 2.5.** *Let  $\{u_n\} \subset S_c$  be a bounded Palais–Smale sequence for  $I|_{S_c}$ , and let  $\{\lambda_n\} \subset \mathbb{R}$  be the associated Lagrange multipliers. Assume that  $\{\lambda_n\}$  is bounded and that*

$$\|u_n^-\|_2 \rightarrow 0.$$

*Then, after passing to a subsequence, there exist  $A \geq 0$ ,  $u_0 \in \mathcal{H}$ ,  $u_0 \geq 0$ , and  $\lambda_0 > 0$  such that*

$$\|\nabla u_n\|_2^2 \rightarrow A, \quad u_n \rightharpoonup u_0 \quad \text{weakly in } \mathcal{H}, \quad \lambda_n \rightarrow \lambda_0,$$

*and*

$$-(a + bA)\Delta u_0 + V(x)u_0 = \lambda_0 u_0 + u_0^5 \quad \text{in } \mathbb{R}^3.$$

*Moreover, exactly one of the following alternatives occurs:*

- (i)  $u_n \rightarrow u_0$  strongly in  $\mathcal{H}$ ,
- (ii)  $u_n \rightharpoonup u_0$  in  $\mathcal{H}$  but not strongly, and

$$I(u_n) \geq J_A(u_0) + \frac{a}{3}K^2\mathcal{S}^{\frac{3}{2}} + \frac{b}{12}K^4\mathcal{S}^3 + o_n(1),$$

*where*

$$J_A(u) = \left(\frac{a}{2} + \frac{bA}{4}\right) \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx.$$

*Proof.* Set

$$G_n := \int_{\mathbb{R}^3} |\nabla u_n|^2 dx.$$

Since  $\{u_n\}$  is bounded in  $\mathcal{H}$ , after passing to a subsequence, we may assume that

$$G_n \rightarrow A := \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx,$$

and that, for some  $u_0 \in \mathcal{H}$ ,

$$u_n \rightharpoonup u_0 \quad \text{weakly in } \mathcal{H}.$$

By the compact embedding

$$\mathcal{H} \hookrightarrow L^p(\mathbb{R}^3), \quad 2 \leq p < 6,$$

we further have

$$u_n \rightarrow u_0 \quad \text{strongly in } L^p(\mathbb{R}^3), \quad 2 \leq p < 6,$$

and, after taking a further subsequence if necessary,

$$u_n(x) \rightarrow u_0(x) \quad \text{for a.e. } x \in \mathbb{R}^3. \quad (2.2)$$

In particular,

$$\|u_0\|_2 = \lim_{n \rightarrow \infty} \|u_n\|_2 = c,$$

so that  $u_0 \in S_c$  and  $u_0 \not\equiv 0$ . Moreover, since  $\|u_n^-\|_2 \rightarrow 0$ , we obtain

$$\|u_0^-\|_2 \leq \|u_0 - u_n\|_2 + \|u_n^-\|_2 \rightarrow 0.$$

Then  $u_0 \geq 0$  a.e. in  $\mathbb{R}^3$ .

Since  $\{\lambda_n\}$  is bounded, up to a subsequence,

$$\lambda_n \rightarrow \lambda_0 \quad \text{for some } \lambda_0 \in \mathbb{R}.$$

Because  $\{(u_n, \lambda_n)\}$  is a Palais–Smale sequence for  $I$  constrained to  $S_c$ , we have

$$\begin{aligned} & (a + bG_n) \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^3} V(x) u_n \varphi \, dx \\ & - \lambda_n \int_{\mathbb{R}^3} u_n \varphi \, dx - \int_{\mathbb{R}^3} |u_n|^4 u_n \varphi \, dx = o_n(1) \|\varphi\|_{\mathcal{H}} \end{aligned} \quad (2.3)$$

for every  $\varphi \in \mathcal{H}$ .

Since  $\{u_n\}$  is bounded in  $L^6(\mathbb{R}^3)$ , (2.2) implies

$$|u_n|^4 u_n \rightharpoonup |u_0|^4 u_0 \quad \text{weakly in } L^{6/5}(\mathbb{R}^3).$$

Letting  $n \rightarrow \infty$  in (2.3), we obtain

$$(a + bA) \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^3} V(x) u_0 \varphi \, dx = \lambda_0 \int_{\mathbb{R}^3} u_0 \varphi \, dx + \int_{\mathbb{R}^3} u_0^5 \varphi \, dx$$

for every  $\varphi \in \mathcal{H}$ . Hence,

$$-(a + bA)\Delta u_0 + V(x)u_0 = \lambda_0 u_0 + u_0^5 \quad \text{in } \mathbb{R}^3. \quad (2.4)$$

We next show that  $\lambda_0 > 0$ . Testing (2.4) with  $u_0$  gives

$$(a + bA) \|\nabla u_0\|_2^2 + \int_{\mathbb{R}^3} V(x) u_0^2 \, dx = \lambda_0 c^2 + \|u_0\|_6^6. \quad (2.5)$$

On the other hand, the Pohozaev identity associated with (2.4) is

$$(a + bA)\|\nabla u_0\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u_0^2 dx = \|u_0\|_6^6. \quad (2.6)$$

Subtracting (2.6) from (2.5), we obtain

$$\lambda_0 c^2 = \int_{\mathbb{R}^3} \left( V(x) + \frac{1}{2} \nabla V(x) \cdot x \right) u_0^2 dx. \quad (2.7)$$

By assumption  $-2V(x) \leq \nabla V(x) \cdot x$ , the integrand on the right-hand side is nonnegative. Thus,  $\lambda_0 \geq 0$ . In fact the inequality is strict. Indeed, by standard elliptic regularity and the strong maximum principle applied to (2.4), since  $u_0 \geq 0$  and  $u_0 \not\equiv 0$ ,

$$u_0(x) > 0 \quad \text{in } \mathbb{R}^3.$$

Moreover,

$$2V(x) + \nabla V(x) \cdot x \geq 0$$

cannot vanish a.e. in  $\mathbb{R}^3$ . Otherwise, for almost every  $\theta \in \mathbb{S}^2$ , the locally absolutely continuous function

$$r \mapsto r^2 V(r\theta)$$

would satisfy

$$\frac{d}{dr}(r^2 V(r\theta)) = r(2V(r\theta) + \nabla V(r\theta) \cdot r\theta) = 0$$

for a.e.  $r > 0$ , which is incompatible with

$$V(x) \longrightarrow +\infty \quad \text{as } |x| \rightarrow \infty.$$

Consequently, the right-hand side of (2.7) is strictly positive, and therefore

$$\lambda_0 > 0.$$

We now establish the compactness alternative. Put

$$w_n := u_n - u_0.$$

Then

$$w_n \rightharpoonup 0 \quad \text{in } \mathcal{H}, \quad w_n \rightarrow 0 \quad \text{in } L^p(\mathbb{R}^3), \quad 2 \leq p < 6.$$

For convenience, denote

$$g_n := \|\nabla w_n\|_2^2, \quad p_n := \int_{\mathbb{R}^3} V(x) w_n^2 dx, \quad q_n := \|w_n\|_6^6.$$

By weak convergence,

$$\|\nabla u_n\|_2^2 = \|\nabla u_0\|_2^2 + g_n + o_n(1), \quad (2.8)$$

and similarly,

$$\int_{\mathbb{R}^3} V(x) u_n^2 dx = \int_{\mathbb{R}^3} V(x) u_0^2 dx + p_n + o_n(1).$$

The Brezis–Lieb lemma gives

$$\|u_n\|_6^6 = \|u_0\|_6^6 + q_n + o_n(1). \quad (2.9)$$

Since  $G_n \rightarrow A$ , from (2.8) we have  $A = \|\nabla u_0\|_2^2 + \lim_{n \rightarrow \infty} g_n$ , whenever the latter limit exists. Furthermore,  $G_n^2 = A G_n + o_n(1)$ . Combining this fact with (2.8)–(2.9), we obtain the energy splitting

$$I(u_n) = J_A(u_0) + J_A(w_n) + o_n(1), \quad (2.10)$$

where

$$J_A(v) = \left( \frac{a}{2} + \frac{bA}{4} \right) \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) v^2 dx - \frac{1}{6} \|v\|_6^6.$$

We next derive the fundamental identity for the remainder  $w_n$ . Taking  $\varphi = w_n$  in (2.3), using

$$G_n \rightarrow A, \quad \lambda_n \rightarrow \lambda_0, \quad \|w_n\|_2 \rightarrow 0,$$

together with

$$\int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla w_n dx = o_n(1), \quad \int_{\mathbb{R}^3} V(x) u_0 w_n dx = o_n(1),$$

and the derivative form of the Brezis–Lieb lemma,

$$|u_n|^4 u_n - |u_0|^4 u_0 - |w_n|^4 w_n \longrightarrow 0 \quad \text{in } L^{6/5}(\mathbb{R}^3),$$

we obtain

$$(a + bA)g_n + p_n = q_n + o_n(1). \quad (2.11)$$

Suppose first that

$$w_n \rightarrow 0 \quad \text{strongly in } \mathcal{H}.$$

Then

$$u_n \rightarrow u_0 \quad \text{strongly in } \mathcal{H},$$

which is alternative (i).

Assume now that  $u_n \not\rightarrow u_0$  strongly in  $\mathcal{H}$ . Passing to a subsequence, there exists  $\delta > 0$  such that

$$\|w_n\|_{\mathcal{H}} \geq \delta \quad \text{for all } n. \quad (2.12)$$

We claim that

$$\liminf_{n \rightarrow \infty} g_n > 0. \quad (2.13)$$

Indeed, if along a subsequence  $g_n \rightarrow 0$ , then by the Sobolev inequality,

$$q_n = \|w_n\|_6^6 \leq \mathcal{S}^{-3} g_n^3 \rightarrow 0.$$

It then follows from (2.11) that  $p_n \rightarrow 0$ . Since  $\|w_n\|_2 \rightarrow 0$ , we would get

$$\|w_n\|_{\mathcal{H}}^2 = ag_n + p_n + \|w_n\|_2^2 \rightarrow 0,$$

contradicting (2.12). Thus, (2.13) holds.

After passing to another subsequence, we may therefore assume

$$g_n \rightarrow g > 0.$$

By (2.8),

$$A = \|\nabla u_0\|_2^2 + g,$$

and consequently

$$A \geq g. \quad (2.14)$$

From (2.11),  $p_n \geq 0$ , and the Sobolev inequality,

$$(a + bA)g_n \leq q_n + o_n(1) \leq \mathcal{S}^{-3}g_n^3 + o_n(1).$$

Passing to the limit and using  $g > 0$ , we find

$$a + bA \leq \mathcal{S}^{-3}g^2.$$

Since  $A \geq g$ ,  $a + bg \leq \mathcal{S}^{-3}g^2$ , and hence  $g^2 - b\mathcal{S}^3g - a\mathcal{S}^3 \geq 0$ . Therefore,

$$g \geq \frac{b\mathcal{S}^3 + \sqrt{b^2\mathcal{S}^6 + 4a\mathcal{S}^3}}{2}.$$

Recalling

$$K^2 = \frac{1}{2} \left( b\mathcal{S}^{3/2} + \sqrt{b^2\mathcal{S}^3 + 4a} \right),$$

we see that

$$\frac{b\mathcal{S}^3 + \sqrt{b^2\mathcal{S}^6 + 4a\mathcal{S}^3}}{2} = K^2\mathcal{S}^{3/2}.$$

Thus,

$$g \geq K^2\mathcal{S}^{3/2}. \quad (2.15)$$

It remains to estimate the energy carried by  $w_n$ . By (2.11),

$$\begin{aligned} J_A(w_n) &= \left( \frac{a}{2} + \frac{bA}{4} \right) g_n + \frac{1}{2}p_n - \frac{1}{6}q_n \\ &= \left( \frac{a}{2} + \frac{bA}{4} \right) g_n + \frac{1}{2}p_n - \frac{1}{6}((a + bA)g_n + p_n) + o_n(1) \\ &= \left( \frac{a}{3} + \frac{bA}{12} \right) g_n + \frac{1}{3}p_n + o_n(1). \end{aligned}$$

Since  $p_n \geq 0$ , using (2.14) and (2.15), we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} J_A(w_n) &\geq \left( \frac{a}{3} + \frac{bA}{12} \right) g \\ &\geq \frac{a}{3} K^2\mathcal{S}^{3/2} + \frac{b}{12} K^4\mathcal{S}^3. \end{aligned} \quad (2.16)$$

Combining (2.10) and (2.16), we finally obtain

$$I(u_n) \geq J_A(u_0) + \frac{a}{3}K^2\mathcal{S}^{3/2} + \frac{b}{12}K^4\mathcal{S}^3 + o_n(1). \quad (2.17)$$

This is precisely alternative (ii). Therefore, after passing to a subsequence, exactly one of the following alternatives occurs:

$$u_n \rightarrow u_0 \quad \text{strongly in } \mathcal{H},$$

or

$$u_n \rightharpoonup u_0 \quad \text{but not strongly in } \mathcal{H},$$

and in the latter case (2.17) holds. This completes the proof.  $\square$

### 3. MOUNTAIN-PASS GEOMETRY

To investigate the geometric structure of  $I$ , we define

$$\mathcal{B}_\alpha := \{u \in S_c : \|\nabla u\|_2^2 < \alpha\} \quad \text{and} \quad \partial\mathcal{B}_\alpha := \{u \in S_c : \|\nabla u\|_2^2 = \alpha\}.$$

We next verify the energy separation that will be used in the local minimization and mountain-pass arguments.

**Lemma 3.1.** *Assume that (1.11) holds and set*

$$\bar{\alpha} = K^2\mathcal{S}^{\frac{3}{2}}.$$

*Then*

$$0 < \inf_{\mathcal{B}_{\bar{\alpha}}} I \leq I(\psi_1) < \frac{a}{3}K^2\mathcal{S}^{\frac{3}{2}} + \frac{b}{12}K^4\mathcal{S}^3 \leq \inf_{u \in \partial\mathcal{B}_{\bar{\alpha}}} I(u).$$

*Proof.* We first verify that  $\psi_1 \in \mathcal{B}_{\bar{\alpha}}$ . Testing (1.10) with  $\psi_1$  gives

$$a\|\nabla\psi_1\|_2^2 + \int_{\mathbb{R}^3} V(x)\psi_1^2 dx = \lambda_1 c^2.$$

Since  $V \geq 0$ ,  $V(x) \rightarrow +\infty$  as  $|x| \rightarrow \infty$ , and  $\psi_1 > 0$ , we have

$$\int_{\mathbb{R}^3} V(x)\psi_1^2 dx > 0.$$

Set

$$x := \frac{\lambda_1 c^2}{a}, \quad \bar{\alpha} = K^2\mathcal{S}^{\frac{3}{2}}, \quad \mathfrak{g}(t) := \frac{a}{2}t + \frac{b}{4}t^2.$$

The assumption (1.11) gives

$$\mathfrak{g}(x) < \frac{a}{3}\bar{\alpha} + \frac{b}{12}\bar{\alpha}^2 < \mathfrak{g}(\bar{\alpha}).$$

Since  $\mathfrak{g}$  is strictly increasing on  $[0, +\infty)$ , we obtain

$$\frac{\lambda_1 c^2}{a} = x < \bar{\alpha}.$$

Therefore,

$$\|\nabla \psi_1\|_2^2 < \frac{\lambda_1 c^2}{a} < K^2 \mathcal{S}^{\frac{3}{2}} = \bar{\alpha},$$

and hence  $\psi_1 \in \mathcal{B}_{\bar{\alpha}}$ . Moreover,

$$\begin{aligned} I(\psi_1) &= \frac{1}{2} \lambda_1 c^2 + \frac{b}{4} \|\nabla \psi_1\|_2^4 - \frac{1}{6} \|\psi_1\|_6^6 \\ &\leq \frac{1}{2} \lambda_1 c^2 + \frac{b \lambda_1^2}{4a^2} c^4 < \frac{a}{3} K^2 \mathcal{S}^{\frac{3}{2}} + \frac{b}{12} K^4 \mathcal{S}^3, \end{aligned}$$

where the last inequality follows from (1.11).

For  $u \in S_c$ , put  $t = \|\nabla u\|_2^2$  and define

$$\mathfrak{f}(t) := \frac{a}{2} t + \frac{b}{4} t^2 - \frac{1}{6} \mathcal{S}^{-3} t^3.$$

By the definition of  $K$ ,

$$K^4 = a + b K^2 \mathcal{S}^{\frac{3}{2}},$$

and consequently

$$\mathcal{S}^{-3} \bar{\alpha}^2 = a + b \bar{\alpha}.$$

It follows that

$$\mathfrak{f}'(t) = \frac{1}{2} (a + bt - \mathcal{S}^{-3} t^2) > 0, \quad 0 < t < \bar{\alpha}.$$

Using the Sobolev inequality, for every  $u \in \mathcal{B}_{\bar{\alpha}}$  we have

$$I(u) \geq \mathfrak{f}(\|\nabla u\|_2^2) + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx > 0.$$

We claim that the infimum is strictly positive. Otherwise, there would exist  $u_n \in \mathcal{B}_{\bar{\alpha}}$  such that  $I(u_n) \rightarrow 0$ . Since both terms on the right-hand side above are nonnegative,

$$\mathfrak{f}(\|\nabla u_n\|_2^2) \rightarrow 0, \quad \int_{\mathbb{R}^3} V(x) u_n^2 dx \rightarrow 0.$$

Because  $\mathfrak{f}$  is strictly increasing on  $[0, \bar{\alpha}]$  and  $\mathfrak{f}(0) = 0$ , we obtain  $\|\nabla u_n\|_2 \rightarrow 0$ . On the other hand, the variational characterization of the first eigenvalue yields

$$a \|\nabla u_n\|_2^2 + \int_{\mathbb{R}^3} V(x) u_n^2 dx \geq \lambda_1 \|u_n\|_2^2 = \lambda_1 c^2,$$

a contradiction. Therefore,

$$\inf_{\mathcal{B}_{\bar{\alpha}}} I > 0.$$

Finally, for every  $u \in \partial\mathcal{B}_{\bar{\alpha}}$ , the Sobolev inequality gives

$$\begin{aligned} I(u) &\geq \frac{a}{2}\bar{\alpha} + \frac{b}{4}\bar{\alpha}^2 - \frac{1}{6}\mathcal{S}^{-3}\bar{\alpha}^3 \\ &= \frac{a}{3}K^2\mathcal{S}^{\frac{3}{2}} + \frac{b}{12}K^4\mathcal{S}^3. \end{aligned}$$

The proof is complete.  $\square$

We now introduce the energy level associated with the local minimization problem:

$$\bar{m} := \inf_{\mathcal{B}_{\bar{\alpha}}} I.$$

From the proof of Lemma 3.1, we have

$$\bar{m} < \frac{a}{3}K^2\mathcal{S}^{\frac{3}{2}} + \frac{b}{12}K^4\mathcal{S}^3. \quad (3.1)$$

On the other hand, Lemma 3.1 also yields

$$I(u) \geq \frac{a}{3}K^2\mathcal{S}^{\frac{3}{2}} + \frac{b}{12}K^4\mathcal{S}^3, \quad u \in \partial\mathcal{B}_{\bar{\alpha}}. \quad (3.2)$$

Thus,  $\partial\mathcal{B}_{\bar{\alpha}}$  provides an energy barrier separating the local minimizing region from the exterior region. We postpone the definition of the mountain-pass class until Section 5, after the local minimizer has been obtained, so that the ground state itself can be chosen as the left endpoint of the mountain-pass paths.

For later use, we record the following elementary scaling property. For every  $u \in S_c$  and  $s > 0$ , define

$$u_s(x) := s^{\frac{3}{2}}u(sx).$$

**Lemma 3.2.** *Let  $u \in S_c$  be fixed. Then  $u_s \in S_c$  for every  $s > 0$ , and*

$$\|\nabla u_s\|_2 \rightarrow +\infty \quad \text{and} \quad I(u_s) \rightarrow -\infty \quad \text{as } s \rightarrow +\infty.$$

*Proof.* Clearly,

$$\|u_s\|_2 = \|u\|_2 = c, \quad \|\nabla u_s\|_2 = s\|\nabla u\|_2.$$

Moreover, by Lemma 2.4,

$$\int_{\mathbb{R}^3} V(x)u_s^2(x) dx = \int_{\mathbb{R}^3} V\left(\frac{x}{s}\right)u^2(x) dx < +\infty.$$

Hence,  $u_s \in \mathcal{H}$  and therefore  $u_s \in S_c$ . Moreover,

$$I(u_s) = \frac{as^2}{2}\|\nabla u\|_2^2 + \frac{bs^4}{4}\|\nabla u\|_2^4 + \frac{1}{2} \int_{\mathbb{R}^3} V\left(\frac{x}{s}\right)u^2(x) dx - \frac{s^6}{6}\|u\|_6^6.$$

For  $s \geq 1$ , the estimate used in the proof of Lemma 2.4 gives

$$V\left(\frac{x}{s}\right) \leq s^2V(x).$$

Consequently,

$$I(u_s) \leq \frac{s^2}{2} \left( a \|\nabla u\|_2^2 + \int_{\mathbb{R}^3} V(x) u^2 dx \right) + \frac{bs^4}{4} \|\nabla u\|_2^4 - \frac{s^6}{6} \|u\|_6^6,$$

and hence  $I(u_s) \rightarrow -\infty$  as  $s \rightarrow +\infty$ . The proof is complete.  $\square$

#### 4. GROUND-STATE SOLUTION

In this section, we prove Theorem 1.3 under assumptions (1.7)–(1.9). We first show the existence of the local minimizer. The local-minimum strategy is analogous to the one used for the critical Schrödinger problem in [35, Theorem 1.2], while compactness here is recovered through Lemma 2.5, which incorporates the effective Kirchhoff coefficient  $a + bA$ . Then, using Lemma 2.3, we verify that the local minimizer is the ground state in the sense of Definition 1.2.

Lemma 3.1 identifies  $\bar{m}$  as the natural local-minimization level. The compactness alternative in Lemma 2.5 allows us to prove that this level is attained.

**Lemma 4.1.** *Under the assumptions of Theorem 1.3, the infimum  $\bar{m}$  is attained.*

*Proof.* Let

$$\bar{\mathcal{B}}_{\bar{\alpha}} := \{u \in S_c : \|\nabla u\|_2^2 \leq \bar{\alpha}\}.$$

Since  $S_c$  is a complete  $C^1$  Hilbert manifold and  $\bar{\mathcal{B}}_{\bar{\alpha}}$  is closed in  $S_c$ , it is a complete metric space with the distance induced by  $\mathcal{H}$ . By Lemma 3.1,

$$\inf_{u \in \partial \bar{\mathcal{B}}_{\bar{\alpha}}} I(u) \geq \frac{a}{3} K^2 \mathcal{S}^{\frac{3}{2}} + \frac{b}{12} K^4 \mathcal{S}^3 > \bar{m}.$$

Consequently,

$$\inf_{\bar{\mathcal{B}}_{\bar{\alpha}}} I = \inf_{\mathcal{B}_{\bar{\alpha}}} I = \bar{m}.$$

Choose a nonnegative minimizing sequence  $\{u_n\} \subset \mathcal{B}_{\bar{\alpha}}$  for  $\bar{m}$ . Applying Ekeland's variational principle to  $I$  on the complete metric space  $\bar{\mathcal{B}}_{\bar{\alpha}}$ , we obtain  $\omega_n \in \bar{\mathcal{B}}_{\bar{\alpha}}$  such that

$$I(\omega_n) \rightarrow \bar{m}, \quad \|u_n - \omega_n\|_{\mathcal{H}} \rightarrow 0,$$

and

$$I(v) \geq I(\omega_n) - \varepsilon_n \|v - \omega_n\|_{\mathcal{H}} \quad \text{for every } v \in \bar{\mathcal{B}}_{\bar{\alpha}},$$

where  $\varepsilon_n \rightarrow 0$ . Since the energy on  $\partial \bar{\mathcal{B}}_{\bar{\alpha}}$  is separated from  $\bar{m}$ , for all sufficiently large  $n$  we have  $\omega_n \in \mathcal{B}_{\bar{\alpha}}$ . Thus, the preceding variational inequality can be tested with arbitrary sufficiently small variations in  $S_c$ , and therefore

$$\|\nabla_{S_c} I(\omega_n)\| \rightarrow 0.$$

Furthermore, since  $u_n \geq 0$ ,

$$\|\omega_n^-\|_2 \leq \|\omega_n - u_n\|_2 \rightarrow 0.$$

Hence,  $\{\omega_n\}$  is a Palais–Smale sequence for  $I|_{S_c}$  at the level  $\bar{m}$  satisfying the negative-part condition in Lemma 2.5.

The sequence  $\{\omega_n\}$  is bounded in  $\mathcal{H}$ . Its associated Lagrange multipliers are

$$\lambda_n := \frac{1}{c^2} \left[ a \int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx + b \left( \int_{\mathbb{R}^3} |\nabla \omega_n|^2 dx \right)^2 + \int_{\mathbb{R}^3} V(x) \omega_n^2 dx - \int_{\mathbb{R}^3} |\omega_n|^6 dx \right],$$

and  $\{\lambda_n\}$  is bounded. Therefore, Lemma 2.5 applies. Up to a subsequence,

$$\omega_n \rightharpoonup \omega_0 \quad \text{in } \mathcal{H}, \quad \omega_n \rightarrow \omega_0 \quad \text{in } L^2(\mathbb{R}^3),$$

and  $\omega_n(x) \rightarrow \omega_0(x)$  for a.e.  $x \in \mathbb{R}^3$ . In particular,  $\omega_0 \in S_c$  and  $\omega_0 \geq 0$ .

We claim that  $\omega_n \rightarrow \omega_0$  strongly in  $\mathcal{H}$ . Suppose otherwise. Set

$$A = \lim_{n \rightarrow \infty} \|\nabla \omega_n\|_2^2,$$

after passing to a further subsequence. The noncompact alternative in Lemma 2.5 gives

$$\bar{m} \geq J_A(\omega_0) + \frac{a}{3} K^2 \mathcal{S}^{\frac{3}{2}} + \frac{b}{12} K^4 \mathcal{S}^3,$$

where

$$J_A(v) = \left( \frac{a}{2} + \frac{bA}{4} \right) \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) v^2 dx - \frac{1}{6} \|v\|_6^6.$$

Moreover,

$$J_A(\omega_0) - I(\omega_0) = \frac{b}{4} \|\nabla \omega_0\|_2^2 (A - \|\nabla \omega_0\|_2^2) \geq 0.$$

Hence, using  $\bar{m} < \frac{a}{3} K^2 \mathcal{S}^{3/2} + \frac{b}{12} K^4 \mathcal{S}^3$ ,

$$I(\omega_0) \leq J_A(\omega_0) \leq \bar{m} - \frac{a}{3} K^2 \mathcal{S}^{\frac{3}{2}} - \frac{b}{12} K^4 \mathcal{S}^3 < 0. \quad (4.1)$$

On the other hand, in the noncompact alternative the proof of Lemma 2.5 gives

$$A = \|\nabla \omega_0\|_2^2 + g \quad \text{for some } g > 0.$$

Since  $\omega_n \in \bar{\mathcal{B}}_{\bar{\alpha}}$ ,  $A \leq \bar{\alpha}$ , and therefore

$$\|\nabla \omega_0\|_2^2 < A \leq \bar{\alpha}.$$

Thus,  $\omega_0 \in \mathcal{B}_{\bar{\alpha}}$ . By the definition of  $\bar{m}$ ,

$$I(\omega_0) \geq \bar{m} > 0,$$

which contradicts (4.1).

Hence,

$$\omega_n \rightarrow \omega_0 \quad \text{strongly in } \mathcal{H}.$$

Passing to the limit in the Palais–Smale equation, we obtain

$$\nabla_{S_c} I(\omega_0) = 0, \quad I(\omega_0) = \bar{m}.$$

Since  $\|\nabla \omega_0\|_2^2 < \bar{\alpha}$  and the boundary energy is strictly larger than  $\bar{m}$ ,  $\omega_0$  is a local minimizer of  $I$  on  $S_c$ .  $\square$

*Proof of Theorem 1.3.* By Lemma 4.1, there exists  $\omega_0 \in \mathcal{B}_{\bar{\alpha}} \cap S_c$  such that

$$I(\omega_0) = \bar{m}, \quad \nabla_{S_c} I(\omega_0) = 0.$$

Thus,  $\omega_0$  is a normalized solution of (1.1) and a local minimizer of  $I$  on  $S_c$ . Since  $\omega_0 \geq 0$  and  $\|\omega_0\|_2 = c > 0$ , the strong maximum principle gives

$$\omega_0 > 0 \quad \text{in } \mathbb{R}^3.$$

We now prove that  $\omega_0$  is a ground state. Let  $u \in S_c$  be any constrained critical point of  $I$ . If  $u \in \mathcal{B}_{\bar{\alpha}}$ , then, by the definition of  $\bar{m}$ ,

$$I(u) \geq \bar{m}.$$

If  $u \notin \mathcal{B}_{\bar{\alpha}}$ , then  $\|\nabla u\|_2^2 \geq \bar{\alpha}$ , and Lemma 2.3 yields

$$\begin{aligned} I(u) &\geq \frac{a}{3} \|\nabla u\|_2^2 + \frac{b}{12} \|\nabla u\|_2^4 \\ &\geq \frac{a}{3} K^2 \mathcal{S}^{\frac{3}{2}} + \frac{b}{12} K^4 \mathcal{S}^3 > \bar{m}. \end{aligned}$$

Therefore,

$$I(\omega_0) = \inf \{I(u) : u \in S_c, \nabla_{S_c} I(u) = 0\},$$

and  $\omega_0$  is a positive normalized ground state. The proof is complete.  $\square$

## 5. MOUNTAIN-PASS SOLUTION

In this section, we prove Theorem 1.4. Throughout, we assume that (1.7)–(1.9) and (1.11) hold. Let  $u_b \in S_c$  be the positive normalized ground state obtained in Theorem 1.3. Thus,

$$I(u_b) = \bar{m}, \quad \|\nabla u_b\|_2^2 < \bar{\alpha}, \tag{5.1}$$

where

$$\bar{\alpha} = K^2 \mathcal{S}^{\frac{3}{2}}.$$

For convenience, set

$$\mathcal{E}_b := \frac{a}{3} K^2 \mathcal{S}^{\frac{3}{2}} + \frac{b}{12} K^4 \mathcal{S}^3.$$

By (3.1),

$$0 < \bar{m} < \mathcal{E}_b. \quad (5.2)$$

We denote by  $\lambda_b$  the Lagrange multiplier associated with  $u_b$ ; hence

$$-(a + b\|\nabla u_b\|_2^2) \Delta u_b + V(x)u_b = \lambda_b u_b + u_b^5 \quad \text{in } \mathbb{R}^3. \quad (5.3)$$

By Lemma 2.2,  $\lambda_b \geq 0$ .

When the dependence on  $b$  needs to be emphasized, we write  $I_b$  and  $\bar{m}_b$  instead of  $I$  and  $\bar{m}$ .

### 5.1. A SHARP UPPER ESTIMATE FOR THE MOUNTAIN-PASS LEVEL

We first establish a compactness property of the family of ground states as  $b \rightarrow 0^+$ .

**Lemma 5.1.** *Let  $b_n \rightarrow 0^+$  be a sequence for which (1.11) holds, and let  $u_{b_n}$  be positive normalized ground states given by Theorem 1.3. Then, up to a subsequence,*

$$u_{b_n} \rightarrow u_0 \quad \text{strongly in } \mathcal{H}$$

for some positive  $u_0 \in S_c$ . Consequently, for every fixed  $R > 0$ , there exist  $b_R > 0$  and  $\kappa_R > 0$  such that

$$u_b(x) \geq \kappa_R \quad \text{for all } x \in \overline{B_{2R}} \quad (5.4)$$

whenever  $0 < b < b_R$  and (1.11) holds.

*Proof.* Set  $u_n = u_{b_n}$ , and write  $K_{b_n}$  for the constant  $K$  corresponding to  $b = b_n$  and

$$\bar{\alpha}_{b_n} := K_{b_n}^2 \mathcal{S}^{\frac{3}{2}}.$$

We first note that  $\psi_1 \in \mathcal{B}_{\bar{\alpha}_{b_n}}$  for every  $n$ . Indeed,

$$a\|\nabla \psi_1\|_2^2 + \int_{\mathbb{R}^3} V(x)\psi_1^2 dx = \lambda_1 c^2,$$

and the potential term is strictly positive. As shown in the proof of Lemma 3.1, the single inequality (1.11) implies

$$\frac{\lambda_1 c^2}{a} < K_{b_n}^2 \mathcal{S}^{\frac{3}{2}} = \bar{\alpha}_{b_n}.$$

Hence,

$$\|\nabla \psi_1\|_2^2 < \frac{\lambda_1 c^2}{a} < \bar{\alpha}_{b_n}.$$

Consequently,

$$I_{b_n}(u_n) = \bar{m}_{b_n} \leq I_{b_n}(\psi_1).$$

From (5.1) and the definition of  $\bar{\alpha}$ , the sequence  $\{\|\nabla u_n\|_2\}$  is bounded. By the Sobolev inequality,  $\{\|u_n\|_6\}$  is bounded. The preceding energy bound then yields

$$\sup_n \int_{\mathbb{R}^3} V(x)u_n^2 dx < +\infty,$$

so  $\{u_n\}$  is bounded in  $\mathcal{H}$ .

By the equation and the Pohozaev identity,

$$\lambda_{b_n} c^2 = \int_{\mathbb{R}^3} \left( V(x) + \frac{1}{2} \nabla V(x) \cdot x \right) u_n^2 dx.$$

In view of (1.9),  $\{\lambda_{b_n}\}$  is bounded. Therefore, up to a subsequence,

$$u_n \rightharpoonup u_0 \quad \text{in } \mathcal{H}, \quad u_n \rightarrow u_0 \quad \text{in } L^2(\mathbb{R}^3),$$

with  $\|u_0\|_2 = c$ .

We next justify the passage to the limiting functional. Define

$$I_0(u) := \frac{a}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx - \frac{1}{6} \|u\|_6^6.$$

Since each  $u_n$  is a constrained critical point of  $I_{b_n}$ , for every  $\varphi \in T_{u_n} S_c$ ,

$$|I'_0(u_n)[\varphi]| = b_n \|\nabla u_n\|_2^2 \left| \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla \varphi dx \right| \leq C b_n \|\varphi\|_{\mathcal{H}}.$$

Thus,

$$\|\nabla_{S_c} I_0(u_n)\| \rightarrow 0.$$

Moreover,

$$I_{b_n}(u_n) - I_0(u_n) = \frac{b_n}{4} \|\nabla u_n\|_2^4 \rightarrow 0.$$

Hence,  $\{u_n\}$  is a bounded Palais–Smale sequence for  $I_0|_{S_c}$ , up to a vanishing perturbation, with bounded multipliers and  $u_n \geq 0$ .

Suppose that  $u_n \not\rightarrow u_0$  strongly in  $\mathcal{H}$ . Applying the splitting argument in the proof of Lemma 2.5 with  $b = 0$ , we obtain

$$\liminf_{n \rightarrow \infty} I_{b_n}(u_n) \geq I_0(u_0) + \frac{1}{3} a^{\frac{3}{2}} \mathcal{S}^{\frac{3}{2}}. \quad (5.5)$$

The limit function  $u_0$  satisfies the limiting Euler–Lagrange equation and its Pohozaev identity. Hence, as in Lemma 2.3,

$$I_0(u_0) \geq \frac{a}{3} \|\nabla u_0\|_2^2 > 0.$$

On the other hand,

$$\begin{aligned} \limsup_{n \rightarrow \infty} I_{b_n}(u_n) &\leq \lim_{n \rightarrow \infty} I_{b_n}(\psi_1) \\ &= \frac{1}{2} \lambda_1 c^2 - \frac{1}{6} \|\psi_1\|_6^6 < \frac{1}{2} \lambda_1 c^2. \end{aligned}$$

Letting  $b_n \rightarrow 0$  in (1.11) gives

$$\frac{1}{3}a^{\frac{3}{2}}\mathcal{S}^{\frac{3}{2}} \geq \frac{1}{2}\lambda_1 c^2,$$

which contradicts (5.5). Therefore,

$$u_n \rightarrow u_0 \quad \text{strongly in } \mathcal{H}.$$

By standard local Brezis–Kato estimates and elliptic regularity, after passing to a subsequence,  $u_n \rightarrow u_0$  locally uniformly in  $\mathbb{R}^3$ . The strong maximum principle gives  $u_0 > 0$  in  $\mathbb{R}^3$ . It remains to prove the uniform lower bound. If (5.4) were false for some fixed  $R > 0$ , there would exist  $b_n \rightarrow 0^+$  and  $x_n \in \overline{B_{2R}}$  such that

$$u_{b_n}(x_n) \rightarrow 0.$$

After passing to a subsequence,  $x_n \rightarrow x_0 \in \overline{B_{2R}}$ , while the preceding compactness and local uniform convergence give

$$u_{b_n}(x_n) \rightarrow u_0(x_0) > 0,$$

a contradiction. This proves (5.4).  $\square$

We next introduce the standard truncated Sobolev bubble

$$U_\epsilon(x) = K\eta(x) \frac{(3\epsilon^2)^{\frac{1}{4}}}{(\epsilon^2 + |x|^2)^{\frac{1}{2}}}, \quad \epsilon > 0,$$

where  $\eta \in C_0^\infty(\mathbb{R}^3)$ ,  $0 \leq \eta \leq 1$ , satisfies

$$\eta \equiv 1 \quad \text{in } B_R, \quad \eta \equiv 0 \quad \text{in } \mathbb{R}^3 \setminus B_{3R/2}.$$

By the standard estimates of Struwe [27, p. 179] and Brézis–Nirenberg [3, (1.13), (1.29)],

$$\|\nabla U_\epsilon\|_2^2 = K^2 \mathcal{S}^{\frac{3}{2}} + O(\epsilon), \quad \|U_\epsilon\|_6^6 = K^6 \mathcal{S}^{\frac{3}{2}} + O(\epsilon^3), \quad (5.6)$$

and

$$\|U_\epsilon\|_2^2 = C_1 \epsilon + o(\epsilon) \quad (5.7)$$

for some  $C_1 > 0$ . More generally,

$$\|U_\epsilon\|_p^p = \begin{cases} D_p \epsilon^{3-\frac{p}{2}} + o(\epsilon^{3-\frac{p}{2}}), & 3 < p < 6, \\ D_3 \epsilon^{\frac{3}{2}} |\log \epsilon| + O(\epsilon^{\frac{3}{2}}), & p = 3, \\ D_p \epsilon^{\frac{p}{2}} + o(\epsilon^{\frac{p}{2}}), & 1 \leq p < 3, \end{cases} \quad (5.8)$$

where  $D_p > 0$  for  $p \neq 3$  and  $D_3 > 0$ . Since  $K \rightarrow a^{1/4}$  as  $b \rightarrow 0^+$ , all constants needed below may be chosen uniformly for sufficiently small  $b$ .

**Lemma 5.2.** *There exists  $b_0 > 0$  such that, for every  $0 < b < b_0$  satisfying the assumptions of Theorem 1.4, there exist  $z_b \in S_c$  and  $\gamma_b \in C([0, 1], S_c)$  such that*

$$\gamma_b(0) = u_b, \quad \gamma_b(1) = z_b, \quad (5.9)$$

$$I(z_b) < 0, \quad \|\nabla z_b\|_2^2 > \bar{\alpha}, \quad (5.10)$$

and

$$\max_{\tau \in [0, 1]} I(\gamma_b(\tau)) < \bar{m} + \mathcal{E}_b. \quad (5.11)$$

*Proof.* The path below is adapted from the normalized truncated-bubble construction in [35, Lemma 5.2]. In the Kirchhoff setting we must additionally control the  $b$ -dependent interaction terms generated by the nonlocal gradient contribution, and these terms are what lead to the smallness restriction on  $b$ .

For  $t \geq 0$ , set

$$v_{t,\epsilon} = u_b + tU_\epsilon, \quad s = s_{t,\epsilon} := \frac{\|v_{t,\epsilon}\|_2}{c},$$

and define

$$\check{v}_{t,\epsilon}(x) = s^{\frac{1}{2}} v_{t,\epsilon}(sx). \quad (5.12)$$

Then  $\|\check{v}_{t,\epsilon}\|_2 = c$ . Since  $u_b > 0$ ,  $U_\epsilon \geq 0$ , and  $t \geq 0$ , we also have

$$\|u_b + tU_\epsilon\|_2 \geq \|u_b\|_2 = c, \quad \text{and hence} \quad s \geq 1.$$

Writing

$$\check{u}_b(x) = s^{\frac{1}{2}} u_b(sx), \quad \check{U}_\epsilon(x) = s^{\frac{1}{2}} U_\epsilon(sx),$$

we have

$$\check{v}_{t,\epsilon} = \check{u}_b + t\check{U}_\epsilon.$$

Moreover,

$$c^2(s^2 - 1) = t^2 \|U_\epsilon\|_2^2 + 2t \int_{\mathbb{R}^3} u_b U_\epsilon dx. \quad (5.13)$$

By Lemma 5.1 and (5.7)–(5.8), for  $t$  in a fixed bounded interval,

$$s - 1 = O(\epsilon^{\frac{1}{2}}) \quad (5.14)$$

uniformly for sufficiently small  $b$ .

Using

$$(A + B)^6 \geq A^6 + B^6 + 6A^5B + 6AB^5, \quad A, B \geq 0,$$

together with the equation satisfied by  $u_b$ , a direct expansion gives

$$I(\check{v}_{t,\epsilon}) \leq \bar{m} + F_\epsilon(t) + H_1 + H_2 + H_3 + H_4, \quad (5.15)$$

where

$$F_\epsilon(t) = \frac{at^2}{2} \|\nabla U_\epsilon\|_2^2 + \frac{bt^4}{4} \|\nabla U_\epsilon\|_2^4 - \frac{t^6}{6} \|U_\epsilon\|_6^6,$$

$$\begin{aligned}
H_1 &= I(\check{u}_b) - I(u_b) + t\lambda_b s^2 \int_{\mathbb{R}^3} \check{u}_b \check{U}_\epsilon dx, \\
H_2 &= t \int_{\mathbb{R}^3} [V(x) - s^2 V(sx)] \check{u}_b \check{U}_\epsilon dx, \\
H_3 &= \frac{t^2}{2} \int_{\mathbb{R}^3} V(x) \check{U}_\epsilon^2 dx - t^5 \int_{\mathbb{R}^3} \check{u}_b \check{U}_\epsilon^5 dx,
\end{aligned}$$

and

$$\begin{aligned}
H_4 &= bt^2 \left( \int_{\mathbb{R}^3} \nabla \check{u}_b \cdot \nabla \check{U}_\epsilon dx \right)^2 \\
&\quad + bt^3 \left( \int_{\mathbb{R}^3} \nabla \check{u}_b \cdot \nabla \check{U}_\epsilon dx \right) \|\nabla \check{U}_\epsilon\|_2^2 \\
&\quad + \frac{bt^2}{2} \|\nabla \check{u}_b\|_2^2 \|\nabla \check{U}_\epsilon\|_2^2.
\end{aligned}$$

By (5.6) and the identity

$$K^4 = a + bK^2 \mathcal{S}^{\frac{3}{2}},$$

we have

$$\max_{t \geq 0} F_\epsilon(t) = \mathcal{E}_b + O(\epsilon). \quad (5.16)$$

We next estimate the error terms. In what follows, all  $o(\epsilon^{1/2})$ -terms are understood uniformly for  $b$  sufficiently small and for  $t$  in the indicated fixed bounded intervals. Set

$$W(x) = V(x) + \frac{1}{2} \nabla V(x) \cdot x.$$

By (5.3) and the Pohozaev identity,

$$\lambda_b c^2 = \int_{\mathbb{R}^3} W(x) u_b^2 dx.$$

Since the gradient and  $L^6$  terms are invariant under the scaling in (5.12),

$$I(\check{u}_b) - I(u_b) = \frac{1}{2} \int_{\mathbb{R}^3} \left[ s^{-2} V\left(\frac{x}{s}\right) - V(x) \right] u_b^2(x) dx.$$

Consequently,

$$I(\check{u}_b) - I(u_b) = \int_1^s \left[ \int_{\mathbb{R}^3} (u_b^2(x) - u_b^2(yx)) W(x) dx - \lambda_b c^2 \right] dy.$$

By (1.9),  $0 \leq W(x) \leq (1 + \frac{C_V}{2}) V(x)$ . We justify the dilation estimate used below. For  $y$  in a fixed neighborhood of 1, Lemma 2.4 gives

$$\int_{\mathbb{R}^3} W(x) |u(yx)|^2 dx \leq C \int_{\mathbb{R}^3} V(x) |u(yx)|^2 dx \leq C \int_{\mathbb{R}^3} V(x) |u(x)|^2 dx,$$

where  $C$  is uniform for such  $y$ . Thus, the dilation operators  $T_y u = u(y \cdot)$  are uniformly bounded from  $\mathcal{H}$  into  $L^2(\mathbb{R}^3, W(x) dx)$  for  $y$  close to 1. For  $u \in C_c^\infty(\mathbb{R}^3)$ ,  $T_y u \rightarrow u$  in this weighted space by dominated convergence; the general case follows by density, using the local boundedness of  $V$ . Hence,

$$T_y u \rightarrow u \quad \text{in } L^2(\mathbb{R}^3, W(x) dx) \quad \text{as } y \rightarrow 1.$$

We next establish the joint small- $b$ /small-dilation limit required below. For arbitrary admissible sequences

$$b_n \rightarrow 0^+, \quad y_n \rightarrow 1,$$

one has

$$\|u_{b_n}(y_n \cdot) - u_{b_n}\|_{L^2(\mathbb{R}^3, W(x) dx)} \rightarrow 0.$$

Indeed, if the limit failed, there would be a subsequence along which the left-hand side remained bounded away from zero. Applying Lemma 5.1 to this subsequence and extracting further if necessary, we obtain  $u_{b_n} \rightarrow u_0$  strongly in  $\mathcal{H}$ , and hence in  $L^2(W dx)$ . The uniform boundedness of  $T_y u = u(y \cdot)$  near  $y = 1$ , together with the strong continuity of  $T_y$  on the fixed function  $u_0$ , gives

$$\begin{aligned} \|T_{y_n} u_{b_n} - u_{b_n}\|_{L^2(W dx)} &\leq \|T_{y_n}(u_{b_n} - u_0)\|_{L^2(W dx)} + \|T_{y_n} u_0 - u_0\|_{L^2(W dx)} \\ &\quad + \|u_0 - u_{b_n}\|_{L^2(W dx)} \rightarrow 0, \end{aligned}$$

a contradiction. Thus, the joint limit holds for the original sequences. Consequently, if  $s_n \rightarrow 1$  and  $1 \leq y_n \leq s_n$ , then  $y_n \rightarrow 1$  and the same joint limit applies. Hence,

$$\sup_{1 \leq y \leq s} \left| \int_{\mathbb{R}^3} (u_b^2(x) - u_b^2(yx)) W(x) dx \right| = o(1) \quad (5.17)$$

as  $b \rightarrow 0^+$  and  $s \rightarrow 1$  jointly. Otherwise, a sequence of near-maximizers would contradict the preceding joint limit and the estimate

$$\left| \int_{\mathbb{R}^3} (u^2 - (T_y u)^2) W dx \right| \leq (\|u\|_{L^2(W dx)} + \|T_y u\|_{L^2(W dx)}) \|u - T_y u\|_{L^2(W dx)}.$$

Hence,

$$I(\check{u}_b) - I(u_b) \leq -\lambda_b c^2 (s - 1) + o(|s - 1|). \quad (5.18)$$

Moreover, the scaling gives

$$s^2 \int_{\mathbb{R}^3} \check{u}_b \check{U}_\epsilon dx = \int_{\mathbb{R}^3} u_b U_\epsilon dx. \quad (5.19)$$

Combining (5.13), (5.18), and (5.19), we obtain

$$H_1 \leq -\frac{\lambda_b t^2}{s+1} \|U_\epsilon\|_2^2 + \lambda_b t \frac{s-1}{s+1} \int_{\mathbb{R}^3} u_b U_\epsilon dx + o(|s-1|).$$

Since

$$\|U_\epsilon\|_2^2 = O(\epsilon), \quad \int_{\mathbb{R}^3} u_b U_\epsilon dx = O(\epsilon^{\frac{1}{2}}),$$

we conclude that

$$H_1 = o(\epsilon^{\frac{1}{2}}). \quad (5.20)$$

Since  $V$  is locally Lipschitz and the support of  $\check{U}_\epsilon$  remains in a fixed bounded set for  $t$  in a fixed bounded interval, (5.14) yields

$$H_2 = o(\epsilon^{\frac{1}{2}}).$$

Furthermore,

$$\int_{\mathbb{R}^3} V(x) \check{U}_\epsilon^2 dx = O(\epsilon).$$

By Lemma 5.1, after decreasing  $b_0$  if necessary,

$$u_b \geq \kappa_R > 0 \quad \text{in } B_R.$$

The scaling invariance of the interaction term gives

$$\int_{\mathbb{R}^3} \check{u}_b \check{U}_\epsilon^5 dx = \int_{\mathbb{R}^3} u_b U_\epsilon^5 dx.$$

Hence, (5.8) with  $p = 5$  implies

$$\int_{\mathbb{R}^3} u_b U_\epsilon^5 dx \geq C_0 \epsilon^{\frac{1}{2}} + o(\epsilon^{\frac{1}{2}})$$

for some  $C_0 > 0$  independent of sufficiently small  $b$ . Thus, for every fixed  $t_0 > 1$  and  $t_0^{-1} \leq t \leq t_0$ ,

$$H_3 \leq -C_2 \epsilon^{\frac{1}{2}} + o(\epsilon^{\frac{1}{2}})$$

for some  $C_2 > 0$  independent of small  $b$ .

Finally, by Cauchy's inequality, Lemma 5.1, and (5.6),

$$|H_4| \leq Cb \quad (5.21)$$

for  $t \in [t_0^{-1}, t_0]$ , with  $C > 0$  independent of sufficiently small  $b$ .

Combining (5.15), (5.16), and (5.20)–(5.21), we find

$$I(\check{v}_{t,\epsilon}) \leq \bar{m} + \mathcal{E}_b - C_2 \epsilon^{\frac{1}{2}} + o(\epsilon^{\frac{1}{2}}) + C(\epsilon + b) \quad (5.22)$$

uniformly for  $t_0^{-1} \leq t \leq t_0$ .

We choose

$$\epsilon = b.$$

Since  $b = o(b^{1/2})$  as  $b \rightarrow 0^+$ , (5.22) gives, after decreasing  $b_0$ ,

$$I(\check{v}_{t,b}) < \bar{m} + \mathcal{E}_b, \quad t_0^{-1} \leq t \leq t_0.$$

It remains to control the two complementary ranges of  $t$ . Since  $\mathcal{E}_b \rightarrow \frac{1}{3}a^{3/2}\mathcal{S}^{3/2} > 0$  as  $b \rightarrow 0^+$ , there exists  $E_* > 0$  such that  $\mathcal{E}_b \geq E_*$  for all sufficiently small  $b$ . For  $0 \leq t \leq 1$ , the estimates above give, uniformly for small  $b$ ,

$$F_b(t) \leq Ct^2, \quad H_1 + H_2 + H_3 + H_4 \leq o_b(1) + Cbt^2.$$

Consequently,

$$I(\check{v}_{t,b}) - \bar{m} \leq Ct^2 + o_b(1) \quad \text{uniformly for } 0 \leq t \leq 1.$$

Choose  $t_0 > 1$  so large that  $Ct_0^{-2} < E_*/2$ , and then decrease  $b_0$  so that  $|o_b(1)| < E_*/2$  for  $0 < b < b_0$ . We obtain

$$I(\check{v}_{t,b}) < \bar{m} + \mathcal{E}_b \quad \text{for } 0 \leq t < t_0^{-1}.$$

For large  $t$ , recall that  $s \geq 1$ . By Lemma 2.4,

$$V\left(\frac{x}{s}\right) \leq s^2 V(x).$$

The definition of  $\check{v}_{t,b}$  therefore gives

$$\frac{1}{2} \int_{\mathbb{R}^3} V(x) \check{v}_{t,b}^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^3} V(x) (u_b + tU_b)^2 dx \leq C(1 + t^2).$$

Moreover,

$$\|\nabla(u_b + tU_b)\|_2^2 \leq C(1 + t^2), \quad b\|\nabla(u_b + tU_b)\|_2^4 \leq Cb(1 + t^4),$$

while, because  $u_b, U_b \geq 0$ ,

$$\|u_b + tU_b\|_6^6 \geq t^6 \|U_b\|_6^6 \geq c_0 t^6$$

for some  $c_0 > 0$  independent of sufficiently small  $b$ . Hence,

$$I(\check{v}_{t,b}) \leq C(1 + t^2) + Cb(1 + t^4) - c_0 t^6.$$

Increasing  $t_0$  if necessary and decreasing  $b_0$  once more, we obtain

$$I(\check{v}_{t,b}) < \bar{m} + \mathcal{E}_b \quad \text{for } t > t_0.$$

Therefore,

$$\sup_{t \geq 0} I(\check{v}_{t,b}) < \bar{m} + \mathcal{E}_b.$$

Moreover,

$$I(\check{v}_{t,b}) \rightarrow -\infty, \quad \|\nabla \check{v}_{t,b}\|_2 \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

Choose  $T_b > t_0$  sufficiently large and set

$$z_b := \check{v}_{T_b,b}.$$

Then (5.10) holds. Reparameterizing the curve  $t \mapsto \check{v}_{t,b}$  on  $[0, T_b]$  gives a path  $\gamma_b \in C([0, 1], S_c)$  satisfying (5.9) and (5.11).  $\square$

We now fix the endpoint  $z_b$  given by Lemma 5.2 and define

$$\Gamma = \{\gamma \in C([0, 1], S_c) : \gamma(0) = u_b, \gamma(1) = z_b\},$$

and

$$\tilde{m} = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)).$$

Since

$$\|\nabla u_b\|_2^2 < \bar{\alpha} < \|\nabla z_b\|_2^2,$$

every path  $\gamma \in \Gamma$  intersects  $\partial \mathcal{B}_{\bar{\alpha}}$ . Therefore, by (3.2),

$$\mathcal{E}_b \leq \tilde{m}. \quad (5.23)$$

On the other hand, Lemma 5.2 gives

$$\tilde{m} < \bar{m} + \mathcal{E}_b. \quad (5.24)$$

Thus,

$$\mathcal{E}_b \leq \tilde{m} < \bar{m} + \mathcal{E}_b.$$

## 5.2. CONSTRUCTION OF A BOUNDED PALAIS–SMALE SEQUENCE

For  $(u, s) \in S_c \times \mathbb{R}$ , define

$$\Phi_s(u)(x) = e^{\frac{3}{2}s} u(e^s x),$$

and

$$\hat{I}(u, s) = I(\Phi_s(u)).$$

Clearly,  $\Phi_s(u) \in S_c$  for every  $(u, s) \in S_c \times \mathbb{R}$ . We also record that

$$\hat{I} \in C^1(S_c \times \mathbb{R}, \mathbb{R}).$$

Indeed, after the change of variables  $y = e^s x$ ,

$$\hat{I}(u, s) = \frac{a}{2} e^{2s} \|\nabla u\|_2^2 + \frac{b}{4} e^{4s} \|\nabla u\|_2^4 + \frac{1}{2} \int_{\mathbb{R}^3} V(e^{-s} x) u^2 dx - \frac{1}{6} e^{6s} \|u\|_6^6.$$

Since  $V$  is locally Lipschitz and  $|\nabla V(x) \cdot x| \leq \max\{2, C_V\}V(x)$  by (1.9), differentiation under the integral sign is justified by Lemma 2.4. In particular,

$$\begin{aligned} \partial_s \hat{I}(u, s) &= a \|\nabla \Phi_s(u)\|_2^2 + b \|\nabla \Phi_s(u)\|_2^4 - \|\Phi_s(u)\|_6^6 \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(x) \cdot x |\Phi_s(u)|^2 dx. \end{aligned}$$

The continuity of this derivative, as well as that of the derivative in the  $u$ -variable, follows from the same weighted-dilation argument used in the proof of (5.17). Hence,  $\hat{I}$  is indeed of class  $C^1$ .

Define

$$\hat{\Gamma} = \{\hat{\gamma} \in C([0, 1], S_c \times \mathbb{R}) : \hat{\gamma}(0) = (u_b, 0), \hat{\gamma}(1) = (z_b, 0)\},$$

and

$$\hat{m} = \inf_{\hat{\gamma} \in \hat{\Gamma}} \max_{t \in [0, 1]} \hat{I}(\hat{\gamma}(t)).$$

**Lemma 5.3.** *We have*

$$\hat{m} = \tilde{m}.$$

*Proof.* For every  $\gamma \in \Gamma$ , the path

$$\hat{\gamma}(t) = (\gamma(t), 0)$$

belongs to  $\hat{\Gamma}$ . Hence,  $\hat{m} \leq \tilde{m}$ .

Conversely, let

$$\hat{\gamma}(t) = (u(t), s(t)) \in \hat{\Gamma}$$

and define

$$\gamma(t) = \Phi_{s(t)}(u(t)).$$

Then  $\gamma \in \Gamma$  and

$$I(\gamma(t)) = \hat{I}(\hat{\gamma}(t)).$$

Taking the infimum over  $\hat{\Gamma}$  yields  $\tilde{m} \leq \hat{m}$ . Therefore,  $\hat{m} = \tilde{m}$ .  $\square$

We use the following minimax principle from [11, Theorem 4.1].

**Lemma 5.4.** *Let  $X$  be a Hilbert manifold and let  $\mathcal{J} \in C^1(X, \mathbb{R})$ . Let  $\mathcal{K} \subset X$  be compact, and let  $\mathcal{G}$  be a homotopy-stable family of compact subsets of  $X$  with boundary  $\mathcal{K}$ . Assume that*

$$\max_{u \in \mathcal{K}} \mathcal{J}(u) < \ell := \inf_{\mathcal{A} \in \mathcal{G}} \max_{u \in \mathcal{A}} \mathcal{J}(u).$$

*If  $\{\mathcal{A}_n\} \subset \mathcal{G}$  satisfies*

$$\max_{u \in \mathcal{A}_n} \mathcal{J}(u) \rightarrow \ell,$$

*then there exists  $\{x_n\} \subset X$  such that*

$$\mathcal{J}(x_n) \rightarrow \ell, \quad \|\nabla_X \mathcal{J}(x_n)\| \rightarrow 0, \quad \text{dist}(x_n, \mathcal{A}_n) \rightarrow 0.$$

**Lemma 5.5.** *There exists a sequence  $\{v_n\} \subset S_c$  such that*

$$I(v_n) \rightarrow \tilde{m}, \quad \|\nabla_{S_c} I(v_n)\| \rightarrow 0, \quad (5.25)$$

and

$$a\|\nabla v_n\|_2^2 + b\|\nabla v_n\|_2^4 - \|v_n\|_6^6 - \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(x) \cdot x v_n^2 dx \rightarrow 0. \quad (5.26)$$

Moreover,

$$\|v_n^-\|_2 \rightarrow 0. \quad (5.27)$$

The sequence  $\{v_n\}$  is bounded in  $\mathcal{H}$ , and there exist associated Lagrange multipliers  $\{\lambda_n\} \subset \mathbb{R}$  such that  $I'(v_n) - \lambda_n v_n \rightarrow 0$  in  $\mathcal{H}^{-1}$ ; moreover,  $\{\lambda_n\}$  is bounded.

*Proof.* Choose  $\zeta_n \in \Gamma$  such that

$$\max_{t \in [0,1]} I(\zeta_n(t)) \rightarrow \tilde{m}.$$

Since  $I(|u|) = I(u)$  and both endpoints  $u_b$  and  $z_b$  are nonnegative, replacing  $\zeta_n$  by  $|\zeta_n|$  if necessary, we may assume that

$$\zeta_n(t) \geq 0 \quad \text{a.e. in } \mathbb{R}^3$$

for every  $t \in [0, 1]$ .

Set

$$\hat{\zeta}_n(t) = (\zeta_n(t), 0).$$

By Lemma 5.3,

$$\max_{t \in [0,1]} \hat{I}(\hat{\zeta}_n(t)) \rightarrow \hat{m} = \tilde{m}.$$

Furthermore, by (5.2), (5.10), and (5.23),

$$\max\{I(u_b), I(z_b)\} < \tilde{m}.$$

We apply Lemma 5.4 on the Hilbert manifold

$$X = S_c \times \mathbb{R}.$$

More precisely, set

$$\mathcal{K} = \{(u_b, 0), (z_b, 0)\},$$

and let

$$\mathcal{G} = \left\{ \hat{\gamma}([0, 1]) : \hat{\gamma} \in \hat{\Gamma} \right\}.$$

Then  $\mathcal{G}$  is homotopy-stable with boundary  $\mathcal{K}$ , and with

$$\mathcal{A}_n = \hat{\zeta}_n([0, 1])$$

we have

$$\max_{x \in \mathcal{A}_n} \hat{I}(x) \rightarrow \hat{m} = \tilde{m}, \quad \max_{x \in \mathcal{K}} \hat{I}(x) < \tilde{m}.$$

Therefore, Lemma 5.4 yields  $(u_n, s_n) \in S_c \times \mathbb{R}$  and  $t_n \in [0, 1]$  such that

$$\begin{aligned}\hat{I}(u_n, s_n) &\rightarrow \tilde{m}, \\ \|\nabla_{S_c \times \mathbb{R}} \hat{I}(u_n, s_n)\| &\rightarrow 0,\end{aligned}\tag{5.28}$$

and

$$\|u_n - \zeta_n(t_n)\|_{\mathcal{H}} + |s_n| \rightarrow 0.\tag{5.29}$$

Define

$$v_n = \Phi_{s_n}(u_n), \quad v_n(x) = e^{\frac{3}{2}s_n} u_n(e^{s_n} x).$$

We now transfer the Palais–Smale information from  $\hat{I}$  to  $I|_{S_c}$ . Let  $\xi \in T_{v_n} S_c$  and set

$$\tilde{\xi}_n := \Phi_{-s_n}(\xi) \in T_{u_n} S_c.$$

Since  $s_n \rightarrow 0$ , Lemma 2.4 implies that the operators  $\Phi_{\pm s_n}$  are uniformly bounded on  $\mathcal{H}$ . By the chain rule,

$$I'(v_n)[\xi] = \partial_u \hat{I}(u_n, s_n)[\tilde{\xi}_n].$$

Therefore, (5.28) gives

$$\|\nabla_{S_c} I(v_n)\| \rightarrow 0, \quad I(v_n) = \hat{I}(u_n, s_n) \rightarrow \tilde{m},$$

which proves (5.25). Finally, the  $s$ -component of (5.28) and the formula for  $\partial_s \hat{I}$  above give (5.26).

Since  $\zeta_n(t_n) \geq 0$ , (5.29) gives

$$\|u_n^-\|_2 \leq \|u_n - \zeta_n(t_n)\|_2 \rightarrow 0.$$

The transformation  $\Phi_s$  preserves the  $L^2$ -norm and the sign, and hence

$$\|v_n^-\|_2 = \|u_n^-\|_2 \rightarrow 0,$$

which proves (5.27).

It remains to prove boundedness. Set

$$\begin{aligned}e_n &= a \|\nabla v_n\|_2^2, \quad \rho_n = b \|\nabla v_n\|_2^4, \\ d_n &= \|v_n\|_6^6, \quad P_n = \int_{\mathbb{R}^3} V(x) v_n^2 dx, \quad h_n = \int_{\mathbb{R}^3} \nabla V(x) \cdot x v_n^2 dx.\end{aligned}$$

From (5.25) and (5.26), we obtain

$$2e_n + \rho_n + 2P_n - \frac{2}{3}d_n = 4\tilde{m} + o_n(1),\tag{5.30}$$

and

$$e_n + \rho_n - \frac{1}{2}h_n = d_n + o_n(1).\tag{5.31}$$

Combining (5.30) and (5.31) gives

$$\frac{1}{3}d_n = 4\tilde{m} - e_n - 2P_n - \frac{1}{2}h_n + o_n(1).$$

By (1.9),

$$h_n \geq -2P_n.$$

Therefore,

$$\frac{1}{3}d_n \leq 4\tilde{m} + o_n(1),$$

so  $\{d_n\}$  is bounded. From (5.30),

$$2e_n + \rho_n + 2P_n = \frac{2}{3}d_n + 4\tilde{m} + o_n(1),$$

and hence  $\{e_n\}$ ,  $\{\rho_n\}$ , and  $\{P_n\}$  are bounded. Thus,  $\{v_n\}$  is bounded in  $\mathcal{H}$ .

We now define the associated Lagrange multiplier by

$$\lambda_n := \frac{1}{c^2} (e_n + \rho_n + P_n - d_n).$$

Equivalently,

$$e_n + \rho_n + P_n = \lambda_n c^2 + d_n. \quad (5.32)$$

The boundedness already established for  $e_n, \rho_n, P_n$ , and  $d_n$  therefore shows that  $\{\lambda_n\}$  is bounded. Moreover, by the definition of  $\lambda_n$  and (5.25),

$$I'(v_n) - \lambda_n v_n \rightarrow 0 \quad \text{in } \mathcal{H}^{-1},$$

so  $\lambda_n$  is precisely the multiplier associated with the constrained Palais–Smale sequence.  $\square$

### 5.3. PROOF OF THEOREM 1.4

*Proof.* Let  $0 < b < b_0$ , where  $b_0$  is given by Lemma 5.2. By Lemma 5.5, there exists a bounded sequence  $\{v_n\} \subset S_c$  satisfying

$$I(v_n) \rightarrow \tilde{m}, \quad \|\nabla_{S_c} I(v_n)\| \rightarrow 0, \quad \|v_n^-\|_2 \rightarrow 0,$$

and the associated Lagrange multipliers  $\{\lambda_n\}$  are bounded.

Passing to a subsequence, we may assume that

$$v_n \rightharpoonup v_0 \quad \text{weakly in } \mathcal{H}, \quad v_n \rightarrow v_0 \quad \text{strongly in } L^2(\mathbb{R}^3),$$

and  $v_n(x) \rightarrow v_0(x)$  for a.e.  $x \in \mathbb{R}^3$ . In particular,

$$\|v_0\|_2 = c, \quad v_0 \geq 0.$$

We claim that

$$v_n \rightarrow v_0 \quad \text{strongly in } \mathcal{H}.$$

Assume by contradiction that the convergence is not strong. Applying Lemma 2.5, after passing to a further subsequence, we may set

$$A = \lim_{n \rightarrow \infty} \|\nabla v_n\|_2^2,$$

with

$$A \geq \|\nabla v_0\|_2^2, \quad (5.33)$$

and the noncompact alternative gives

$$\tilde{m} \geq J_A(v_0) + \mathcal{E}_b, \quad (5.34)$$

where

$$J_A(v) = \left(\frac{a}{2} + \frac{bA}{4}\right) \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)v^2 dx - \frac{1}{6} \|v\|_6^6. \quad (5.35)$$

Moreover,  $v_0$  satisfies

$$-(a + bA)\Delta v_0 + V(x)v_0 = \lambda_0 v_0 + v_0^5 \quad \text{in } \mathbb{R}^3 \quad (5.36)$$

for some  $\lambda_0 > 0$ .

We distinguish two cases.

*Case 1.* Assume that

$$\|\nabla v_0\|_2^2 < \bar{\alpha}.$$

Then  $v_0 \in \mathcal{B}_{\bar{\alpha}} \cap S_c$ , and hence, by the definition of  $\bar{m}$ ,

$$I(v_0) \geq \bar{m}.$$

Furthermore, by (5.33),

$$J_A(v_0) - I(v_0) = \frac{b}{4} \|\nabla v_0\|_2^2 (A - \|\nabla v_0\|_2^2) \geq 0.$$

Therefore,  $J_A(v_0) \geq \bar{m}$ , and (5.34) yields

$$\tilde{m} \geq \bar{m} + \mathcal{E}_b,$$

which contradicts (5.24).

*Case 2.* Assume that

$$\|\nabla v_0\|_2^2 \geq \bar{\alpha}.$$

The Pohozaev identity associated with (5.36) is

$$(a + bA)\|\nabla v_0\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^3} \nabla V(x) \cdot x v_0^2 dx = \|v_0\|_6^6. \quad (5.37)$$

Substituting (5.37) into (5.35), we obtain

$$J_A(v_0) = \left(\frac{a}{3} + \frac{bA}{12}\right) \|\nabla v_0\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) v_0^2 dx + \frac{1}{12} \int_{\mathbb{R}^3} \nabla V(x) \cdot x v_0^2 dx.$$

Using (1.9), we infer that

$$J_A(v_0) \geq \left(\frac{a}{3} + \frac{bA}{12}\right) \|\nabla v_0\|_2^2.$$

Since

$$A \geq \|\nabla v_0\|_2^2 \geq \bar{\alpha},$$

it follows that

$$J_A(v_0) \geq \frac{a}{3} \bar{\alpha} + \frac{b}{12} \bar{\alpha}^2 = \mathcal{E}_b.$$

Thus, (5.34) gives

$$\tilde{m} \geq 2\mathcal{E}_b.$$

However, by (5.2) and (5.24),

$$\tilde{m} < \bar{m} + \mathcal{E}_b < 2\mathcal{E}_b,$$

a contradiction.

Both cases are impossible. Hence,

$$v_n \rightarrow v_0 \quad \text{strongly in } \mathcal{H}.$$

Passing to the limit in the constrained Euler–Lagrange equation, we obtain

$$-(a + b\|\nabla v_0\|_2^2) \Delta v_0 + V(x)v_0 = \lambda_0 v_0 + v_0^5 \quad \text{in } \mathbb{R}^3,$$

and

$$I(v_0) = \tilde{m}.$$

Since  $v_0 \geq 0$  and  $\|v_0\|_2 = c > 0$ , the strong maximum principle yields

$$v_0 > 0 \quad \text{in } \mathbb{R}^3.$$

Finally, by (5.2), (5.23), and the identity  $I(v_0) = \tilde{m}$ ,

$$I(v_0) = \tilde{m} \geq \mathcal{E}_b > \bar{m} = I(u_b).$$

Therefore,  $v_0 \neq u_b$ . Hence,  $v_0$  is a second positive normalized solution of (1.1) at the mountain-pass level  $\tilde{m}$ . The proof of Theorem 1.4 is complete.  $\square$

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