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NORMALIZED SOLUTIONS FOR p-KIRCHHOFF EQUATION WITH GENERAL NONLINEARITIES ON BOUNDED DOMAIN

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Abstract. In this paper, we study the existence of the solutions for the following p-Kirchhoff equation on bounded domain

$$\begin{cases} -\left(a + b\int\limits_{\Omega} |\nabla u|^p dx\right)\Delta_p u + \lambda |u|^{p-2}u = h(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

with prescribed mass

$$\int\limits_{\Omega} |u|^p dx = m^p,$$

where $2 \leq p < 3$, $a > 0$, $b > 0$ are positive constants, $h(u)$ is a general nonlinearity with Sobolev subcritical growth, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset \mathbb{R}^3$ is a bounded domain, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. First, we prove that the equation admits a positive solution which is a local minimizer when h is L^p-subcritical or L^p-critical at infinity by Brezis–Nirenberg technique. Moreover, we get the multiplicity result by using the genus theory. Next, we prove that the equation has a local minimizer when h is L^p-supercritical. Besides, by using the Pohozaev analysis, we prove that in this case, if Ω is a star-shaped domain with respect to the origin, the equation admits a second solution which is mountain pass type. To the best of our knowledge, this work seems to be the first contribution on the existence of normalized solutions for p-Kirchhoff equation on bounded domain.

Keywords: p-Laplacian Kirchhoff equation, normalized solutions, existence and multiplicity, bounded domain.

Mathematics Subject Classification: 35A15, 35B33, 35J20, 35J60.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the existence of normalized solution for the following p-Kirchhoff equation on bounded domain

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^p dx\right)\Delta_p u + \lambda|u|^{p-2}u = h(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

with prescribed mass

$$\int_\Omega |u|^p dx = m^p, \tag{1.2}$$

where $2 \le p < 3$, $a > 0$, $b > 0$ are positive constants, $h(u)$ is a general nonlinearity with Sobolev subcritical growth, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset \mathbb{R}^3$ is a bounded domain and $0 \in \Omega$, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier.

The Kirchhoff equation of (1.1) is originated from the D'Alembert wave equation

$$\begin{cases} u_{tt} - \left(a+b\displaystyle\int_\Omega |\nabla u|^p dx\right)\Delta u = f(t,x,u), & x \in \Omega, t > 0, \\ u(\cdot,t)|_{\partial\Omega} = 0, \quad t \ge 0, \end{cases} \tag{1.3}$$

which is used to describe the effects of the length of strings during vibrations, where u stands for the displacement, $\Omega \subset \mathbb{R}^3$ is a smooth domain, $f(x;u)$ denotes the external force, the number b is the initial tension and the number a is related to the intrinsic properties of the string. Such problems are often referred as nonlocal since the appearance of the nonlocal term $(\int_\Omega |\nabla u|^2 dx)\Delta u$, which leads the study of (1.3) more interesting and challenging. It is well-known that (1.2) was proposed by Kirchhoff in [15] as an extension of the classical D'Alembert's wave equations for free vibration of elastic strings, particularly, taking into account the changes in string length produced by transverse vibrations. We refer to Arosio-Panizzi [2] for more physical background. In equilibrium steady state, i.e., $u_t \equiv 0$, and problem (1.3) reduces to the following form:

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^p dx\right)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$. In the past decades, there are many studies on the existence and multiplicity solutions to (1.4) by applying the variational methods, we refer to the papers [1, 6, 8, 9, 22, 37] and references therein.

We notice that, in the past few years, there is an increasing interesting in the study of solutions to the Schrödinger equations, possessing a prescribed L^2-norm, which are commonly referred to as normalized solutions, we refer to [12–14, 18] and references therein. When $p = 2$, there are many studies for problem (1.1)–(1.2) with $\Omega = \mathbb{R}^3$.

For instance, in a recent paper [5], Chen and Tang investigated the existence of normalized solutions for the following Kirchhoff equation

$$\begin{cases} -\left(a + b \int\limits_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u = \lambda u + |u|^4 u + \mu |u|^{q-2} u, \quad x \in \mathbb{R}^3, \\ \int\limits_{\mathbb{R}^3} |u|^2 dx = c, \end{cases} \tag{1.5}$$

by using the Pohozaev manifold decomposition method and analytical skills. For more earlier results for the studies of problem (1.5), we refer to Zhang and Han [35], Li, Nie and Zhang [17], Li, Luo and Yang [16] and references therein. For more results on normalized solutions for quasilinear problems and Kirchhoff equations, we refer to [19, 32–34, 36, 39, 40] and references therein.

On the other hand, the existence of normalized solutions for problems on bounded domain has recently attracted much interest. Recently, Pierotti, Verzini and Yu [23] studied the existence and multiplicity of positive solutions with prescribed L^2-norm for the Sobolev critical Schrödinger equation on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, i.e.

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega, \\ \int\limits_{\Omega} u^2 dx = \rho^2, \ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

and obtained the existence of a mountain pass solution on the L^2-sphere for $\rho > 0$ small enough. In [27], Song and Zou studied the existence of multiple sign-changing normalized solutions to (1.6) by using a new kind of linking contained in an open set of a Hilbert–Riemannian manifold below a level set and the estimates of energies of the sign-changing critical points. In [24], Qi and Zou considered the normalized solutions of nonhomogeneous mass supercritical Schrödinger equations

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{p-2}u + \mu |u|^{q-2} u & \text{in } \Omega, \\ \int\limits_{\Omega} |u|^2 dx = \alpha^2, \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

with potential $V(x)$. In [10], Jin and Zhang considered the following problem

$$\begin{cases} -\Delta u + \lambda u = \mu u^{p-1} + u^{q-1}, \ u > 0 & \text{in } \Omega, \\ \int\limits_{\Omega} u^2 dx = \rho^2, \ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and they proved the existence of local minimizer solution, and showed the existence of mountain pass solution when $\mu \geq 0$ and Ω is a star-shaped domain, by using the monotonicity trick and Pohozaev method. We also notice that, the following p-Laplace equations with prescribed L^p-norm

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u = f(u), \quad \text{in } \Omega \subset \mathbb{R}^N, \\ \int\limits_{\Omega} |u|^p dx = \rho, \end{cases} \tag{1.7}$$

have been studied by several authors, where f is a general nonlinearity. When $f(u) = \mu|u|^{q-2}u + g(u)$ and $\Omega = \mathbb{R}^N$, Zhang and Zhang [38] used a fountain theorem type argument, with suitable $\mu \in \mathbb{R}$, to obtain infinitely many radial solutions for any $N \geq 2$ and established the existence of infinitely many nonradial sign-changing solutions for $N = 4$ or $N \geq 6$. When $\Omega \subset \mathbb{R}^N$ is a bounded domain, in [7], Gao and Guo studied the existence and multiplicity result, and also proved the nonexistence of finite Morse index solution by means of blow-up analysis. In [35], Zhang considered the following Kirchhoff equation on bounded domain

$$\begin{cases} -(a + b \int\limits_{\Omega} |\nabla u|^2 dx)\Delta u = \lambda u + |u|^{p-2}u, & x \in \Omega, \\ \int\limits_{\Omega} u^2 dx = c^2, \quad u = 0, & x \in \partial\Omega, \end{cases}$$

where $\frac{14}{3} < p \leq 6$. By the minimization technique and the mountain pass theory, the author proved the existence of local minimizer and the existence of mountain pass solution. For more results on the constrained variational problems on bounded domain, we refer to [20, 21, 29, 30] and references therein.

As far as we know, there are few papers on the study of the normalized solutions for the p-Kirchhoff equations in the whole space $\mathbb{R}^N (N \geq 3)$, after a careful literature review. In fact, we are only aware of two papers [25, 26] considering this topic. In [25], Ren and Lan studied the p-Kirchhoff equation

$$-\left(a + b \int\limits_{\mathbb{R}^N} |\nabla u|^p dx\right)\Delta_p u = f(u) - \mu u - V(x)u^{p-1}, \quad u \in H^1(\mathbb{R}^N),$$

and they obtained that any energy ground state normalized solutions of (1.7) has constant sign and is radially symmetric monotone with respect to some point in $\mathbb{R}^N$ by using some energy estimates; and the paper [26] is concerned with the following p-Kirchhoff equation

$$\begin{cases} \left(a + b \int\limits_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) dx\right)(-\Delta_p u + |u|^{p-2}u) = \mu u + |u|^{s-2}u, & x \in \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} |u|^2 dx = \rho, \end{cases}$$

where $a > 0$, $b \geq 0$, $\rho > 0$ are constants, $s \in \left(\frac{2(N+2)}{N}p - 2, p^*\right)$. The authors showed that the p-Kirchhoff equation has a normalized solution using the mountain pass lemma and some analysis techniques.

Motivated by all of the work above, in this paper we focuss our attention on the normalized solutions for the p-Kirchhoff problem (1.1)–(1.2) with general nonlinearity h. Before delving into the results to motivate our research, let us highlight some novel aspects in the study of (1.1)–(1.2) with an L^p-constraint in the next arguments. Specifically, when h is at most L^p-critical at infinity, by the minimization methods, we proved the existence of solutions which is a local minimizer. In addition, we obtained the multiplicity of solutions by means of genus theory. While if h is L^p-supercritical, we proved the existence of solutions as local minimizer. Finally, by means of Pohozaev analysis, we found the second solution of mountain pass type when Ω is a star-shaped domain with respect to 0.

In order to find solutions to the problem (1.1)–(1.2), we turn to look for critical points to the functional

$$I(u) = \frac{a}{p}\int\limits_{\Omega} |\nabla u|^p dx + \frac{b}{2p}\Bigg(\int\limits_{\Omega} |\nabla u|^p dx\Bigg)^2 - \int\limits_{\Omega} H(u)dx,$$

on the constrained manifold

$$S(m) = \Bigg\{u \in W_0^{1,p}(\Omega) : \int\limits_{\Omega} |u|^p dx = m^p\Bigg\},$$

where $H(u) = \int_0^u h(t)dt$.

Before presenting our main results, we introduce some necessary notations and assumptions. Our workspace $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \Bigg(\int\limits_{\Omega} |\nabla u|^p dx\Bigg)^{\frac{1}{p}}.$$

The first Dirichlet eigenvalue of $-\Delta_p$ on Ω is denoted by $\lambda_1(\Omega)$ and φ_1 is the first eigenfunction of the problem

$$\begin{cases} -\Delta_p \varphi_1 = \lambda_1(\Omega)\varphi_1^{p-1}, & x \in \Omega, \\ \varphi_1 = 0, & x \in \partial\Omega, \end{cases} \tag{1.8}$$

with $\varphi_1 > 0$, $\|\varphi_1\|_p^p = m^p$. Moreover, $p^* = \frac{3p}{3-p}$ is the Sobolev critical exponent and $\bar{p} = p + \frac{p^2}{3}$ is the L^p-critical exponent. Throughout this paper, we introduce the following assumptions on h:

(H_1) $h \in C^1(\mathbb{R})$, $h'(t) > 0$ for $t > 0$ and $h(-t) = h(t)$ holds for all $t > 0$,

(H_2) there exist $p < \alpha, \beta < p^*$, and constants $\mu_1, \mu_2 > 0$ such that

$$\lim_{t\to 0+} \frac{h'(t)}{t^{\alpha-2}} = \mu_1(\alpha - 2) > 0 \quad \text{and} \quad \lim_{t\to +\infty} \frac{h'(t)}{t^{\beta-2}} = \mu_2(\beta - 2) > 0,$$

(H_3) there exists $\gamma > p + \frac{2p^2}{3}$, such that $th(t) > \gamma H(t)$ for any $t \neq 0$.

Our main results can be formulated as follows. When h is L^p-subcritical growth or L^p-critical growth at infinity, we have the following result:

Theorem 1.1. *Let $2 \le p < 3$, h satisfies (H_1), (H_2) and one of the conditions holds:*

(a_1) *$p < \beta < \bar{p}$ and $m > 0$,*
(a_2) *$\beta = \bar{p}$ and m satisfies*

$$0 < \max\left\{\mathcal{K}C(\bar{p})m^{\frac{p^2}{3}}, \mathcal{Q}C(\bar{p})m^{\frac{p^2}{3}}\right\} < \frac{a}{p}, \tag{1.9}$$

where $\mathcal{K}$, $\mathcal{Q}$ and $C(\bar{p})$ will be defined in Section 2. Then the following conclusions hold:

(i) *the problem* (1.1)–(1.2) *admits a global minimizer for $I(u)$ constrained on $S(m)$,*
(ii) *the problem* (1.1)–(1.2) *admits infinitely many solutions.*

When h is L^p-supercritical growth, we have the following result:

Theorem 1.2. *Let $2 \le p < 3$ and $p + \frac{2p^2}{3} < \alpha, \beta < p^*$, h satisfies (H_1), (H_2) and m satisfies either*

$$0 < \mathcal{K}C(\alpha)(2\lambda_1(\Omega))^{\frac{\alpha\delta_\alpha - p}{p}} m^{\alpha-p} + \mathcal{K}C(\beta)(2\lambda_1(\Omega))^{\frac{\beta\delta_\beta - p}{p}} m^{\beta-p} < \frac{a}{2p}, \tag{1.10}$$

or

$$0 < \mathcal{K}C(\alpha)(2\lambda_1(\Omega))^{\frac{\alpha\delta_\alpha - 2p}{p}} m^{\alpha-2p} + \mathcal{K}C(\beta)(2\lambda_1(\Omega))^{\frac{\beta\delta_\beta - 2p}{p}} m^{\beta-2p} < \frac{b}{4p}, \tag{1.11}$$

where $C(\alpha)$, $C(\beta)$ will be defined in Section 2, $\delta_i = \frac{3(i-p)}{pi}$, $i \in \{\alpha, \beta\}$. Then the following conclusions hold:

(i) *the problem* (1.1)–(1.2) *admits a local minimizer;*
(ii) *if Ω is a star-shaped domain with respect to 0 and h satisfies (H_3), then the problem* (1.1)–(1.2) *admits a second solution which is mountain pass type.*

Our paper is organized as follows. In Section 2, we present some lemmas and results which are necessary to the proof of Theorem 1.1 and Theorem 1.2. In Section 3, we focus on the case that h is L^p-subcritical or L^p-critical at infinity, and finish the proof of Theorem 1.1. In Section 4, we discuss on the cast that h is L^p-supercritical and finish the proof of Theorem 1.2.

For notation convenience, throughout this paper, we use the following notations.

- $\|\cdot\|$ denotes the norm in $W_0^{1,p}(\Omega)$ and $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$.
- $o_n(1)$ denotes a quantity which converges to 0 when $n \to \infty$.
- $B_r(x)$ denote the open ball with center x and radius r, B_r denotes the open ball with center 0 and radius r.
- $\to$ and $\rightharpoonup$ denotes the strong and weak convergence in the relevant function spaces, respectively.

2. PRELIMINARY MATERIAL

In this section, we present some preliminary lemmas and results which play the key role in proving our main results. First, we introduce the Gagliardo–Nirenberg inequality, which is crucial in our arguments.

Lemma 2.1. *Let $1 < p < 3$, $p < q < p^*$, $\Omega \subset \mathbb{R}^3$, then there exists a constant $C(q) > 0$, such that for any $u \in L^q(\Omega)$, the following inequality holds:*

$$\|u\|_q^q \le C(q)\|\nabla u\|_p^{q\delta_q}\|u\|_p^{q-q\delta_q}, \tag{2.1}$$

where $\delta_q = \frac{3(q-p)}{pq}$.

In the following, we comment on the properties of h. As in [12, 13], by direct calculation, we know that if h satisfies the condition (H_1), (H_2), we have the following conclusion.

Lemma 2.2. *Let (H_1), (H_2) hold. Then:*

(i) $\lim_{|t|\to 0} \frac{h(t)}{|t|^{\alpha-2}t} = \mu_1$ *and* $\lim_{t\to+\infty} \frac{h(t)}{|t|^{\beta-2}t} = \mu_2$,

(ii) *there exists $C_1 > 0$, such that*

$$|h(t)| \le C_1(|t|^{\alpha-1} + |t|^{\beta-1}), \quad \forall t \in \mathbb{R}, \tag{2.2}$$

and for any $M > 0$, there exist $K_{M,1} \ge K_{M,2} > 0$ such that

$$K_{M,2}|t|^{\alpha-2}t \le h(t) \le K_{M,1}|t|^{\alpha-2}t, \quad \forall t \in [0, M],$$

(iii) *there exists $\mathcal{K} > 0$ such that*

$$H(t) \le \mathcal{K}(|t|^\alpha + |t|^\beta), \quad \forall t \in \mathbb{R}, \tag{2.3}$$

and for any $M > 0$, there exist $K_{M,1} \ge K_{M,2} > 0$ such that

$$K_{M,2}|t|^\alpha \le H(t) \le K_{M,1}|t|^\alpha, \quad \forall t \in [0, M],$$

(iv) *if $\alpha \ge \beta$, there exists $\mathcal{Q} > 0$, such that*

$$H(t) \le \mathcal{Q}|t|^\beta, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

To obtain the multiplicity result of Theorem 1.1, we need the following results about genus theory.

Definition 2.3. Let V be a Banach space, and a family of subsets $\mathcal{A}$ is defined as

$$\mathcal{A} = \{A \subset V : A \text{ is closed}, A = -A\}.$$

Then for $A \subset \mathcal{A}$, we define

$$\gamma(A) := \inf\{m \in \mathbb{N} : \exists h \in C^0(A; R^m\backslash\{0\})\, \forall u \in A\ h(-u) = -h(u)\},$$

which is called the genus of A.

The following assertion is crucial in the proof of Theorem 1.1.

Theorem 2.4 ([28]). *Let V be a Banach space, $I \in C^1(M)$ is an even functional on a complete symmetric $C^{1,1}$-manifold $M \subset V\backslash\{0\}$. Suppose that I is bounded from below and satisfies the (P.S.) condition. Let $\hat{\gamma}(M) = \sup\{\gamma(A) : A \subset M,\ A$ is compact and symmetric$\}$. Then the functional I admits at least $\hat{\gamma}(M) \le \infty$ pairs of critical points. Moreover, for any $k \in \mathbb{N}$, let*

$$\beta_k = \inf_{A \in \mathcal{A}_k} \sup_{u \in A} I(u),$$

where

$$\mathcal{A}_k := \{A \in \mathcal{A} : A \subset M,\ \gamma(A) \ge k\}.$$

Then I admits a critical point u_k at level β_k.

In order to finish the proof of Theorem 1.2, we need the following lemma.

Lemma 2.5 ([3]). *Let W be a Banach space equipped with the norm $\|\cdot\|$, $S(m) \subset W$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family I_μ of C^1-functionals on W of the form*

$$I_\mu(u) = A(u) - \mu B(u),\ \mu \in J,$$

where $B(u) \ge 0$ for all $u \in W$ and such that $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$. Assume that here are two points w_0, w_1 in $S(m)$, setting

$$\Gamma = \{\gamma \in C([0,1], S(m)) : \gamma(0) = w_0, \gamma(1) = w_1\},$$

there holds, for all $\mu \in J$,

$$\sigma_\mu := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(w_0), I_\mu(w_1)\}.$$

Then for almost every $\mu \in J$, there exists a sequence $\{u_n\} \subset S(m)$ such that

(i) *$\{u_n\}$ is bounded in W,*
(ii) *$I_\mu(u_n) \to \sigma_\mu$ and $I'_\mu|_{S(m)}(u_n) \to 0$ in W^*,*

where W^* is the dual space of W.

Finally, we present the Pohozaev identity, which is important in the proof of Theorem 1.2.

Lemma 2.6. *Let $2 \le p < 3$, $\Omega \subset \mathbb{R}^3$ is a bounded domain, u is a solution to the problem (1.1)–(1.2), then the following identity holds*

$$(a + b\|\nabla u\|_p^p)\left(\frac{p-1}{p}\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^p dS + \frac{3-p}{p}\|\nabla u\|_p^p\right) + \frac{3}{p}\lambda\|u\|_p^p = 3\int_{\Omega} H(u)dx, \tag{2.5}$$

where ν denotes the unit outer normal vector.

Proof. Firstly, since u is a solution to (1.1), multiply by $x \cdot \nabla u$ and integrating on Ω, we have

$$-(a+b\|\nabla u\|_p^p)\int_\Omega \Delta_p u(x\cdot\nabla u)dx = \int_\Omega h(u)(x\cdot\nabla u)dx - \lambda\int_\Omega |u|^{p-2}u(x\cdot\nabla u)dx. \tag{2.6}$$

Since $u = 0$ on $\partial\Omega$, by direct calculation, we have

$$\begin{aligned}\int_\Omega h(u)(x\cdot\nabla u)dx &= \int_\Omega x\cdot\nabla(H(u))dx\\ &= \int_{\partial\Omega} H(u)(x\cdot\nu)dS - 3\int_\Omega H(u)dx\\ &= -3\int_\Omega H(u)dx,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |u|^{p-2}u(x\cdot\nabla u)dx &= \frac{1}{p}\int_\Omega x\cdot\nabla(|u|^p)dx\\ &= \frac{1}{p}\left(\int_{\partial\Omega}|u|^p(x\cdot\nu)dS - 3\int_\Omega |u|^p dx\right)\\ &= -\frac{3}{p}\int_\Omega |u|^p dx\end{aligned} \tag{2.7}$$

Next, we deal with the left hand side of (2.6). For convenience, we define

$$V(x) = (x\cdot\nabla u)|\nabla u|^{p-2}\nabla u - \frac{1}{p^*}|\nabla u|^p x.$$

By simple calculation, we have

$$\mathrm{div}V(x) = (x\cdot\nabla u)\Delta_p u + \frac{1}{3}x\cdot\nabla(|\nabla u|^p) + \frac{2p-3}{p}|\nabla u|^p. \tag{2.8}$$

Thus, combining with (2.6), (2.7), (2.8), we have

$$\begin{aligned}&(a+b\|\nabla u\|_p^p)\left(-\int_\Omega(\mathrm{div}V + \frac{1}{3}x\cdot\nabla(|\nabla u|^p))dx + \frac{2p-3}{p}\|\nabla u\|_p^p\right)\\ &= -3\int_\Omega H(u)dx + \frac{3}{p}\lambda\|u\|_p^p.\end{aligned} \tag{2.9}$$

By simple calculation, we have

$$\int_\Omega x\cdot\nabla(|\nabla u|^p)dx = \int_{\partial\Omega}(x\cdot\nu)|\nabla u|^p dS - \int_\Omega |\nabla u|^p \mathrm{div}(x)dx$$
$$= \int_{\partial\Omega}|\nabla u|^p(x\cdot\nu)dS - 3\int_\Omega |\nabla u|^p dx.$$

Now, we notice that since $u = 0$ on $\partial\Omega$, $\nabla u = |\nabla u|\nu$, then, we have

$$\begin{aligned} -\int_\Omega \mathrm{div}V(x)dx &= -\int_{\partial\Omega} V\cdot\mu dS \\ &= -\int_{\partial\Omega}(x\cdot\nabla u)|\nabla u|^{p-2}(\nabla u\cdot\nu)dS + \frac{1}{p^*}\int_{\partial\Omega}|\nabla u|^p(x\cdot\nu)dS \\ &= -\frac{p^*-1}{p^*}\int_{\partial\Omega}|\nabla u|^p(x\cdot\nu)dS. \end{aligned} \tag{2.10}$$

Combining (2.9)–(2.10), by simple calculation, we get

$$\begin{aligned} &(a+b\|\nabla u\|_p^p)\left(\frac{p-1}{p}\int_{\partial\Omega}|\nabla u|^p(x\cdot\nu)dS + \frac{3-p}{p}\int_\Omega |\nabla u|^p dx\right) \\ &= 3\int_\Omega H(u)dx - \frac{3}{p}\lambda\int_\Omega |u|^p dx, \end{aligned}$$

which show that (2.5) holds. □

3. PROOF OF THEOREM1.1

In this section, we consider the case that h is at most L^p-critical growth at infinity, i.e. $p < \alpha, \beta < \bar{p}$ or $\beta = \bar{p}$ and complete the proof of Theorem 1.1.

Lemma 3.1. *Let $2 \le p < 3$, h satisfies (H_1), (H_2), and one of the following two conditions holds:*

(i) *$p < \beta < \bar{p}$ and $m > 0$,*
(ii) *$\beta = \bar{p}$ and m satisfies*

$$0 < \max\left\{\mathcal{K}C(\bar{p})m^{\frac{p^2}{3}}, \mathcal{Q}C(\bar{p})m^{\frac{p^2}{3}}\right\} < \frac{a}{p}. \tag{3.1}$$

Then $I|_{S(m)}(u)$ admits a global minimizer.

Proof. If case (i) holds, if $p < \alpha < \bar{p}$, by the Gagliardo–Nirenberg inequality (2.1) and (2.3), we have that for any $u \in S(m)$,

$$\begin{aligned} I(u) &= \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \int_\Omega H(u)dx \\ &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{K}\int_\Omega (|u|^\alpha + |u|^\beta)dx \\ &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{K}\left(C(\alpha)m^{\alpha(1-\delta_\alpha)}\|\nabla u\|_p^{\alpha\delta_\alpha} + C(\beta)m^{\beta(1-\delta_\beta)}\|\nabla u\|_p^{\beta\delta_\beta}\right) \\ &=: g(\|\nabla u\|_p), \end{aligned}$$

where

$$g(t) = \frac{a}{p}t^p - \mathcal{K}C(\alpha)m^{\alpha(1-\delta_\alpha)}t^{\alpha\delta_\alpha} - \mathcal{K}C(\beta)m^{\beta(1-\delta_\beta)}t^{\beta\delta_\beta}, \ \forall t \geq 0.$$

From the fact that $p < \alpha < \bar{p}$ and $p < \beta < \bar{p}$, by direct calculation, we know that $\alpha\delta_\alpha < p$ and $\beta\delta_\beta < p$. Thus, we have $g(t) \to 0-$ as $t \to 0+$ and $g(t) \to +\infty$ as $t \to +\infty$. Then we know that g is bounded from below. Noticing that $I(u) \geq g(\|\nabla u\|_p)$, we know that $I(u)$ is bounded from below and coercive on $S(m)$, then $I(u)$ admits a global minimizer on $S(m)$.

If $\bar{p} \leq \alpha < p^*$, from (2.4), arguing similarly as above, we can also deduce that

$$\begin{aligned} I(u) &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{Q}\int_{\mathbb{R}^3} |u|^\beta dx \\ &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{Q}C(\beta)m^{\beta(1-\delta_\beta)}\|\nabla u\|_p^{\beta\delta_\beta}. \end{aligned}$$

Noting that $\beta\delta_\beta < p$, we know that $I(u)$ is coercive and bounded from below on $S(m)$, then $I(u)$ admits a global minimizer on $S(m)$.

If case (ii) holds, then by $\beta = \bar{p}$, from simple calculation, we know that $\beta\delta_\beta = p$ and $\beta(1-\delta_\beta) = \frac{p^2}{3}$.

If $p < \alpha < \beta = \bar{p}$, from (2.1) and (2.3), we have

$$\begin{aligned} I(u) &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{K}\int_\Omega (|u|^\alpha + |u|^\beta)dx \\ &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{K}\left(C(\alpha)m^{\alpha(1-\delta_\alpha)}\|\nabla u\|_p^{\alpha\delta_\alpha} + C(\beta)m^{\beta(1-\delta_\beta)}\|\nabla u\|_p^{\beta\delta_\beta}\right) \\ &= \left(\frac{a}{p} - \mathcal{K}C(\bar{p})m^{\frac{p^2}{3}}\right)\|\nabla u\|_p^p - \mathcal{K}C(\alpha)m^{\alpha(1-\delta_\alpha)}\|\nabla u\|_p^{\alpha\delta_\alpha}. \end{aligned}$$

From (3.1), we know that $\frac{a}{p} - \mathcal{K}C(\bar{p})m^{\frac{p^2}{3}} > 0$, thus, arguing as in case (i), we know that $I(u)$ admits a global minimizer on $S(m)$.

If $\bar{p} = \beta \leq \alpha < p^*$, then from the Gagliardo–Nirenberg inequality (2.1) and (2.4), we know that

$$\begin{aligned} I(u) &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{Q}\int_\Omega |u|^\beta dx \\ &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{Q}C(\beta)\|u\|_p^{\beta(1-\delta_\beta)}\|\nabla u\|_p^{\beta\delta_\beta} \\ &\geq \frac{a}{p}\|\nabla u\|_p^p - \mathcal{Q}C(\bar{p})m^{\frac{p^2}{3}}\|\nabla u\|_p^p \\ &= \left(\frac{a}{p} - \mathcal{Q}C(\bar{p})m^{\frac{p^2}{3}}\right)\|\nabla u\|_p^p. \end{aligned}$$

Since (3.1) holds, we know that $\frac{a}{p} - \mathcal{Q}C(\bar{p})m^{\frac{p^2}{3}} > 0$, then, arguing as case (i), we deduce that $I(u)$ admits a global minimizer on $S(m)$. □

Now, we define the infimum:

$$c_m := \inf_{u\in S(m)} I(u),$$

and are ready to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. To show item (i), we note that, if one of conditions (a_1) and (a_2) occurs, then by Lemma 3.1, we know that $I(u)$ admits a global minimizer on $S(m)$. Thus, there exists a sequence $\{u_n\} \subset S(m)$ such that $I(u_n) \to c_m$ and $I'|_{S(m)}(u_n) \to 0$ as $n \to \infty$. Since I is coercive, we know that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, thus, we assume that up to a subsequence,

$$\begin{cases} u_n \rightharpoonup \tilde{u} & \text{in } W_0^{1,p}(\Omega), \\ u_n \to \tilde{u} & \text{in } L^q(\Omega), \quad \forall q \in [1, p^*), \\ u_n \to \tilde{u} & \text{a.e. in } \Omega. \end{cases}$$

Since $I'|_{S(m)}(u_n) \to 0$ as $n \to \infty$, we know that there exists $\{\lambda_n\} \subset \mathbb{R}$ such that $I'(u_n) + \lambda_n|u_n|^{p-2}u_n \to 0$ in $(W_0^{1,p}(\Omega))^*$. Thus, from (2.2), we know that

$$\begin{aligned} |\lambda_n|m^p &= |\langle I'(u_n), u_n\rangle| + o_n(1) \\ &\leq a\|\nabla u_n\|_p^p + b\|\nabla u_n\|_p^{2p} + \left|\int_\Omega h(u_n)u_n dx\right| \\ &\leq a\|\nabla u_n\|_p^p + b\|\nabla u_n\|_p^{2p} + \mathcal{K}\int_\Omega (|u_n|^\alpha + |u_n|^\beta)dx \\ &\leq a\|\nabla u_n\|_p^p + b\|\nabla u_n\|_p^{2p} \\ &\quad + \mathcal{K}\left(C(\alpha)\|\nabla u_n\|_p^{\alpha\delta_\alpha}\|u_n\|_p^{\alpha-\alpha\delta_\alpha} + C(\beta)\|\nabla u_n\|_p^{\beta\delta_\beta}\|u_n\|_p^{\beta-\beta\delta_\beta}\right) \end{aligned}$$

from the Gagliardo–Nirenberg inequality (2.1), it is easy to show that $\{\lambda_n\}$ is bounded in $\mathbb{R}$, then, up to a subsequence, we have $\lambda_n \to \tilde{\lambda}$ as $n \to \infty$. Moreover, from (2.2) and Sobolev embedding, we know that $h(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, then up to a subsequence, $h(u_n) \rightharpoonup h(\tilde{u})$ in $(W_0^{1,p}(\Omega))^*$. Assume that $\|\nabla u_n\|_p^p \to A^p$ as $n \to \infty$, we have

$$(a + bA^p) \int\limits_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla \phi dx + \tilde{\lambda} |\tilde{u}|^{p-2} \tilde{u} \phi = \int\limits_\Omega h(\tilde{u}) \phi dx, \quad \forall \phi \in W_0^{1,p}(\Omega). \tag{3.2}$$

Let

$$J_A(u) = \frac{a + bA^p}{p} \|\nabla u\|_p^p + \frac{\tilde{\lambda}}{p} \int\limits_\Omega |u|^p dx - \int\limits_{\mathbb{R}^3} H(u) dx.$$

Then we deduce that $J'_A(\tilde{u}) = 0$ and $\{u_n\}$ is a bounded Palais–Smale sequence of J_A.

From the dominated convergence theorem, it's easy to see that

$$\int\limits_\Omega h(u_n) u_n dx = \int\limits_\Omega h(\tilde{u}) \tilde{u} dx + o_n(1). \tag{3.3}$$

Since $I'(u_n) + \lambda_n |u_n|^{p-2} u_n \to 0$, one has

$$(a + b\|\nabla u_n\|_p^p) \|\nabla u_n\|_p^p + \lambda_n \|u_n\|_p^p - \int\limits_\Omega h(u_n) u_n dx = o_n(1),$$

passing to limit as $n \to \infty$, combing with (3.3), one has

$$(a + bA^p) A^p + \tilde{\lambda} \|\tilde{u}\|_p^p - \int\limits_\Omega h(\tilde{u}) \tilde{u} dx = 0. \tag{3.4}$$

Moreover, from (3.2), we have

$$(a + bA^p) \|\nabla \tilde{u}\|_p^p + \tilde{\lambda} \|\tilde{u}\|_p^p - \int\limits_\Omega h(\tilde{u}) \tilde{u} dx = 0. \tag{3.5}$$

Thus, from (3.4) and (3.5), we have

$$(a + bA^p)(A^p - \|\nabla \tilde{u}\|_p^p) = 0,$$

which shows that $A^p + o_n(1) = \|\nabla u_n\|_p^p \to \|\nabla \tilde{u}\|_p^p = A^p$ as $n \to \infty$. Thus, we have that $u_n \to \tilde{u}$ in $W_0^{1,p}(\Omega)$. Moreover, $I(\tilde{u}) = c_m = \inf_{u \in S(m)} I(u)$.

To check item (ii), it's easy to show that $I(u)$ is an even functional and $S(m)$ is a complete symmetric manifold in $W_0^{1,p}(\Omega)$. Thus, if we take $V = W_0^{1,p}(\Omega)$ and $M = S(m)$, from Theorem 2.4, we know that I admits infinitely many critical points $u_{m,k}$ at level β_k, $k \in \mathbb{N}$, where

$$\beta_k = \inf_{A \in \mathcal{A}_k} \sup_{u \in A} I(u).$$

Then, for any $k \in \mathbb{N}$, take a Palais–Smale sequence $\{u_n^k\}$ such that $I(u_n^k) \to \beta_k$ and $I'|_{S(m)}(u_n^k) \to 0$ as $n \to \infty$, arguing as in the proof of Theorem 1.1 for case (i), we know that the problem (1.1)–(1.2) admits a solution u_k at level β_k. Thus, we can conclude that (1.1)–(1.2) admits infinitely many solutions, $u^k \in W_0^{1,p}(\Omega)$ with $\|u^k\|_p = m$ for all $k \in \mathbb{N}$. □

4. PROOF OF THEOREM 1.2

In this section, we consider the L^p-critical case, i.e. $p + \frac{2p^2}{3} < \alpha,\ \beta < p^*$, and complete the proof of Theorem 1.2. To this aim, we introduce the following perturbed p-Kirchhoff problem

$$\begin{cases} -\left(a + b\int\limits_\Omega |\nabla u|^p dx\right)\Delta_p u + \lambda |u|^{p-2}u = \mu h(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ \int\limits_\Omega |u|^p dx = m^p. \end{cases} \tag{4.1}$$

The energy functional of (4.1) is defined as

$$I_\mu(u) = \frac{a}{p}\int\limits_\Omega |\nabla u|^p dx + \frac{b}{2p}\left(\int\limits_\Omega |\nabla u|^p dx\right)^2 - \mu \int\limits_\Omega H(u)dx.$$

Based on the monotonicity method originated from [3], we will show that the problem admits a mountain pass-type normalized solution for almost every $\mu \in \left[\frac{1}{2}, 1\right]$.

Lemma 4.1. *Let $2 \le p < 3$, h satisfies (H_1), (H_2), $p + \frac{2p^2}{3} < \alpha,\ \beta < p^*$, and m satisfies either* (1.10) *or* (1.11), *then we have*

$$\inf_{u \in D_{\rho_0}} I_\mu(u) < \inf_{u \in \partial D_{\rho_0}} I_\mu(u),$$

where $\rho_0 = 2\lambda_1(\Omega)m^p$, and

$$D_{\rho_0} = \{u \in S(m) : \|\nabla u\|_p^p < \rho_0\}.$$

Proof. From direct calculation, we have α, β, $\alpha\delta_\alpha$, $\beta\delta_\beta > 2p$. Recall that φ_1 is the eigenfunction of the problem (1.8) with the norm $\|\varphi_1\|_p^p = m^p$, thus, we know that $\|\nabla\varphi_1\|_p^p = \lambda_1(\Omega)m^p < \rho_0$, which implies that $\varphi_1 \in D_{\rho_0}$. Consequently, we have

$$\begin{aligned} I_\mu(\varphi_1) &= \frac{a}{p}\|\nabla\varphi_1\|_p^p + \frac{b}{2p}\|\nabla\varphi_1\|_p^{2p} - \mu\int\limits_\Omega H(\varphi_1)dx \\ &\le \frac{a}{p}\|\nabla\varphi_1\|_p^p + \frac{b}{2p}\|\nabla\varphi_1\|_p^{2p} \\ &= \frac{a}{p}\lambda_1(\Omega)m^p + \frac{b}{2p}(\lambda_1(\Omega))^2 m^{2p}. \end{aligned} \tag{4.2}$$

If the condition (1.10) holds, by (2.1) and (2.3), we have that for any $u \in D_{\rho_0}$,

$$
\begin{aligned}
I_\mu(u) &\geq \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \int_\Omega H(u)dx \\
&\geq \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \mathcal{K}\int_\Omega (|u|^\alpha + |u|^\beta)dx \\
&\geq \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} \\
&\quad - \mathcal{K}\Big(C(\alpha)\|u\|_p^{\alpha-\alpha\delta_\alpha}\|\nabla u\|_p^{\alpha\delta_\alpha} + C(\beta)\|u\|_p^{\beta-\beta\delta_\beta}\|\nabla u\|_p^{\beta\delta_\beta}\Big) \\
&= \frac{a}{p}(2\lambda_1(\Omega)m^p) + \frac{b}{2p}(2\lambda_1(\Omega)m^p)^2 - \mathcal{K}C(\alpha)(2\lambda_1(\Omega)m^p)^{\frac{\alpha\delta_\alpha}{p}} m^{\alpha(1-\delta_\alpha)} \\
&\quad - \mathcal{K}C(\beta)(2\lambda_1(\Omega)m^p)^{\frac{\beta\delta_\beta}{p}} m^{\beta(1-\delta_\beta)} \\
&= \frac{a}{p}(2\lambda_1(\Omega)m^p) + \frac{b}{2p}(2\lambda_1(\Omega)m^p)^2 - \mathcal{K}C(\alpha)(2\lambda_1(\Omega))^{\frac{\alpha\delta_\alpha}{p}} m^{\alpha} \\
&\quad - \mathcal{K}C(\beta)(2\lambda_1(\Omega))^{\frac{\beta\delta_\beta}{p}} m^{\beta} \\
&= 2\lambda_1(\Omega)m^p\Big(\frac{a}{p} - \mathcal{K}C(\alpha)(2\lambda_1(\Omega))^{\frac{\alpha\delta_\alpha-p}{p}} m^{\alpha-p} \\
&\qquad\qquad - \mathcal{K}C(\beta)(2\lambda_1(\Omega))^{\frac{\beta\delta_\beta-p}{p}} m^{\beta-p}\Big) + \frac{2b}{p}(\lambda_1(\Omega))^2 m^{2p} \\
&> \frac{a}{p}\lambda_1(\Omega)m^p + \frac{2b}{p}(\lambda_1(\Omega))^2 m^{2p}.
\end{aligned}
$$

Thus, together with (4.2), we have

$$
\inf_{u\in D_{\rho_0}} I_\mu(u) \leq I_\mu(\varphi_1) < \inf_{u\in \partial D_{\rho_0}} I_\mu(u).
$$

If the condition (1.11) holds, by (2.1) and (2.3), arguing like above, we have

$$
\begin{aligned}
I_\mu(u) &\geq \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \int_\Omega H(u)dx \\
&\geq \frac{a}{p}(2\lambda_1(\Omega)m^p) + \frac{b}{2p}(2\lambda_1(\Omega))^2 m^{2p} - \mathcal{K}C(\alpha)(2\lambda_1(\Omega))^{\frac{\alpha\delta_\alpha}{p}} m^{\alpha} \\
&\quad - \mathcal{K}C(\beta)(2\lambda_1(\Omega))^{\frac{\beta\delta_\beta}{p}} m^{\beta} \\
&= \frac{2a}{p}\lambda_1(\Omega)m^p + (2\lambda_1(\Omega))^2 m^{2p}\Bigg(\frac{b}{2p} - \mathcal{K}C(\alpha)(2\lambda_1(\Omega))^{\frac{\alpha\delta_\alpha-2p}{p}} m^{\alpha-2p} \\
&\qquad\qquad - \mathcal{K}C(\beta)(2\lambda_1(\Omega))^{\frac{\beta\delta_\beta-2p}{p}} m^{\beta-2p}\Bigg) \\
&> \frac{2a}{p}\lambda_1(\Omega)m^p + \frac{b}{2p}(\lambda_1(\Omega))^2 m^{2p}.
\end{aligned}
$$

Together with (4.2), we have

$$\inf_{u\in D_{\rho_0}} I_\mu(u) \le I_\mu(\varphi_1) < \inf_{u\in \partial D_{\rho_0}} I_\mu(u). \qquad \square$$

In the following, we show that the functional $I_\mu(u)$ possess a mountain pass geometry.

Lemma 4.2. *Let* $2 \le p < 3$*,* h *satisfies* (H_1)*,* (H_2)*, and* m *satisfies either* (1.10) *or* (1.11)*. Then there exists* $\phi \in S(m)$ *such that* $\|\nabla\phi\|_p^p > \rho_0$ *and*

$$\sigma_\mu := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\mu(\gamma(t)) > \max\{I_\mu(\varphi_1), I_\mu(\phi)\}, \quad \forall \mu \in \left[\frac{1}{2}, 1\right],$$

where $\Gamma = \{\gamma \in C([0,1], S(m)) : \ \gamma(0) = \varphi_1, \gamma(1) = \phi\}$*.*

Proof. From Lemma 4.1, we have

$$I_\mu(\varphi_1) < \inf_{u\in\partial D_{\rho_0}} I_\mu(u), \quad \forall \mu \in \left[\frac{1}{2}, 1\right].$$

Now, we take $w \in S(m) \cap C_0^\infty(\Omega)$ and $w > 0$. We set

$$w_t(x) = t^{\frac{3}{p}} w(tx), \quad \forall t \ge 1.$$

Then, it is easy to see that $w_t \in S(m)$. From (H_2), we have that for a.e. $x \in \Omega$,

$$\lim_{t\to+\infty} \frac{H(w_t(x))}{w_t(x)^\beta} = \frac{\mu_2}{\beta},$$

thus, we deduce that for t large enough,

$$\int_\Omega H(w_t(x))dx > \frac{\mu_2}{2\beta}\int_\Omega |w_t(x)|^\beta dx = \frac{\mu_2}{2\beta} t^{\frac{3(\beta-p)}{p}} \int_\Omega |w(x)|^\beta dx.$$

Thus, we infer to

$$\begin{aligned}
I_\mu(w_t) &= \frac{a}{p}\|\nabla w_t\|_p^p + \frac{b}{2p}\|\nabla w_t\|_p^{2p} - \int_\Omega H(w_t(x))dx \\
&< \frac{a}{p} t^p \|\nabla w\|_p^p + \frac{b}{2p} t^{2p}\|\nabla u\|_p^{2p} - \frac{\mu_2}{2\beta} t^{\frac{3(\beta-p)}{p}} \int_\Omega |w(x)|^\beta dx \\
&\to -\infty,
\end{aligned}$$

if we notice that $\beta > p + \frac{2p^2}{3}$ implies $\frac{3(\beta-p)}{p} > 2p$. Hence, if we take $\phi = w_t(x)$ for t large enough, such that $I_\mu(\phi) < 0$ and $\|\nabla\phi\|_p^p > \rho_0$. Then, by the continuity of the norm, we know that there for any $\gamma \in \Gamma$, there exist $t_0 \in (0,1)$ such that $\gamma(t_0) \in \partial D_{\rho_0}$. Therefore, we have

$$\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\mu(\gamma(t)) \ge \inf_{u\in\partial D_{\rho_0}} I_\mu(u) > \max\{I_\mu(\varphi_1), I_\mu(\phi)\}.$$

This completes the proof. $\square$

Lemma 4.3. *Let $2 \le p < 3$, m satisfies either (1.10) or (1.11), h satisfies (H_1), (H_2). Then, for a.e. $\mu \in [\frac{1}{2}, 1]$, there exists a critical point u_μ of I_μ constrained on $S(m)$, which is a solution to (4.1) with some $\lambda = \lambda_\mu$. Moreover, if Ω is a star-shaped domain with respect to 0, there holds*

$$(a + b\|\nabla u_\mu\|_p^p)\|\nabla u_\mu\|_p^p > \frac{3\mu}{p}\int\limits_\Omega (h(u_\mu)u_\mu - pH(u_\mu))dx. \tag{4.3}$$

Proof. If we take $A(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p}$ and $B(u) = \int_\Omega H(u)dx$ in Lemma 2.5, then, combing with Lemma 4.2, we know that for almost everywhere $\mu \in [\frac{1}{2}, 1]$, there exist a bounded Palais–Smale sequence $\{u_n\} \subset S(m)$ such that $I_\mu(u_n) \to \sigma_\mu$ and $I'_\mu|_{S(m)} \to 0$ as $n \to \infty$. Up to a subsequence, there exists $u_0 \in S(m)$ such that

$$\begin{cases} u_n \rightharpoonup u_\mu, & \text{in } W_0^{1,p}(\Omega), \\ u_n \to u_\mu, & \text{in } L^q(\Omega),\ \forall 1 \le q < p^*, \\ u_n \to u_\mu, & \text{a.e. in } \Omega. \end{cases}$$

Arguing as in the proof of Theorem 1.1 for case (i), we know that for some $u_\mu \in W_0^{1,p}(\Omega)$, up to a subsequence, $u_n \to u_\mu$ in $W_0^{1,p}(\Omega)$. Moreover, u_μ is a solution to (4.1) with $\lambda = \lambda_\mu$.

Similar to Lemma 2.6, we have

$$\begin{aligned} &(a + b\|\nabla u_\mu\|_p^p)\left(\frac{p-1}{p}\int\limits_{\partial\Omega}(x\cdot\nu)|\nabla u_\mu|^p dS + \frac{3-p}{p}\|\nabla u_\mu\|_p^p\right) + \frac{3}{p}\lambda_\mu\|u_\mu\|_p^p \\ &= 3\mu\int\limits_\Omega H(u_\mu)dx. \end{aligned} \tag{4.4}$$

Since u_μ is the solution to (4.1), we have

$$(a + b\|\nabla u_\mu\|_p^p)\|\nabla u_\mu\|_p^p + \lambda_\mu\|u_\mu\|_p^p = \mu\int\limits_\Omega h(u_\mu)u_\mu dx. \tag{4.5}$$

Combing (4.4) and (4.5), we have

$$\begin{aligned} &\Big(a + b\|\nabla u_\mu\|_p^p\Big)\left(-\frac{p-1}{p}\int\limits_{\partial\Omega}(x\cdot\nu)|\nabla u_\mu|^p dS + \|\nabla u_\mu\|_p^p\right) \\ &= 3\mu\int\limits_\Omega \Big(\frac{1}{p}h(u_\mu)u_\mu - H(u_\mu)\Big)dx. \end{aligned}$$

Since Ω is a star-shaped domain with respect to 0, we know that

$$\int_{\partial\Omega} (x\cdot\nu)|\nabla u_\mu|^p dS > 0,$$

then we get

$$(a+b\|\nabla u_\mu\|_p^p)\|\nabla u_\mu\|_p^p > \frac{3\mu}{p}\int_\Omega (h(u_\mu)u_\mu - pH(u_\mu))dx,$$

which completes the proof. □

With all of the argument above, we are ready to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. To deal with item (i), by Lemma 4.1, we can define the infimum

$$\theta_m := \inf_{u\in D_{\rho_0}} I(u),$$

then, arguing like in Section 3, we know that there exist a Palais–Smale sequence $\{u_n\}\subset D_{\rho_0}$ such that

$$I(u_n)\to\theta_m,\quad I'|_{S(m)}\to 0,\ \text{as } n\to\infty.$$

Up to a subsequence, there exist $u_0\in S(m)$ such that

$$\begin{cases} u_n \rightharpoonup u_0, & in\ W_0^{1,p}(\Omega),\\ u_n\to u_0, & in\ L^q(\Omega),\quad \forall 1\le q<p^*,\\ u_n\to u_0, & a.e\ in\ \Omega. \end{cases}$$

Arguing as in the proof of Theorem 1.1 for case (i), we know that $u_n\to u_0$ in $W_0^{1,p}(\Omega)$. Hence, u_0 is a solution for the problem (1.1)–(1.2). Moreover,

$$I(u_0) = \inf_{u\in D_{\rho_0}} I(u).$$

To prove item (ii), by Lemma 4.3, we take u_μ solving (4.1) with $\lambda=\lambda_\mu$. Noticing that $I(u)\le I_\mu(u)\le I_{\frac12}(u)$, from the definition of σ_μ, we have

$$\sigma_1\le\sigma_\mu = I_\mu(u_\mu)\le\sigma_{\frac12}, \tag{4.6}$$

from which we derive that

$$\frac{a}{p}\|\nabla u_\mu\|_p^p + \frac{b}{2p}\|\nabla u_\mu\|_p^{2p} \le \sigma_{\frac12} + \mu\int_\Omega H(u_\mu)dx.$$

Thus, combining (4.3) and (4.6), we have

$$\frac{3}{p}\mu\int_\Omega (h(u_\mu)u_\mu - pH(u_\mu))dx < 2a\|\nabla u_\mu\|_p^p + b\|\nabla u_\mu\|_p^{2p} \le 2p\sigma_{\frac12} + 2p\mu\int_\Omega H(u_\mu)dx. \tag{4.7}$$

Since h satisfies (H_3), we infer to

$$\frac{3(\gamma-p)}{p}\mu\int_\Omega H(u_\mu)dx < \frac{3}{p}\mu\int_\Omega (h(u_\mu)u_\mu - pH(u_\mu))dx < 2p\sigma_{\frac{1}{2}} + 2p\mu\int_\Omega H(u_\mu)dx.$$

Note that $\frac{3(\gamma-p)}{p} = \gamma\delta_\gamma$. We have

$$(\gamma\delta_\gamma - 2p)\mu\int_\Omega H(u_\mu)dx < 2p\sigma_{\frac{1}{2}},$$

and

$$\begin{aligned} a\|\nabla u_\mu\|_p^p &\le p\sigma_{\frac{1}{2}} + p\mu\int_\Omega H(u_\mu)dx \\ &< p\sigma_{\frac{1}{2}} + \frac{2p^2}{\gamma\delta_\gamma - 2p}\sigma_{\frac{1}{2}}. \end{aligned} \tag{4.8}$$

Now, we take $\mu_n \to 1^-$ as $n \to \infty$ and set $u_n := u_{\mu_n}$ solving (4.1), from (4.7), we have

$$\begin{aligned} \lim_{n\to+\infty} I(u_n) &= \lim_{n\to+\infty}(I_{\mu_n}(u_n) + (\mu_n - 1)\int_\Omega H(u_n)dx) \\ &= \lim_{n\to+\infty} I_{\mu_n}(u_n) \\ &= \lim_{n\to+\infty}\sigma_{\mu_n} < \sigma_{\frac{1}{2}}. \end{aligned} \tag{4.9}$$

From [31], we have

$$\begin{aligned} &\lim_{n\to+\infty}\|I'|_{S(m)}(u_n)\|_* \\ &= \lim_{n\to+\infty}\sup_{v\in T_{u_n},\|v\|=1}\left(\langle I'_{\mu_n}(u_n), v\rangle + (\mu_n - 1)\int_\Omega h(u_n)vdx\right) = 0, \end{aligned} \tag{4.10}$$

where

$$T_u = \{v \in W_0^{1,p}(\Omega) | \int_\Omega |u|^{p-2}uvdx = 0\}.$$

Thus, from (4.8), (4.10) and (4.9), we derive that $\{u_n\}$ is a bounded Palais–Smale sequence for the functional $I(u)$ constrained on $S(m)$. Similar to the argument in the proof of Theorem 1.1(i), we can derive that $u_n \to \hat{u}_0$ in $W_0^{1,p}(\Omega)$. Moreover, $\hat{u}_0$ is a solution to (1.1)–(1.2). From Lemma 4.1 and (4.6), we have

$$I(\hat{u}_0) \ge \sigma_1 > I(\varphi_1) \ge \inf_{u\in D_{\rho_0}} I(u) = I(u_0),$$

which shows that $u_0 \ne \hat{u}_0$. $\square$

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