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GROUND STATES FOR THE PLANAR INDEFINITE CHOQUARD EQUATION WITH SUPERCRITICAL EXPONENTIAL GROWTH

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Abstract. This paper investigates a class of strongly indefinite Choquard equations formulated on the two-dimensional unit ball B. Specifically, consider the following equation:

$$\begin{cases} -\Delta u + V(x)u = (I_\mu * F(u))f(u), & x \in B, \\ u \in H^1_{0,\mathrm{rad}}(B), \end{cases}$$

where $\mu \in (0,2)$ and I_μ represents the classical Riesz potential. A fundamental hypothesis is that zero lies within a spectral gap of the Schrödinger operator $-\Delta + V$. Furthermore, the continuous nonlinearity $f(t)$ is characterized by a supercritical exponential growth governed by $\exp[(\beta + |x|^\alpha)t^2]$ with $\beta, \alpha > 0$. Most existing works on this problem are limited to the subcritical or critical exponential growth cases. However, dealing with supercritical growth is much more difficult, especially when estimating the exact upper bound of the minimax levels. To overcome this, we use an approximation method and fine estimates to prove that this indefinite problem has a positive ground state solution. Our approach can also be applied to other elliptic problems with supercritical growth.

Keywords: strongly indefinite, Trudinger–Moser inequality, supercritical exponential growth.

Mathematics Subject Classification: 35J20, 35J62, 35Q55.

1. INTRODUCTION

We consider the following class of Choquard equations characterized by a supercritical exponential growth condition:

$$\begin{cases} -\Delta u + V(x)u = (I_\mu * F(u))f(u), & x \in B, \\ u \in H^1_{0,\mathrm{rad}}(B). \end{cases} \tag{1.1}$$

Here, $B := B_1(0)$ denotes the standard unit open ball in $\mathbb{R}^2$ centered at the origin, and for a given parameter $\mu \in (0,2)$, the classical Riesz potential $I_\mu : \mathbb{R}^2 \to \mathbb{R}$ takes the following form:

$$I_\mu(x) = \frac{\Gamma\left(\frac{2-\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right) 2^\mu \pi |x|^{2-\mu}} := \frac{A_\mu}{|x|^{2-\mu}}, \quad x \in \mathbb{R}^2 \setminus \{0\}.$$

Furthermore, we assume that zero is located strictly within a spectral gap of the associated Schrödinger operator $-\Delta + V$. Specifically, the potential function satisfies:

(V) $V \in \mathcal{C}(B, \mathbb{R})$, $V(x)$ is 1-periodic in x_1 and x_2 and

$$\sup\left[\sigma\left(-\Delta + V\right) \cap (-\infty, 0)\right] < 0 < \inf\left[\sigma\left(-\Delta + V\right) \cap (0, \infty)\right].$$

As a result, the operator $-\Delta + V$ is strongly indefinite. It follows that its spectrum in $L^2(B)$ is purely continuous, comprising mutually disjoint closed intervals. Additionally, the nonlinearity $f(t)$ is assumed to be continuous and to possess supercritical exponential growth.

Nonlocal elliptic equations of Choquard type have long played an important role in mathematical models arising from quantum mechanics and interaction systems. The origin of this model can be traced back to Pekar's description of the stationary polaron in quantum theory [36]. Later, in connection with the Hartree–Fock approximation for a one-component plasma, Lieb [23] pointed out that Choquard used such an equation to describe an electron interacting with the hole generated by itself. Owing to the presence of the convolution term, the equation exhibits essentially nonlocal features, which make its analytical treatment substantially different from that of local semilinear elliptic equations. Over the past decades, variational methods have become one of the main approaches to the study of Choquard-type problems. We refer, for instance, to the classical works of Lions [25], Ma and Zhao [26], and Moroz and Van Schaftingen [28–30], as well as the references therein, for related developments.

Recently, elliptic problems involving critical exponential growth received much attention, and this kind of growth means that the nonlinearities f should satisfy

($\bar{\text{F}}$1) there exists $\beta_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{uniformly on } x \in B \text{ for all } \beta > \beta_0$$

and

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\beta t^2}} = +\infty, \quad \text{uniformly on } x \in B \text{ for all } \beta < \beta_0.$$

The theoretical basis of this strictly critical exponential growth is the following well-known Moser–Trudinger inequality [8, 31, 38, 43], which ensures

$$\mathrm{MT}_\beta := \sup_{u \in H^1_{0,\mathrm{rad}}(B): \|\nabla u\|_2 \le 1} \int_B \exp\left(\beta |u|^2\right) \mathrm{d}x < +\infty \tag{1.2}$$

for any $\beta \leq 4\pi$, where $H^1_{0,\mathrm{rad}}(B)$ consists of radially symmetric functions and is the subspace of the Sobolev space $H^1_0(B)$. Moreover, the inequality (1.2) is sharp (or critical): for any growth $\exp\left(\beta|u|^2\right)$ with $\beta > 4\pi$, the corresponding MT_β is $+\infty$. More general, for any $\beta > 0$,

$$\overline{\mathrm{MT}}_\beta := \sup_{u\in H^1_{0,\mathrm{rad}}(B)} \int_B \exp\left(\beta|u|^2\right) \mathrm{d}x < +\infty. \tag{1.3}$$

Because inequality (1.2) and its modified forms serve as fundamental tools across multiple analytical domains, extending these results to broader elliptic equations has been extensively investigated. For a comprehensive overview, see [1–3, 6, 11, 17, 21, 27, 33, 35, 41, 44, 45].

Considering the Choquard equation involved in the exponential critical case, that is when f satisfies $(\bar{F}1)$, Alves *et al.* [5] utilized the frameworks of (1.2) and (1.3) to demonstrate the existence and concentration behavior of semiclassical ground states for the planar problem. Expanding on this framework, Qin-Tang [40] investigated systems featuring indefinite potentials under critical exponential growth, established several existence results. In the fractional setting, Clemente *et al.* [12] combined variational arguments with minimax estimates to deduce the existence of solutions for the one-dimensional Choquard problem, provided the nonlinear term features Trudinger–Moser critical growth. Furthermore, by imposing weak growth assumptions on the critical exponential term f rather than relying on strict monotonicity, Yuan *et al.* [47] confirmed the existence of ground state solutions for the fractional (N/s)-Laplacian Choquard problem. For broader discussions and additional findings regarding Choquard equations constrained by critical exponential frameworks, one may consult [5, 12, 18, 20, 40, 47] along with the references cited therein.

In contrast to the aforementioned critical case, we introduce a new hypothesis assuming the continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfies a supercritical exponential growth:

(F1) For a certain constant $\beta_0 > 0$, the following limits hold uniformly with respect to $x \in B$:

$$\lim_{|t|\to\infty} \frac{|f(t)|}{\exp[(\beta + |x|^\alpha)t^2]} = 0 \quad \text{for any } \beta > \beta_0,$$

and

$$\lim_{|t|\to\infty} \frac{|f(t)|}{\exp[(\beta + |x|^\alpha)t^2]} = +\infty \quad \text{for any } \beta < \beta_0.$$

The primary difference between the critical growth discussed in earlier literature and the supercritical case is the inclusion of the weight term $|x|^\alpha$. To handle supercritical conditions, Ngô-Nguyen [32] generalized the classical Moser–Trudinger inequality (1.2). Specifically, for $\beta \leq 4\pi$, they established the following bound:

$$\mathrm{MT}^1_\alpha(\beta) := \sup_{u\in H^1_{0,\mathrm{rad}}(B):\|\nabla u\|_2\leq 1} \int_B \exp\left[\left(\beta + |x|^\alpha\right)|u|^2\right] \mathrm{d}x < +\infty.$$

Furthermore, the supremum $\mathrm{MT}^1_\alpha(\beta)$ is critical, as the integral diverges whenever $\beta > 4\pi$. This generalized inequality enables the application of variational techniques to various elliptic equations. For instance, Alves and Shen [4] studied elliptic problems with supercritical exponential growth, including a periodic Schrödinger-type framework, and obtained existence results by variational methods. Temgoua [42] obtained solutions for supercritical problems defined on nonradial exterior domains. Additionally, for a slightly supercritical Neumann problem on a ball, Pistoia *et al.* [37] verified the existence of multiple concentrating solutions.

Another typical feature of the current work is that the potential V is indefinite potential, which requires the use of the generalized link theorem. Initiated by Kryszewski and Szulkin[19], this tool was later refined by Li and Szulkin [22], and Ding [13, 16]. As a primary point of reference for such indefinite problems, Qin and Tang [40] investigated equation (1.1) but restricted their framework to the critical exponential case. To handle this critical growth, they imposed the following assumptions on f:

(F2) $f(t) = o(|t|^{\mu/2})$ as $|t| \to 0$,

(F3) there exists $\theta > 1$ such that

$$tf(t) \geq \theta F(t) > 0, \quad \forall\, t \in \mathbb{R} \setminus \{0\},$$

(F4) there exist $M_0 > 0$ and $t_0 > 0$ such that

$$F(t) \leq M_0 \left| f(t) \right|, \quad \forall\, |t| \geq t_0,$$

where $F(x,t) = \int_0^t f(s)\mathrm{d}s$,

(F5′)

$$\liminf_{t\to+\infty} \frac{f(t)}{\exp\left(\beta_0 t^2\right)} = \kappa_0 > \frac{\sqrt{\mu(1+\mu)}(2+\mu)}{\sqrt{2\pi A_\mu}\rho^{1+\mu/2}} e^{4(2+\mu)\pi\mathcal{C}_0^2(1+\rho)^2/(2+\rho)},$$

where $\rho > 0$ satisfies $2(2+\mu)\pi\rho^2\mathcal{C}_0^2 < 1$ and $\mathcal{C}_0 > 0$ is an embedding constant, see (3.1).

To be specific, they constructed the ground state of problem (1.1) by combining periodic function approximations, the linking theorem, and refined analytical estimates. Regarding the corresponding local framework of the planar Schrödinger equation, we refer to the work in [10], which advances the findings of de Figueiredo *et al.* [16]. By introducing novel techniques to sharply bound the minimax energy levels, they verified the existence of Nehari-type ground states in the indefinite setting. Consequently, an open problem remains: whether ground state solutions exist for the strongly indefinite system (1.1) involving supercritical exponential growth. To the best of our knowledge, this specific problem remains unexplored in the literature.

In this work, we investigate problem (1.1) under the assumptions that the operator $-\Delta+V$ is strongly indefinite and the nonlinearity $f(t)$ exhibits supercritical exponential growth. As far as we are aware, the existence of ground state solutions in this specific setting has not been previously established under the assumption:

(F5)

$$\liminf_{t\to+\infty} \frac{tF(t)}{\exp\left[(\beta_0+|x|^\alpha)\,t^2\right]} = \kappa > \frac{1}{2e\pi\beta_0^2 A_\mu B_\mu},$$

where $B_\mu := \rho^{2+\mu}/[\mu(1+\mu)(2+\mu)]$.

which is essential for deriving a sharp upper bound on the minimax energy level. Furthermore, (F5) naturally follows from the condition:

(F5$''$)

$$\begin{aligned}\liminf_{t\to+\infty} \frac{f(t)}{\exp\left[(\beta_0+|x|^\alpha)\,t^2\right]} &= \kappa_1 \\ &> \frac{\sqrt{\mu(1+\mu)}(2+\mu)}{\sqrt{2\pi A_\mu}\rho^{1+\mu/2}} e^{4(2+\mu)\pi\mathcal{C}_0^2(1+\rho)^2/(2+\rho)},\end{aligned}$$

where $\rho > 0$ is chosen that $2(2+\mu)\pi\rho^2\mathcal{C}_0^2 < 1$, and $\mathcal{C}_0 > 0$ denotes the relevant embedding constant defined in (3.1). This implication is verified by the following sequence of inequalities:

$$\begin{aligned}\liminf_{t\to\infty} \frac{tF(t)}{\exp\left[(\beta_0+|x|^\alpha)\,t^2\right]} &\geq \liminf_{t\to\infty} \frac{f(t)}{2\left(\beta_0+|x|^\alpha\right)\exp\left[(\beta_0+|x|^\alpha)\,t^2\right]} \\ &\geq \frac{\kappa_1}{2\left(\beta_0+|x|^\alpha\right)}.\end{aligned}$$

In particular, when the weight $|x|^\alpha$ vanishes, the lower bound reduces to:

$$\frac{\kappa_1}{2\beta_0} > \frac{\sqrt{\mu(1+\mu)}(2+\mu)}{2\beta_0\sqrt{2\pi A_\mu}\rho^{1+\mu/2}} e^{4(2+\mu)\pi\mathcal{C}_0^2(1+\rho)^2/(2+\rho)} > \frac{1}{2e\pi\beta_0^2 A_\mu B_\mu}.$$

(F5$''$) actually is a generalization of (F5$'$) in [40] under the supercritical exponential growth. The weak version hypothesis (F5) means that we need to develop the energy estimation process to meet more nonlinearities.

We emphasize that coupling the strongly indefinite features with supercritical exponential growth engenders novel analytical complexities, which need to use new ideas and more technical estimates to overcome and can be summarized as follows:

(i) Restricting the minimax energy below a precise threshold poses a primary analytical obstacle induced by the supercritical nonlinearity. Overcoming this is mandatory for the successful proof of Theorem 1.1.

(ii) To legitimately apply $\mathrm{MT}^1_\alpha(\beta)$ and derive the ground state, one must first ensure $f(u) \in L^q(B)$ with $q \in (1,2)$. Because the standard Trudinger–Moser inequality $\overline{\mathrm{MT}}_\beta$ for $\beta > 0$ becomes invalid under supercritical growth conditions, we are compelled to rigorously estimate the delicate balance between the $H^1_{0,\mathrm{rad}}(B)$ and $L^2(B)$ norms. Securing this relationship is a prerequisite for employing $\mathrm{MT}^1_\alpha(\beta)$, which strictly demands an exponent bounded by 4π.

(iii) Refining the estimation techniques for the minimax energy level associated with the functional Φ. Specifically, let the working space $H^1_{0,\mathrm{rad}}(B) := E$ be equipped with the orthogonal splitting $E = E^- \oplus E^+$, under which the quadratic form $\int_B (|\nabla u|^2 + V(x)u^2)\,dx$ is negative definite on E^- and positive definite on E^+. Utilizing the critical point theorem from [7], we analyze the functional Φ restricted to the subspaces $E_k = E_k^- \oplus E^+ \subset E$, where $\dim E_k^- = k$ for each $k \in \mathbb{N}$. We then demonstrate that Φ admits a Cerami sequence $\{u_n^k\}_n \subset E_k$ at the level $c_k > 0$, ensuring that both c_k and the sequence norms $\{\|u_n^k\|_E\}_n$ remain bounded for all $k \in \mathbb{N}$.

Our results read as follows:

Theorem 1.1. *Under the hypotheses* (V) *and* (F1)–(F5)*, problem* (1.1) *admits a positive ground state solution.*

Corollary 1.2. *Under the same framework as Theorem 1.1, replacing* (F5) *with* (F5$''$)*, problem* (1.1) *possesses a positive ground state solution.*

Remark 1.3. Since (F5$''$) is a stronger condition than (F5), Corollary 1.2 follows as a direct consequence of Theorem 1.1. More precisely, the argument hinges on establishing the existence of some $\bar{n} \in \mathbb{N}$ which satisfies

$$\max_{s \geq 0, v \in E^-} \Phi\,(v + s w_n) < \frac{(2+\mu)\pi}{\beta_0},$$

where E^- will be shown in Section 2. For the convenience of the reader, the detailed derivation is provided in the Appendix.

The structure of this study is outlined below. Section 2 presents essential preliminaries and auxiliary lemmas. In Section 3, we develop the minimax energy estimates linked to $\mathrm{MT}^1_\alpha(\beta)$, while the verification of Theorem 1.1 is carried out in Section 4. For clarity, throughout the paper, $C_1, C_2, \ldots$ denote positive constants whose values may vary from line to line; for $x \in B$ and $r > 0$, we write $B_r(x) = \{y \in B : |y - x| < r\}$ and $B_r = B_r(0)$; moreover, each $u \in H^1_{0,\mathrm{rad}}(B)$ is understood to be radially symmetric, so that $u(x) = u(|x|)$.

2. PRELIMINARIES

Consider the operator $\mathcal{A} = -\Delta + V$ with $V \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. It is well-established that $\mathcal{A}$ is self-adjoint in $L^2(\mathbb{R}^2)$ having the domain $\mathfrak{D}(\mathcal{A}) = H^2(\mathbb{R}^2)$ (refer to [15, Theorem 4.26]). Let $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ and $|\mathcal{A}|$ denote the spectral family and the absolute value of $\mathcal{A}$, respectively. Defining $|\mathcal{A}|^{1/2}$ as the square root of $|\mathcal{A}|$ and setting $U = \mathrm{id} - \mathcal{E}(0) - \mathcal{E}(0-)$, one observes that U commutes with $\mathcal{A}$, $|\mathcal{A}|$, and $|\mathcal{A}|^{1/2}$. Furthermore, $\mathcal{A} = U|\mathcal{A}|$ represents the polar decomposition of $\mathcal{A}$ (see [14, Theorem IV 3.3]). Let

$$E = \mathfrak{D}(|\mathcal{A}|^{1/2}), \quad E^- = \mathcal{E}(0-)E, \quad E^+ = [\mathrm{id} - \mathcal{E}(0)]E. \tag{2.1}$$

Under condition (V), the space E admits the orthogonal decomposition $E = E^- \oplus E^+$. For any $u \in E$, we can uniquely write $u = u^- + u^+$, where

$$u^- := \mathcal{E}(0-)u \in E^-, \quad u^+ := [\mathrm{id} - \mathcal{E}(0)]u \in E^+. \tag{2.2}$$

Consequently, the following relations

$$\mathcal{A}u^- = -|\mathcal{A}|u^-, \quad \mathcal{A}u^+ = |\mathcal{A}|u^+ \tag{2.3}$$

hold for $u \in E \cap \mathfrak{D}(\mathcal{A})$. We equip the space E with the inner product defined by

$$(u, v) = \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_{L^2}, \quad u, v \in E \tag{2.4}$$

which induces the corresponding norm

$$\|u\| = \left\||\mathcal{A}|^{1/2}u\right\|_2, \quad u \in E, \tag{2.5}$$

where and in the sequel, $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(B)$, $\|\cdot\|_s$ denotes the norm of $L^s(B)$. Moreover, the orthogonal splitting $E = E^- \oplus E^+$ holds simultaneously under the $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$ inner products. Note that E^- reduces to the trivial space $\{0\}$ provided that the spectrum $\sigma(-\Delta + V)$ is entirely contained in $(0, \infty)$; in any other case, E^- remains infinite-dimensional.

Assume $\Omega \subset \mathbb{R}^2$ is an open set. To simplify the notation, we introduce the functionals:

$$\Psi_\Omega(u) = \frac{1}{2}\int\limits_\Omega [I_\mu * F(u)]F(u)\,dx, \quad \langle \Psi'_\Omega(u), v\rangle = \int\limits_\Omega [I_\mu * F(u)]f(u)v\,dx,$$

for any $u, v \in H^1(\mathbb{R}^2)$. Throughout this paper, when Ω coincides with the ball B, the notations $\Psi_\Omega(u)$ and $\langle \Psi'_\Omega(u), v\rangle$ are abbreviated as $\Psi(u)$ and $\langle \Psi'(u), v\rangle$, respectively.

Let $E = H^1(B)$. Based on the hypotheses (V), (F1), and (F2), the weak solutions to problem (1.1) correspond to the critical points of the following energy functional:

$$\Phi(u) = \frac{1}{2}\int\limits_B \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \Psi(u), \quad \forall\, u \in E. \tag{2.6}$$

Consequently, the Sobolev embedding $E \hookrightarrow L^s(B)$ is continuous for all $s \in [2, \infty)$, implying the existence of a constant $\gamma_s > 0$ satisfying

$$\|u\|_s \le \gamma_s \|u\|, \quad \forall\, u \in E. \tag{2.7}$$

Taking (2.3) and (2.5) into account, the functional Φ can be decomposed as

$$\Phi(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\right) - \Psi(u), \quad \forall\, u = u^- + u^+ \in E^- \oplus E^+ = E. \tag{2.8}$$

According to (F1) and (F2), for any fixed $\varepsilon > 0$, we can determine constants $\beta > 0$ and $C_\varepsilon > 0$ such that the primitive F satisfies the growth estimate:

$$|F(t)| \le \varepsilon|t|^2 + C_\varepsilon|t| \exp\left[\left(\beta + |x|^\alpha\right)t^2\right], \quad \forall\, (x, t) \in B \times \mathbb{R}. \tag{2.9}$$

According to (2.9), it is standard to verify that Φ is of class $\mathcal{C}^1(E,\mathbb{R})$ under (F1), and

$$\langle \Phi'(u), v\rangle = \int\limits_B (\nabla u \nabla v + V(x)u\ v)\mathrm{d}x - \langle \Psi'(u), v\rangle\,, \quad \forall\, u, v \in E. \tag{2.10}$$

In particular, it follows from (2.3) and (2.5) that

$$\langle \Phi'(u), u\rangle = \left\|u^+\right\|^2 - \left\|u^-\right\|^2 - \langle \Psi'(u), u\rangle\,, \quad \forall\, u \in E.$$

Define

$$\mathcal{M} = \left\{u \in E\backslash E^- : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0, \forall\, v \in E^-\right\}. \tag{2.11}$$

Lemma 2.1. *Under the assumptions* (V) *and* (F1)–(F3), *there exists a radius* $\bar{\rho} > 0$ *ensuring that*

$$\kappa_0 := \inf\left\{\Phi(u) : u \in E^+, \|u\| = \bar{\rho}\right\} > 0. \tag{2.12}$$

Proof. Invoking (F1) and (F2), we can select constants $\beta \in (\beta_0, 2\pi]$ and $C_1 > 0$ such that the following pointwise estimate holds:

$$|F(t)| \le |t|^{\frac{2+\mu}{2}} + C_1|t|^3 \exp\left[(\beta + |x|^\alpha)\, |t|^2\right], \quad \forall\, t \in \mathbb{R}. \tag{2.13}$$

By virtue of the inequality (2.13) together with the Hölder inequality, it follows that

$$\begin{aligned}
\int\limits_B F(u)^{\frac{4}{2+\mu}}\mathrm{d}x &\le \int\limits_B \left\{|u|^{\frac{2+\mu}{2}} + C_1|u|^3\exp\left[(\beta + |x|^\alpha)\,|u|^2\right]\right\}^{\frac{4}{2+\mu}}\mathrm{d}x \\
&\le 2\int\limits_B \left\{|u|^2 + C_1^{\frac{4}{2+\mu}}|u|^{\frac{12}{2+\mu}}\exp\left[\frac{4\,(\beta + |x|^\alpha)\,u^2}{2+\mu}\right]\right\}\mathrm{d}x \\
&\le 2\|u\|_2^2 + C_1^{\frac{4}{2+\mu}}\|u\|_6^{\frac{12}{2+\mu}}\left\{\int\limits_B \exp\left[\frac{4\,(\beta + |x|^\alpha)\,u^2}{\mu}\right]\mathrm{d}x\right\}^{\frac{\mu}{2+\mu}} \\
&\le 2\gamma_2^2\|u\|^2 + C_1^{\frac{4}{2+\mu}}\|u\|_6^{\frac{12}{2+\mu}}\left\{\int\limits_B \exp\left[2\,(\beta + |x|^\alpha)\,u^2\right]\mathrm{d}x\right\}^{\frac{\mu}{2+\mu}} \\
&\le 2\gamma_2^2\|u\|^2 + \left(C_0^{\frac{\mu}{4}}C_1\gamma_6^3\right)^{\frac{4}{2+\mu}}\|u\|^{\frac{12}{2+\mu}}, \quad \forall\, \|u\| \le \sqrt{\pi\mu/(1+\beta_0)}.
\end{aligned} \tag{2.14}$$

Applying the Hardy–Littlewood–Sobolev (HLS) inequality as presented in [24] and (2.14), we obtain the following estimate for the nonlocal term:

$$\begin{aligned}
\Psi(u) &\le \frac{\mathcal{B}_0}{2}\left(\int\limits_B |F(u)|^{\frac{4}{2+\mu}}\mathrm{d}x\right)^{\frac{2+\mu}{2}} \\
&\le \frac{\mathcal{B}_0}{2}\left[2\gamma_2^2\|u\|^2 + \left(C_0^{\frac{\mu}{4}}C_1\gamma_6^3\right)^{\frac{4}{2+\mu}}\|u\|^{\frac{12}{2+\mu}}\right]^{\frac{2+\mu}{2}} \\
&\le 2\mathcal{B}_0\gamma_2^2\|u\|^{2+\mu} + \mathcal{B}_0\left(C_0^{\frac{\mu}{4}}C_1\gamma_6^3\right)^2\|u\|^6, \quad \forall\, \|u\| \le \sqrt{\pi\mu/(1+\beta_0)},
\end{aligned} \tag{2.15}$$

where $\mathcal{B}_0 = \mathcal{B}(2-\mu, 4/(2+\mu), 4/(2+\mu))$. By substituting the upper bound (2.15) into the definition of the energy functional (2.6), it follows that for any $u \in E^+$ with $\|u\| \le \sqrt{\pi\mu}/(1+\beta_0)$,

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \Psi(u) \ge \frac{1}{2}\|u\|^2 - 2\mathcal{B}_0\gamma_2^2\|u\|^{2+\mu} - \mathcal{B}_0\left(C_0^{\frac{\mu}{4}}C_1\gamma_6^3\right)^2\|u\|^6.$$

Consequently, for a sufficiently small radius $\bar{\rho} \in (0,1)$, we conclude that the lower bound in (2.12) is satisfied. □

Following the results in [39], we state the next auxiliary lemma regarding the geometric structure of the functional.

Lemma 2.2. *Suppose that conditions* (V) *and* (F1)–(F3) *are satisfied. For any fixed* $e \in E^+$*, there exists a radius* $r_0 > \rho$ *such that* $\sup \Phi(\partial Q) \le 0$*, where*

$$Q = \left\{v + se : v \in E^-, 0 \le s \le r_0, \|v\| \le r_0\right\}.$$

To address the strongly indefinite nature of the planar problem, we adopt the approximation framework established in [10, 40]. Let $\{e_k\}$ denote a total orthonormal sequence in E^-. For each $k \in \mathbb{N}$, we define the finite-dimensional subspaces $E_k^- := \operatorname{span}\{e_1, e_2, \dots, e_k\}$ and set $E_k := E_k^- \oplus E^+$.

As a direct consequence of Lemma 2.2, the following property for the truncated subspaces holds.

Lemma 2.3. *Assume that* (V), (F1), (F2) *and* (F3) *hold. For* $e \in \partial B \cap E^+$*, there exists* $r_0 > \bar{\rho}$ *(with* $\bar{\rho}$ *given by Lemma 2.1) such that* $\sup \Phi(\partial Q_k) \le 0$*, where*

$$Q_k = \left\{v + se : v \in E_k^-, 0 \le s \le r_0, \|v\| \le r_0\right\}, \quad k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, let

$$\Gamma_k := \left\{\gamma \in \mathcal{C}(Q_k, E) : \gamma|_{\partial Q_k} = \mathrm{id}\right\}$$

and define the minimax levels as

$$c_k := \inf_{\gamma \in \Gamma_k} \sup_{u \in Q_k} \Phi(\gamma(u)). \tag{2.16}$$

By combining the definition of c_k with the estimates from Lemmas 2.1 and 2.3, the following bounds can be established.

Lemma 2.4. *Assume that* (V), (F1), (F2) *and* (F3) *hold. Then*

$$\kappa_0 \le c_k \le \frac{r_0^2}{2}, \quad \forall\, k \in \mathbb{N},$$

where κ_0 *is given by Lemma 2.1.*

By invoking the critical point theorem of Bartolo *et al.* [7] in conjunction with the geometric properties established in 2.1 and 2.3, we obtain Cerami sequences for the functional Φ on the subspaces E_k as follows.

Lemma 2.5. *Under the hypotheses* (V), (F1) , (F2) *and* (F3)*, for each $k \in \mathbb{N}$, there exists a sequence $\{u_n^k\} \subset E_k$ satisfying*

$$\Phi(u_n^k) \to c_k, \quad \|\Phi'(u_n^k)\|_{E_k^*}(1 + \|u_n^k\|) \to 0, \quad n \to \infty, \tag{2.17}$$

where c_k is defined by (2.16)*.*

In view of (F1) and (F2), there exists a constant $\delta_2 > 0$ such that

$$\frac{|f(t)|}{|t|^{\mu/2}} \leq \frac{1}{4\gamma_2^{(2+\mu)/2}} \left(\frac{\theta - 1}{r_0^2 \mathcal{B}_0}\right)^{\frac{1}{2}}, \quad \forall\, |t| \leq \delta_2. \tag{2.18}$$

By combining (2.18) with the argument in [40, Lemma 4.5], we establish the uniform boundedness of the sequences $\{u_n^k\}$ as follows.

Lemma 2.6. *Suppose that the assumptions* (V) *and* (F1)–(F3) *hold. If $\{u_n^k\}$ is a sequence satisfying* (2.17)*, then*

$$\|u_n^k\| \leq \max\left\{\frac{2\theta r_0(\delta_2 + 2\mathcal{K}_0\gamma_2)}{(\theta - 1)\delta_2}, 1\right\} + o_n(1), \quad \forall\, k \in \mathbb{N},$$

where γ_2 and δ_2 are given by (2.7) *and* (2.18)*, respectively.*

3. MINIMAX LEVEL ESTIMATES

This section is devoted to establishing the minimax estimates for the energy functional associated with problem (1.1) in the framework of MT_α^1. Following [10, Lemma 2.3], the following inequality holds for any $v \in E^-$:

$$\|\nabla v\|_\infty + \|v\|_\infty \leq \mathcal{C}_0 \|v\|, \quad \forall\, v \in E^-. \tag{3.1}$$

Assuming $V(0) < 0$ without loss of generality, condition (V) allows us to determine a constant $\rho \in (0, 4/\|V\|_\infty)$ such that

$$V(x) \leq 0, \quad |x| \leq \rho.$$

Following the construction in [34], we introduce the Moser-type functions $w_n(x)$ supported in $B_\rho \subset B$ as follows:

$$w_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leq |x| \leq \rho/n, \\ \frac{\log(\rho/|x|)}{\sqrt{\log n}}, & \rho/n \leq |x| \leq \rho, \\ 0, & \rho \leq |x| \leq 1. \end{cases} \tag{3.2}$$

It is straightforward to verify that $w_n \in E$. Moreover, a direct calculation yields

$$\left\|w_n^+\right\|^2 - \left\|w_n^-\right\|^2 < \int_B |\nabla w_n|^2 \,\mathrm{d}x = 1. \tag{3.3}$$

Lemma 3.1. *Under the assumptions* (V), (F1)–(F3) *and* (F5), *one can find an index* $\bar{n} \in \mathbb{N}$ *ensuring that*

$$\max_{s\geq 0, v\in E^-} \Phi(v + sw_n) < \frac{(2+\mu)\pi}{\beta_0}.$$

Proof. Suppose, for the sake of contradiction, that

$$\max_{s\geq 0, v\in E^-} \Phi(v + sw_n) \geq \frac{(2+\mu)\pi}{\beta_0}, \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

For each $n \in \mathbb{N}$, let $v_n \in E^-$ and $s_n > 0$ be the sequences that attain the maximum $\Phi(v_n + s_n w_n) = \max_{s\geq 0, v\in E^-} \Phi(v + sw_n)$. Then, the following conditions must hold:

$$\Phi(v_n + s_n w_n) \geq (2+\mu)\pi/\beta_0 \quad \text{and} \quad \langle \Phi'(v_n + s_n w_n), v_n + s_n w_n\rangle = 0.$$

Specifically, we have

$$\frac{1}{2}\left(s_n^2 \left\|w_n^+\right\|^2 - \left\|v_n + s_n w_n^-\right\|^2\right) - \Psi(x, v_n + s_n w_n) \geq \frac{(2+\mu)\pi}{\beta_0}.$$

In light of the definitions and estimates in (2.2), (2.4), (3.1) and (3.2), it follows that

$$\begin{aligned} \left|\left(w_n^-, v_n\right)\right| = |(w_n, v_n)| &= \left|\int\limits_B [\nabla w_n \nabla v_n + V(x) w_n v_n]\,\mathrm{d}x\right| \\ &\leq \|\nabla v_n\|_\infty \int\limits_B |\nabla w_n|\,\mathrm{d}x + \|V\|_\infty \|v_n\|_\infty \int\limits_B |w_n|\,\mathrm{d}x \leq \frac{\sqrt{2\pi}\mathcal{C}_0\rho}{\sqrt{\log n}}\|v_n\|. \end{aligned} \tag{3.5}$$

By combining (2.2), (2.4), (2.5), (3.3) and (3.5), we obtain the following expansion:

$$\begin{aligned} &s_n^2 \left\|w_n^+\right\|^2 - \left\|v_n + s_n w_n^-\right\|^2 \\ &= s_n^2\left(\left\|w_n^+\right\|^2 - \left\|w_n^-\right\|^2\right) - \|v_n\|^2 - 2s_n\left(v_n, w_n^-\right) \\ &\leq s_n^2 - \|v_n\|^2 + \frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\| \leq s_n^2\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right). \end{aligned} \tag{3.6}$$

Applying (3.1) and (3.2) leads to the following pointwise lower bound:

$$\begin{aligned} v_n(x) + s_n w_n(x) &\geq -\|v_n\|_\infty + s_n w_n \\ &\geq -\mathcal{C}_0\|v_n\| + s_n w_n \\ &\geq s_n\left(w_n - 2\mathcal{C}_0\right). \end{aligned} \tag{3.7}$$

More specifically, by setting $M_n = \frac{1}{\sqrt{2\pi}}\sqrt{\log n}$, we infer from (3.7) that

$$v_n(x) + s_n w_n(x) \geq s_n\left(M_n - 2\mathcal{C}_0\right), \quad \forall\, x \in B_{\rho/n}.$$

According to the growth condition (F5), for any $\varepsilon > 0$, there corresponds a threshold $t_\varepsilon > 0$ such that

$$tF(t) \geq (\kappa - \varepsilon) \exp\left[\left(\beta_0 + |x|^\alpha\right) t^2\right], \quad \forall\, x \in B,\ |t| \geq t_\varepsilon. \tag{3.8}$$

Recalling the definition $B_\mu := \rho^{2+\mu}/[\mu(1+\mu)(2+\mu)]$, and utilizing the estimate (3.8) together with the integral identity

$$\int\limits_{B_{\rho/n}} \mathrm{d}x \int\limits_{B_{\rho/n}} \frac{\mathrm{d}y}{|x-y|^{2-\mu}} \geq \frac{4\pi^2 B_\mu}{n^{2+\mu}},$$

we can obtain

$$\begin{aligned}
&\Psi\left(v_n + s_n w_n\right)\\
&\geq \frac{A_\mu}{2} \int\limits_{B_{\rho/n}} \left[\int\limits_{B_{\rho/n}} \frac{1}{|x-y|^{2-\mu}} F(v_n(y) + s_n w_n(y))\mathrm{d}y\right]\\
&\quad \times F(v_n(x) + s_n w_n(x))\mathrm{d}x\\
&\geq \frac{A_\mu}{2}\left[F(-\mathcal{C}_0\|v_n\| + s_n M_n)\right]^2 \int\limits_{B_{\rho/n}} \left(\int\limits_{B_{\rho/n}} \frac{\mathrm{d}y}{|x-y|^{2-\mu}}\right)\mathrm{d}x\\
&\geq \frac{2\pi^2 A_\mu B_\mu}{n^{2+\mu}}\left[F(-\mathcal{C}_0\|v_n\| + s_n M_n)\right]^2\\
&\geq \frac{2\pi^2(\kappa-\varepsilon)^2 A_\mu B_\mu}{(-\mathcal{C}_0\|v_n\| + s_n M_n)^2 n^{2+\mu}} \exp\left[2\left(\beta_0 + |x|^\alpha\right)\left(-\mathcal{C}_0\|v_n\| + s_n M_n\right)^2\right],
\end{aligned}$$

which together with (2.8), (3.2), (3.6) implies that

$$\begin{aligned}
&\Phi\left(v_n + s_n w_n\right)\\
&= \frac{1}{2}\left(s_n^2\left\|w_n^+\right\|^2 - \left\|v_n + s_n w_n^-\right\|^2\right) - \Psi\left(v_n + s_n w_n\right)\\
&\leq \frac{s_n^2}{2} - \frac{1}{2}\|v_n\|^2 + \frac{\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\| - \frac{2\pi^2 A_\mu B_\mu}{n^{2+\mu}}\left[F(-\mathcal{C}_0\|v_n\| + s_n M_n)\right]^2\\
&\leq \frac{s_n^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\\
&\quad - \frac{2\pi^2(\kappa-\varepsilon)^2 A_\mu B_\mu}{(-\mathcal{C}_0\|v_n\| + s_n M_n)^2 n^{2+\mu}} \exp\left[2\left(\beta_0 + |x|^\alpha\right)\left(-\mathcal{C}_0\|v_n\| + s_n M_n\right)^2\right].
\end{aligned} \tag{3.9}$$

Since κ satisfies the strict inequality in (F5), one can fix a sufficiently small $\varepsilon > 0$ such that the following two conditions are simultaneously satisfied:

$$\frac{\pi(\kappa-\varepsilon)A_\mu B_\mu}{(1+\varepsilon)^2} > \frac{1}{2e\beta_0^2} \tag{3.10}$$

and

$$\log \frac{(1+\varepsilon)^2}{2\pi(\kappa-\varepsilon)\beta_0^2 A_\mu B_\mu} < 1. \tag{3.11}$$

Defining $P(n) = \sqrt{2\pi}[\beta_0 + (\rho/n)^\alpha]^{-1/2}$, we simply write P hereafter for brevity. We consider four distinct cases. In the following analysis, all estimates are assumed to hold for sufficiently large $n \in \mathbb{N}$.

Case 1. $s_n \in [0, P]$. By substituting (F4), (2.6) and (3.6) into the expression of the functional, we obtain

$$\begin{aligned}\Phi(v_n + s_n w_n) &= \frac{1}{2}\left(s_n^2\left\|w_n^+\right\|^2 - \left\|v_n + s_n w_n^-\right\|^2\right) - \Psi(v_n + s_n w_n)\\ &\leq \frac{1}{2}\left(s_n^2 - \|v_n\|^2 + \frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|\right)\\ &\leq \frac{1}{2}s_n^2\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \leq \frac{(2+\mu)\pi}{\beta_0} + O\left(\frac{1}{\log n}\right),\end{aligned}$$

which results in a contradiction with the assumption (3.4).

Case 2. $s_n \in \left[P, \sqrt{2}P\right]$. In this case, $s_n w_n(x) \geq t_\varepsilon$ for $x \in B_{\rho/\sqrt{n}}$ whenever n is large. It follows from (2.6) and (3.9) that

$$\begin{aligned}&\Phi(v_n + s_n w_n)\\ &\leq \frac{s_n^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\\ &\quad - \frac{2\pi^2(\kappa-\varepsilon)^2 A_\mu B_\mu}{(-\mathcal{C}_0\|v_n\| + s_n M_n)^2 n^{2+\mu}}\exp\left[2\left(\beta_0 + |x|^\alpha\right)\left(-\mathcal{C}_0\|v_n\| + s_n M_n\right)^2\right]\\ &\leq \frac{s_n^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\\ &\quad - \frac{\pi^2(\kappa-\varepsilon)^2 A_\mu B_\mu}{n^{2+\mu}\log n}\left[\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right]\exp\left[\pi^{-1}\left(\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right)s_n^2\log n\right]\\ &:= \varphi_n(s_n).\end{aligned} \tag{3.12}$$

Let $\bar{s}_n > 0$ be the critical point such that $\varphi_n'(\bar{s}_n) = 0$. This yields

$$\begin{aligned}1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n} &= \frac{2\pi(\kappa-\varepsilon)^2 A_\mu B_\mu}{n^{2+\mu}}\left[\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right]^2\\ &\qquad \times \exp\left[\pi^{-1}\left(\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right)\bar{s}_n^2\log n\right].\end{aligned} \tag{3.13}$$

It follows (3.13) that

$$\lim_{n\to\infty}\bar{s}_n^2 = \lim_{n\to\infty}(2+\mu)\pi\left(\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right)^{-1} = \frac{(2+\mu)\pi}{\beta_0}. \tag{3.14}$$

Let

$$D_1 = \log\frac{1+\varepsilon}{2\pi(\kappa-\varepsilon)A_\mu B_\mu\beta_0^2}, \tag{3.15}$$

then (3.10) and (3.11) imply

$$\max\{D_1, 0\} - 1 < 0. \tag{3.16}$$

From (3.14) and (3.15), we have

$$\begin{aligned}\bar{s}_n^2 = &\ \frac{(2+\mu)\pi}{\beta_0+(\rho/n)^\alpha} \\ &\times\left[1+\frac{\log\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)-\log\left[2\pi(\kappa-\varepsilon)^2A_\mu B_\mu\left(\beta_0+\left(\frac{\rho}{n}\right)^\alpha\right)^2\right]}{(2+\mu)\log n}\right] \\ \le&\ \frac{(2+\mu)\pi}{\beta_0+(\rho/n)^\alpha}+\frac{\pi D_1}{(\beta_0+(\rho/n)^\alpha)\log n},\end{aligned}$$

which together with (3.12), (3.13) implies that

$$\begin{aligned}\varphi_n(s_n)&\le\varphi_n\left(\bar{s}_n\right) \\ &\le\frac{(2+\mu)\pi}{2\left(\beta_0+(\rho/n)^\alpha\right)}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\ &\quad+\frac{\pi\left[\max\{D_1,0\}-1\right]}{2\left(\beta_0+(\rho/n)^\alpha\right)\log n}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\ &\le\frac{(2+\mu)\pi(1+\varepsilon)}{2\left(\beta_0+(\rho/n)^\alpha\right)}+\frac{\pi\left[\max\{D_1,0\}-1\right]}{2\left(\beta_0+n^{-\alpha}\right)\log n}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right).\end{aligned}\tag{3.17}$$

Then, from (3.12), (3.16) and (3.17), one has

$$\begin{aligned}\Phi\left(v_n+s_nw_n\right)&\le\frac{(2+\mu)\pi(1+\varepsilon)}{2\left(\beta_0+(\rho/n)^\alpha\right)}+\frac{\pi\left[\max\{D_1,0\}-1\right]}{2\left(\beta_0+n^{-\alpha}\right)\log n}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\ &<\frac{(2+\mu)\pi}{\beta_0}.\end{aligned}$$

This contradicts with (3.4).

Case 3. $t\in\left[\sqrt{2}P,\sqrt{2(1+\varepsilon)}P\right]$. Then $tw_n(x)\ge t_\varepsilon$ for $x\in B_{\rho/\sqrt{n}}$ and for large $n\in\mathbb{N}$. It follows from (F4) and (3.8) that

$$\begin{aligned}&\Phi\left(v_n+s_nw_n\right) \\ &\le\frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\ &\quad-\frac{2\pi^2(\kappa-\varepsilon)^2A_\mu B_\mu}{(-\mathcal{C}_0\|v_n\|+s_nM_n)^2n^{2+\mu}}\exp\left[2\left(\beta_0+|x|^\alpha\right)(-\mathcal{C}_0\|v_n\|+s_nM_n)^2\right] \\ &\le\frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\ &\quad-\frac{\pi^2(\kappa-\varepsilon)^2A_\mu B_\mu}{(1+\varepsilon)n^{2+\mu}\log n}\left[\beta_0+\left(\frac{\rho}{n}\right)^\alpha\right]\exp\left[\pi^{-1}\left(\beta_0+\left(\frac{\rho}{n}\right)^\alpha\right)s_n^2\log n\right] \\ &:=\psi_n(s_n).\end{aligned}$$

Let $\tilde{s}_n > 0$ such that $\psi_n'(\tilde{s}_n) = 0$. Then

$$1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n} = \frac{2\pi(\kappa-\varepsilon)^2 A_\mu B_\mu}{(1+\varepsilon)n^{2+\mu}} \left[\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right]^2 \times \exp\left[\pi^{-1}\left(\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right)\tilde{s}_n^2 \log n\right]. \tag{3.18}$$

It follows (3.18) that

$$\lim_{n\to\infty} \tilde{s}_n^2 = \lim_{n\to\infty} (2+\mu)\pi\left(\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right)^{-1} = \frac{(2+\mu)\pi}{\beta_0}. \tag{3.19}$$

Let

$$D_2 = \log\frac{(1+\varepsilon)^2}{2\pi(\kappa-\varepsilon)A_\mu B_\mu \beta_0^2}, \tag{3.20}$$

then (3.10) and (3.11) imply

$$\max\{D_2, 0\} - 1 < 0. \tag{3.21}$$

From (3.19) and (3.20), we have

$$\begin{aligned}\tilde{s}_n^2 &= \frac{(2+\mu)\pi}{\beta_0 + (\rho/n)^\alpha}\left[1 + \frac{\log(1+\varepsilon)\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{(2+\mu)\log n}\right. \\ &\quad \left. - \frac{\log\left[2\pi(\kappa-\varepsilon)^2 A_\mu B_\mu\left(\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right)^2\right]}{(2+\mu)\log n}\right] \\ &\le \frac{(2+\mu)\pi}{\beta_0 + (\rho/n)^\alpha} + \frac{\pi D_2}{(\beta_0 + (\rho/n)^\alpha)\log n},\end{aligned}$$

which together with (3.18) implies that

$$\begin{aligned}\psi_n(s_n) &\le \psi_n(\tilde{s}_n) \\ &= \frac{1}{2}\tilde{s}_n^2\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) - \frac{\pi\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{2\left(\beta_0 + (\rho/n)^\alpha\right)\log n} \\ &\le \frac{(2+\mu)\pi(1+\varepsilon)}{2\left(\beta_0 + (\rho/n)^\alpha\right)} + \frac{\pi\left[\max\{D_2,0\} - 1\right]}{2\left(\beta_0 + n^{-\alpha}\right)\log n}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right).\end{aligned} \tag{3.22}$$

Then, from (3.21) and (3.22), one has

$$\begin{aligned}\Phi(v_n + s_n w_n) &\le \frac{(2+\mu)\pi(1+\varepsilon)}{2\left(\beta_0 + (\rho/n)^\alpha\right)} + \frac{\pi\left[\max\{D_2,0\} - 1\right]}{2\left(\beta_0 + n^{-\alpha}\right)\log n}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\ &< \frac{(2+\mu)\pi}{\beta_0}.\end{aligned}$$

This contradicts with (3.4).

Case 4. $t \in \left(\sqrt{2(1+\varepsilon)}P, +\infty\right)$. For sufficiently large $n \in \mathbb{N}$, the estimate $tw_n(x) \geq t_\epsilon$ holds for $x \in B_{1/n}$. It then follows from (2.6) and (3.3) that

$$\begin{aligned}
&\Phi\left(v_n + s_n w_n\right) \\
&\leq \frac{s_n^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) - \frac{2\pi^2(\kappa-\varepsilon)^2 A_\mu B_\mu}{s_n^2 M_n^2 n^{2+\mu}} \exp\left[2\left(\beta_0 + |x|^\alpha\right) s_n^2 M_n^2\right] \\
&\leq \frac{2\pi(1+\varepsilon)}{\beta_0 + (\rho/n)^\alpha}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\
&\quad - \frac{\pi^2(\kappa-\varepsilon)^2 A_\mu B_\mu}{(1+\varepsilon)n^{2+\mu}\log n}\left[\beta_0 + \left(\frac{\rho}{n}\right)^\alpha\right] \\
&\quad \times \exp\left[4(1+\varepsilon)\left(\beta_0 + |x|^\alpha\right)\left(\beta_0 + (\rho/n)^\alpha\right)^{-1}\right] \\
&\leq \frac{(2+\mu)\pi}{\beta_0}.
\end{aligned}$$

This leads to a contradiction with (3.4). The preceding derivation exploits the fact that the function

$$\frac{s_n^2}{2}\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) - \frac{2\pi^2(\kappa-\varepsilon)^2 A_\mu B_\mu}{s_n^2 M_n^2 n^{2+\mu}} \exp\left[2\left(\beta_0 + |x|^\alpha\right) s_n^2 M_n^2\right]$$

is monotonically decreasing on the interval $t \in (\sqrt{2(1+\epsilon)}P, \infty)$, since its stationary points converge to P as $n \to \infty$. $\square$

By setting $e = w_n^+/\|w_n^+\|$ in the context of 2.3, the following result is established as a consequence of Lemma 3.1.

Lemma 3.2. *Assume that hypotheses* (V), (F1), (F2), (F3) *and* (F5) *are satisfied. Then we have*

$$\sup_{k\in\mathbb{N}} c_k < (2+\mu)\pi/\beta_0.$$

4. THE PROOF OF MAIN RESULT

Proof of Theorem 1.1. Combining Lemmas 2.4 and 3.2, there exists a subsequence $\{c_{k_n}\} \subset \{c_k\}$ and a corresponding limit $\tilde{c} \in [\kappa_0, (2+\mu)\pi/\beta_0)$ satisfying $\lim_{n\to\infty} c_{k_n} = \tilde{c}$. According to Lemma 2.5, we can select a subsequence $\{u_{j_n}^{k_n}\}$ with $u_{j_n}^{k_n} \in E_{k_n}$ such that

$$\Phi(u_{j_n}^{k_n}) \to \tilde{c}, \quad \|\Phi'(u_{j_n}^{k_n})\|_{E_{k_n}^*}(1 + \|u_{j_n}^{k_n}\|) \to 0. \tag{4.1}$$

To simplify the notation, we hereafter denote $u_{j_n}^{k_n}$ by $\tilde{u}_n$. Invoking (4.1), Lemma 2.4 and [10, Lemma 4.5], it follows that $\{\tilde{u}_n\}$ is bounded in E and

$$\Phi\left(\tilde{u}_n\right) \to \tilde{c}, \quad \|\Phi'\left(\tilde{u}_n\right)\|_{E_{k_n}^*}\left(1 + \|\tilde{u}_n\|\right) \to 0. \tag{4.2}$$

Consequently, there exists a constant $C_5 > 0$ such that $\|\tilde{u}_n\|_2 \leq C_5$. In light of (F3), (2.6), (2.10), Lemmas 2.4 and 2.5, we obtain the estimate

$$\int_B [I_\mu * F(\tilde{u}_n)]\, F(\tilde{u}_n)\mathrm{d}x \leq \int_B [I_\mu * F(\tilde{u}_n)]\, f(\tilde{u}_n)\tilde{u}_n\mathrm{d}x \leq C_6. \tag{4.3}$$

Suppose that the vanishing condition holds, i.e.,

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} |\tilde{u}_n|^2 \ \mathrm{d}x = 0.$$

Then, Lions' concentration compactness principle [46, Lemma 1.21] implies that $\tilde{u}_n \to 0$ in $L^s(\mathbb{R}^2)$ for all $s \in (2,\infty)$. For any fixed $\epsilon > 0$, by choosing $M_\epsilon > M_0C_5/\epsilon$, it follows from (F4) and (4.3) that

$$\begin{aligned}\int_{|\tilde{u}_n|\geq M_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, F(\tilde{u}_n)\mathrm{d}x &\leq M_0 \int_{|\tilde{u}_n|\geq M_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, |f(\tilde{u}_n)|\mathrm{d}x \\ &\leq \frac{M_0}{M_\varepsilon} \int_{|\tilde{u}_n|\geq M_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, f(\tilde{u}_n)\tilde{u}_n\mathrm{d}x \\ &< \varepsilon.\end{aligned} \tag{4.4}$$

Furthermore, invoking (F2) and (F3), there exists $N_\epsilon \in (0,1)$ ensuring that

$$\begin{aligned}&\int_{|\tilde{u}_n|\leq N_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, F(\tilde{u}_n)\mathrm{d}x \\ &\leq \int_{|\tilde{u}_n|\leq N_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, f(\tilde{u}_n)\tilde{u}_n\mathrm{d}x \\ &\leq \frac{\varepsilon}{C_5^{(2+\mu)/2}} \left(\int_B [I_\mu * F(\tilde{u}_n)]\, F(\tilde{u}_n)\mathrm{d}x\right)^{\frac{1}{2}} \\ &\quad\times \left(\int_B \left(I_\mu * |\tilde{u}_n|^{(2+\mu)/2}\right) |\tilde{u}_n|^{(2+\mu)/2}\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{C_6\mathcal{C}_0}\varepsilon}{C_5^{(2+\mu)/2}} \|\tilde{u}_n\|_2^{(2+\mu)/2} \leq \sqrt{C_6\mathcal{B}_0}\varepsilon.\end{aligned} \tag{4.5}$$

By taking (F1), (F3), and (4.3) into account, we have

$$\begin{aligned}
&\int\limits_{N_\varepsilon \le |\tilde{u}_n| \le M_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, F(\tilde{u}_n) \mathrm{d}x \\
&\le \int\limits_{N_\varepsilon \le |\tilde{u}_n| \le M_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, f(\tilde{u}_n) \tilde{u}_n \mathrm{d}x \\
&\le C_7 \int\limits_{N_\varepsilon \le |\tilde{u}_n| \le M_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, |\tilde{u}_n|^3 \mathrm{d}x \qquad (4.6)\\
&\le C_7 \left(\int\limits_B [I_\mu * F(\tilde{u}_n)]\, F(\tilde{u}_n) \mathrm{d}x \right)^{\frac{1}{2}} \left(\int\limits_B \left(I_\mu * |\tilde{u}_n|^3\right) |\tilde{u}_n|^3 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\le \sqrt{C_6} C_7 \|\tilde{u}_n\|^3_{12/(2+\mu)} = o(1).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude from (4.4)–(4.6) that

$$\int\limits_B [I_\mu * F(\tilde{u}_n)]\, F(\tilde{u}_n)\, \mathrm{d}x = o(1). \qquad (4.7)$$

Thus, by (2.8), (4.7) and (4.2), one has

$$\left\|\tilde{u}_n^+\right\|^2 - \left\|\tilde{u}_n^-\right\|^2 = 2\tilde{c} + o(1). \qquad (4.8)$$

From (F1), (F2), (2.10), (4.3), (4.5) and the arguments in [40, (4.72)–(4.73)], we derive

$$\begin{aligned}
\left\|\tilde{u}_n^-\right\|^2 &= -\int\limits_B [I_\mu * F(\tilde{u}_n)]\, f(\tilde{u}_n)\, \tilde{u}_n^- \mathrm{d}x + o(1) \\
&\le \int\limits_{|\tilde{u}_n| \le N_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, |f(\tilde{u}_n)\, |\tilde{u}_n^-| \mathrm{d}x \\
&\quad + \int\limits_{N_\varepsilon \le |\tilde{u}_n| \le M_\varepsilon} |\, [I_\mu * F(\tilde{u}_n)]\, f(\tilde{u}_n)\, |\tilde{u}_n^-| \mathrm{d}x \qquad (4.9)\\
&\quad + \frac{\|\tilde{u}_n^-\|_\infty}{M_\varepsilon} \int\limits_{|\tilde{u}_n| \ge M_\varepsilon} [I_\mu * F(\tilde{u}_n)]\, f(\tilde{u}_n)\, \tilde{u}_n \ \mathrm{d}x + o(1) \\
&\le \sqrt{C_6 \mathcal{C}_0}\varepsilon + \frac{\mathcal{C}_0}{M_0} \left\|\tilde{u}_n^-\right\| \varepsilon + o(1) \\
&\le C_8 \varepsilon + o(1),
\end{aligned}$$

yielding

$$\left\|\tilde{u}_n^-\right\|^2 = o(1). \qquad (4.10)$$

Integrating the estimates from (4.8) and (4.10), it follows that

$$\|\tilde{u}_n\|^2 = \left\|\tilde{u}_n^+\right\|^2 + \left\|\tilde{u}_n^-\right\|^2 = 2\tilde{c} + o(1) := \frac{(2+\mu)\pi}{\beta_0}(1-3\bar{\varepsilon}) + o(1).$$

Adopting the strategy from [9], we select $\mu > 0$ such that $\|V\|_\infty/(\mu - \|V\|_\infty) < \bar{\epsilon}$. Let $\tilde{u}_n^+$ be decomposed as $v_n + z_n$, where $v_n \in \mathcal{E}(\mu)E$ and $z_n \in [\mathrm{id} - \mathcal{E}(\mu)]E$. Analogous to the derivation of (4.9), an application of (F1), (F2), (2.10), (4.2) and (4.3) implies that

$$\|v_n\|^2 = \langle \Phi'(\tilde{u}_n), v_n\rangle + \int\limits_B [I_\mu * F(\tilde{u}_n)]\, f(\tilde{u}_n) v_n \mathrm{d}x = o(1). \tag{4.11}$$

Hence, it follows from (4.10) and (4.11) that

$$\|\tilde{u}_n - z_n\|^2 = o(1), \quad \|\nabla \tilde{u}_n\|_2^2 = \|\nabla z_n\|_2^2 + o(1).$$

Due to $z_n \in [id - \mathcal{E}(\mu)]E$, one has

$$\|z_n\|^2 = \int\limits_{\mathbb{R}^2} \left[|\nabla z_n|^2 + V(x) z_n^2\right] \mathrm{d}x = (\mathcal{A} z_n, z_n)_{L^2} \geq \mu \|z_n\|_2^2, \tag{4.12}$$

which means

$$\|\nabla z_n\|_2^2 \geq (\mu - \|V\|_\infty) \|z_n\|_2^2. \tag{4.13}$$

In view of (4.12) and (4.13), we have

$$\begin{aligned} \|z_n\|^2 &\geq \|\nabla z_n\|_2^2 - \|V\|_\infty \|z_n\|_2^2 \\ &\geq \left(1 - \frac{\|V\|_\infty}{\mu - \|V\|_\infty}\right) \|\nabla z_n\|_2^2 \geq (1-\bar{\varepsilon}) \|\nabla z_n\|_2^2. \end{aligned} \tag{4.14}$$

From (4.8) and (4.14), we get

$$\|\tilde{u}_n\|^2 = \|z_n\|^2 + o(1) \geq (1-\bar{\varepsilon}) \|\nabla z_n\|_2^2 + o(1) = (1-\bar{\varepsilon}) \|\nabla \tilde{u}_n\|_2^2 + o(1).$$

We then fix a parameter $q \in (1,2)$ to satisfy

$$\frac{(1+\bar{\varepsilon})^2(1-3\bar{\varepsilon})q^2}{1-\bar{\varepsilon}} < 1. \tag{4.15}$$

In view of (F1), one can find a positive constant C_9 ensuring that

$$|f(t)|^q \leq C_9 \exp\left[(1+\bar{\varepsilon})q\left(\beta_0 + |x|^\alpha\right) t^2\right], \quad \forall\, t \in \mathbb{R},$$

which together with (1.2), (4.15) and the Young's inequality implies that

$$
\begin{aligned}
&\int\limits_{|\tilde{u}_n|\geq 1} |f(\tilde{u}_n)|^{\frac{4q}{2+\mu}}\,\mathrm{d}x \\
&\leq C_9 \int\limits_{|\tilde{u}_n|\geq 1} \exp\left[\frac{4(1+\bar{\varepsilon})q\,(\beta_0+|x|^\alpha)\,\tilde{u}_n^2}{2+\mu}\right]\mathrm{d}x \\
&\leq C_9 \left\{\int\limits_{|\tilde{u}_n|\geq 1} \exp\left[\frac{4(1+\bar{\varepsilon})^2 q^2\beta_0\|\tilde{u}_n\|^2}{2+\mu}\left(\frac{\tilde{u}_n}{\|\tilde{u}_n\|}\right)^2\right]\mathrm{d}x\right\}^{1/q} \\
&\quad\times\left\{\int\limits_{|\tilde{u}_n|\geq 1} \exp\left[\frac{4(1+\bar{\varepsilon})^2 qq'|x|^\alpha u_n^2}{2+\mu}\right]\mathrm{d}x\right\}^{1/q'} \qquad (4.16)\\
&\leq (\mathrm{MT}_{\check{q}})^{1/q}\left\{2\pi\int_0^\delta \exp\left[\frac{4(1+\bar{\varepsilon})^2 qq' r^\alpha\tilde{u}_n^2}{2+\mu}\right] r\mathrm{d}r\right. \\
&\quad\left.+2\pi\int_\delta^1 r(1-r)^{-\bar{\varepsilon}\rho q\hat{q}}\mathrm{d}r\right\}^{1/q'} \\
&\leq (\mathrm{MT}_{\check{q}})^{1/q}\left[\delta\exp\left(\bar{\varepsilon}q\hat{q}\sup_{r\in(0,\delta)} g(r)\right)+\frac{2\pi(1-\delta)^{1-\bar{\varepsilon}\rho q\hat{q}}}{1-\bar{\varepsilon}\rho q\hat{q}}\right]^{1/q'},
\end{aligned}
$$

where q' denotes the conjugate exponent $q/(q-1)$ and

$$
\hat{q} = \frac{4(1+\bar{\varepsilon})^2 q}{\bar{\varepsilon}(2+\mu)(q-1)}, \qquad \check{q} := \frac{4(1+\bar{\varepsilon})^2 q^2\beta_0\|\tilde{u}_n\|^2}{2+\mu}, \qquad g(r) = \frac{-r^\alpha\log r}{2\pi}.
$$

Recalling (4.16), we obtain

$$
\begin{aligned}
&\int\limits_{|\tilde{u}_n|\geq 1} [I_\mu * F(\tilde{u}_n)]\,f(\tilde{u}_n)\tilde{u}_n\mathrm{d}x \\
&\leq \sqrt{C_3\mathcal{C}_0}\left[\int\limits_{|\tilde{u}_n|\geq 1} |f(\tilde{u}_n)|^{4q/(2+\mu)}\right]^{\frac{2+\mu}{4q}}\left[\int\limits_{|\tilde{u}_n|\geq 1} |\tilde{u}_n|^{4q'/(2+\mu)}\right]^{\frac{2+\mu}{4q'}} \qquad (4.17)\\
&\leq C_{10}\|\tilde{u}_n\|_{4q'/(2+\mu)} = o(1).
\end{aligned}
$$

Consequently, merging the results from (2.10), (4.5), (4.6), (4.2) and (4.17), one finds that

$$\begin{aligned}\tilde{c} + o(1) &= \Phi(\tilde{u}_n) - \frac{1}{2}\langle \Phi'(\tilde{u}_n), \tilde{u}_n\rangle \\ &= \frac{1}{2}\int\limits_B [I_\mu * F(\tilde{u}_n)]\,[f(\tilde{u}_n)\tilde{u}_n - F(\tilde{u}_n)]\,\mathrm{d}x < \varepsilon + o(1).\end{aligned}$$

This contradiction establishes that $\delta > 0$. By passing to a subsequence if necessary, there exists a sequence of translations $\{y_n\} \subset \mathbb{Z}^2$ such that

$$\int\limits_{B_{1+\sqrt{2}}(y_n)} |\tilde{u}_n|^2 \ \mathrm{d}x > \frac{\delta}{2}.$$

Let $\tilde{v}_n(x) = \tilde{u}_n(x + y_n)$ denote the translated sequence, which satisfies

$$\int\limits_{B_{1+\sqrt{2}}(0)} |\tilde{v}_n|^2 \ \mathrm{d}x > \frac{\delta}{2}. \tag{4.18}$$

The 1-periodicity of $V(x)$ and $f(x,u)$ with respect to x implies that $\|\tilde{v}_n\| = \|\tilde{u}_n\|$ and

$$\Phi(\tilde{v}_n) \to \tilde{c}, \quad \|\Phi'(\tilde{v}_n)\|_{E^*_{k_n}}(1 + \|\tilde{v}_n\|) \to 0. \tag{4.19}$$

By taking a further subsequence, it follows that $\tilde{v}_n \rightharpoonup \tilde{v}$ in E, with $\tilde{v}_n \to \tilde{v}$ in $L^s(B)$ for $s \in [2, \infty)$ and $\tilde{v}_n \to \tilde{v}$ almost everywhere on B. In particular, (4.18) ensures that the limit $\tilde{v}$ is non-trivial. For an arbitrary test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, consider the decomposition

$$\phi = \phi^+ + \sum_{j=1}^{\infty} (\phi, e_j)\, e_j, \quad \left\|\phi^-\right\|^2 = \sum_{j=1}^{\infty} |(\phi, e_j)|^2 . \tag{4.20}$$

Set

$$\phi_n = \phi^+ + \sum_{j=1}^{k_n} (\phi, e_j)\, e_j, \quad \tilde{\phi}_n = \sum_{k_n+1}^{\infty} (\phi, e_j)\, e_j. \tag{4.21}$$

For each fixed $\epsilon > 0$, it holds that

$$\begin{aligned}&\int\limits_{|\tilde{v}_n| \geq C_6 \mathcal{K}_0 \gamma_2 \|\phi^-\| \varepsilon^{-1}} [I_\mu * F(\tilde{v}_n)] |f(\tilde{v}_n)\tilde{\phi}_n| \mathrm{d}x \\ &\leq \frac{\varepsilon}{C_6} \int\limits_{|\tilde{v}_n| \geq C_6 \mathcal{K}_0 \gamma_2 \|\phi^-\| \varepsilon^{-1}} [I_\mu * F(\tilde{v}_n)] f(\tilde{v}_n)\tilde{v}_n \mathrm{d}x < \varepsilon.\end{aligned} \tag{4.22}$$

On the other hand, in view of (F1)–(F3) and the decompositions in (4.20) and (4.21), it follows that

$$\begin{aligned}
&\int\limits_{|\tilde{v}_n|<C_6\mathcal{K}_0\gamma_2\|\phi^-\|\varepsilon^{-1}} [I_\mu * F(\tilde{v}_n)]|f(\tilde{v}_n)\tilde{\phi}_n|\mathrm{d}x \\
&\le \left[\int\limits_B [I_\mu * F(\tilde{v}_n)]F(\tilde{v}_n)\mathrm{d}x\right]^{\frac{1}{2}} \\
&\quad \times \left[\int\limits_{|\tilde{v}_n|<C_6\mathcal{K}_0\gamma_2\|\phi^-\|\varepsilon^{-1}} [I_\mu * |f(\tilde{v}_n)\tilde{\phi}_n|]|f(\tilde{v}_n)\tilde{\phi}_n|\mathrm{d}x\right]^{\frac{1}{2}} \qquad (4.23)\\
&\le C_{11}\left(\int\limits_{|\tilde{v}_n|<C_6\mathcal{K}_0\gamma_2\|\phi^-\|\varepsilon^{-1}} |\tilde{v}_n|^{2\mu/(2+\mu)}|\tilde{\phi}_n|^{4/(2+\mu)}\mathrm{d}x\right)^{\frac{2+\mu}{4}} \\
&\le C_{11}\|\tilde{v}_n\|_2^{\mu/2}\|\tilde{\phi}_n\|_2 = o(1).
\end{aligned}$$

By virtue of the arbitrariness of $\epsilon > 0$, together with (4.22) and (4.23), we conclude that

$$\lim_{n\to\infty}\int\limits_B [I_\mu * F(\tilde{v}_n)]f(\tilde{v}_n)\tilde{\phi}_n\mathrm{d}x = 0.$$

Combining this with (2.10), (4.19), and [16, Lemma 2.1] we obtain

$$\begin{aligned}
\langle\Phi'(\tilde{v}),\phi\rangle &= \int\limits_{\mathbb{R}^2}(\nabla\tilde{v}\cdot\nabla\phi + V(x)\tilde{v}\phi)\,\mathrm{d}x - \int\limits_{\mathbb{R}^2}[I_\mu * F(\tilde{v})]f(\tilde{v})\phi\mathrm{d}x \\
&= \lim_{n\to\infty}\left\{\int\limits_{\mathbb{R}^2}(\nabla\tilde{v}_n\nabla\phi + V(x)\tilde{v}_n\phi)\,\mathrm{d}x - \int\limits_{\mathbb{R}^2}[I_\mu * F(\tilde{v}_n)]f(\tilde{v}_n)\phi\mathrm{d}x\right\} \\
&= \lim_{n\to\infty}\left[\langle\Phi'(\tilde{v}_n),\phi_n\rangle - \int\limits_{\mathbb{R}^2}[I_\mu * F(\tilde{v}_n)]f(\tilde{v}_n)\tilde{\phi}_n\mathrm{d}x\right] \\
&= -\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}[I_\mu * F(\tilde{v}_n)]f(\tilde{v}_n)\tilde{\phi}_n\mathrm{d}x = 0.
\end{aligned}$$

This establishes that $\tilde{v}$ constitutes a nontrivial solution to problem (1.1). $\square$

5. APPENDIX

In this section, we maintain the assumption that $V(0) < 0$ without loss of generality. Condition (V) allows us to determine a constant $\rho \in (0, 4/\|V\|_\infty)$ ensuring that

$$2(2+\mu)\pi\mathcal{C}_0^2\rho^2 < 1 \text{ and } V(x) \leq 0, \quad |x| \leq \rho, \tag{5.1}$$

where the embedding constant $\mathcal{C}_0 > 0$ is derived from the results in [9, Proposition 2.2 and Proposition 2.4] and [10, Lemma 2.3].

Lemma 5.1. *Suppose that the hypotheses* (V)*,* (F1)–(F4)*, and* (F5″) *are satisfied. Then there exists an index* $\bar{n} \in \mathbb{N}$ *such that*

$$\max_{s\geq 0, v\in E^-} \Phi\left(v + sw_n\right) < \frac{(2+\mu)\pi}{\beta_0}. \tag{5.2}$$

Proof. Suppose, for the sake of contradiction, that

$$\max_{s\geq 0, v\in E^-} \Phi\left(v + sw_n\right) \geq \frac{(2+\mu)\pi}{\beta_0}, \quad \forall\, n \in \mathbb{N}. \tag{5.3}$$

For each $n \in \mathbb{N}$, let $v_n \in E^-$ and $s_n > 0$ be the sequences achieving the maximum $\Phi(v_n + s_n w_n) = \max_{s\geq 0, v\in E^-} \Phi(v + sw_n)$. It then follows that

$$\Phi(v_n + s_n w_n) \geq (2+\mu)\pi/\beta_0 \quad \text{and} \quad \langle \Phi'(v_n + s_n w_n), v_n + s_n w_n\rangle = 0,$$

which yields

$$\begin{aligned} &\frac{1}{2}\left(s_n^2\left\|w_n^+\right\|^2 - \left\|v_n + s_n w_n^-\right\|^2\right) \\ &- \frac{1}{2}\int_B [I_\mu * F(v_n + s_n w_n)] F\left(x, v_n + s_n w_n\right) \mathrm{d}x \geq \frac{(2+\mu)\pi}{\beta_0} \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &s_n^2\left\|w_n^+\right\|^2 - \left\|v_n + s_n w_n^-\right\|^2 \\ &= \int_B [I_\mu * F(v_n + s_n w_n)] f\left(x, v_n + s_n w_n\right)\left(v_n + s_n w_n\right) \mathrm{d}x. \end{aligned} \tag{5.5}$$

Integrating (3.5), (5.4) with (5.5), we arrive at

$$\frac{(2+\mu)\pi}{\beta_0} \leq s_n^2 - \|v_n\|^2 + \frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\| \leq s_n^2\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \tag{5.6}$$

and

$$\begin{aligned} &s_n^2\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\ &\geq \int_B [I_\mu * F(v_n + s_n w_n)] f\left(x, v_n + s_n w_n\right)\left(v_n + s_n w_n\right) \mathrm{d}x. \end{aligned} \tag{5.7}$$

Moreover, (5.6) implies

$$s_n^2 \geq \frac{(2+\mu)\pi}{\beta_0}\left(1-\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right), \quad \frac{\|v_n\|}{s_n} \leq 1+\frac{2\sqrt{2\pi}\mathcal{C}_0\rho}{\sqrt{\log n}}. \tag{5.8}$$

From (3.1), (3.2) and (5.8), one has $v_n(x)+s_nw_n(x) \geq s_n\left(M_n-2\mathcal{C}_0\right)$ for all $x \in B_{\rho/n}$. According to the hypothesis $(F5'')$, one can find a threshold $t_\epsilon > 0$ ensuring that

$$tF(t) \geq \frac{\kappa_1-\varepsilon}{2\left(\beta_0+|x|^\alpha\right)}\exp\left[\left(\beta_0+|x|^\alpha\right)t^2\right], \quad \forall\, t \geq t_\varepsilon \tag{5.9}$$

and

$$f(t) \geq (\kappa_1-\varepsilon)\exp\left[\left(\beta_0+|x|^\alpha\right)t^2\right]. \tag{5.10}$$

In the following analysis, we assume that $n \in \mathbb{N}$ is sufficiently large. By combining (5.7) and (5.10), we obtain

$$\begin{aligned} & s_n^2\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\ & \geq \frac{2\pi^2(\kappa_1-\varepsilon)^2A_\mu B_\mu}{\left(\beta_0+|x|^\alpha\right)} \\ & \quad \times \exp\left\{\log n\left[\frac{\left(\beta_0+|x|^\alpha\right)s_n^2}{\pi}\left(1-\frac{4\mathcal{C}_0}{M_n}\right)-(2+\mu)\right]\right\}, \end{aligned} \tag{5.11}$$

from which we infer the existence of a constant $A > 0$ such that

$$s_n^2 \leq \frac{(2+\mu)\pi}{\beta_0+|x|^\alpha}\left(1-\frac{4\mathcal{C}_0}{M_n}\right)^{-1}\left(1+\frac{A}{\log n}\right), \tag{5.12}$$

which together with (5.8) yield

$$\lim_{n\to\infty} s_n^2 = \frac{(2+\mu)\pi}{\beta_0+|x|^\alpha}, \quad \lim_{n\to\infty}\|v_n\| = 0. \tag{5.13}$$

Similar to (5.11), one has

$$\begin{aligned} & \int_B [I_\mu * F(v_n+s_nw_n)]F\left(x, v_n+s_nw_n\right)\mathrm{d}x \\ & \geq \frac{\pi^2(\kappa_1-\varepsilon)^2A_\mu B_\mu}{n^{2+\mu}\left(\beta_0+|x|^\alpha\right)^2\left(-\mathcal{C}_0\|v_n\|+s_nM_n\right)^2} \\ & \quad \times \exp\left[2\left(\beta_0+|x|^\alpha\right)\left(-\mathcal{C}_0\|v_n\|+s_nM_n\right)^2\right]. \end{aligned}$$

Hence, it follows from (2.8), (3.2), (3.5) and (5.9), we get

$$\begin{aligned} \Phi\left(v_n+s_nw_n\right) \leq\ & \frac{s_n^2}{2}-\frac{1}{2}\|v_n\|^2+\frac{\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\| \\ & -\frac{\pi^2(\kappa_1-\varepsilon)^2A_\mu B_\mu}{2n^{2+\mu}\left(\beta_0+|x|^\alpha\right)^2}\frac{\exp\left[2\left(\beta_0+|x|^\alpha\right)\left(-\mathcal{C}_0\|v_n\|+s_nM_n\right)^2\right]}{\left(-\mathcal{C}_0\|v_n\|+s_nM_n\right)^2}. \end{aligned} \tag{5.14}$$

In view of (5.8) and (5.12), the following estimates for s_n^2 are established:

$$\frac{(2+\mu)\pi}{\beta_0+|x|^\alpha}(1-\varepsilon) \leq s_n^2 \leq \frac{(2+\mu)\pi}{\beta_0+|x|^\alpha}(1+\varepsilon).$$

We distinguish the following cases.

Case 1.

$$\frac{(2+\mu)\pi}{\beta_0+|x|^\alpha}(1-\varepsilon) \leq s_n^2 \leq \frac{(2+\mu)\pi}{\beta_0+|x|^\alpha}.$$

It follows from (5.6) that

$$\frac{\|v_n\|}{s_n} \leq \frac{4\pi\mathcal{C}_0\rho M_n}{\log n}. \tag{5.15}$$

In conjunction with (5.6),(5.14) and (5.15), we have

$$\begin{aligned}\Phi\left(v_n+s_n w_n\right) \leq \frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)&-\frac{\pi^2(\kappa_1-\varepsilon)^2 A_\mu B_\mu}{(2+\mu)n^{2+\mu}\left(\beta_0+|x|^\alpha\right)\log n}\\ &\times \exp\left[\pi^{-1}\left(\beta_0+|x|^\alpha\right)s_n^2(\log n-8\pi\mathcal{C}_0^2\rho)\right].\end{aligned}$$

We introduce an auxiliary function $\varphi_n(s)$ defined by

$$\begin{aligned}\varphi_n(s)=\frac{s^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)&-\frac{\pi^2(\kappa_1-\varepsilon)^2 A_\mu B_\mu}{(2+\mu)n^{2+\mu}\left(\beta_0+|x|^\alpha\right)\log n}\\ &\times \exp\left[\pi^{-1}\left(\beta_0+|x|^\alpha\right)s^2(\log n-8\pi\mathcal{C}_0^2\rho)\right].\end{aligned}$$

Setting $C_1 = 8(2+\mu)\pi\mathcal{C}_0^2$ and letting $\hat{s}_n > 0$ be the stationary point satisfying $\varphi_n'(\hat{s}_n)=0$, we have

$$\hat{s}_n^2=\frac{(2+\mu)\pi}{\beta_0+|x|^\alpha}\left[1+\frac{\mathcal{C}_1+\log(2+\mu)-\log(2\pi(\kappa_1-\varepsilon)^2A_\mu B_\mu)}{(2+\mu)(\log n-8\pi\mathcal{C}_0^2\rho)}\right]+O\left(\frac{1}{\log^2 n}\right)$$

and

$$\varphi_n(s_n)\leq\varphi_n(\hat{s}_n)=\frac{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}}{2}\hat{s}_n^2-\frac{\pi\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{2\left(\beta_0+|x|^\alpha\right)(\log n-8\pi\mathcal{C}_0^2\rho)}. \tag{5.16}$$

Thus we have

$$\begin{aligned}&\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\hat{s}_n^2\\ &=\frac{(2+\mu)\pi}{\beta_0+|x|^\alpha}\left[1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right.\\ &\quad\left.+\frac{\mathcal{C}_1+\log(2+\mu)-\log(2\pi(\kappa_1-\varepsilon)^2A_\mu B_\mu)}{(2+\mu)(\log n-8\pi\mathcal{C}_0^2\rho)}\right]+O\left(\frac{1}{\log^2 n}\right).\end{aligned} \tag{5.17}$$

Hence, it follows from (F5$''$), (5.16) and (5.17), we derive

$$\Phi(v_n + s_n w_n) \leq \frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}\left[\frac{1}{2} - \frac{1 - 2(2+\mu)\pi\mathcal{C}_0^2\rho^2}{2(2+\mu)\log n}\right] + O\left(\frac{1}{\log^2 n}\right).$$

This result contradicts (5.3) in light of (5.1).

Case 2.

$$\frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1 + 2\mathcal{C}_0\|v_n\|/s_n M_n) \leq s_n^2 \leq \frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1+\varepsilon).$$

Then, assumptions (F5$''$) together with (5.11) and (5.13) imply that

$$\frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1+\varepsilon/2) \geq \frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1+\varepsilon)\exp\left[4\pi(2+\mu)\mathcal{C}_0^2\right],$$

giving rise to a contradiction.

Case 3.

$$\frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1 + 2\mathcal{C}_0\|v_n\|/[(2+\rho)s_n M_n)] \leq s_n^2 \leq \frac{(2+\mu)\pi}{\beta_0}(1 + 2\mathcal{C}_0\|v_n\|/s_n M_n).$$

According to (5.6), it follows that $\frac{\|v_n\|}{s_n} \leq \frac{2\mathcal{C}_0(1+\rho)}{M_n}$, which together with (F5$''$), (5.11) and (5.13) yield that

$$\frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1+\varepsilon/2) \geq \frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1+\varepsilon) - o(1),$$

which is a contradiction.

Case 4.

$$\frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha} \leq s_n^2 \leq \frac{(2+\mu)\pi}{\beta_0}(1 + 2\mathcal{C}_0\|v_n\|/[(2+\rho)s_n M_n)].$$

By virtue of (5.6), we obtain

$$\frac{\|v_n\|}{s_n} \leq \frac{2\mathcal{C}_0(1+\rho)^2}{(2+\rho)M_n}. \tag{5.18}$$

Taking (F5$''$), (5.11), (5.13) and (5.18) into account, we arrive at

$$\begin{aligned}
\frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1+\varepsilon/2) &\geq s_n^2\left(1 + \frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\
&\geq \frac{2\pi^2(\kappa_1 - \varepsilon)^2 A_\mu B_\mu}{\beta_0 + |x|^\alpha}\exp\left[-8\pi(2+\mu)\mathcal{C}_0^2(1+\rho)^2/(2+\rho)\right] \\
&\geq \frac{(2+\mu)\pi}{\beta_0 + |x|^\alpha}(1+\varepsilon),
\end{aligned}$$

which yields a contradiction. Combining the analysis of the aforementioned four cases, we conclude that (5.2) holds. □

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