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NONLINEAR HYBRID SECOND-ORDER NEUTRAL DELAY DIFFERENTIAL EQUATION WITH MIXED SIGN TERMS: OSCILLATION VIA SYMMETRY TRANSFORM

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Abstract. This paper investigates the oscillatory behavior of solutions to a class of second-order nonlinear neutral delay differential equations with both positive and negative terms of the form

$$(a(\theta)z'(\theta))' - p(\theta)x(\theta) + q(\theta)x^{\alpha}(\sigma(\theta)) = 0, \quad \theta \geq \theta_0,$$

where $z(\theta) = x(\theta) + b(\theta)x(\tau(\theta))$. To facilitate the analysis, the equation is transformed, via a positive solution of an auxiliary second-order ordinary differential equation, into a binomial form. By employing the comparison and integral averaging techniques together with the arithmetic-geometric mean inequality, we establish new sufficient conditions for the oscillation of all solutions. The results obtained extend and improve several existing criteria in the literature. Finally, illustrative examples are presented to demonstrate the effectiveness, novelty, and applicability of the proposed oscillation criteria.

Keywords: oscillation, second-order, neutral, delay differential equation, symmetry transform.

Mathematics Subject Classification: 34C10, 34K11.

1. INTRODUCTION

Second-order neutral differential equations with both positive and negative terms are important tools for modeling complex systems in engineering and science, particularly in situations where delays occur in the system dynamics, such as high-speed electrical networks, biological population models, and mechanical systems. The inclusion of both positive (growth) and negative (decay) coefficients allows for more realistic and intricate descriptions of these phenomena, leading to diverse behaviors such as oscillation or stability. These properties are crucial for understanding and predicting the system's long-term behavior (see [10, 18]).

Motivated by these considerations, in this paper we focus on the oscillatory properties of solutions to the second-order nonlinear hybrid-type neutral delay differential equation of the form

$$(a(\theta)z'(\theta))' - p(\theta)x(\theta) + q(\theta)x^{\alpha}(\sigma(\theta)) = 0, \quad \theta \geq \theta_0, \tag{E}$$

where $z(\theta) = x(\theta) + b(\theta)x(\tau(\theta))$, subject to the following conditions:

$(H1)$ $p(\theta), q(\theta), b(\theta) \in C([\theta_0, \infty), \mathbb{R})$, with $0 \leq b(\theta) < 1$, $p(\theta) > 0$, and $q(\theta) \geq 0$ for all $\theta \geq \theta_0$;
$(H2)$ $a(\theta) \in C^1([\theta_0, \infty), (0, \infty))$, and α is a ratio of odd positive integers;
$(H3)$ $\sigma(\theta), \tau(\theta) \in C([\theta_0, \infty), \mathbb{R})$, with $\sigma(\theta) \leq \theta$, $\tau(\theta) \leq \theta$, and

$$\lim_{\theta\to\infty} \sigma(\theta) = \lim_{\theta\to\infty} \tau(\theta) = \infty.$$

Let $\delta(\theta) = \min\{\tau(\theta), \sigma(\theta)\}$. As usual, by a *proper solution* of equation (E) we mean a function that satisfies (E) for all sufficiently large θ and for which $\sup\{|x(\theta)| : \theta \geq T\} > 0$ for all $\theta \geq \theta_0$. A proper solution is called *oscillatory* if it has infinitely many zeros; otherwise, it is called *nonoscillatory*.

Note that if $p(\theta) = 0$, then (E) reduces to the second-order nonlinear neutral delay differential equation

$$(a(\theta)z'(\theta))' + q(\theta)x^{\alpha}(\sigma(\theta)) = 0, \quad \theta \geq \theta_0, \tag{E_1}$$

and if $q(\theta) = 0$, then we obtain the linear neutral differential equation

$$(a(\theta)z'(\theta))' - p(\theta)x(\theta) = 0, \quad \theta \geq \theta_0. \tag{E_2}$$

Therefore, we may refer to equation (E) as a *hybrid-type nonlinear neutral differential equation.*

Equations of type (E) and their variants model systems in which the state at a given time depends not only on its current behavior but also on its past states. The “neutral” term, in which the derivative also depends on delayed versions of the unknown function, captures these memory effects. The presence of both positive and negative coefficients enables the modeling of feedback mechanisms within the system. This leads to a richer variety of behaviors, including stability, instability, or oscillations, as observed in many natural and engineered systems (see [9, 22] and the references therein).

A central focus in the study of such equations is the analysis of the oscillatory behavior of their solutions. The presence of positive and negative terms can give rise to solutions that oscillate around an equilibrium point, a property of critical importance in many physical and biological phenomena. Due to this practical significance, numerous researchers have studied oscillation criteria for second-order neutral differential equations of various forms.

In recent years, there has been increasing interest in developing and improving oscillation results for second-order functional differential equations of different types (see the monographs [2, 3], the papers [6, 11, 15, 20] and the references cited therein). In particular, oscillation criteria have been reported for second-order differential equations with linear neutral terms in [1, 4, 9, 11, 26, 30], sublinear neutral terms in [5, 16, 17, 31] and superlinear neutral terms in [7, 12, 13, 25, 29, 31]. Notice that all the results mentioned in these references pertain to the case when all the non-neutral terms are either positive or negative. This is due to the fact that for such equations, the structure of the set of nonoscillatory solutions can be readily characterized. Consequently, various techniques can be applied to eliminate all nonoscillatory solutions, thereby providing criteria for the oscillation of all solutions.

However, oscillation criteria for second-order differential equations involving both positive and negative terms remain an active area of research. The presence of both types of terms complicates the analysis, since the structure of the set of nonoscillatory solutions of the hybrid neutral equation (E) is not well understood. As a result, only a limited number of criteria have been developed to study the oscillatory behavior of such equations.

In [9, 28, 33], the authors examined the oscillation and nonoscillation of the first-order neutral differential equation

$$(x(\theta) - R(\theta)x(\theta - r))' + P(\theta)x(\theta - \tau) - Q(\theta)x(\theta - \sigma) = 0, \tag{E_3}$$

using the integral averaging method and fixed-point theorems. Furthermore, second-order neutral differential equations of the form

$$(r(\theta)(x(\theta) \pm C(\theta)h(x(\theta - \tau)))')' + P(\theta)f(x(\theta - \delta)) - Q(\theta)g(x(\theta - \sigma)) = 0, \tag{E_4}$$

and their generalizations were studied in [8, 14, 21, 32, 35] via the comparison method and fixed-point theorems.

More recently, in [36], the authors investigated oscillation criteria for the second-order linear neutral differential equation

$$(a(\theta)z'(\theta))' + p(\theta)z(\theta) + q(\theta)x(\sigma(\theta)) = 0, \quad \theta \geq \theta_0, \tag{E_5}$$

where $z(\theta) = x(\theta) + b(\theta)x(\tau(\theta))$, using the Riccati transformation technique and the integral averaging method.

Observe that the equations considered in [8, 9, 14, 21, 32, 35] involve deviating arguments in both the positive and negative terms, while the equation studied in [36] is linear and contains only positive terms. By contrast, our equation (E) is nonlinear and contains a positive delayed term together with a negative non-delayed term. Therefore, equation (E) is fundamentally and structurally different from those previously investigated in the literature.

Moreover, we employ a symmetry transform method, in which a positive solution of a second-order auxiliary differential equation is used to convert the trinomial hybrid equation (E) into a binomial form. This transformation allows a clear identification of the structure of nonoscillatory solutions of the transformed equation. The symmetry transform is crucial since it not only converting equation (E) into a binomial form, but also it preserves the oscillatory character of both the original and transformed equations.

By employing the comparison method, the integral averaging technique, and the arithmetic-geometric mean inequality, we derive new sufficient conditions ensuring the oscillation of all solutions of the transformed binomial-type equation, which in turn guarantees the oscillation of all solutions of equation (E). Three illustrative examples are presented to demonstrate the applicability and novelty of the main results. In particular, for one example, an explicit solution is obtained and shown to be oscillatory.

2. PRELIMINARY RESULTS

In this section, we present some preliminary results that will be needed to establish our main theorems. We first transform the hybrid equation (E) into a binomial-type equation using a positive solution of the related auxiliary second-order linear ordinary differential equation

$$Lu = (a(\theta)u'(\theta))' - p(\theta)u(\theta) = 0, \quad \theta \geq \theta_0. \tag{2.1}$$

Our first lemma is a generalization of the well-known result (see Theorem 2.45 of [34]).

Lemma 2.1. *Let* (H_1) *and* (H_2) *hold. Then equation* (2.1) *has a positive decreasing solution on* $[\theta_0, \infty)$.

Proof. The idea of the proof is to reduce the equation (2.1) to a simpler form $u'' - p(\theta)u = 0$ and then apply Kneser Theorem 2.45 of [34]. Define the transformation

$$s = \phi(\theta) = \int_{\theta_0}^{\theta} \frac{1}{a(x)} dx,$$

then ϕ is a strictly increasing function of θ. It follows that $\frac{d}{ds} = a(\theta)\frac{d}{d\theta}$, and hence

$$a(\theta)Lu = \frac{d^2}{ds^2} - p(\theta)a(\theta)u.$$

Consequently, u satisfies $Lu = 0$ if and only if u (as a function of s) satisfies

$$\frac{d^2u}{ds^2} - p_1(s)u = 0, \tag{2.2}$$

where

$$p_1(s) = p[\phi^{-1}(s)]a[\phi^{-1}(s)] \geq 0,$$

with ϕ^{-1} the inverse of the (monotone) function ϕ. Equation (2.2) has a positive nonincreasing solution $u = h(s)$ by Kneser's Theorem 2.45 of [34], and since $s = \phi(\theta)$ is strictly increasing, it follows that (2.1) also has a positive nonincreasing solution. This ends the proof. $\square$

Next, note that equation (E) can be rewritten in the equivalent form

$$(a(\theta)z'(\theta))' - p(\theta)z(\theta) + p(\theta)b(\theta)x(\tau(\theta)) + q(\theta)x^{\alpha}(\sigma(\theta)) = 0. \tag{2.3}$$

Our next result is based on an alternative representation of the differential operator

$$L(z(\theta)) = (a(\theta)z'(\theta))' - p(\theta)z(\theta), \tag{2.4}$$

in terms of a positive solution $u(\theta)$ of (2.1).

Lemma 2.2. *Let $u(\theta)$ be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$. *Then the operator* (2.4) *can be expressed as*

$$L(z(\theta)) = \frac{1}{u(\theta)} \left(a(\theta)u^2(\theta) \left(\frac{z(\theta)}{u(\theta)} \right)' \right)'. \tag{2.5}$$

Proof. Differentiating the right-hand side of (2.5), we obtain

$$\begin{aligned} L(z(\theta)) &= \frac{1}{u(\theta)} (a(\theta)u(\theta)z'(\theta) - a(\theta)z(\theta)u'(\theta))' \\ &= (a(\theta)z'(\theta))' - (a(\theta)u'(\theta))' \frac{z(\theta)}{u(\theta)} \\ &= (a(\theta)z'(\theta))' - p(\theta)z(\theta), \end{aligned}$$

where we have used the fact that $u(\theta)$ is a positive solution of (2.1). This completes the proof. □

Lemma 2.3. *Let $u(\theta)$ be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$. *Then equation* (E) *can be written in the form*

$$(\eta(\theta)\mu'(\theta))' + u(\theta)p(\theta)b(\theta)x(\tau(\theta)) + u(\theta)q(\theta)x^{\alpha}(\sigma(\theta)) = 0, \tag{2.6}$$

where

$$\eta(\theta) = a(\theta)u^2(\theta), \quad \mu(\theta) = \frac{z(\theta)}{u(\theta)}.$$

Proof. Combining (2.3) with (2.5), we immediately obtain (2.6). This completes the proof. □

For our investigation, it is convenient to assume that (2.6) is in canonical form, i.e.

$$\int_{\theta_0}^{\infty} \frac{1}{\eta(\theta)}\, d\theta = \int_{\theta_0}^{\infty} \frac{1}{a(\theta)u^2(\theta)}\, d\theta = \infty. \tag{2.7}$$

Next, we establish the structure of nonoscillatory solutions of (2.6) under condition (2.7).

Lemma 2.4. *Let $u(\theta)$ be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$ *satisfying* (2.7)*. If $x(\theta)$ is a positive solution of (E), then the corresponding function $\mu(\theta)$ satisfies*

$$\mu(\theta) > 0, \quad \eta(\theta)\mu'(\theta) > 0, \quad (\eta(\theta)\mu'(\theta))' \leq 0 \tag{2.8}$$

for all $\theta \geq \theta_1 \geq \theta_0$.

Proof. Let $x(\theta)$ be an eventually positive solution of (E), i.e. $x(\theta) > 0$, $x(\sigma(\theta)) > 0$, and $x(\tau(\theta)) > 0$ for all $\theta \geq \theta_1$ for some $\theta_1 \geq \theta_0$. Then, by definition, $z(\theta) > 0$ for all $\theta \geq \theta_1$. Since $u(\theta) > 0$ for all $\theta \geq \theta_0$, it follows that $\mu(\theta) = \frac{z(\theta)}{u(\theta)} > 0$ for all $\theta \geq \theta_1$. From (2.6), we have $(\eta(\theta)\mu'(\theta))' \leq 0$ for all $\theta \geq \theta_1$. Moreover, by condition (2.7), it follows that $\eta(\theta)\mu'(\theta) > 0$ for all $\theta \geq \theta_1$. The proof of the lemma is complete. □

In view of $(H1)$, let us make an additional assumption that

$$b(\theta) < \frac{u(\theta)}{u(\tau(\theta))} < 1, \tag{2.9}$$

where $u(\theta)$ is a positive decreasing solution of (2.1) on $[\theta_0, \infty)$.

Lemma 2.5. *Let $u(\theta)$ be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$ *satisfying* (2.7) *and* (2.9)*. Then*

$$x(\theta) \geq u(\theta)F(\theta)\mu(\theta), \quad \theta \geq \theta_1 \geq \theta_0, \tag{2.10}$$

where

$$F(\theta) = 1 - b(\theta)\frac{u(\tau(\theta))}{u(\theta)} > 0.$$

Proof. From the definition of $\mu(\theta)$ and its monotonicity, we have

$$u(\theta)\mu(\theta) = z(\theta) = x(\theta) + b(\theta)x(\tau(\theta)).$$

Thus,

$$x(\theta) \geq u(\theta)\mu(\theta) - b(\theta)u(\tau(\theta))\mu(\tau(\theta))$$

and therefore,

$$x(\theta) \geq u(\theta)\left(1 - b(\theta)\frac{u(\tau(\theta))}{u(\theta)}\right)\mu(\theta),$$

which proves the result. □

Before stating and proving the next result, let us define

$$Q_1(\theta) = u(\theta)b(\theta)p(\theta)u(\tau(\theta))F(\tau(\theta)), \quad Q_2(\theta) = u(\theta)q(\theta)u^\alpha(\sigma(\theta))F^\alpha(\sigma(\theta)),$$

$$Q_3(\theta) = Q_1(\theta) + Q_2(\theta), \quad \Omega(\theta) = \int_{\theta_1}^{\theta} \frac{1}{\eta(s)}\, ds, \quad F(\theta) > 0,$$

where $\theta_1 \geq \theta_0$.

Lemma 2.6. *Let $u(\theta)$ be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$ *satisfying* (2.7) *and* (2.9). *Then, equation* (E) *is oscillatory provided that*

$$(\eta(\theta)\mu'(\theta))' + Q_1(\theta)\mu(\tau(\theta)) + Q_2(\theta)\mu^{\alpha}(\sigma(\theta)) = 0, \quad \theta \geq \theta_0, \tag{2.11}$$

is oscillatory.

Proof. For the sake of contradiction, assume that $x(\theta)$ is an eventually positive solution of (E), i.e. $x(\theta) > 0$, $x(\tau(\theta)) > 0$, and $x(\sigma(\theta)) > 0$ for all $\theta \geq \theta_1 \geq \theta_0$. Then the corresponding function $z(\theta) > 0$ for all $\theta \geq \theta_1$. By Lemma 2.4, we have $\mu(\theta) = \frac{z(\theta)}{u(\theta)} > 0$, and it satisfies condition (2.8) for all $\theta \geq \theta_1$. Using (2.10) in (2.6), we obtain that $\mu(\theta)$ is a positive and increasing solution of

$$(\eta(\theta)\mu'(\theta))' + Q_1(\theta)\mu(\tau(\theta)) + Q_2(\theta)\mu^{\alpha}(\sigma(\theta)) \leq 0.$$

However, by Theorem 2 of [23], the corresponding equation (2.11) also has a positive solution, which yields a contradiction. This completes the proof. □

3. OSCILLATION RESULTS

In this section, we derive oscillation criteria for equation (E) with the help of equation (2.11). We begin with the following theorem.

Theorem 3.1. *Let $u(\theta)$ be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$ *satisfying* (2.7)*and* (2.9). *If*

$$\int_{\theta_0}^{\infty} Q_1(\theta)\, d\theta = \infty \quad \text{or} \quad \int_{\theta_0}^{\infty} Q_2(\theta)\, d\theta = \infty, \tag{3.1}$$

then equation (E) *is oscillatory.*

Proof. Assume, for the sake of contradiction, that $x(\theta)$ is an eventually positive solution of (E), i.e. $x(\theta) > 0$, $x(\tau(\theta)) > 0$, and $x(\sigma(\theta)) > 0$ for all $\theta \geqslant \theta_1$, for some $\theta_1 \geqslant \theta_0$. Then the corresponding function satisfies $z(\theta) > 0$, $z(\tau(\theta)) > 0$, and $z(\sigma(\theta)) > 0$ for all $\theta \geqslant \theta_1$.

From Lemma 2.4, the function $\mu(\theta) = \frac{z(\theta)}{u(\theta)} > 0$ satisfies condition (2.8). Since $\mu(\theta)$ is increasing, there exists a constant $M > 0$ such that $\mu(\theta) \geqslant M$ for all $\theta \geqslant \theta_2 \geqslant \theta_1$.

Using this in (2.11) and integrating from θ_2 to θ, we obtain

$$\int_{\theta_2}^{\theta} (MQ_1(s) + M^{\alpha}Q_2(s))\, ds \leq \eta(\theta_2)\, \mu'(\theta_2) - \eta(\theta)\mu'(\theta).$$

As $\theta \to \infty$, it follows that

$$\int_{\theta_2}^{\infty} (MQ_1(\theta) + M^{\alpha}Q_2(\theta))\, d\theta \leq \eta(\theta_2)\, \mu'(\theta_2) < \infty,$$

which contradicts (3.1). Hence, the proof is complete. □

Remark 3.2. Note that the condition (3.1) in Theorem 3.1 is independent of α and the deviating arguments $\tau(\theta)$, $\sigma(\theta)$ and therefore, it applies to linear, sublinear, and superlinear equations, as well as to both delay and advanced-type differential equations.

In the following, we present several oscillation criteria for equation (E) when condition (3.1) fails to hold.

Theorem 3.3. *Assume $\alpha = 1$ and let $u(\theta)$ be a positive decreasing solution of* (2.1) *on $[\theta_0, \infty)$ satisfying* (2.7) *and* (2.9). *If*

$$\liminf_{\theta\to\infty} \int_{\delta(\theta)}^{\theta} Q_3(s)\Omega(\delta(s))\,ds > \frac{1}{e}, \tag{3.2}$$

then equation (E) is oscillatory.

Proof. Assume, for the sake of contradiction, that $x(\theta)$ is an eventually positive solution of (E), i.e. $x(\theta) > 0$, $x(\tau(\theta)) > 0$, and $x(\sigma(\theta)) > 0$ for all $\theta \geqslant \theta_1 \geqslant \theta_0$. Then the corresponding function satisfies $z(\theta) > 0$, $z(\sigma(\theta)) > 0$, and $z(\tau(\theta)) > 0$ for all $\theta \geqslant \theta_1$.

From Lemma 2.4, the function $\mu(\theta) = \dfrac{z(\theta)}{u(\theta)} > 0$ satisfies condition (2.8). From the monotonicity of $\eta(\theta)\mu'(\theta)$, we obtain

$$\mu(\theta) \geqslant \int_{\theta_1}^{\theta} \frac{\eta(s)\mu'(s)}{\eta(s)}\,ds \geqslant \Omega(\theta)\eta(\theta)\mu'(\theta), \tag{3.3}$$

and hence

$$\left(\frac{\mu(\theta)}{\Omega(\theta)}\right)' = \frac{\Omega(\theta)\eta(\theta)\mu'(\theta) - \mu(\theta)}{\eta(\theta)\Omega^2(\theta)} \leqslant 0,$$

which implies that $\frac{\mu(\theta)}{\Omega(\theta)}$ is decreasing.

Now, from (2.11), we have

$$(\eta(\theta)\mu'(\theta))' + Q_1(\theta)\mu(\tau(\theta)) + Q_2(\theta)\mu(\sigma(\theta)) = 0. \tag{3.4}$$

Since $\delta(\theta) = \min\{\tau(\theta), \sigma(\theta)\}$ and $\mu(\theta)$ is increasing, it follows from (3.4) that

$$(\eta(\theta)\mu'(\theta))' + Q_3(\theta)\mu(\delta(\theta)) \leqslant 0. \tag{3.5}$$

Combining (3.3) with (3.5) and letting $\omega(\theta) = \eta(\theta)\mu'(\theta)$, we see that $\omega(\theta)$ is a positive solution of the inequality

$$\omega'(\theta) + Q_3(\theta)\Omega(\delta(\theta))\omega(\delta(\theta)) \leqslant 0. \tag{3.6}$$

However, by Theorem 2.2.6(i) in [18], condition (3.2) implies that inequality (3.6) has no eventually positive solutions, a contradiction. Hence, the proof is complete. □

Theorem 3.4. *Assume* $\alpha = 1$ *and let* $u(\theta)$ *be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$ *satisfying* (2.7) *and* (2.9). *If*

$$\liminf_{\theta\to\infty} \Omega(\delta(\theta)) \int_{\theta}^{\infty} Q_3(s)\, ds > \frac{1}{4}, \tag{3.7}$$

then equation (E) *is oscillatory.*

Proof. Suppose, on the contrary, that $x(\theta)$ is an eventually positive solution of (E), that is, $x(\theta) > 0$, $x(\tau(\theta)) > 0$, and $x(\sigma(\theta)) > 0$ for all $\theta \geqslant \theta_1 \geqslant \theta_0$. Proceeding as in the proof of Theorem 3.3, we see that

$$\mu(\theta) = \frac{z(\theta)}{u(\theta)} > 0$$

satisfies condition (2.1) and the inequality

$$\big(\eta(\theta)\mu'(\theta)\big)' + Q_3(\theta)\, \mu(\delta(\theta)) \leq 0. \tag{3.8}$$

However, by Theorem 3 of [24], condition (3.7) implies that $\mu(\theta)$ is oscillatory, which is a contradiction. Hence, the proof is complete. □

Theorem 3.5. *Assume that* $\alpha = 1$ *and let* $u(\theta)$ *be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$ *satisfying* (2.7) *and* (2.9). *If*

$$\limsup_{\theta\to\infty} \left\{ \frac{1}{\Omega(\delta(\theta))} \int_{\theta_1}^{\delta(\theta)} Q_3(s)\, \Omega(s)\, \Omega(\delta(s))\, ds + \int_{\delta(\theta)}^{\theta} Q_3(s)\, \Omega(\delta(s))\, ds + \Omega(\delta(\theta)) \int_{\theta}^{\infty} Q_3(s)\, ds \right\} > 1, \tag{3.9}$$

for some $\theta_1 \geqslant \theta_0$, *then equation* (E) *is oscillatory.*

Proof. Suppose, to the contrary, that $x(\theta)$ is an eventually positive solution of (E), that is, $x(\theta) > 0$, $x(\tau(\theta)) > 0$, and $x(\sigma(\theta)) > 0$ for all $\theta \geqslant \theta_1$, for some $\theta_1 \geqslant \theta_0$. Proceeding as in the proof of Theorem 3.4, we see that

$$\mu(\theta) = \frac{z(\theta)}{u(\theta)} > 0$$

satisfies condition (2.8) and inequality (3.5) for all $\theta \geqslant \theta_1$. Integrating (3.5), we obtain

$$\mu'(\theta) \geqslant \frac{1}{\eta(\theta)} \int_{\theta}^{\infty} Q_3(s)\, \mu(\delta(s))\, ds.$$

Integrating once more yields

$$\begin{aligned}\mu(\theta) &\geqslant \int_{\theta_1}^{\theta} \frac{1}{\eta(s)} \int_{s}^{\infty} Q_3(s_1)\,\mu(\delta(s_1))\,ds_1\,ds\\ &= \int_{\theta_1}^{\theta} \frac{1}{\eta(s)} \int_{s}^{\theta} Q_3(s_1)\,\mu(\delta(s_1))\,ds_1\,ds + \int_{\theta_1}^{\theta} \frac{1}{\eta(s)} \int_{\theta}^{\infty} Q_3(s_1)\,\mu(\delta(s_1))\,ds_1\,ds.\end{aligned}$$

By integration by parts, we obtain

$$\mu(\theta) \geqslant \int_{\theta_1}^{\theta} Q_3(s)\,\Omega(s)\,\mu(\delta(s))\,ds + \Omega(\theta) \int_{\theta}^{\infty} Q_3(s)\,\mu(\delta(s))\,ds.$$

Hence,

$$\begin{aligned}\mu(\delta(\theta)) \geqslant \int_{\theta_1}^{\delta(\theta)} Q_3(s)\,\Omega(s)\,\mu(\delta(s))\,ds &+ \Omega(\delta(\theta)) \int_{\delta(\theta)}^{\theta} Q_3(s)\,\mu(\delta(s))\,ds\\ &+ \Omega(\delta(\theta)) \int_{\theta}^{\infty} Q_3(s)\,\mu(\delta(s))\,ds.\end{aligned}$$

Since $\mu(\theta)$ is increasing and $\mu(\theta)/\Omega(\theta)$ is decreasing, the inequality becomes

$$\begin{aligned}\mu(\delta(\theta)) \geqslant &\frac{\mu(\delta(\theta))}{\Omega(\delta(\theta))} \int_{\theta_1}^{\delta(\theta)} Q_3(s)\,\Omega(s)\,\Omega(\delta(s))\,ds\\ &+ \mu(\delta(\theta)) \int_{\delta(\theta)}^{\theta} Q_3(s)\,\Omega(\delta(s))\,ds + \Omega(\delta(\theta))\,\mu(\delta(\theta)) \int_{\theta}^{\infty} Q_3(s)\,ds.\end{aligned}$$

Dividing through by $\mu(\delta(\theta))$ yields

$$1 \geqslant \frac{1}{\Omega(\delta(\theta))} \int_{\theta_1}^{\delta(\theta)} Q_3(s)\,\Omega(s)\,\Omega(\delta(s))\,ds + \int_{\delta(\theta)}^{\theta} Q_3(s)\,\Omega(\delta(s))\,ds + \Omega(\delta(\theta)) \int_{\theta}^{\infty} Q_3(s)\,ds,$$

which contradicts (3.9). Hence, the proof is complete. □

Next, we derive an oscillation condition for equation (E) in the case $\alpha > 1$.

Theorem 3.6. *Assume that β is a ratio of odd positive integers such that $\alpha > \beta > 1$, and let $u(\theta)$ be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$ *satisfying* (2.7) *and* (2.9). *If*

$$\int_{\theta_0}^{\infty} \frac{1}{\eta(\theta)} \int_{\theta}^{\infty} Q_4(s) \frac{\Omega^{\beta}(\delta(s))}{\Omega^{\beta}(s)}\, ds\, d\theta = \infty, \tag{3.10}$$

where

$$Q_4(\theta) = \left(\frac{Q_1(\theta)}{\eta_1}\right)^{\eta_1} \left(\frac{Q_2(\theta)}{\eta_2}\right)^{\eta_2}, \quad \eta_1 = \frac{\alpha-\beta}{\alpha-1}, \quad \eta_2 = \frac{\beta-1}{\alpha-1}, \tag{3.11}$$

then equation (E) *is oscillatory.*

Proof. Assume, on the contrary, that $x(\theta)$ is an eventually positive solution of (E), i.e. $x(\theta) > 0$, $x(\tau(\theta)) > 0$, and $x(\sigma(\theta)) > 0$ for all $\theta \geqslant \theta_1$, for some $\theta_1 \geqslant \theta_0$. Proceeding as in the proof of Lemma 2.6, we see that $\mu(\theta) = \frac{z(\theta)}{u(\theta)} > 0$ satisfies condition (2.8) and the inequality

$$(\eta(\theta)\mu'(\theta))' + Q_1(\theta)\mu(\tau(\theta)) + Q_2(\theta)\mu^{\alpha}(\sigma(\theta)) \leqslant 0.$$

Since μ is increasing and $\delta(\theta) = \min\{\tau(\theta), \sigma(\theta)\}$, we obtain

$$(\eta(\theta)\mu'(\theta))' + Q_1(\theta)\mu(\delta(\theta)) + Q_2(\theta)\mu^{\alpha}(\delta(\theta)) \leqslant 0. \tag{3.12}$$

Using the arithmetic-geometric mean inequality

$$\eta_1 y_1 + \eta_2 y_2 \geqslant y_1^{\eta_1} y_2^{\eta_2}, \quad y_1, y_2 > 0, \tag{3.13}$$

with η_1 and η_2 as in (3.11), and taking

$$y_1 = \eta_1^{-1} Q_1(\theta)\mu^{1-\beta}(\delta(\theta)), \quad y_2 = \eta_2^{-1} Q_2(\theta)\mu^{\alpha-\beta}(\delta(\theta)),$$

we obtain

$$Q_1(\theta)\mu^{1-\beta}(\delta(\theta)) + Q_2(\theta)\mu^{\alpha-\beta}(\delta(\theta)) \geqslant Q_4(\theta). \tag{3.14}$$

Substituting (3.14) into (3.12), we get

$$(\eta(\theta)\mu'(\theta))' + Q_4(\theta)\mu^{\beta}(\delta(\theta)) \leqslant 0. \tag{3.15}$$

Since $\mu(\theta)/\Omega(\theta)$ is decreasing, inequality (3.15) implies

$$(\eta(\theta)\mu'(\theta))' + Q_4(\theta)\frac{\Omega^{\beta}(\delta(\theta))}{\Omega^{\beta}(\theta)}\mu^{\beta}(\theta) \leqslant 0, \quad \theta \geqslant \theta_1. \tag{3.16}$$

Integrating (3.16) from θ to ∞, we obtain

$$\int_{\theta}^{\infty} Q_4(s)\frac{\Omega^{\beta}(\delta(s))}{\Omega^{\beta}(s)}\mu^{\beta}(s)\,ds \leqslant \eta(\theta)\mu'(\theta),$$

or equivalently,

$$\frac{1}{\eta(\theta)}\int_{\theta}^{\infty} Q_4(s)\frac{\Omega^{\beta}(\delta(s))}{\Omega^{\beta}(s)}\,ds \leqslant \frac{\mu'(\theta)}{\mu^{\beta}(\theta)}.$$

Integrating this inequality from θ_1 to θ gives

$$\begin{aligned}\int_{\theta_1}^{\theta}\frac{1}{\mu(s)}\int_{s}^{\infty} Q_4(s_1)\frac{\Omega^{\beta}(\delta(s_1))}{\Omega^{\beta}(s_1)}\,ds_1\,ds &\leqslant \int_{\theta_1}^{\theta}\frac{\mu'(s)}{\mu^{\beta}(s)}\,ds\\ &\leqslant \frac{\mu^{1-\beta}(\theta_1)}{\beta-1}-\frac{\mu^{1-\beta}(\theta)}{\beta-1}.\end{aligned}$$

As $\theta \to \infty$, we obtain

$$\int_{\theta_1}^{\infty}\frac{1}{\mu(\theta)}\int_{\theta}^{\infty} Q_4(s)\frac{\Omega^{\beta}(\delta(s))}{\Omega^{\beta}(s)}\,ds\,d\theta \leq \frac{\mu^{1-\beta}(\theta_1)}{\beta-1} < \infty,$$

which contradicts (3.10). Hence, the proof is complete. □

We conclude this section with the following theorem for the case $\alpha < 1$.

Theorem 3.7. *Assume that γ is a ratio of odd positive integers such that $1 > \gamma > \alpha$, and let $u(\theta)$ be a positive decreasing solution of* (2.1) *on* $[\theta_0, \infty)$ *satisfying* (2.7) *and* (2.9)*. If*

$$\int_{\theta_0}^{\infty} \Omega^{\gamma}(\delta(\theta))Q_5(\theta)\,d\theta = \infty, \tag{3.17}$$

where

$$Q_5(\theta) = \left(\frac{Q_1(\theta)}{\eta_1}\right)^{\eta_1}\left(\frac{Q_2(\theta)}{\eta_2}\right)^{\eta_2}, \quad \eta_1 = \frac{\gamma-\alpha}{1-\alpha}, \quad \eta_2 = \frac{1-\gamma}{1-\alpha}, \tag{3.18}$$

then equation (E) *is oscillatory.*

Proof. Assume, on the contrary, that $x(\theta)$ is an eventually positive solution of (E); that is, $x(\theta) > 0$, $x(\tau(\theta)) > 0$, and $x(\sigma(\theta)) > 0$ for all $\theta \geqslant \theta_1$, for some $\theta_1 \geqslant \theta_0$. Proceeding as in the proof of Theorem 3.6, we arrive at (3.12) for all $\theta \geqslant \theta_1$.

Recall the well-known arithmetic-geometric mean inequality

$$\eta_1 y_1 + \eta_2 y_2 \geqslant y_1^{\eta_1} y_2^{\eta_2}, \quad y_1, y_2 > 0, \tag{3.19}$$

where η_1 and η_2 are chosen to satisfy (3.18).

Now, setting

$$y_1 = \eta_1^{-1} Q_1(\theta)\, \mu^{1-\gamma}(\delta(\theta)), \quad y_2 = \eta_2^{-1} Q_2(\theta)\, \mu^{\alpha-\gamma}(\delta(\theta)),$$

in (3.19), we obtain

$$Q_1(\theta)\, \mu^{1-\gamma}(\delta(\theta)) + Q_2(\theta)\, \mu^{\alpha-\gamma}(\delta(\theta)) \geqslant \left(\frac{Q_1(\theta)}{\eta_1}\right)^{\eta_1} \left(\frac{Q_2(\theta)}{\eta_2}\right)^{\eta_2}. \tag{3.20}$$

Using (3.20) in (3.12), we get

$$(\eta(\theta)\mu'(\theta))' + Q_5(\theta)\, \mu^{\gamma}(\delta(\theta)) \leqslant 0. \tag{3.21}$$

Now, applying (3.3) in (3.21), we find

$$(\eta(\theta)\mu'(\theta))' + Q_5(\theta)\, \Omega^{\gamma}(\delta(\theta)) \big(\eta(\delta(\theta))\mu'(\delta(\theta))\big)^{\gamma} \leqslant 0, \quad \theta \geqslant \theta_1.$$

Since $\delta(\theta) \leqslant \theta$ and $\eta(\theta)\mu'(\theta)$ is decreasing, it follows that

$$(\eta(\theta)\mu'(\theta))' + Q_5(\theta)\, \Omega^{\gamma}(\delta(\theta)) \big(\eta(\theta)\mu'(\theta)\big)^{\gamma} \leqslant 0, \quad \theta \geqslant \theta_1.$$

Dividing by $\big(\eta(\theta)\mu'(\theta)\big)^{\gamma}$ and integrating from θ_1 to θ, we obtain

$$\begin{aligned}\int_{\theta_1}^{\theta} Q_5(s)\, \Omega^{\gamma}(\delta(s))\, ds &\leqslant -\int_{\theta_1}^{\theta} \frac{\big(\eta(s)\mu'(s)\big)'}{\big(\eta(s)\mu'(s)\big)^{\gamma}}\, ds \\ &\leqslant \frac{\big(\eta(\theta_1)\mu'(\theta_1)\big)^{1-\gamma}}{1-\gamma} - \frac{\big(\eta(\theta)\mu'(\theta)\big)^{1-\gamma}}{1-\gamma}.\end{aligned} \tag{3.22}$$

Letting $\theta \to \infty$ in (3.22) yields a contradiction with (3.17). This completes the proof. □

Remark 3.8. Theorems 3.3 to 3.7 are new and complement several known oscillation criteria for second-order nonlinear neutral delay differential equations. In particular, they unify oscillation results for both linear and nonlinear equations with mixed arguments.

4. EXAMPLES

In this section, we present three examples to demonstrate the importance and novelty of the main results. In the first example, we provide an explicit solution for the considered equation.

Example 4.1. Consider the second-order hybrid neutral delay differential equation

$$z''(\theta) - \frac{2}{\theta^2}x(\theta) + \left(\frac{\theta^2+4}{2\theta^2}\right)x(\theta-2\pi) = 0, \quad \theta \geqslant 3\pi, \tag{4.1}$$

where $z(\theta) = x(\theta) + \frac{1}{2}x(\theta-\pi)$.

The corresponding auxiliary equation (2.1) takes the form

$$u''(\theta) - \frac{2}{\theta^2}u(\theta) = 0, \quad \theta \geqslant 3\pi. \tag{4.2}$$

The function $u(\theta) = \theta^{-1}$ is a positive decreasing solution of (4.2), and $\eta(\theta) = \frac{1}{\theta^2}$ satisfies condition (2.7). Further calculations show that

$$F(\theta) = \frac{\theta-2\pi}{2(\theta-\pi)}, \quad \delta(\theta) = \theta - 2\pi, \quad Q_1(\theta) \approx \frac{1}{\theta^4}, \quad Q_2(\theta) \approx \frac{1}{4\theta^2} + \frac{1}{\theta^4}, \quad \Omega(\theta) = \frac{\theta^3}{3}.$$

It is easy to verify that condition (3.7) becomes

$$\lim_{\theta\to\infty} \inf \frac{(\theta-2\pi)^3}{3} \int_\theta^\infty \left(\frac{1}{4s^2} + \frac{2}{s^4}\right) ds = \infty > \frac{1}{4},$$

which implies that condition (3.7) holds. Therefore, by Theorem 3.4, equation (4.1) is oscillatory. In fact, $x(\theta) = \sin\theta$ is one such oscillatory solution.

Example 4.2. Consider the second-order hybrid neutral delay differential equation

$$\left(\theta^{3/2}z'(\theta)\right)' - \frac{1}{2\sqrt{\theta}}x(\theta) + q_0\theta\, x^{5/3}(\theta/3) = 0, \quad \theta \geqslant 1, \tag{4.3}$$

where $z(\theta) = x(\theta) + \frac{1}{3}x(\theta/2)$ and $q_0 > 0$ is a constant.

The corresponding auxiliary equation (2.1) takes the form

$$\left(\theta^{3/2}u'(\theta)\right)' - \frac{1}{2\sqrt{\theta}}u(\theta) = 0, \quad \theta \geqslant 1. \tag{4.4}$$

The function $u(\theta) = 1/\theta$ is a positive decreasing solution of (4.4), and $\eta(\theta) = \frac{1}{\sqrt{\theta}}$ satisfies condition (2.7). Further calculations show that

$$F(\theta) = \tfrac{1}{3}, \quad \delta(\theta) = \theta/3, \quad Q_1(\theta) = \tfrac{1}{9}\theta^{-5/2}, \quad Q_2(\theta) = q_0\theta^{-5/3}.$$

Choose $\beta = 7/5$ so that $5/3 > 7/5 > 1$. Then $\eta_1 = 2/5$, $\eta_2 = 3/5$, and

$$Q_4(\theta) = q_0^{3/5}\left(\frac{5}{3}\right)\left(\frac{1}{6}\right)^{3/5}\frac{1}{\theta^2}, \quad \Omega(\theta) \approx \tfrac{2}{3}\theta^{3/2}.$$

Condition (3.10) becomes

$$q_0^{3/5}\int\limits_1^\infty \sqrt{\theta}\int\limits_\theta^\infty \frac{5}{27\sqrt{3}}\,(1/24)^{1/5}\,\frac{1}{s^2}ds\,d\theta = \frac{5}{27\sqrt{3}}\,(1/24)^{1/5}\,q_0^{3/5}\int\limits_1^\infty \frac{1}{\sqrt{\theta}}d\theta = \infty,$$

which shows that (3.10) holds for all $q_0 > 0$. Hence, by Theorem 3.6, equation (4.3) is oscillatory for all $q_0 > 0$.

Example 4.3. Consider the second-order hybrid neutral delay differential equation

$$(\theta z'(\theta))' - \frac{1}{\theta}x(\theta) + q_0 x^{1/3}(\theta/2) = 0, \quad \theta \geqslant 1, \tag{4.5}$$

where $z(\theta) = x(\theta) + \frac{1}{4}x(\theta/3)$ and $q_0 > 0$ is a constant.

The corresponding auxiliary equation (2.1) takes the form

$$(\theta u'(\theta))' - \frac{1}{\theta}u(\theta) = 0, \quad \theta \geqslant 1. \tag{4.6}$$

The function $u(\theta) = \frac{1}{\theta}$ is a positive decreasing solution of (4.6), and $\eta(\theta) = \frac{1}{\theta}$ satisfies condition (2.7). Further calculations show that

$$F(\theta) = \frac{1}{4}, \quad \delta(\theta) = \theta/3, \quad Q_1(\theta) = \frac{3}{16\theta^3}, \quad Q_2(\theta) = \frac{q_0}{2^{1/3}}\,\theta^{-1/3}.$$

Choose $\gamma = \frac{3}{5}$ so that $1 > \frac{3}{5} > \frac{1}{3}$. Then $\eta_1 = \frac{2}{5}$, $\eta_2 = \frac{3}{5}$, and

$$Q_5(\theta) = q_0^{3/5}\left(\tfrac{1}{2}\right)^{1/5}\left(\tfrac{15}{32}\right)^{2/5}\left(\frac{5}{3}\right)^{3/5}\theta^{-7/5}, \quad \Omega(\theta) \approx \tfrac{\theta^2}{2}.$$

Condition (3.17) becomes

$$\int\limits_1^\infty q_0^{3/5}\left(\frac{1}{2}\right)^{4/5}\left(\frac{15}{32}\right)^{2/5}\left(\frac{5}{3}\right)^{3/5}\theta^{-1/5}\,d\theta = \infty,$$

which shows that condition (3.17) holds whenever $q_0 > 0$. Therefore, by Theorem 3.7, equation (4.5) is oscillatory.

5. CONCLUSION

In this paper, we established new criteria for testing the oscillation of solutions of hybrid second-order nonlinear neutral delay differential equations. These criteria were obtained by first transforming the studied equation into a binomial form and then applying the integral averaging method together with the arithmetic-geometric mean inequality. Through illustrative examples and appropriate remarks, we have shown that the obtained results are novel and complement the existing oscillation theory of neutral differential equations. Furthermore, none of the known results in [8, 14, 21, 32, 33, 35] are directly applicable to equations (4.1), (4.3), and (4.5) in order to deduce their oscillatory behavior. Finally, we propose the following possible directions for future research:

1. Extend the results of this work to the case when condition (2.7) fails to hold.
2. Develop oscillation criteria similar to those in this paper without relying on the solution of the auxiliary equation (2.1).
3. Investigate the oscillatory behavior of the following hybrid nonlinear neutral equation:
$$\big(a(\theta)z'(\theta)\big)' + p(\theta)x(\theta) - q(\theta)x^{\alpha}(\sigma(\theta)) = 0,$$
where $z(\theta) = x(\theta) + b(\theta)x(\tau(\theta))$.

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