

Dedicated to Professor Makhmud Sadybekov
on the occasion of his 63rd birthday and retirement

MULTIPLICITY RESULTS FOR SUBELLIPTIC PROBLEMS WITH HARDY POTENTIAL ON STRATIFIED LIE GROUPS

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Abstract. In this paper, we study the existence and multiplicity of solutions to the following subelliptic problem with Hardy potential

$$\begin{cases} -\Delta_{\mathbb{G}}u(\xi) - \mu \frac{u(\xi)}{|x'|^2} = u^q(\xi) + \lambda u^r(\xi) & \text{in } \Omega, \\ u(\xi) > 0 & \text{in } \Omega, \\ u(\xi) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{G}$ is a bounded domain containing the origin with smooth boundary $\partial\Omega$, $0 < \mu < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$ with $N > 2$, and $\lambda > 0$. Here, the Hardy potential is defined on the first stratum of a stratified Lie group $\mathbb{G}$. Specifically, we have $|x'| = \sqrt{x_1^2 + \dots + x_N^2}$, $x' = (x_1, \dots, x_N) \in \mathbb{R}^N$, where N is the dimension of the first stratum of the stratified Lie group $\mathbb{G}$. Under the assumptions $1 < r+1 < 2 < q+1 < 2^* := \frac{2Q}{Q-2}$, we prove the existence of at least two positive solutions provided that λ is sufficiently small.

Keywords: subelliptic problem, Hardy potential, Palais–Smale condition, concentration-compactness principle.

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1. INTRODUCTION

A great deal of attention has been devoted to the study of the existence and non-existence of solutions to elliptic equations with Hardy potential, both in linear and nonlinear settings. This interest is driven by the potential's importance in physics, as it is closely related to Heisenberg's uncertainty principle in quantum mechanics. From a mathematical perspective, the Hardy potential given by the reciprocal of the square

function provides an excellent setting to study the existence of solutions to nonlinear partial differential equations, certain eigenvalue problems, and other related issues.

In a seminal paper [8] Brezis, Dupaigne, and Tesei studied the existence and nonexistence of solutions to the following equation in a ball $B_R = B(0, R)$ of $\mathbb{R}^n$, $n \geq 3$,

$$-\Delta u - \frac{\mu}{|x|^2} u = u^p,$$

where $1 < p < \frac{n+2}{n-2}$ and the coefficient μ satisfies the inequality $0 < \mu \leq \bar{\mu}$, with $\bar{\mu} := \frac{(n-2)^2}{4}$. Since then, many authors have considered similar problems involving the Hardy potential. We can refer to the following papers and the references therein, [1, 4, 11, 18, 19], among others.

In [17], Dupaigne considered the following equation

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^2} u = u^p + tf & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a smooth bounded open set of $\mathbb{R}^n$ ($n \geq 3$) containing the origin, $\mu > 0$, $p > 1$, $t > 0$ are constants, $0 < \mu \leq \bar{\mu} := \frac{(n-2)^2}{4}$ and $f \not\equiv 0$ is a smooth, bounded, nonnegative function. In equations like (1.1), the crucial role is played by the constant $\bar{\mu}$, which is the best constant in the Hardy inequality:

$$\int_{\Omega} |\nabla u|^2 \geq \bar{\mu} \int_{\Omega} \frac{u^2}{|x|^2} \quad \text{for all } u \in H_0^1(\Omega).$$

According to the result in [10, Proposition 5.2], if $\mu > \bar{\mu}$, then there is no nontrivial nonnegative solution to the problem

$$\begin{cases} -\Delta u - \frac{\mu}{|x|^2} u = f(x) & \text{in } \Omega \setminus \{0\}, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain containing the origin and f is a measurable nonnegative function.

In [24], Ghoussoub and Yuan considered the following problem with the Hardy–Sobolev potential:

$$\begin{cases} -\Delta_p u(x) - \mu \frac{|u|^{q-2} u(x)}{|x|^s} = \lambda |u|^{r-2} u & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_p is the p -Laplacian, Ω is a smooth bounded domain in $\mathbb{R}^n$ containing 0 in its interior, λ and μ are positive parameters. Under the assumptions $1 < p < n$, $0 \leq s \leq p$, $p \leq q \leq \frac{n-s}{n-p}p$ and $0 \leq s \leq p$, $p \leq r \leq p^* = \frac{np}{n-p}$ they obtained existence and multiplicity of weak solutions.

The study of subelliptic equations with Hardy potential on stratified Lie groups has received great interest in the last few years. Many results in this direction have been established regarding the existence of solutions to semilinear equations, beginning with the pioneering works on the Heisenberg group (see e.g., [25]).

In [13], Citti considered the following problem with the critical exponent

$$\begin{cases} -\Delta_{\mathbb{H}^n} u(\xi) + au(\xi) = u^{\frac{Q+2}{Q-2}}(\xi) & \text{in } \Omega, \\ u(\xi) = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where Ω is a smooth bounded domain in $\mathbb{H}^n$, $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, and $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian on $\mathbb{H}^n$. Since the equation (1.2) involves a nonlinearity of critical growth, Citti [13] established a representation formula for the Palais–Smale sequence and subsequently demonstrated the existence of positive solutions to (1.2) under appropriate assumptions on a . The reader is referred to [12, 14, 22, 27, 28] and the references therein for further reading on this topic.

In [26], Kassymov and Suragan considered the following semilinear problem

$$\begin{cases} -\Delta_{\mathbb{H}^n} u(\xi) - \lambda \frac{u(\xi)}{|z|^2} = u^p(\xi), & \xi = (z, t) \in \Omega \subset \mathbb{H}^n, \\ u(\xi) = 0, & \xi \in \partial\Omega, \end{cases} \quad (1.3)$$

where $|z| = \sqrt{x_1^2 + \dots + x_n^2 + y_1^2 + \dots + y_n^2}$, $z = (x, y) \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$, Ω is a smooth bounded domain in $\mathbb{R}^{2n+1}$ with smooth enough boundary $\partial\Omega$, and $0 < \lambda < \frac{(Q-4)^2}{4}$ with $Q > 4$. When $1 < p < \frac{Q+2}{Q-2}$, they proved the existence of at least two positive weak solutions to the problem (1.3).

Motivated by the above studies, we aim to extend these results to the sub-Laplacian operator on stratified Lie groups. Specifically, we consider the following semilinear subelliptic problem involving the sub-Laplacian and the Hardy potential associated with the first stratum of the stratified Lie group $\mathbb{G}$:

$$\begin{cases} -\Delta_{\mathbb{G}} u(\xi) - \mu \frac{u(\xi)}{|x'|^2} = u^q(\xi) + \lambda u^r(\xi) & \text{in } \Omega, \\ u(\xi) > 0 & \text{in } \Omega, \\ u(\xi) = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{G}$ is a bounded domain with smooth boundary $\partial\Omega$ containing the origin, $0 < \mu < \bar{\mu} := \left(\frac{N-2}{2}\right)^2$ with $N > 2$, and $\lambda > 0$. Moreover, $|x'| = \sqrt{x_1^2 + \dots + x_N^2}$, $x' = (x_1, \dots, x_N) \in \mathbb{R}^N$, with N being the dimension of the first stratum of a stratified Lie group $\mathbb{G}$.

The main existence result for the problem (1.4) is stated as follows.

Theorem 1.1. *Let $1 < r + 1 < 2 < q + 1 < 2^* := \frac{2Q}{Q-2}$. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ the problem (1.4) admits at least two positive solutions.*

To prove our main results, we employ a variational approach based on the first stratum Hardy inequality in the stratified Lie groups. For $1 < p < n$, the classical Hardy inequality in $\mathbb{R}^n$ takes the following form:

$$\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p dx, \quad u \in C_0^\infty(\mathbb{R}^n)$$

where the constant $\left(\frac{n-p}{p}\right)^p$ is sharp. Different versions of the Hardy inequality were established in the context of stratified Lie groups in [15, 16, 21, 29] and [34]. In particular, the following first stratum Hardy inequality on stratified Lie groups was proved in [31, Theorem 3.1]:

$$\frac{N-p}{p} \left\| \frac{f}{|x'|} \right\|_{L^p(\mathbb{G})} \leq \|\nabla_H f\|_{L^p(\mathbb{G})}, \quad \forall f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\}), \quad 1 < p < N, \quad (1.5)$$

where N is the dimension of the first stratum. We note that the first stratum Hardy inequality (1.5) is valid for the general class of stratified groups, including, in particular, the Heisenberg group and Euclidean space.

The paper is organized as follows. In Section 2, we recall the definition of a stratified Lie group, as well as the horizontal gradient and sub-Laplacian. In addition, we introduce the main inequalities and function spaces relevant to our study. In Section 3, we prove the main result of this paper, Theorem 1.1. The proof involves several steps. First, we introduce the variational framework for the problem (1.4), associating it with the corresponding Euler–Lagrange functional. Next, using the Sobolev–Folland–Stein embedding theorem along with the first stratum Hardy inequality, we show that this functional meets the assumptions of the mountain pass lemma and satisfies the Palais–Smale condition. The most crucial step is proving the weak lower semicontinuity of the truncated functional, for which we employ a concentration-compactness argument developed by Smets [36]. Finally, the proof of the main result follows by using the mountain pass lemma and a minimization argument. The other question is to show the positivity of solutions, which relies on the strong maximum principle. Some versions of the strong maximum principle have been proved, for example, in [33] for nonlinear sub-Laplacian equations on stratified Lie groups, and in [5] for fully nonlinear subelliptic equations involving vector fields that satisfy the Hörmander rank condition. We apply the strong maximum principle from [38], which is most relevant to our setting.

2. PRELIMINARIES

2.1. STRATIFIED LIE GROUPS

Let us briefly recall the definition of a stratified Lie group (homogeneous Carnot group). We refer to the recent books [6, 20, 32] for extensive studies on stratified Lie groups and their properties.

Definition 2.1. A Lie group $\mathbb{G} = (\mathbb{R}^m, \circ)$ is called a stratified Lie group (or a homogeneous Carnot group) if the following two conditions are satisfied:

- (a) For some natural numbers $N_1 + \dots + N_r = m$ the decomposition $\mathbb{R}^m = \mathbb{R}^{N_1} \times \dots \times \mathbb{R}^{N_r}$ holds, and for every $\lambda > 0$ there exists a dilation $\delta_\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^m$ given by

$$\delta_\lambda(\xi) = \delta_\lambda(x^{(1)}, \dots, x^{(r)}) := (\lambda x^{(1)}, \dots, \lambda^r x^{(r)}),$$

which is an automorphism of the group $\mathbb{G}$. Here, $\xi = (x^{(1)}, \dots, x^{(r)}) \in \mathbb{R}^m$ with $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 1, \dots, r$.

- (b) Let $X_1, \dots, X_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k} \Big|_0$ for $k = 1, \dots, N_1$ with N_1 as in the above decomposition of $\mathbb{R}^m$. Then the Hörmander rank condition

$$\text{rank}(\text{Lie}\{X_1, \dots, X_{N_1}\}) = m$$

holds for every $\xi \in \mathbb{R}^m$. In other words, the iterated commutators of $X_1, \dots, X_{N_1}$ span the whole Lie algebra of $\mathbb{G}$.

In Definition 2.1, r is called the step of the stratified Lie group $\mathbb{G}$. The number

$$Q := \sum_{k=1}^r k N_k$$

is called the homogeneous dimension of $\mathbb{G}$.

Throughout this paper, we denote the dimension of the first stratum by $N := N_1$, and express its coordinates as $x' := x^{(1)} = (x_1, \dots, x_N) \in \mathbb{R}^N$.

The horizontal gradient on $\mathbb{G}$ is given by

$$\nabla_H := (X_1, \dots, X_N).$$

For any C^1 vector field $V = (V_1, V_2, \dots, V_N)$, the horizontal divergence is defined by

$$\text{div}_H V := \nabla_H \cdot V = \sum_{k=1}^N X_k V_k.$$

Then, the sub-Laplacian (or horizontal Laplacian) on $\mathbb{G}$ is defined by

$$\Delta_{\mathbb{G}} := \nabla_H \cdot \nabla_H = \sum_{k=1}^N X_k^2.$$

We denote by $d\xi$ the Haar measure on $\mathbb{G}$, which is a usual Lebesgue measure on $\mathbb{R}^m$. For any Haar measurable subset $\Omega \subset \mathbb{G}$, we have $|\delta_\lambda(\Omega)| = \lambda^Q |\Omega|$, where $|\Omega|$ is the Haar measure of Ω .

2.2. HEISENBERG GROUP

Let us recall the definition of the Heisenberg group. For a complete treatment, we refer to the book [9]. The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$ is the simplest example of a stratified Lie group endowed with the non-Abelian group law

$$\xi \circ \xi' = (x, y, t) \circ (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2} (x \cdot y' - x' \cdot y) \right)$$

defined for any $\xi, \xi' \in \mathbb{H}^n$. The left-invariant vector fields

$$X_j = \partial_{x_j} - \frac{y_j}{2} \partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2} \partial_t, \quad T = \partial_t, \quad 1 \leq j \leq n$$

form a canonical basis for the Lie algebra $\mathfrak{h}_n$ of the Heisenberg group $\mathbb{H}^n$. They satisfy the commutation relations $[X_i, Y_j] = \delta_{ij} T$ and $[X_i, T] = [Y_i, T] = 0$, $i, j = 1, 2, \dots, n$, while all commutators of length greater than two vanish. Hence, $\mathfrak{h}_n$ admits the stratification

$$\mathfrak{h}_n = \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where $\mathfrak{g}_1 = \text{span}\{X_j, Y_j\}_{j=1}^n$, and $\mathfrak{g}_2 = \text{span}\{T\}$. Therefore, $\mathbb{H}^n$ is a stratified Lie group of step two. The corresponding group dilations are defined as

$$\delta_\lambda(\xi) := (\lambda x, \lambda y, \lambda^2 t), \quad \forall \lambda > 0.$$

The homogeneous dimension of $\mathbb{H}^n$ is $Q := 2n + 2$.

Consider a horizontal vector field $V \in \text{span}\{X_j, Y_j\}_{j=1}^n$. Then the horizontal divergence of V is defined as

$$\text{div}_H V := \sum_{j=1}^n (X_j V_j + Y_j V_{n+j}).$$

The horizontal gradient of a C^1 function $u : \mathbb{H}^n \rightarrow \mathbb{R}$ is defined by

$$\nabla_H u := (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u).$$

The sub-Laplacian (Kohn—Spencer Laplacian) of $u \in C^2(\mathbb{H}^n)$ is defined by

$$\Delta_{\mathbb{H}^n} u := \sum_{j=1}^n (X_j^2 + Y_j^2) u.$$

2.3. INEQUALITIES AND FUNCTION SPACES ON STRATIFIED LIE GROUPS

Let us now provide a brief overview of the main tools.

For $p \in (1, +\infty)$, the Folland–Stein–Sobolev space $S_0^{1,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{S_0^{1,p}(\Omega)} := \left(\int_{\Omega} |\nabla_H u(\xi)|^p d\xi \right)^{\frac{1}{p}}.$$

By the result of [23], if Ω is a bounded domain in $\mathbb{G}$, then the embedding

$$S_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega) \quad (2.1)$$

is compact for all $r \in [1, p^*)$, and continuous for all $r \in [1, p^*]$, where $p^* := \frac{pQ}{Q-p}$ is the critical Sobolev exponent.

In particular, for $p = 2$, we denote the space as $S_0^1(\Omega) := S_0^{1,2}(\Omega)$ and its norm simply as

$$\|u\| := \|u\|_{S_0^1(\Omega)} = \left(\int_{\Omega} |\nabla_H u(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

In this case, the embedding

$$S_0^1(\Omega) \hookrightarrow L^s(\Omega) \quad (2.2)$$

is continuous for all $s \in [1, 2^*]$ and compact for all $s \in [1, 2^*)$, where $2^* := \frac{2Q}{Q-2}$.

Consider the Euclidean distance on the first stratum of a stratified Lie group, defined by

$$|x'| = \sqrt{x_1'^2 + \dots + x_N'^2}.$$

Then we have the first stratum L^p -Caffarelli–Kohn–Nirenberg inequality on stratified Lie groups.

Theorem 2.2 ([31, Theorem 3.1]). *Let $\mathbb{G}$ be a homogeneous stratified group with N being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$, and all $1 < p < \infty$, we have*

$$\frac{|N - \gamma|}{p} \left\| \frac{f}{|x'|^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{G})}^p \leq \left\| \frac{1}{|x'|^\alpha} \nabla_H f \right\|_{L^p(\mathbb{G})} \left\| \frac{f}{|x'|^{\frac{\beta}{p-1}}} \right\|_{L^p(\mathbb{G})}^{p-1}, \quad (2.3)$$

where $\gamma = \alpha + \beta + 1$ and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. If $\gamma \neq N$ then the constant $\frac{|N-\gamma|}{p}$ is sharp.

In particular, when $\alpha = 0$, $\beta = p - 1$, and $1 < p < N$, inequality (2.3) reduces to the following first stratum Hardy inequality

$$\int_{\mathbb{G}} \frac{|f|^p}{|x'|^p} d\xi \leq \frac{1}{\bar{\mu}} \int_{\mathbb{G}} |\nabla_H f|^p d\xi, \quad \forall f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\}) \quad (2.4)$$

with $\bar{\mu} := \left(\frac{N-p}{p}\right)^p$ being the best constant.

We recall the following strong maximum principle from [38, Theorem 5]. Consider the following problem

$$-\operatorname{div}_* \mathcal{A}(x, \nabla_{\mathcal{L}} u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^m, \quad (2.5)$$

where $\nabla_{\mathcal{L}} u = (X_1 u, X_2 u, \dots, X_N u)$, and $X_1, X_2, \dots, X_N$ are C^∞ -vector fields in $\mathbb{R}^m$ satisfying the Hörmander hypoellipticity condition. Note that the sub-Laplacian on $\mathbb{G}$ is a particular case of the operator $\operatorname{div}_* \mathcal{A}(x, \nabla_{\mathcal{L}} u)$; see [38, p. 248].

Theorem 2.3. *If u is a nonnegative supersolution of equation (2.5) in the domain Ω , then the inequality*

$$\left(\int_B u^s d\mu \right)^{1/s} \leq c_2 \operatorname{ess\,inf}_{\frac{1}{2}B} u$$

is valid for every $0 < s < s_0(p, \beta/\alpha, C_\mu)$ and every ball B ($2B \subset \Omega$). Here c_2 is a positive constant depending on the domain Ω and the constants $p, \beta/\alpha, C_\mu$, and s .

3. PROOF OF THEOREM 1.1

In this section, we prove the main result given by Theorem 1.1. Taking into account that we are looking for positive solutions, we consider the following problem

$$\begin{cases} -\Delta_{\mathbb{G}} u(\xi) - \mu \frac{u_+(\xi)}{|x'|^2} = u_+^q(\xi) + \lambda u_+^r(\xi) & \text{in } \Omega, \\ u(\xi) = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where $u_+ := \max\{u, 0\}$. We will also denote $u_- := \min\{u, 0\}$. Note that if the solutions of (3.1) are strictly positive in Ω , then they also satisfy problem (1.4).

Definition 3.1. We say that $u \in S_0^1(\Omega)$ is a weak solution of (3.1) if and only if

$$\int_{\Omega} \nabla_H u \cdot \nabla_H \varphi d\xi - \mu \int_{\Omega} \frac{u_+ \varphi}{|x'|^2} d\xi - \int_{\Omega} u_+^q \varphi d\xi - \lambda \int_{\Omega} u_+^r \varphi d\xi = 0$$

for any function $\varphi \in S_0^1(\Omega)$.

We define the Euler–Lagrange functional $I_{\mu, \lambda} : S_0^1(\Omega) \rightarrow \mathbb{R}$, associated to the problem (3.1), as

$$I_{\mu, \lambda}(u) := \frac{1}{2} \int_{\Omega} |\nabla_H u|^2 d\xi - \frac{\mu}{2} \int_{\Omega} \frac{u_+^2}{|x'|^2} d\xi - \frac{1}{q+1} \int_{\Omega} u_+^{q+1} d\xi - \frac{\lambda}{r+1} \int_{\Omega} u_+^{r+1} d\xi. \quad (3.2)$$

Note that $I_{\mu, \lambda} \in C^1(S_0^1(\Omega), \mathbb{R})$, and we have

$$\langle I'_{\mu, \lambda}(u), \varphi \rangle = \int_{\Omega} \nabla_H u \cdot \nabla_H \varphi d\xi - \mu \int_{\Omega} \frac{u_+ \varphi}{|x'|^2} d\xi - \int_{\Omega} u_+^q \varphi d\xi - \lambda \int_{\Omega} u_+^r \varphi d\xi$$

for any $u, \varphi \in S_0^1(\Omega)$.

From the above definitions, it follows that $u \in S_0^1(\Omega)$ is a weak solution to the problem (3.1) if and only if $\langle I'_{\mu, \lambda}(u), \varphi \rangle = 0$ for all $\varphi \in S_0^1(\Omega)$. In other words, the critical points of the functional $I_{\mu, \lambda}$ correspond to weak solutions of the problem (3.1).

Let us now recall the definition of a Palais–Smale sequence.

Definition 3.2. Let E be a Banach space. A sequence $\{u_k\} \subset E$ is a Palais–Smale sequence for a functional $F \in C^1(E, \mathbb{R})$ at some level $c \in \mathbb{R}$, if

$$F(u_k) \rightarrow c \quad \text{as } k \rightarrow +\infty,$$

and

$$F'(u_k) \rightarrow 0 \quad \text{as } k \rightarrow +\infty \text{ in } E^*,$$

where E^* is the dual space of E .

We will use the following version of the classical mountain pass lemma [2] (see also [3]).

Lemma 3.3. Let F be a functional on a Banach space E , $F \in C^1(E, \mathbb{R})$. Assume that there exist positive constants $\rho, \alpha > 0$, such that

- (i) $F(u) \geq \alpha$, $\forall u \in E$ with $\|u\| = \rho$,
- (ii) $F(0) = 0$, and there exists $v \in E$ with $\|v\| > \rho$ such that $F(v) < \alpha$.

Define the set of paths joining $u = 0$ with $u = v$:

$$\Gamma := \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = v\}$$

and set

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} F(\gamma(t)).$$

Then, there exists a sequence $\{u_k\} \subset E$, such that $F(u_k) \rightarrow c$, and $F'(u_k) \rightarrow 0$ in E^* (dual of E).

Now we show that the functional $I_{\mu, \lambda}$ defined in (3.2) verifies the assumptions of Lemma 3.3.

Lemma 3.4. Let Ω be a bounded domain in $\mathbb{G}$. Then there exist positive constants $\rho, \alpha > 0$ such that the following assertions hold:

- (i) there exists $\lambda^* > 0$ such that $I_{\mu, \lambda}(u) \geq \alpha$ for all $\lambda \in (0, \lambda^*)$ and all $u \in S_0^1(\Omega)$ with $\|u\| = \rho$,
- (ii) for any $\lambda > 0$ there exists $v \in S_0^1(\Omega)$ positive such that $\|v\| > \rho$ and $I_{\mu, \lambda}(v) < \alpha$.

Proof. (i) Since $1 < r + 1 < 2$, applying the embedding (2.2), we obtain

$$\|u\|_{L^{r+1}(\Omega)} \leq C\|u\|. \quad (3.3)$$

Similarly, for $q + 1 \in (2, 2^*)$, we have

$$\|u\|_{L^{q+1}(\Omega)} \leq C\|u\|. \quad (3.4)$$

Then, using the inequalities (3.3), (3.4), and the first stratum Hardy inequality (2.4) with $p = 2$, we get

$$\begin{aligned}
 I_{\mu,\lambda}(u) &= \frac{1}{2} \int_{\Omega} |\nabla_H u|^2 d\xi - \frac{\mu}{2} \int_{\Omega} \frac{u_+^2}{|x'|^2} d\xi - \frac{1}{q+1} \int_{\Omega} u_+^{q+1} d\xi - \frac{\lambda}{r+1} \int_{\Omega} u_+^{r+1} d\xi \\
 &\geq \frac{1}{2} \|u\|^2 - \frac{\mu}{2\bar{\mu}} \|u\|^2 - \frac{1}{q+1} \int_{\Omega} u_+^{q+1} d\xi - \frac{\lambda}{r+1} \int_{\Omega} u_+^{r+1} d\xi \\
 &\geq \left(\frac{1}{2} - \frac{\mu}{2\bar{\mu}} \right) \|u\|^2 - \frac{C_2}{q+1} \|u\|^{q+1} - \lambda \frac{C_3}{r+1} \|u\|^{r+1} \\
 &= \frac{C_1}{2} \|u\|^2 - \frac{C_2}{q+1} \|u\|^{q+1} - \lambda \frac{C_3}{r+1} \|u\|^{r+1},
 \end{aligned} \tag{3.5}$$

where $C_1 = 1 - \frac{\mu}{\bar{\mu}} > 0$, and $C_2, C_3 > 0$. Since $1 < r+1 < 2 < q+1$, we can choose a radius $\rho > 0$ and a sufficiently small threshold $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, we have

$$\alpha := \frac{C_1 \rho^2}{2} - \frac{C_2 \rho^{q+1}}{q+1} - \lambda \frac{C_3 \rho^{r+1}}{r+1} > 0.$$

Consequently, for any $u \in S_0^1(\Omega)$ with $\|u\| = \rho$, we have $I_{\mu,\lambda}(u) \geq \alpha > 0$.

(ii) Fix $v_0 \in S_0^1(\Omega)$ positive such that $\|v_0\| = 1$ and take $t > 0$. Then we have

$$\begin{aligned}
 I_{\mu,\lambda}(tv_0) &= \frac{1}{2} \|tv_0\|^2 - \frac{\mu}{2} \int_{\Omega} \frac{(tv_0)^2}{|x'|^2} d\xi - \frac{1}{q+1} \int_{\Omega} (tv_0)^{q+1} d\xi - \frac{\lambda}{r+1} \int_{\Omega} (tv_0)^{r+1} d\xi \\
 &\leq \frac{1}{2} \|tv_0\|^2 - \frac{1}{q+1} \int_{\Omega} (tv_0)^{q+1} d\xi \\
 &= \frac{t^2}{2} - \frac{t^{q+1}}{q+1} \int_{\Omega} v_0^{q+1} d\xi.
 \end{aligned}$$

Since $2 < q+1$, we get $I_{\mu,\lambda}(tv_0) \rightarrow -\infty$ as $t \rightarrow +\infty$. Therefore, by setting $v = t_0 v_0$, with t_0 sufficiently large, it follows that $\|v\| > \rho$ and $I_{\mu,\lambda}(v) < \alpha$. $\square$

According to Lemma 3.3, Lemma 3.4 ensures the existence of a Palais–Smale sequence for the functional $I_{\mu,\lambda}$. To obtain the existence of a critical point, we must additionally verify that the functional $I_{\mu,\lambda}$ satisfies the Palais–Smale condition. We say that a functional satisfies the Palais–Smale condition if every Palais–Smale sequence of this functional admits a convergent subsequence.

Next, we show that the functional $I_{\mu,\lambda}$ defined in (3.2) satisfies the Palais–Smale condition.

Lemma 3.5. *Let $\{u_k\}$ be a Palais–Smale sequence at level $c \in \mathbb{R}$ for a functional $I_{\mu,\lambda}$ defined in (3.2). Then there exists $u \in S_0^1(\Omega)$ such that, up to a subsequence, we have*

$$\lim_{k \rightarrow +\infty} \|u_k - u\| = 0.$$

Proof. Let $\{u_k\}$ be a Palais–Smale sequence at level $c \in \mathbb{R}$ for the functional $I_{\mu,\lambda}$. Then, by Definition 3.2 we have

$$I_{\mu,\lambda}(u_k) \rightarrow c \quad (3.6)$$

and

$$I'_{\mu,\lambda}(u_k) \rightarrow 0 \text{ in } (S_0^1(\Omega))^* \quad (3.7)$$

as $k \rightarrow +\infty$.

First, we show that the sequence $\{u_k\}$ is bounded in $S_0^1(\Omega)$. Using inequality (2.4) with $p = 2$ and the embedding (2.2) for $r + 1 \in (1, 2)$, we get

$$\begin{aligned} & I_{\mu,\lambda}(u_k) - \frac{1}{q+1} \langle I'_{\mu,\lambda}(u_k), u_k \rangle \\ &= \left(\frac{1}{2} - \frac{1}{q+1} \right) \|u_k\|^2 - \mu \left(\frac{1}{2} - \frac{1}{q+1} \right) \int_{\Omega} \frac{(u_k)_+^2}{|x'|^2} d\xi \\ &\quad - \lambda \left(\frac{1}{r+1} - \frac{1}{q+1} \right) \int_{\Omega} (u_k)_+^{r+1} d\xi \\ &\geq \left(\frac{1}{2} - \frac{1}{q+1} \right) \|u_k\|^2 - \frac{\mu}{\bar{\mu}} \left(\frac{1}{2} - \frac{1}{q+1} \right) \|u_k\|^2 - \lambda \left(\frac{1}{r+1} - \frac{1}{q+1} \right) C_3 \|u_k\|^{r+1} \\ &= \left(\frac{1}{2} - \frac{1}{q+1} \right) \left(1 - \frac{\mu}{\bar{\mu}} \right) \|u_k\|^2 - \lambda \left(\frac{1}{r+1} - \frac{1}{q+1} \right) C_3 \|u_k\|^{r+1} \\ &= C_4 \|u_k\|^2 - C_5 \|u_k\|^{r+1}, \end{aligned}$$

with $C_4, C_5 > 0$. On the other hand, for k sufficiently large, (3.6) and (3.7) imply that

$$I_{\mu,\lambda}(u_k) - \frac{1}{q+1} \langle I'_{\mu,\lambda}(u_k), u_k \rangle \leq c + 1 + \|u_k\|.$$

Combining the estimates above, we obtain $C_4 \|u_k\|^2 - C_5 \|u_k\|^{r+1} \leq c + 1 + \|u_k\|$. Since $r + 1 < 2$, we conclude that $\|u_k\| \leq C$ for some constant $C > 0$. Hence, the sequence $\{u_k\}$ is bounded in $S_0^1(\Omega)$. Consequently, since $S_0^1(\Omega)$ is a reflexive space, there exists $u \in S_0^1(\Omega)$ such that, up to a subsequence,

$$u_k \rightharpoonup u \text{ in } S_0^1(\Omega) \quad \text{as } k \rightarrow +\infty, \quad (3.8)$$

and the compact embedding in (2.2) yields

$$u_k \rightarrow u \text{ in } L^s(\Omega) \quad 1 \leq s < 2^*, \quad u_k \rightarrow u \text{ a.e. in } \Omega \quad \text{as } k \rightarrow +\infty. \quad (3.9)$$

Using (3.7) and the boundedness of the sequence $\{u_k\}$ in $S_0^1(\Omega)$, we deduce that

$$\langle I'_{\mu,\lambda}(u_k), u_k - u \rangle \rightarrow 0$$

as $k \rightarrow +\infty$. Moreover, using (3.8), it follows that

$$\langle I'_{\mu,\lambda}(u), u_k - u \rangle \rightarrow 0$$

as $k \rightarrow +\infty$.

Hence,

$$\langle I'_{\mu,\lambda}(u_k) - I'_{\mu,\lambda}(u), u_k - u \rangle = o(1) \quad (3.10)$$

as $k \rightarrow +\infty$.

On the other hand, we have

$$\begin{aligned} \langle I'_{\mu,\lambda}(u_k) - I'_{\mu,\lambda}(u), u_k - u \rangle &= \|u_k - u\|^2 - \mu \int_{\Omega} \frac{((u_k)_+ - u_+)(u_k - u)}{|x'|^2} d\xi \\ &\quad - \int_{\Omega} ((u_k)_+^q - u_+^q)(u_k - u) d\xi - \lambda \int_{\Omega} ((u_k)_+^r - u_+^r)(u_k - u) d\xi. \end{aligned} \quad (3.11)$$

Applying Hölder's inequality with exponents $\frac{q+1}{q}$ and $q+1$ and (3.9) with $q+1 < 2^*$ yields

$$\left| \int_{\Omega} ((u_k)_+^q - u_+^q)(u_k - u) d\xi \right| \leq (\|(u_k)_+\|_{L^{q+1}}^q + \|u_+\|_{L^{q+1}}^q) \|u_k - u\|_{L^{q+1}} \rightarrow 0 \quad (3.12)$$

as $k \rightarrow +\infty$. Using the same argument, we also obtain

$$\left| \int_{\Omega} ((u_k)_+^r - u_+^r)(u_k - u) d\xi \right| \leq (\|(u_k)_+\|_{L^{r+1}}^r + \|u_+\|_{L^{r+1}}^r) \|u_k - u\|_{L^{r+1}} \rightarrow 0 \quad (3.13)$$

as $k \rightarrow +\infty$.

Note that $|(u_k)_+ - u_+| \leq |u_k - u|$ pointwise a.e. in Ω . Using this, together with the Cauchy-Schwarz inequality and the first stratum Hardy inequality (2.4) with $p = 2$, it follows that

$$\begin{aligned} \left| \int_{\Omega} \frac{((u_k)_+ - u_+)(u_k - u)}{|x'|^2} d\xi \right| &\leq \left(\int_{\Omega} \frac{((u_k)_+ - u_+)^2}{|x'|^2} d\xi \right)^{\frac{1}{2}} \left(\int_{\Omega} \frac{(u_k - u)^2}{|x'|^2} d\xi \right)^{\frac{1}{2}} \\ &\leq \int_{\Omega} \frac{(u_k - u)^2}{|x'|^2} d\xi \leq \frac{1}{\bar{\mu}} \|u_k - u\|^2. \end{aligned} \quad (3.14)$$

Combining (3.10)–(3.14), we get

$$o(1) \geq \|u_k - u\|^2 - \frac{\mu}{\bar{\mu}} \|u_k - u\|^2 + o(1) = \left(1 - \frac{\mu}{\bar{\mu}}\right) \|u_k - u\|^2 + o(1),$$

and since $1 - \mu/\bar{\mu} > 0$ we conclude $\|u_k - u\| \rightarrow 0$ as $k \rightarrow +\infty$. This completes the proof. $\square$

We now define the functional $\mathcal{H}_{\mu,p} : S_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ as follows

$$\mathcal{H}_{\mu,p}(u) := \int_{\Omega} |\nabla_H u|^p d\xi - \mu \int_{\Omega} \frac{|u|^p}{|x'|^p} d\xi. \quad (3.15)$$

Before proving the main result, we first need to show that the functional $\mathcal{H}_{\mu,p}$ is weakly lower semicontinuous on $S_0^{1,p}(\Omega)$. For this purpose, we employ the concentration-compactness argument based on the decomposition technique (see, e.g., Smets [36] or Willem [39]). In the setting of the Heisenberg group, certain versions of the concentration-compactness principle related to the Hardy–Sobolev embedding were established by Bordonì *et al.* in [7] for bounded domains, and by Pucci and Temperini in [30] for the entire group $\mathbb{H}^n$. However, the following result differs slightly from these, as we consider the first-stratum Hardy potential on a general stratified Lie group $\mathbb{G}$. We have the following decomposition lemma, which follows the arguments given in the Euclidean setting in [35].

Lemma 3.6. *Let $\{u_k\} \subset S_0^{1,p}(\Omega)$ be such that $u_k \rightharpoonup u$ in $S_0^{1,p}(\Omega)$ as $k \rightarrow +\infty$. Let us define*

$$\begin{aligned} \Omega_R &:= \{x \in \Omega : |x'| < R^{-1}\}, \\ \mu_0 &:= \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H u_k|^p d\xi, \\ \nu_0 &:= \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} \frac{|u_k|^p}{|x'|^p} d\xi. \end{aligned}$$

Then we have

$$\overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} \frac{|u_k|^p}{|x'|^p} d\xi = \int_{\Omega} \frac{|u|^p}{|x'|^p} d\xi + \nu_0, \quad (3.16)$$

$$\overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H u_k|^p d\xi \geq \int_{\Omega} |\nabla_H u|^p d\xi + \mu_0, \quad (3.17)$$

$$\bar{\mu}\nu_0 \leq \mu_0, \quad (3.18)$$

where $\bar{\mu}$ is the best constant in inequality (2.4).

Proof. Since $u_k \rightarrow u$ in $L^p(\Omega \setminus \Omega_R)$ by the compact embedding (2.2), and $1/|x'|$ is bounded in $\Omega \setminus \Omega_R$, we have, for every $R > 0$,

$$\lim_{k \rightarrow +\infty} \int_{\Omega \setminus \Omega_R} \frac{|u_k|^p}{|x'|^p} d\xi = \int_{\Omega \setminus \Omega_R} \frac{|u|^p}{|x'|^p} d\xi.$$

Splitting $\Omega = (\Omega \setminus \Omega_R) \cup \Omega_R$ we obtain, for every $R > 0$,

$$\begin{aligned} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} \frac{|u_k|^p}{|x'|^p} d\xi &= \lim_{k \rightarrow +\infty} \int_{\Omega \setminus \Omega_R} \frac{|u_k|^p}{|x'|^p} d\xi + \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} \frac{|u_k|^p}{|x'|^p} d\xi \\ &= \int_{\Omega \setminus \Omega_R} \frac{|u|^p}{|x'|^p} d\xi + \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} \frac{|u_k|^p}{|x'|^p} d\xi. \end{aligned}$$

Therefore, taking the limit as $R \rightarrow +\infty$ yields (3.16).

Since the mapping $u \mapsto |\nabla_H u|^p$ is convex, the functional $\int_{\Omega \setminus \Omega_R} |\nabla_H u|^p d\xi$ is weakly lower semicontinuous. Therefore, using the property of the limit superior, we get

$$\begin{aligned} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H u_k|^p d\xi &\geq \underline{\lim}_{k \rightarrow +\infty} \int_{\Omega \setminus \Omega_R} |\nabla_H u_k|^p d\xi + \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H u_k|^p d\xi, \\ &\geq \int_{\Omega \setminus \Omega_R} |\nabla_H u|^p d\xi + \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H u_k|^p d\xi. \end{aligned}$$

Then, taking the limit as $R \rightarrow +\infty$, we obtain (3.17).

Next, we prove (3.18). Using the fact that for each $\varepsilon > 0$, there exists a constant $c(\varepsilon, p) > 0$ such that

$$||a + b|^p - |a|^p| \leq \varepsilon |a|^p + c(\varepsilon, p) |b|^p, \quad (3.19)$$

we can verify that

$$\mu_0 = \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H(u_k - u)|^p d\xi, \quad (3.20)$$

$$\nu_0 = \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} \frac{|u_k - u|^p}{|x'|^p} d\xi. \quad (3.21)$$

Indeed, if we take $a = \nabla_H(u_k - u)$ and $b = \nabla_H u$ in (3.19), so that $a + b = \nabla_H u_k$, then we have

$$||\nabla_H u_k|^p - |\nabla_H(u_k - u)|^p| \leq \varepsilon |\nabla_H(u_k - u)|^p + c(\varepsilon, p) |\nabla_H u|^p. \quad (3.22)$$

Using this inequality and integrating over Ω_R , we derive

$$\int_{\Omega_R} |\nabla_H u_k|^p d\xi \leq (1 + \varepsilon) \int_{\Omega_R} |\nabla_H(u_k - u)|^p d\xi + c(\varepsilon, p) \int_{\Omega_R} |\nabla_H u|^p d\xi.$$

Taking the $\lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty}$ on both sides, we get

$$\lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H u_k|^p d\xi \leq (1 + \varepsilon) \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H(u_k - u)|^p d\xi,$$

which implies that

$$\mu_0 \leq (1 + \varepsilon) \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H(u_k - u)|^p d\xi.$$

Now, using the inequality (3.22) again and integrating over Ω_R , we obtain

$$(1 - \varepsilon) \int_{\Omega_R} |\nabla_H(u_k - u)|^p d\xi \leq \int_{\Omega_R} |\nabla_H u_k|^p d\xi + c(\varepsilon, p) \int_{\Omega_R} |\nabla_H u|^p d\xi.$$

Taking the $\lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty}$ on both sides, we have

$$(1 - \varepsilon) \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H(u_k - u)|^p d\xi \leq \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H u_k|^p d\xi,$$

which gives

$$(1 - \varepsilon) \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H(u_k - u)|^p d\xi \leq \mu_0.$$

Therefore, since ε is arbitrary, we conclude that

$$\mu_0 = \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega_R} |\nabla_H(u_k - u)|^p d\xi.$$

The proof for ν_0 proceeds similarly.

Next, we introduce a cutoff function $\phi_R \in C^\infty(\mathbb{G})$ with $0 \leq \phi_R \leq 1$, such that

$$\phi_R(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus \Omega_R, \\ 1, & \text{if } x \in \Omega_{R+1}. \end{cases}$$

By the construction of ϕ_R , there exists a constant $C_R > 0$ such that $|\nabla_H \phi_R| \leq C_R$ in Ω . Moreover, by (2.4), we have for all $v \in S_0^{1,p}(\Omega)$ that

$$\bar{\mu} \int_{\Omega} \frac{|\phi_R v|^p}{|x'|^p} d\xi \leq \int_{\Omega} |\nabla_H(\phi_R v)|^p d\xi. \quad (3.23)$$

Using the truncation ϕ_R we obtain from (3.20) and (3.21) that

$$\mu_0 = \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H(u_k - u)|^p \phi_R^p d\xi, \quad (3.24)$$

and

$$\nu_0 = \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} \frac{|u_k - u|^p \phi_R^p}{|x'|^p} d\xi. \quad (3.25)$$

By applying (3.19) once more and using the compact embedding (2.1), we get

$$\mu_0 = \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H ((u_k - u) \phi_R)|^p d\xi. \quad (3.26)$$

Indeed, since

$$\nabla_H ((u_k - u) \phi_R) = \phi_R \nabla_H (u_k - u) + (u_k - u) \nabla_H \phi_R,$$

taking $a = \phi_R \nabla_H (u_k - u)$ and $b = (u_k - u) \nabla_H \phi_R$ in (3.19), we obtain

$$\begin{aligned} & |\nabla_H ((u_k - u) \phi_R)|^p - |\phi_R \nabla_H (u_k - u)|^p \\ & \leq \varepsilon |\phi_R \nabla_H (u_k - u)|^p + c(\varepsilon, p) |(u_k - u) \nabla_H \phi_R|^p. \end{aligned} \quad (3.27)$$

Then, integrating over Ω gives

$$\begin{aligned} & \int_{\Omega} |\nabla_H ((u_k - u) \phi_R)|^p d\xi \\ & \leq (1 + \varepsilon) \int_{\Omega} |\phi_R \nabla_H (u_k - u)|^p d\xi + c(\varepsilon, p) \int_{\Omega} |(u_k - u) \nabla_H \phi_R|^p d\xi \\ & \leq (1 + \varepsilon) \int_{\Omega} |\phi_R \nabla_H (u_k - u)|^p d\xi + c(\varepsilon, p) C_R^p \int_{\Omega} |(u_k - u)|^p d\xi. \end{aligned}$$

Taking the $\lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty}$ on both sides of the inequality above and applying the compact embedding $S_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, we derive

$$\lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H ((u_k - u) \phi_R)|^p d\xi \leq (1 + \varepsilon) \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\phi_R \nabla_H (u_k - u)|^p d\xi.$$

Thus, using (3.24), we have

$$\lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H ((u_k - u) \phi_R)|^p d\xi \leq (1 + \varepsilon) \mu_0.$$

Again, applying inequality (3.27) and integrating over Ω yields

$$\begin{aligned} & (1 - \varepsilon) \int_{\Omega} |\phi_R \nabla_H (u_k - u)|^p d\xi \\ & \leq \int_{\Omega} |\nabla_H ((u_k - u) \phi_R)|^p d\xi + c(\varepsilon, p) \int_{\Omega} |(u_k - u) \nabla_H \phi_R|^p d\xi \\ & \leq \int_{\Omega} |\nabla_H ((u_k - u) \phi_R)|^p d\xi + c(\varepsilon, p) C_R^p \int_{\Omega} |(u_k - u)|^p d\xi. \end{aligned}$$

Then, arguing as above, we obtain

$$(1 - \varepsilon)\mu_0 \leq \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H ((u_k - u) \phi_R)|^p d\xi.$$

Consequently, since ε is arbitrary, we conclude that

$$\mu_0 = \lim_{R \rightarrow +\infty} \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H ((u_k - u) \phi_R)|^p d\xi.$$

Finally, by substituting $v = u_k - u$ into (3.23) and applying (3.25) and (3.26), we obtain (3.18). The proof is complete. $\square$

As a direct consequence of Lemma 3.6, we prove that the functional $\mathcal{H}_{\mu,p}$, defined in (3.15), is weakly lower semicontinuous on $S_0^{1,p}(\Omega)$. Note that, although we only need weak lower semicontinuity for $\mathcal{H}_{\mu} := \mathcal{H}_{\mu,2}$, which also follows from the Hilbert space structure of $S_0^1(\Omega)$, we provide the proof for all $1 < p < N$, as we believe that the result is interesting in its own right.

Lemma 3.7. *If $0 \leq \mu < \bar{\mu}$, then the functional $\mathcal{H}_{\mu,p}$ is weakly lower semicontinuous on $S_0^{1,p}(\Omega)$.*

Proof. Let $\{u_k\}$ be such that $u_k \rightharpoonup u$ in $S_0^{1,p}(\Omega)$. Going if necessary to a subsequence, we can assume that $\mathcal{H}_{\mu,p}(u_k)$ is convergent. By Lemma 3.6, we have

$$\begin{aligned} \lim_{k \rightarrow +\infty} \mathcal{H}_{\mu,p}(u_k) &= \lim_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H u_k|^p d\xi - \mu \lim_{k \rightarrow +\infty} \int_{\Omega} \frac{|u_k|^p}{|x'|^p} d\xi \\ &= \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} |\nabla_H u_k|^p d\xi - \mu \overline{\lim}_{k \rightarrow +\infty} \int_{\Omega} \frac{|u_k|^p}{|x'|^p} d\xi \\ &\geq \mathcal{H}_{\mu,p}(u) + \mu_0 - \mu \nu_0 \\ &\geq \mathcal{H}_{\mu,p}(u) + (\bar{\mu} - \mu) \nu_0 \\ &\geq \mathcal{H}_{\mu,p}(u), \end{aligned}$$

concluding the proof. $\square$

We are now in a position to prove our main result.

Proof of Theorem 1.1. To prove the existence of the first solution, we apply the mountain pass lemma. As we already proved in Lemmas 3.4 and 3.5, the functional $I_{\mu,\lambda}$ defined in (3.2) has a mountain pass geometry and satisfies the Palais-Smale condition. Moreover, we have $I_{\mu,\lambda}(0) = 0$. Therefore, if we take v as in Lemma 3.4 and define

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_{\mu,\lambda}(\gamma(t)),$$

with

$$\Gamma := \left\{ \gamma \in C([0,1], S_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = v \right\},$$

then, by the mountain pass lemma, c is a critical value of $I_{\mu,\lambda}$. Consequently, there exists $u_1 \in S_0^1(\Omega)$ satisfying

$$I_{\mu,\lambda}(u_1) = c \geq \alpha > 0,$$

which is a solution to the problem (3.1).

We construct the second solution using a minimization technique. For this, we define the truncation of the functional $I_{\mu,\lambda}$. First, using the estimate in (3.5), we obtain, for all $u \in S_0^1(\Omega)$,

$$I_{\mu,\lambda}(u) \geq \frac{C_1}{2} \|u\|^2 - \frac{C_2}{q+1} \|u\|^{q+1} - \frac{\lambda C_3}{r+1} \|u\|^{r+1},$$

where $C_1, C_2, C_3 > 0$ are positive constants. For $t \geq 0$, we define the following function

$$h(t) := \frac{C_1}{2} t^2 - \frac{C_2}{q+1} t^{q+1} - \frac{\lambda C_3}{r+1} t^{r+1}.$$

Let us write

$$h(t) = t^{r+1} (g(t) - c_0) \quad (3.28)$$

with

$$g(t) := \frac{C_1}{2} t^{1-r} - \frac{C_2}{q+1} t^{q-r} \quad \text{and} \quad c_0 := \frac{\lambda C_3}{r+1}.$$

Since $1 < r+1 < 2 < q+1$ we have $1-r > 0$, $q-1 > 0$ and $q-r > 1-r$. Hence, $g(0) = 0$, $g(t) \rightarrow -\infty$ as $t \rightarrow +\infty$, and g is strictly increasing on $(0, t_0)$ and strictly decreasing on (t_0, ∞) , where

$$t_0 = \left(\frac{C_1(1-r)(q+1)}{2C_2(q-r)} \right)^{\frac{1}{q-1}}.$$

Let us denote $M_0 := g(t_0) > 0$ and set $\lambda^{**} := (r+1)M_0/C_3$. Then, for all $\lambda \in (0, \lambda^{**})$, it holds that $0 < c_0 < M_0$. Hence, the equation $g(t) = c_0$ has exactly two roots R_1 and R_2 , with $0 < R_1 < t_0 < R_2$ such that $g(t) > c_0$ on (R_1, R_2) . Finally, by (3.28), we conclude that $h(R_1) = h(R_2) = 0$ and

$$h(t) > 0 \text{ in } (R_1, R_2), \quad h(t) \leq 0 \text{ in } [0, R_1] \cup [R_2, \infty).$$

Now, we take a nonincreasing cutoff function $\sigma \in C^\infty([0, +\infty), [0, 1])$ such that

$$\sigma(t) := \begin{cases} 1, & 0 \leq t \leq R_1, \\ 0, & t \geq R_2. \end{cases} \quad (3.29)$$

Consider the following truncated functional

$$J_{\mu,\lambda}(u) := \frac{1}{2} \int_{\Omega} |\nabla_H u|^2 d\xi - \frac{\mu}{2} \int_{\Omega} \frac{u_+^2}{|x'|^2} d\xi - \frac{\sigma(\|u\|)}{q+1} \int_{\Omega} u_+^{q+1} d\xi - \frac{\lambda}{r+1} \int_{\Omega} u_+^{r+1} d\xi.$$

Then, arguing as in (3.5), we obtain

$$J_{\mu,\lambda}(u) \geq \frac{C_1}{2} \|u\|^2 - \frac{\sigma(\|u\|)}{q+1} C_2 \|u\|^{q+1} - \frac{\lambda C_3}{r+1} \|u\|^{r+1} \quad \forall u \in S_0^1(\Omega),$$

where $C_1, C_2, C_3 > 0$ are positive constants. We define the function

$$\bar{h}(t) := \frac{C_1}{2} t^2 - \frac{C_2 \sigma(t)}{q+1} t^{q+1} - \frac{\lambda C_3}{r+1} t^{r+1}, \quad \forall t \geq 0.$$

Then, by (3.29), $\bar{h}(t) \rightarrow +\infty$ as $t \rightarrow +\infty$ and $\bar{h}(t) \geq h(t)$ for all $t \geq 0$. Furthermore, the following properties hold

$$\begin{aligned} \bar{h}(t) &= h(t) \text{ in } (0, R_1), \quad \bar{h}(R_1) = h(R_1) = 0, \\ \bar{h}(t) &\geq h(t) > 0 \text{ in } (R_1, R_2), \quad \bar{h}(R_2) > 0. \end{aligned} \quad (3.30)$$

Observe also that $\bar{h}(t) > 0$ for all $t \in (R_2, +\infty)$.

Consequently, $J_{\mu,\lambda}(u) \geq \bar{h}(\|u\|)$ for all $u \in S_0^1(\Omega)$. Moreover, by (3.29) and the definition of the truncated functional, we have

$$I_{\mu,\lambda}(u) = J_{\mu,\lambda}(u), \quad \forall u \in S_0^1(\Omega) \text{ with } 0 \leq \|u\| \leq R_1.$$

Since both σ and $I_{\mu,\lambda}$ are C^1 , it follows that $J_{\mu,\lambda} \in C^1(S_0^1(\Omega), \mathbb{R})$.

Observe that if $\|u\| \geq R_1$, then $J_{\mu,\lambda}(u) \geq \bar{h}(\|u\|) \geq 0$. Therefore, if $J_{\mu,\lambda}(u) < 0$, then we necessarily have $\|u\| < R_1$. Moreover, by the continuity of $J_{\mu,\lambda}$ and (3.30) we conclude that, if $J_{\mu,\lambda}(u) < 0$, then $I_{\mu,\lambda}(v) = J_{\mu,\lambda}(v)$ for all v in a small enough neighborhood of u .

Now we will analyse the properties of the functional $J_{\mu,\lambda}$. Since $r+1 < 2$, the functional $J_{\mu,\lambda}$ is coercive. Moreover, the functional $J_{\mu,\lambda}$ is weakly lower semicontinuous on $S_0^1(\Omega)$. Indeed, applying Lemma 3.7 to $\mathcal{H}_\mu := \mathcal{H}_{\mu,2}$ and using (3.9), the weak lower semicontinuity of the norm $\|\cdot\|$, and the fact that σ is continuous and non-increasing, we obtain

$$\begin{aligned} & \lim_{k \rightarrow +\infty} J_{\mu,\lambda}(u_k) \\ &= \lim_{k \rightarrow +\infty} \left(\frac{1}{2} \|u_k\|^2 - \frac{\mu}{2} \int_{\Omega} \frac{(u_k)_+^2}{|x'|^2} d\xi - \frac{\sigma(\|u_k\|)}{q+1} \int_{\Omega} (u_k)_+^{q+1} d\xi - \frac{\lambda}{r+1} \int_{\Omega} (u_k)_+^{r+1} d\xi \right) \\ &= \lim_{k \rightarrow +\infty} \left(\frac{1}{2} \|(u_k)_+\|^2 - \frac{\mu}{2} \int_{\Omega} \frac{(u_k)_+^2}{|x'|^2} d\xi + \frac{1}{2} \|(u_k)_-\|^2 - \frac{\sigma(\|u_k\|)}{q+1} \int_{\Omega} (u_k)_+^{q+1} d\xi \right. \\ & \quad \left. - \frac{\lambda}{r+1} \int_{\Omega} (u_k)_+^{r+1} d\xi \right) \\ &\geq \frac{1}{2} \|u\|^2 - \frac{\mu}{2} \int_{\Omega} \frac{u_+^2}{|x'|^2} d\xi - \frac{\sigma(\|u\|)}{q+1} \int_{\Omega} u_+^{q+1} d\xi - \frac{\lambda}{r+1} \int_{\Omega} u_+^{r+1} d\xi \\ &= J_{\mu,\lambda}(u). \end{aligned}$$

Thus, by [37, Theorem 1.2], we can conclude that the infimum

$$m := \inf_{v \in S_0^1(\Omega)} J_{\mu,\lambda}(v)$$

is attained at some $u_0 \in S_0^1(\Omega)$.

Now we show that $m < 0$. Fix $v_0 \in S_0^1(\Omega)$ with $v_0 > 0$ and $\|v_0\| = 1$. By (3.29), we have that $\sigma(t) = 1$ for $0 \leq t \leq R_1$. Hence,

$$\begin{aligned} J_{\mu,\lambda}(tv_0) &= \frac{t^2}{2} \|v_0\|^2 - \frac{\mu t^2}{2} \int_{\Omega} \frac{v_0^2}{|x'|^2} d\xi - \frac{t^{q+1}}{q+1} \int_{\Omega} v_0^{q+1} d\xi - \frac{\lambda t^{r+1}}{r+1} \int_{\Omega} v_0^{r+1} d\xi \\ &\leq \frac{t^2}{2} - \frac{\lambda t^{r+1}}{r+1} \int_{\Omega} v_0^{r+1} d\xi < 0 \end{aligned}$$

for sufficiently small $t > 0$, since $r+1 < 2$. This implies that $m = J_{\mu,\lambda}(u_0) < 0$. Consequently, we have $\|u_0\| < R_1$. From this, we conclude that u_0 is a local minimum of $I_{\mu,\lambda}$, and thus $I'_{\mu,\lambda}(u_0) = J'_{\mu,\lambda}(u_0) = 0$. Hence, u_0 is a critical point of $I_{\mu,\lambda}$, i.e., a weak solution of (3.1). Finally, $I_{\mu,\lambda}(u_0) = m < 0 < \alpha \leq c = I_{\mu,\lambda}(u_1)$, where u_1 is a mountain pass solution. Therefore, the two solutions are distinct.

As a result, for all $\lambda \in (0, \lambda_0)$, where $\lambda_0 := \min\{\lambda^*, \lambda^{**}\}$, problem (3.1) admits two solutions. Let u be any of these two solutions. Then $\langle I'_{\mu,\lambda}(u), \varphi \rangle = 0$ for any $\varphi \in S_0^1(\Omega)$. In particular, $\langle I'_{\mu,\lambda}(u), u_- \rangle = \|u_-\|^2 = 0$, which implies $u_- = 0$. Consequently, we have $u = u_+ \geq 0$.

Note that any weak solution u of problem (3.1) is also a weak supersolution of the homogeneous problem $-\Delta_{\mathbb{G}} u(\xi) = 0$ in Ω . Indeed, using the weak formulation of problem (3.1), we have

$$\int_{\Omega} \nabla_H u \cdot \nabla_H \varphi d\xi = \mu \int_{\Omega} \frac{u_+ \varphi}{|x'|^2} d\xi + \int_{\Omega} u_+^q \varphi d\xi + \lambda \int_{\Omega} u_+^r \varphi d\xi \geq 0$$

for all $0 \leq \varphi \in S_0^1(\Omega)$. Therefore, by applying Theorem 2.3, we deduce that any nontrivial weak solution u to problem (3.1) is strictly positive in Ω .

Thus, we conclude that problem (1.4) admits two positive solutions. $\square$

Corollary 3.8. *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain containing the origin, with smooth boundary $\partial\Omega$. Assume that $0 < \mu < \bar{\mu} = \left(\frac{Q-4}{2}\right)^2$ with $Q > 4$, $\lambda > 0$ and $1 < r+1 < 2 < q+1 < 2^* = \frac{2Q}{Q-2}$, where $Q = 2n+2$. Consider the following problem:*

$$\begin{cases} -\Delta_{\mathbb{H}^n} u(\xi) - \mu \frac{u(\xi)}{|z|^2} = u^q(\xi) + \lambda u^r(\xi) & \text{in } \Omega, \\ u(\xi) > 0 & \text{in } \Omega, \\ u(\xi) = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.31)$$

where $\xi = (z, t) = (x, y, t) \in \Omega$ and $|z| = \sqrt{x_1^2 + \dots + x_n^2 + y_1^2 + \dots + y_n^2}$. Then, there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ the problem (3.31) admits at least two positive solutions.

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REFERENCES

- [1] B. Abdellaoui, I. Peral, *Some results for semilinear elliptic equations with critical potential*, Proc. R. Soc. Edinb., Sect. A. Math. **132** (2002), 1–24.
- [2] A. Ambrosetti, P. Rabinowitz, *Dual methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
- [3] J.G. Azorero, I.P. Alonso, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc. **323** (1991), 877–895.
- [4] B. Barrios, M. Medina, I. Peral, *Some remarks on the solvability of non-local elliptic problems with the Hardy potential*, Commun. Contemp. Math. **16** (2014), 1350046.
- [5] T. Bhattacharya, A. Mohammed, *On a strong maximum principle for fully nonlinear subelliptic equations with Hörmander condition*, Calc. Var. Partial Differential Equations **60** (2021), Paper No. 9.
- [6] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer, Berlin, 2007.
- [7] S. Bordon, R. Filippucci, P. Pucci, *Existence problems on Heisenberg groups involving Hardy and critical terms*, J. Geom. Anal. **30** (2020), 1887–1917.
- [8] H. Brezis, L. Dupaigne, A. Tesei, *On a semilinear elliptic equation with inverse-square potential*, Selecta Math. (N.S.) **11** (2005), Paper No. 1.
- [9] L. Capogna, D. Danielli, S.D. Pauls, J.T. Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Progress in Mathematics, vol. 259, Birkhäuser Verlag, Basel, 2007.
- [10] H. Chen, A. Quaas, F. Zhou, *On nonhomogeneous elliptic equations with the Hardy–Leray potentials*, J. Anal. Math. **144** (2021), 305–334.
- [11] J. Chen, *Existence of solutions for a nonlinear PDE with an inverse square potential*, J. Differ. Equ. **195** (2003), 497–519.
- [12] J. Chen, E.M. Rocha, *Existence of solution of subelliptic equations on the Heisenberg group with critical growth and double singularities*, Opuscula Math. **33** (2013), 237–254.
- [13] G. Citti, *Semilinear Dirichlet problem involving critical exponent for the Kohn Laplacian*, Ann. Mat. Pura Appl. **169** (1995), 375–392.

- [14] G. Citti, F. Uguzzoni, *Critical semilinear equations on the Heisenberg group: the effect of the topology of the domain*, Nonlinear Anal. **46** (2001), 399–417.
- [15] L. D’Ambrosio, *Some Hardy inequalities on the Heisenberg group*, Differ. Equ. **40** (2004), 552–564.
- [16] L. D’Ambrosio, *Hardy-type inequalities related to degenerate elliptic differential operators*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 451–486.
- [17] L. Dupaigne, *A nonlinear elliptic PDE with the inverse square potential*, J. Anal. Math. **86** (2002), 359–398.
- [18] M. Fall, *Semilinear elliptic equations for the fractional Laplacian with Hardy potential*, Nonlinear Anal. **193** (2020), 111311.
- [19] A. Ferrero, F. Gazzola, *Existence of solutions for singular critical growth semilinear elliptic equations*, J. Differ. Equ. **177** (2001), 494–522.
- [20] V. Fischer, M. Ruzhansky, *Quantization on Nilpotent Lie Groups*, Progress in Mathematics, vol. 314, Birkhäuser, 2016.
- [21] N. Garofalo, E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*, Ann. Inst. Fourier (Grenoble) **40** (1990), 313–356.
- [22] N. Garofalo, E. Lanconelli, *Existence and nonexistence results for semilinear equations on the Heisenberg group*, Indiana Univ. Math. J. **41** (1992), 71–98.
- [23] N. Garofalo, D.-M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces*, Commun. Pure Appl. Math. **49** (1996), 1081–1144.
- [24] N. Ghoussoub, C. Yuan, *Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents*, Trans. Am. Math. Soc. **12** (2000), 5703–5743.
- [25] D.S. Jerison, J.M. Lee, *Extremals of the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc. **1** (1988), 1–13.
- [26] A. Kassymov, D. Suragan, *Multiplicity of positive solutions for a nonlinear equation with a Hardy potential on the Heisenberg group*, Bull. Sci. Math. **165** (2020), 102916.
- [27] A. Loiudice, *Critical problems with Hardy potential on Stratified Lie Groups*, Adv. Differential Equations **28** (2023), 1–33.
- [28] H. Mokrani, *Semilinear subelliptic equations on the Heisenberg group with a singular potential*, Comm. Pure Appl. Anal. **8** (2009), 1619–1636.
- [29] P. Niu, H. Zhang, Y. Wang, *Hardy type and Rellich type inequalities on the Heisenberg group*, Proc. Amer. Math. Soc. **129** (2001), 3623–3630.
- [30] P. Pucci, L. Temperini, *On the concentration-compactness principle for Folland–Stein spaces and for fractional horizontal Sobolev spaces*, Math. Eng. **5** (2023), Paper No. 007.

- [31] M. Ruzhansky, D. Suragan, *On horizontal Hardy, Rellich, Caffarelli–Kohn–Nirenberg and p -sub-Laplacian inequalities on stratified groups*, J. Differential Equations **262** (2017), 1799–1821.
- [32] M. Ruzhansky, D. Suragan, *Hardy inequalities on homogeneous groups*, Progress in Math., vol. 327, Birkhäuser, 2019, 71–127.
- [33] M. Ruzhansky, D. Suragan, *Green’s identities, comparison principle and uniqueness of positive solutions for nonlinear p -sub-Laplacian equations on stratified Lie groups*, Potential Anal. **53** (2020), 645–658.
- [34] M. Ruzhansky, B. Sabitbek, D. Suragan, *Weighted L^p -Hardy and L^p -Rellich inequalities with boundary terms on stratified Lie groups*, Rev. Math. Complutense **32** (2019), 19–35.
- [35] P. Sintzoff, *Existence of minimizer of some functionals involving Hardy-type inequalities*, Topol. Methods Nonlinear Anal. **28** (2006), 105–115.
- [36] D. Smets, *A concentration-compactness lemma with applications to singular eigenvalue problems*, J. Funct. Anal. **167** (1999), 463–480.
- [37] M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, Berlin, 1990.
- [38] S.K. Vodop’yanov, *Weighted Sobolev spaces and the boundary behavior of solutions of degenerate hypoelliptic equations*, Siberian Math. J. **36** (1995), 246–264 (translated from Sibirsk. Mat. Zh. **36** (1995), 278–300).
- [39] M. Willem, *A decomposition lemma and critical minimization problems*, [in:] Proceedings of the Workshop on Morse Theory, Minimax Theory and Their Applications, Chinese Academy of Sciences, Beijing, 2003, 247–256.

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