

PURE-SWIRL LOSS OF BOUNDEDNESS  
UNDER  $L_t^1 L_x^2$  FORCING:  
EXACT INTEGRABILITY OF  
AN EXPLICIT FAMILY

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**Abstract.** We give an explicit pure-swirl family of forced Navier–Stokes solutions in a bounded circular cylinder with no slip on the lateral wall and free slip on the horizontal faces. For each  $T > 0$  and each finite pair  $p, q \geq 1$  with  $1/p + 1/q > 1$ , we construct a corresponding force in  $L^q(0, T; L^p(D)) \cap L^1(0, T; L^2(D))$  whose velocity loses  $L^\infty$  boundedness as  $t \uparrow T$ . The construction refines the  $k = 1$  case of our previous weighted ansatz: we correct the mixed-norm calculation and the convection identity, and impose an annular cancellation to ensure smoothness at the axis. For this family we prove exact mixed-norm membership criteria, including the spatial force endpoint  $p = 1$ , and two-sided rates for the force, velocity supremum norm and enstrophy. The velocity has a strong  $L^2$  trace at  $T$  and, with this trace, is the unique Leray–Hopf solution on  $[0, T]$  and satisfies the energy equality. The same velocity and force solve Stokes with zero pressure.

**Keywords:** Navier–Stokes equations, external force, loss of boundedness, pure swirl, mixed norms, Leray–Hopf solution.

**Mathematics Subject Classification:** 35Q30, 76D03, 35A01.

## 1. INTRODUCTION AND MAIN RESULTS

We study the forced incompressible Navier–Stokes system

$$\begin{cases} \partial_t v - \Delta v + (v \cdot \nabla)v + \nabla P = f & \text{in } D \times (0, T), \\ \nabla \cdot v = 0 & \text{in } D \times (0, T), \end{cases} \quad (1.1)$$

where

$$D = B_2(0, 1) \times (0, 1) \subset \mathbb{R}^3$$

is the unit circular cylinder. Write

$$\Gamma_V = \partial B_2(0, 1) \times (0, 1), \quad \Gamma_H = B_2(0, 1) \times \{0, 1\}.$$

On the vertical boundary we impose no slip, and on the horizontal boundary we impose the free-slip (zero tangential stress) condition

$$v = 0 \quad \text{on } \Gamma_V, \quad v \cdot n = 0, \quad ((\nabla v + \nabla v^\top)n)_\tau = 0 \quad \text{on } \Gamma_H. \quad (1.2)$$

For the pure-swirl fields constructed below, the condition on  $\Gamma_H$  reduces simply to

$$v_3 = 0, \quad \partial_{x_3} v_1 = \partial_{x_3} v_2 = 0.$$

For the classical energy formulation, see [7–9, 12]; for conditional regularity and weak–strong uniqueness criteria, see [4, 5, 11]. Forced weak solutions may nevertheless display nonuniqueness or loss of boundedness; see [1, 6]. These phenomena are distinct from the unforced smooth-data regularity problem.

A direct forced blow-up construction in a finite cylinder was recently given by Zhang [14]. His velocity is purely azimuthal, and the system reduces to a scalar parabolic equation; he obtains a force in  $L_t^q L_x^1$  for every finite  $q > 1$ . The self-similar scalar profile is inspired by Leray’s classical ansatz, although nontrivial unforced backward self-similar solutions are ruled out under natural integrability assumptions by Nečas, Růžička and Šverák [10]. Thus, the singular forcing is an essential part of Zhang’s mechanism.

**Remark 1.1** (The pure-swirl reduction). For an axisymmetric field  $v = W(r, t)e_\theta$  independent of  $x_3$ ,

$$(v \cdot \nabla)v = -\frac{W^2}{r}e_r.$$

This term is cancelled by the pressure gradient when  $P_r = W^2/r$ . The azimuthal equation is therefore linear. The same velocity and force solve the Stokes problem with a spatially constant pressure, which we take to be zero. This corrects the statement in [2, Remark 2.1] that the convective term itself vanishes.

Our purpose is to give a direct and quantitative account of the  $k = 1$  weighted construction in [2]. The annular profile below makes the velocity and force smooth at the axis for every  $0 < \alpha < 1$ . Its explicit outer tail gives matching upper and lower bounds, exact integrability criteria, and an explicit terminal trace. We describe the corrections to [2] and compare the result with [6, 13] after the main statement.

We now state the main result; the mixed-boundary Leray–Hopf class is recalled below.

**Theorem 1.2.** *Let  $T > 0$  and let  $1 \leq p, q < \infty$  satisfy*

$$\frac{1}{p} + \frac{1}{q} > 1. \quad (1.3)$$

*Then there exist a force*

$$f \in L^q(0, T; L^p(D)) \quad (1.4)$$

*and a solution  $(v, P)$  of (1.1)–(1.2), with  $v, P, f$  smooth on  $\overline{D} \times [0, T']$  for every  $T' < T$ , including at the boundary edges, such that*

$$v \in L^\infty(0, T; L^2(D)) \cap L^2(0, T; H^1(D)) \quad (1.5)$$

and

$$\lim_{t \uparrow T} \|v(t)\|_{L^\infty(D)} = \infty. \quad (1.6)$$

In addition, the force constructed above always satisfies

$$f \in L^1(0, T; L^2(D)). \quad (1.7)$$

Consequently, the solution extends to a function  $v \in C([0, T]; L^2(D))$ . This extension is the unique Leray–Hopf solution on  $[0, T]$  with the same force, initial datum and boundary conditions, and it satisfies the energy equality on  $[0, T]$ .

More precisely, one may choose a parameter

$$\max \left\{ 0, 3 - \frac{2}{p} - \frac{2}{q} \right\} < \alpha < 1 \quad (1.8)$$

such that, as  $t \uparrow T$ ,

$$\|f(t)\|_{L^p(D)} \asymp (T - t)^{-\frac{3-\alpha}{2} + \frac{1}{p}}, \quad (1.9)$$

$$\|v(t)\|_{L^\infty(D)} \asymp (T - t)^{-\frac{1-\alpha}{2}}, \quad (1.10)$$

$$\|\nabla v(t)\|_{L^2(D)}^2 \asymp (T - t)^{\alpha-1}. \quad (1.11)$$

For each fixed  $\alpha$  in this constructed family, the velocity has the following exact mixed-norm classification. For  $1 \leq m, \sigma < \infty$ ,

$$v \in L^\sigma(0, T; L^m(D)) \iff \alpha > 1 - \frac{2}{m} - \frac{2}{\sigma}. \quad (1.12)$$

Moreover,

$$\begin{aligned} v \in L^\infty(0, T; L^m(D)) &\iff \alpha > 1 - \frac{2}{m}, \quad 1 \leq m < \infty, \\ v \in L^\sigma(0, T; L^\infty(D)) &\iff \alpha > 1 - \frac{2}{\sigma}, \quad 1 \leq \sigma < \infty. \end{aligned} \quad (1.13)$$

**Corollary 1.3.** For every finite  $p \geq 1$ , one may take  $q = 1$ ; in particular the construction reaches  $f \in L^1(0, T; L^2(D))$ . Moreover, if  $1 \leq p \leq 2$  and

$$q = \frac{4p}{7p-6}, \quad \frac{2}{q} + \frac{3}{p} = \frac{7}{2},$$

then the conclusion of Theorem 1.2 holds, including  $(p, q) = (2, 1)$ . For any admissible force exponents  $p, q$  and any prescribed finite velocity exponents  $m, \sigma \geq 1$ , one may choose the construction so that  $v \in L_t^\sigma L_x^m$  as well, while  $\|v(t)\|_\infty \rightarrow \infty$ .

The curve  $2/q + 3/p = 7/2$  is retained for comparison with [2, Corollary 3.3], which treats  $1 \leq p < 2$ . The present construction also covers its endpoint  $(p, q) = (2, 1)$ . This curve is not an additional critical threshold: the whole segment  $1 \leq p \leq 2$  lies strictly inside the range (1.3).

**Remark 1.4** (Scope of the exponent condition). The condition  $1/p + 1/q > 1$  is the exact feasibility region for choosing  $0 < \alpha < 1$  in this family. It is not asserted to be a universal threshold for forced Navier–Stokes solutions. The line  $2/q + 3/p = 2$  is the boundary of the strict heat-potential boundedness range  $2/q + 3/p < 2$ ; endpoints require separate analysis. In contrast,  $2/q + 3/p = 3$  is the scale-invariant line for the force under  $f_\lambda(x, t) = \lambda^3 f(\lambda x, \lambda^2 t)$ . Our force becomes unbounded as  $t \uparrow T$ . The example therefore concerns loss of boundedness under singular forcing.

### 1.1. CORRECTIONS AND REFINEMENTS RELATIVE TO THE PREVIOUS PAPER

The relation with [2] is summarized in Table 1. We restrict the revised existence and classification statements to  $k = 1$  and  $0 < \alpha < 1$ . The earlier weighted construction was first posted as arXiv:2411.10823v1 on 16 November 2024, before the first preprint of Galdi–Gazzola. Its  $k = 1$  statements and argument, in Theorem 4.1 and Corollary 4.2 of that preprint, correspond to Theorem 3.1 and Corollary 3.2 of the published paper. The corrections below concern this shared  $k = 1$  construction.

**Table 1**  
Corrections relevant to the  $k = 1$  construction in [2]

| Earlier statement                           | Status                                                                  | Present treatment                                                         |
|---------------------------------------------|-------------------------------------------------------------------------|---------------------------------------------------------------------------|
| Mixed-norm conditions (3.1), (3.3)          | stronger than necessary in the positive- $\alpha$ range considered here | retain the spatial root before time integration; obtain (1.14) and (1.12) |
| Convection in Remark 2.1                    | the zero-convection identity is incorrect                               | use $(v \cdot \nabla)v = -W^2 e_r/r$ and distinguish the two pressures    |
| Smoothness in Theorem 3.1 and Corollary 3.2 | a noninteger radial weight need not preserve axis smoothness            | impose annular cancellation; then $W = \beta r$ near the axis             |
| The endpoint $\alpha = 0$                   | a nonzero $r^{-1}$ terminal tail is outside $L^2(D)$                    | require $\alpha > 0$ for the energy-class conclusions                     |

For the force, an estimate of the form

$$\|f(t)\|_{L^p}^p \lesssim (T - t)^{-((3-\alpha)p-2)/2}$$

must be raised to the power  $q/p$  before time integration. The resulting condition is

$$((3 - \alpha)p - 2)q < 2p, \quad \text{equivalently} \quad \alpha > 3 - \frac{2}{p} - \frac{2}{q}. \quad (1.14)$$

For  $p \geq 1$  the earlier inequality with right-hand side 2 remains sufficient for this temporal estimate, but gives a smaller range. With the axis-compatible profile below, that subrange is retained. The assertion in [2] that  $L_t^1 L_x^2$  was not reached is superseded. The same root correction applies to the earlier sufficient velocity condition; we prove the full classification for this family in Lemma 4.3.

To see the smoothness issue directly, suppose an unweighted profile has  $\phi(r, t) = c(t)r + O(r^3)$  with  $c(t) \neq 0$ . Then

$$r^\alpha \phi(r, t) e_\theta = (c(t)r^\alpha + O(r^{\alpha+2}))(-x_2, x_1, 0).$$

For  $0 < \alpha < 1$  this is not  $C^\infty$  at the axis. If the unweighted vector field is smooth there, multiplication by  $r^{2j}$ ,  $j = 0, 1, \dots$ , preserves smoothness; arbitrary noninteger powers require additional vanishing. Thus, for a generic profile with  $c(t) \neq 0$ , no parameter  $\alpha \in (0, 1)$  preserves the earlier unqualified smoothness claim without a modification of the profile. Our profile vanishes in a neighborhood of the axis and hence removes this obstruction for every parameter used here. Thus, the earlier unqualified smoothness assertion requires this correction. This note does not re-establish the higher-order assertions for  $k \geq 2$ .

There is a separate reason to exclude  $\alpha = 0$ . For the nonzero-tail profile used here, the outer formula at that value would give  $W = \beta(r - r^{-1})$ , and hence

$$\|v(t)\|_2^2 \asymp |\log(T - t)|, \quad \|\nabla v(t)\|_2^2 \asymp (T - t)^{-1} \quad (t \uparrow T).$$

Thus, axis smoothness at each  $t < T$  does not itself imply a uniform energy bound up to  $T$ . The strict range  $0 < \alpha < 1$  avoids this endpoint and still gives every force pair in (1.3).

## 1.2. COMPARISON WITH RELATED CONSTRUCTIONS

Galdi–Gazzola [6, Theorem 2.5] construct unique global Leray–Hopf solutions on  $\mathbb{R}^3$  with energy equality and countably many singular times for a broader force range. Their theorem includes  $L_t^1 L_x^2$ ; their Remark 3.8 treats Stokes. Consequently, these qualitative properties alone are not a novelty claim here. Table 2 records the distinctions. Their formulas (4.13) and (4.25) give exact velocity-norm and enstrophy rates. Our quantitative statements concern the present cylinder family, including its logarithmic endpoints and terminal trace.

The relevant qualitative overlap is already present in version v1 (13 June 2026): Theorems 4.5 and 4.10 give unique global Leray–Hopf solutions with energy equality and divergent  $L^3$  norms for  $f \in L_{\text{loc}}^{5/4}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \subset L_{\text{loc}}^1(\mathbb{R}_+; L^2(\mathbb{R}^3))$ . Their bounded  $L^2$  norms then imply  $L^\infty$  divergence through  $\|V(t)\|_3^3 \leq \|V(t)\|_\infty \|V(t)\|_2^2$ . The full finite-exponent statement of Theorem 2.5 is already present in version v2 (10 July 2026). Theorem 2.5 in the cited version v4 retains the force condition (2.11), uniqueness, energy equality and  $L^\infty$  blow-up conclusions of version v3 (21 July 2026). The detailed comparison below uses version v4 (15 August 2026) and its numbering.

The spatial endpoint  $p = 1$  is a distinction from that theorem, not a new existence phenomenon by itself: Zhang’s earlier cylinder construction [14] already treats  $L_t^q L_x^1$ . Here it is included in the exact classification of the annular weighted family.

Zhang’s critical-force preprint [13], version v3, Theorem 1.1(a), treats compactly supported solutions in general open domains with a force having a principal term of order  $-2$ . Theorem 1.1(b) gives an  $L_t^\infty L_x^{3/2}$  force on  $\mathbb{R}^3$ , and Theorem 1.2 prescribes an explicit small force in that class with zero initial data on  $\mathbb{R}^3$ . These are distinct

statements about the domains and force classes. Our contribution is the elementary bounded-cylinder construction, its axis-smoothness mechanism, and the complete finite-exponent classification of this particular family.<sup>1)</sup>

**Table 2**  
Comparison with [6, Theorem 2.5], in version v4

| Feature                       | Galdi–Gazzola                                                     | Present family                                                                             |
|-------------------------------|-------------------------------------------------------------------|--------------------------------------------------------------------------------------------|
| Domain and boundary           | $\mathbb{R}^3$ ; no boundary                                      | bounded cylinder; lateral no slip, horizontal free slip                                    |
| Force exponents               | in our notation, finite $p > 1$ , $q \geq 1$ with $2/q + 3/p > 2$ | finite $p, q \geq 1$ ; $1/p + 1/q > 1$ , including $p = 1$                                 |
| Singularity and time interval | global solution; countably many singular instants and points      | one prescribed terminal time; unbounded trace near the axis                                |
| Smoothness                    | smooth away from the singular set                                 | smooth on $\overline{D} \times [0, T']$ for every $T' < T$                                 |
| Quantitative information      | explicit norm calculations and rates                              | two-sided force and velocity rates, logarithmic endpoints, explicit nonzero terminal trace |

## 2. LERAY–HOPF SETTING AND PURE-SWIRL REDUCTION

Let

$$\begin{aligned} \mathcal{V} &= \{\psi \in C^\infty(\overline{D}; \mathbb{R}^3) : \nabla \cdot \psi = 0, \quad \psi|_{\Gamma_V} = 0, \quad \psi_3|_{\Gamma_H} = 0\}, \\ H &= \overline{\mathcal{V}}^{L^2(D)}, \quad V = \overline{\mathcal{V}}^{H^1(D)}. \end{aligned}$$

Here  $C^\infty(\overline{D})$  means restriction from a neighborhood of the closed cylinder. Thus, the smooth fields are compatible at the circular edges. The space  $V$  imposes only the essential boundary conditions; the tangential stress condition is natural. For a smooth solution  $u$ , it reduces on each flat horizontal face to  $\partial_{x_3} u_\tau = 0$ . The edges have surface measure zero in the boundary integrations.

For  $f \in L^1(0, T; L^2(D))$  and  $v_0 \in H$ , a Leray–Hopf solution is a field

$$v \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap C_w([0, T]; H), \quad v(0) = v_0,$$

satisfying the weak formulation below and the energy inequality

$$\frac{1}{2} \|v(t)\|_2^2 + \int_s^t \|\nabla v(\tau)\|_2^2 \, d\tau \leq \frac{1}{2} \|v(s)\|_2^2 + \int_s^t (f(\tau), v(\tau))_{L^2} \, d\tau$$

<sup>1)</sup> The annular construction and main integrability criteria already appeared in our preprint [3] (arXiv:2608.11553v1, 12 August 2026), under the original subtitle “exact mixed-norm ranges.” The revised subtitle clarifies that the exact ranges concern this explicit family.

for  $s = 0$ , for almost every  $s \in (0, T)$ , and for every  $t \in [s, T]$ .

For every

$$\varphi \in C^\infty([0, T] \times \overline{D}; \mathbb{R}^3), \quad \varphi(t) \in \mathcal{V} \quad (0 \leq t \leq T), \quad \varphi(T) = 0,$$

the weak formulation is

$$\begin{aligned} & - \int_0^T \int_D v \cdot \partial_t \varphi + \int_0^T \int_D \nabla v : \nabla \varphi - \int_0^T \int_D (v \otimes v) : \nabla \varphi \\ & = \int_D v_0 \cdot \varphi(0) + \int_0^T \int_D f \cdot \varphi. \end{aligned}$$

This closure-based definition supplies the spatial approximation used below. Indeed, the trace theorem preserves the essential conditions under  $H^1$  convergence, and the trilinear form  $b(a, b, c) = \int_D (a \cdot \nabla) b \cdot c$  satisfies

$$|b(a, b, c)| \leq \|a\|_4 \|\nabla b\|_2 \|c\|_4 \leq C \|a\|_{H^1} \|b\|_{H^1} \|c\|_{H^1}.$$

Consequently, the integration-by-parts identities for  $\mathcal{V}$  extend to  $V$ . In the uniqueness proof the comparison field itself belongs to  $\mathcal{V}$  at every time before  $T$ , so no identification with a different mixed-boundary Sobolev space or additional density theorem is needed.

Use cylindrical coordinates  $(r, \theta, x_3)$  and write

$$e_r = (\cos \theta, \sin \theta, 0), \quad e_\theta = (-\sin \theta, \cos \theta, 0).$$

Consider

$$v(x, t) = W(r, t)e_\theta, \quad f(x, t) = F(r, t)e_\theta. \quad (2.1)$$

Then  $\nabla \cdot v = 0$  and

$$(v \cdot \nabla)v = -\frac{W^2}{r}e_r.$$

If

$$P(r, t) = \int_0^r \frac{W(\ell, t)^2}{\ell} d\ell, \quad (2.2)$$

then the radial equation is satisfied, whereas the azimuthal equation becomes

$$\partial_t W - \left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \right) W = F. \quad (2.3)$$

A field of the form (2.1) is independent of  $x_3$ , has zero normal component on  $\Gamma_H$ , and satisfies the free-slip condition there. Thus, only the condition

$$W(1, t) = 0$$

remains to be imposed.

With (2.2),

$$(v \cdot \nabla)v + \nabla P = 0,$$

and hence  $\partial_t v - \Delta v = f$ . Thus,  $(v, \Pi, f)$  solves the corresponding Stokes system with  $\Pi \equiv 0$ . The Navier–Stokes pressure  $P$  in (2.2) is not retained in the Stokes equation.

**Remark 2.1** (Enstrophy). We use enstrophy to mean  $\|\omega(t)\|_{L^2(D)}^2$ , where  $\omega = \nabla \times v$  is the vorticity. For the pure-swirl family considered here,  $\omega = (0, 0, \partial_r W + W/r)$  and  $|\nabla v|^2 = |\partial_r W|^2 + |W|^2/r^2$ . Thus, for  $t < T$ ,

$$\|\omega(t)\|_2^2 - \|\nabla v(t)\|_2^2 = 4\pi \int_0^1 W(r, t) \partial_r W(r, t) \, dr = 2\pi [W(r, t)^2]_{r=0}^{r=1} = 0.$$

Here we used  $x_3$ -independence, smoothness at the axis, and the lateral no-slip condition, which give  $W(0, t) = W(1, t) = 0$ . Consequently, the gradient rates below are also enstrophy rates for this family with the stated mixed boundary conditions. The convention  $\frac{1}{2}\|\omega(t)\|_2^2$  changes only a constant and leaves all rates and integrability statements unchanged.

### 3. ANNULAR PROFILE AND CONSTRUCTION

Fix the prescribed  $T > 0$ . Choose

$$0 < a < b < \min \left\{ 1, \frac{1}{\sqrt{2T}} \right\} \quad (3.1)$$

and a nonnegative, nonzero function

$$A \in C_c^\infty((a, b)).$$

Set

$$k(\rho) = -e^{\rho^2/2} A'(\rho) \quad (3.2)$$

and

$$\phi_0(\rho) = -\frac{1}{\rho} \int_0^\rho s e^{s^2/2} A(s) \, ds, \quad \rho > 0, \quad (3.3)$$

with  $\phi_0(0) = 0$ . Finally, define

$$\beta = \int_0^\infty s e^{s^2/2} A(s) \, ds > 0. \quad (3.4)$$

Since

$$A(\rho) = \int_\rho^\infty e^{-\ell^2/2} k(\ell) \, d\ell, \quad \int_0^\infty e^{-\ell^2/2} k(\ell) \, d\ell = 0,$$

formula (3.3) is the profile formula used in Zhang's construction. The displayed cancellation, generated by the annular choice of  $A$ , forces the profile to vanish near the symmetry axis and is the key to smoothness for noninteger  $\alpha$ .

**Lemma 3.1.** *The function  $\phi_0$  belongs to  $C^\infty([0, \infty))$  and satisfies*

$$\phi_0'' + \frac{1}{\rho}\phi_0' - \frac{1}{\rho^2}\phi_0 - \phi_0 - \rho\phi_0' = k(\rho). \quad (3.5)$$

Moreover,

$$\phi_0(\rho) = 0 \quad (0 \leq \rho \leq a), \quad \phi_0(\rho) = -\frac{\beta}{\rho} \quad (\rho \geq b), \quad (3.6)$$

and

$$|\phi_0(\rho)| \leq C \frac{\rho}{1+\rho^2}, \quad |\phi_0'(\rho)| \leq \frac{C}{1+\rho^2}. \quad (3.7)$$

*Proof.* Let  $g(\rho) = \rho\phi_0(\rho)$ . From (3.3),

$$g'(\rho) = -\rho e^{\rho^2/2} A(\rho).$$

Using (3.2), direct differentiation gives

$$g'' - \left(\frac{1}{\rho} + \rho\right)g' = \rho k(\rho).$$

Substituting  $g = \rho\phi_0$  and dividing by  $\rho$  gives (3.5). Since  $A$  is supported in  $(a, b)$ , the integral in (3.3) vanishes for  $\rho \leq a$  and equals  $\beta$  for  $\rho \geq b$ . This proves (3.6) and smoothness across the joining points. The bounds follow from the exact inner and outer formulas and smoothness on  $[a, b]$ .  $\square$

For  $0 \leq t < T$ , write

$$R(t) = \sqrt{2(T-t)}$$

and define

$$\phi(r, t) = \frac{1}{R(t)}\phi_0\left(\frac{r}{R(t)}\right), \quad h(r, t) = \frac{1}{R(t)^3}k\left(\frac{r}{R(t)}\right). \quad (3.8)$$

A direct computation from (3.5) gives

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\phi - \partial_t\phi = h. \quad (3.9)$$

The profile bounds imply

$$|\phi(r, t)| \leq C \frac{r}{r^2 + T - t}, \quad |\partial_r\phi(r, t)| \leq \frac{C}{r^2 + T - t}. \quad (3.10)$$

Furthermore,

$$\phi(r, t) = h(r, t) = 0 \quad \text{for } 0 \leq r \leq aR(t), \quad (3.11)$$

and

$$\phi(r, t) = -\frac{\beta}{r}, \quad \partial_r\phi(r, t) = \frac{\beta}{r^2} \quad \text{for } r \geq bR(t). \quad (3.12)$$

## 4. CONSTRUCTION AND ESTIMATES

Fix  $0 < \alpha < 1$  and set

$$W(r, t) = r^\alpha \phi(r, t) + \beta r. \quad (4.1)$$

Define  $v = We_\theta$ , define  $P$  by (2.2), and set

$$F = -r^\alpha h - \alpha^2 r^{\alpha-2} \phi - 2\alpha r^{\alpha-1} \partial_r \phi, \quad f = Fe_\theta. \quad (4.2)$$

Indeed, the product rule and (3.9) yield

$$\partial_t W - \left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \right) W = F,$$

so Section 2 gives a solution of (1.1).

For each  $t < T$ , (3.11) gives  $W = \beta r$  and  $f = 0$  near the axis, so the noninteger factor  $r^\alpha$  causes no loss of smoothness. In Cartesian coordinates this reads  $v = \beta(-x_2, x_1, 0)$  near the axis. Moreover,  $x_3$ -independence gives the condition on  $\Gamma_H$ , while  $1/R(t) > b$  and (3.12) imply  $W(1, t) = 0$ . Thus,  $v, P, f$  are smooth on  $\bar{D} \times [0, T']$  for every  $T' < T$ , including the circular edges, and satisfy (1.2).

**Lemma 4.1** (Exact force rate). *For every  $1 \leq m < \infty$  there exist constants  $c_m, C_m > 0$  and  $t_m < T$  such that*

$$c_m(T-t)^{-\frac{3-\alpha}{2}+\frac{1}{m}} \leq \|f(t)\|_{L^m(D)} \leq C_m(T-t)^{-\frac{3-\alpha}{2}+\frac{1}{m}} \quad (4.3)$$

for  $t_m < t < T$ . Consequently, for this fixed profile and finite exponents  $1 \leq \sigma, m < \infty$ ,

$$f \in L^\sigma(0, T; L^m(D)) \iff \alpha > 3 - \frac{2}{m} - \frac{2}{\sigma}. \quad (4.4)$$

In particular,  $f \in L^1(0, T; L^2(D))$  for every  $0 < \alpha < 1$ .

*Proof.* Set  $s = T - t$  and

$$B_m = \frac{(3-\alpha)m-2}{2} > 0.$$

Because  $k$  is supported in  $(a, b)$ , cylindrical coordinates give

$$\int_D |r^\alpha h|^m \, dx \leq CR(t)^{-3m} \int_0^{bR(t)} r^{\alpha m+1} \, dr \leq Cs^{-B_m}. \quad (4.5)$$

By (3.10),

$$|\alpha^2 r^{\alpha-2} \phi + 2\alpha r^{\alpha-1} \partial_r \phi| \leq C \frac{r^{\alpha-1}}{r^2 + s}.$$

The left-hand side vanishes for  $r \leq aR(t)$ . Hence, after  $r = \sqrt{s}y$ ,

$$\int_D |\alpha^2 r^{\alpha-2} \phi + 2\alpha r^{\alpha-1} \partial_r \phi|^m \, dx \leq Cs^{-B_m} \int_{a\sqrt{2}}^\infty \frac{y^{1+\alpha m-m}}{(1+y^2)^m} \, dy \leq Cs^{-B_m}. \quad (4.6)$$

The final integral is finite because  $m(3 - \alpha) > 2$ , which follows from  $m \geq 1$  and  $\alpha < 1$ . This proves the upper bound in (4.3).

For the lower bound, fix  $r_0 \in (0, 1)$ . When  $bR(t) \leq r \leq r_0$ , (3.12) and (4.2) give the exact identity

$$F(r, t) = -\beta\alpha(2 - \alpha)r^{\alpha-3}. \quad (4.7)$$

Therefore,

$$\|f(t)\|_{L^m(D)}^m \geq c \int_{bR(t)}^{r_0} r^{1-m(3-\alpha)} dr \geq cs^{-B_m}$$

for  $t$  sufficiently close to  $T$ . This proves (4.3). The equivalence (4.4) follows by integrating  $s^{-\sigma((3-\alpha)/2-1/m)}$  near  $s = 0$ . At equality the integral diverges logarithmically. Taking  $(\sigma, m) = (1, 2)$  gives the last assertion.  $\square$

**Lemma 4.2** (Velocity and energy rates). *There exist  $c, C > 0$  and  $t_0 < T$  such that, for  $t_0 < t < T$ ,*

$$c(T - t)^{-\frac{1-\alpha}{2}} \leq \|v(t)\|_{L^\infty(D)} \leq C(T - t)^{-\frac{1-\alpha}{2}} \quad (4.8)$$

and

$$c(T - t)^{\alpha-1} \leq \|\nabla v(t)\|_{L^2(D)}^2 \leq C(T - t)^{\alpha-1}. \quad (4.9)$$

Moreover,

$$\sup_{0 \leq t < T} \|v(t)\|_{L^2(D)} < \infty. \quad (4.10)$$

*Proof.* From (3.10),

$$|W(r, t)| \leq C \frac{r^{\alpha+1}}{r^2 + s} + Cr, \quad s = T - t.$$

Writing  $r = \sqrt{s}y$  shows that

$$\sup_{r \in [0, 1]} \frac{r^{\alpha+1}}{r^2 + s} \leq Cs^{(\alpha-1)/2},$$

because  $y^{\alpha+1}/(1+y^2)$  is bounded for  $0 < \alpha < 1$ . This proves the upper bound in (4.8). For any fixed  $\rho_0 > b$ , put  $r(t) = \rho_0 R(t)$ . For  $t$  close to  $T$ ,  $r(t) < 1$ , and (3.12) yields

$$W(r(t), t) = -\beta\rho_0^{\alpha-1}R(t)^{\alpha-1} + \beta\rho_0 R(t).$$

The first term dominates, proving the lower bound.

For a pure swirl independent of  $x_3$ ,

$$|\nabla v|^2 = |\partial_r W|^2 + \frac{|W|^2}{r^2}. \quad (4.11)$$

The pointwise bounds on  $\phi$  and  $\partial_r \phi$  give

$$|\partial_r W| + \frac{|W|}{r} \leq C \frac{r^\alpha}{r^2 + s} + C.$$

Consequently,

$$\|\nabla v(t)\|_2^2 \leq C \int_0^1 \frac{r^{2\alpha+1}}{(r^2+s)^2} dr + C \leq Cs^{\alpha-1} \int_0^\infty \frac{y^{2\alpha+1}}{(1+y^2)^2} dy + C \leq Cs^{\alpha-1}.$$

For the reverse inequality, in the outer region one has

$$W(r, t) = \beta(r - r^{\alpha-1}).$$

Choosing a fixed small  $r_0 > 0$ , the term  $|W|/r$  is bounded below by  $cr^{\alpha-2}$  on  $bR(t) \leq r \leq r_0$  for all  $t$  close enough to  $T$ . Thus,

$$\|\nabla v(t)\|_2^2 \geq c \int_{bR(t)}^{r_0} r^{2\alpha-3} dr \geq cs^{\alpha-1},$$

which proves (4.9).

Finally,

$$\|v(t)\|_2^2 \leq C \int_0^1 \frac{r^{2\alpha+3}}{(r^2+s)^2} dr + C \quad (4.12)$$

$$= Cs^\alpha \int_0^{1/\sqrt{s}} \frac{y^{2\alpha+3}}{(1+y^2)^2} dy + C \leq C, \quad (4.13)$$

where the last estimate follows by splitting the  $y$ -integral at 1. This proves (4.10).  $\square$

**Lemma 4.3** (Exact mixed norms of the velocity). *For every  $1 \leq m < \infty$ , as  $t \uparrow T$ ,*

$$\|v(t)\|_{L^m(D)} \asymp \begin{cases} 1, & m(1-\alpha) < 2, \\ |\log(T-t)|^{1/m}, & m(1-\alpha) = 2, \\ (T-t)^{-\frac{1-\alpha}{2} + \frac{1}{m}}, & m(1-\alpha) > 2. \end{cases} \quad (4.14)$$

Consequently, for  $1 \leq m, \sigma < \infty$ ,

$$v \in L^\sigma(0, T; L^m(D)) \iff \alpha > 1 - \frac{2}{m} - \frac{2}{\sigma}. \quad (4.15)$$

Furthermore,

$$v \in L^\infty(0, T; L^m(D)) \iff \alpha > 1 - \frac{2}{m}, \quad 1 \leq m < \infty, \quad (4.16)$$

and, for  $1 \leq \sigma < \infty$ ,

$$v \in L^\sigma(0, T; L^\infty(D)) \iff \alpha > 1 - \frac{2}{\sigma}. \quad (4.17)$$

In particular, the terminal profile belongs to  $L^m(D)$  precisely when  $\alpha > 1 - 2/m$ ; in this case  $v(t) \rightarrow v_T$  strongly in  $L^m(D)$ .

*Proof.* Put  $s = T - t$ . The pointwise estimate used in the proof of Lemma 4.2 gives

$$\|v(t)\|_{L^m(D)}^m \leq C \int_0^1 \frac{r^{m(\alpha+1)+1}}{(r^2 + s)^m} dr + C. \quad (4.18)$$

After the change of variables  $r = \sqrt{s}y$ , the integral on the right is

$$s^{1-\frac{m(1-\alpha)}{2}} \int_0^{1/\sqrt{s}} \frac{y^{m(\alpha+1)+1}}{(1+y^2)^m} dy.$$

The large- $y$  behavior of the integrand is  $y^{1-m(1-\alpha)}$ . It follows that the right-hand side of (4.18) is bounded respectively by a constant, by  $C|\log s|$ , or by  $Cs^{1-m(1-\alpha)/2}$  according as  $m(1-\alpha) < 2$ ,  $m(1-\alpha) = 2$ , or  $m(1-\alpha) > 2$ .

For the reverse estimates, choose  $r_0 \in (0, 1)$  sufficiently small. In the outer region  $bR(t) \leq r \leq r_0$  one has

$$W(r, t) = \beta(r - r^{\alpha-1}),$$

and hence  $|W(r, t)| \geq cr^{\alpha-1}$  for all  $t$  sufficiently close to  $T$ . Therefore,

$$\|v(t)\|_{L^m(D)}^m \geq c \int_{bR(t)}^{r_0} r^{1-m(1-\alpha)} dr.$$

The three alternatives in (4.14) now follow by evaluating this elementary integral. Integrating the resulting power, or logarithm, in time proves (4.15). The endpoint statements (4.16) and (4.17) follow from (4.14) and (4.8), respectively. Finally,  $v_T(r) = \beta(r - r^{\alpha-1})e_\theta$  belongs to  $L^m(D)$  exactly when  $m(1-\alpha) < 2$ . In that case the domination  $|W(r, t)| \leq C(r^{\alpha-1} + r)$  yields strong  $L^m$  convergence by dominated convergence.  $\square$

The rate is realized on the scale  $r \asymp R(t)$ , whereas for fixed  $r > 0$  we have  $W(r, t) \rightarrow W_T(r) = \beta(r - r^{\alpha-1})$ ; thus the terminal loss of boundedness is concentrated on the symmetry axis.

**Proposition 4.4** (Final trace, energy equality and uniqueness). *The solution constructed above extends to  $v \in C([0, T]; L^2(D))$ , belongs to  $L^2(0, T; V)$ , satisfies (2.1) and the energy equality on  $[0, T]$ , and is the unique Leray–Hopf solution with the same data.*

*Proof.* For every fixed  $r > 0$ , the outer formula applies for all  $t$  sufficiently close to  $T$ , so

$$W(r, t) \longrightarrow W_T(r) := \beta(r - r^{\alpha-1}). \quad (4.19)$$

Moreover, (3.10) implies

$$|W(r, t)| \leq C(r^{\alpha-1} + r).$$

The right-hand side belongs to  $L^2(D)$  precisely because  $\alpha > 0$ . Dominated convergence gives

$$v(t) \longrightarrow v_T := W_T e_\theta \quad \text{strongly in } L^2(D). \quad (4.20)$$

Set  $v(T) = v_T$ . Smoothness before  $T$  and (4.20) imply  $v \in C([0, T]; L^2(D))$ . By (4.9) and  $\alpha > 0$ ,  $v \in L^2(0, T; V)$ .

For every  $t < T$ , multiplying (1.1) by  $v$  and integrating gives

$$\frac{1}{2} \|v(t)\|_2^2 + \int_0^t \|\nabla v(s)\|_2^2 \, ds = \frac{1}{2} \|v(0)\|_2^2 + \int_0^t \int_D f \cdot v \, dx \, ds. \quad (4.21)$$

The pressure and transport terms vanish by incompressibility and the normal boundary conditions. The viscous boundary term vanishes because  $v = 0$  on  $\Gamma_V$ , while  $\partial_{x_3} v = 0$  on  $\Gamma_H$ . The work term is absolutely integrable because

$$\int_0^T |(f(s), v(s))_{L^2}| \, ds \leq \|v\|_{L_t^\infty L_x^2} \|f\|_{L_t^1 L_x^2} < \infty.$$

The dissipation converges since  $\nabla v \in L_{t,x}^2$ , and the kinetic energy converges by (4.20). Passing  $t \uparrow T$  in (4.21) therefore proves the equality at  $T$ . Subtracting two such identities gives the energy equality on every subinterval of  $[0, T]$ .

For a test field from Section 2, the classical formulation on  $[0, t]$  contains the terminal term  $(v(t), \varphi(t))$ , which tends to zero since  $\varphi(T) = 0$ . The nonlinear integral converges, for example, by

$$\int_0^T \int_D |(v \otimes v) : \nabla \varphi| \leq T \|v\|_{L_t^\infty L_x^2}^2 \|\nabla \varphi\|_{L_{t,x}^\infty} < \infty.$$

The other terms converge by the energy bounds and the force integrability. This proves (2.1), so the extension is a Leray–Hopf solution.

Let  $u$  be another Leray–Hopf solution with the same data. Fix  $T' < T$  and set  $w = u - v$ . On  $[0, T']$  the field  $v$  is smooth up to the boundary. Testing the weak equation for  $u$  with a smooth time cutoff times  $v$  gives the cross identity for  $(u(t), v(t))$ . Inserting the classical equation for  $v$ , and combining this identity with the energy inequality for  $u$  and the energy equality for  $v$ , gives

$$\frac{1}{2} \|w(t)\|_2^2 + \int_0^t \|\nabla w(s)\|_2^2 \, ds \leq - \int_0^t b(w, v, w) \, ds \leq \int_0^t \|\nabla v(s)\|_\infty \|w(s)\|_2^2 \, ds. \quad (4.22)$$

Here the spatial integrations against  $u \in V$  follow by the closure and trilinear continuity described in Section 2; the time cutoffs converge using weak continuity. The boundary terms vanish on  $\Gamma_V$  because  $u = 0$ , and on  $\Gamma_H$  because  $u \cdot n = 0$  and  $\partial_{x_3} v = 0$ . Gronwall's lemma gives  $u = v$  almost everywhere on  $[0, T']$ . Weak continuity then

gives equality at every time in  $[0, T']$ . Since  $T' < T$  is arbitrary, equality holds for all  $t < T$ . For each  $\psi \in H$ ,

$$(u(T), \psi) = \lim_{t \uparrow T} (u(t), \psi) = \lim_{t \uparrow T} (v(t), \psi) = (v_T, \psi).$$

Thus,  $u(T) = v_T$ , which proves uniqueness on the closed interval.  $\square$

*Proof of Theorem 1.2.* Condition (1.3) is equivalent to

$$3 - \frac{2}{p} - \frac{2}{q} < 1.$$

Choose  $\alpha$  as in (1.8) and perform the construction in Sections 3, 4. The preceding construction gives a classical solution with the required boundary conditions. Applying Lemma 4.1 with  $(\sigma, m) = (q, p)$  gives  $f \in L_t^q L_x^p$ , while applying it with  $(\sigma, m) = (1, 2)$  gives  $f \in L_t^1 L_x^2$ . The rates and the loss of boundedness follow from Lemmas 4.1, 4.2, while Lemma 4.3 gives (1.12)–(1.13). Finally, Proposition 4.4 gives the Leray–Hopf extension, the energy equality and uniqueness.  $\square$

*Proof of Corollary 1.3.* For  $q = 1$ , condition (1.3) holds for every finite  $p \geq 1$ . On the displayed curve,

$$\frac{1}{p} + \frac{1}{q} = \frac{7}{4} - \frac{1}{2p} \geq \frac{5}{4} > 1 \quad (1 \leq p \leq 2),$$

so Theorem 1.2 applies, including at  $(p, q) = (2, 1)$ . For the last assertion choose

$$\max \left\{ 0, 3 - \frac{2}{p} - \frac{2}{q}, 1 - \frac{2}{m} - \frac{2}{\sigma} \right\} < \alpha < 1$$

and apply (1.12).  $\square$

## 5. CONCLUSION

For the annular weighted family with  $0 < \alpha < 1$ , the exact conditions for finite exponents  $p, q, m, \sigma \geq 1$  are

$$f \in L_t^q L_x^p \iff \alpha > 3 - \frac{2}{p} - \frac{2}{q}, \quad v \in L_t^\sigma L_x^m \iff \alpha > 1 - \frac{2}{m} - \frac{2}{\sigma}.$$

The velocity cases with one infinite exponent are stated separately in Lemma 4.3. These classifications, together with the two-sided rates and explicit terminal trace, give a quantitative description of this bounded-cylinder example. They do not characterize all possible forced singularities. The same velocity and force solve Stokes after the Navier–Stokes pressure is replaced by zero.

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*The authors used ChatGPT 5.6 for exploratory discussion and editorial feedback during the preparation of this manuscript. The authors are responsible for verifying all mathematical statements, proofs, and references and take full responsibility for the final manuscript.*

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