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# ONE COMPLEX VARIABLE VERSUS SEVERAL COMPLEX VARIABLES: THE HARTOGS EXTENSION PHENOMENON IN CONTEXT

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**Abstract.** We explicate the difference between the function theory of one complex variable and the function theory of several complex variables. In particular, we use the inhomogeneous Cauchy–Riemann equations to explain why there is a Hartogs extension phenomenon in several complex variables but not in one complex variable.

**Keywords:** domain, Hartogs extension phenomenon, Cauchy–Riemann equations.

**Mathematics Subject Classification:** 30-02, 30A99, 32A99, 32T05.

## 1. INTRODUCTION

The deep difference between real analysis of one variable and real analysis of several variables is not apparent. In fact Hardy and Littlewood thought that the distinction was merely a matter of bookkeeping. It was not until the school of Calderón and Zygmund in the 1950s that it became clear how truly different the two subjects really are.

Just so, in complex analysis the difference between one variable and several variables is subtle. The first substantial result to explicate this distinction was the theorem of Poincaré in 1906 to the effect that the unit ball in $\mathbb{C}^n$ and the unit polydisc in $\mathbb{C}^n$ are biholomorphically inequivalent. It is worth reviewing his proof, because it shows what deep insight the man had.

Seeking a contradiction, suppose that there is a biholomorphic map $\Phi : B \to D^n$. Since both domains have a transitive automorphism group, we may suppose that $\Phi(0) = 0$. Now let $G$ be the connected component of the identity in the isotropy group of the origin in $B$ and let $G'$ be the connected component of the identity in the isotropy group of the origin in $D^n$. Of course, $\Phi$ maps $G$ to $G'$ isomorphically. But $G$ is the unitary group, which is non-abelian. While $G'$ is the rotation map in each variable separately together with the map that switches variables and that is abelian. Thus, we have a contradiction.

Around the same time, Hartogs produced his remarkable result that says, informally, that in sevaral variables a holomorphic function defined near the boundary of a domain

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extends analytically to the entire domain. (That such a result fails in one complex variable is illustrated by the function $f(z) = 1/z$ on the unit disc.) This result can be proved on the domain $D^2(0,2) \setminus \overline{D}^2(0,1)$ with a simple analysis of Laurent series. We shall instead, in the discussion below, give a proof using partial differential equations that explicates why matters are different in one and several complex variables.

Hartogs's theorem very naturally raises the question of which domains are the natural support of some holomorphic function and which are not. If a domain $\Omega$ has defined on it a holomorphic function that cannot be analytically continued to some larger domain, then we call $\Omega$ a domain of holomorphy. It is natural to want to have an extrinsic geometric characterization of domains of holomorphy. These considerations led, historically, to the definition of a pseudoconvex domain. It was conjectured, by Levi and others, that a domain is pseudoconvex if and only if it is a domain of holomorphy. This question became known as the Levi problem, and occupied complex analysis for much of the first half of the twentieth century.

To put this discussion in perspective, it is worth noting that every domain in one complex variable is a domain of holomorphy. To see this, let $\Omega$ in the complex plane be a domain (a connected open set). Let $K \subseteq \Omega$ be a discrete set (that is, a set with no accumulation points) in $\Omega$ which contains $2^j$ points having distance $c \cdot 2^{-j}$ to the boundary and about distance $c' \cdot 2^{-j}$ from each other for each $j = 1, 2, \ldots$. Then $K$ has no interior accumulation point but in fact does accumulate at every point of $\partial\Omega$ (just because there are about $2^j$ points at distance $2^{-j}$ from the boundary). By the theorem of Weierstrass, there is a holomorphic function $f$ on $\Omega$ which vanishes at each point of $K$ but at no other point.

It is easy to see that $f$ cannot be analytically continued to any larger open set; if it did so continue, then it would have an interior accumulation point of the zero set and would therefore be identically 0. That would be a contradiction.

The solution of the Levi problem turned out to be rather complicated. The problem was first cracked by Oka (see [10–13]) in the 1940s using the Cousin problems. That approach was later generalized using sheaf theory (see [6]). And the more modern method of examining the situation relies on partial differential equations. See [9] for a detailed consideration of these matters.

## 2. BACKGROUND

Speaking informally, in several complex variables the Hartogs extension theorem says that a holomorphic function defined near the boundary of a domain extends to the interior of the domain. A first version of this theorem was proved by Friedrich Hartogs [7]. It is also called the Osgood/Brown theorem, acknowledging later work by Arthur Barton Brown [2] and William Fogg Osgood [14].

Hartogs's original 1906 proof uses Cauchy's integral formula for functions of several complex variables (see [9, Ch. 1]). Today, there are proofs that rely either the Bochner–Martinelli–Koppelman formula [9, Ch. 1] or the solution of the inhomogeneous Cauchy–Riemann equations with compact support ([4], [9, Ch. 0]). In fact, it is Ehrenpreis's approach that we discuss and develop in this paper. Yet another proof of this result was given by Gaetano Fichera [5] using his solution of the Dirichlet problem for holomorphic functions of several variables. His ideas were later further explored by Giuliano Bratti [1]. In addition, there are contributions by Akira Kaneko [8]. It is also possible to prove the Hartogs result using analytic continuation via analytic discs.

## 3. THE KEY IDEA

Now we present a rigorous discussion of the ideas in this paper. Our purpose is to explain why there is a Hartogs phenomenon in several complex variables but not in one complex variable.

Throughout this discussion a *domain* in the plane or in space is a connected, open set.

Let $D$ be the unit disc in the complex plane. It is obvious that the function $f(z) = 1/z$ is holomorphic on $D \setminus \{0\}$ but cannot be analytically continued past the origin. It is striking that there is no such example in the function theory of two or more complex variables.

More precisely, let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain, $n > 1$. Let $K$ be a compact subset of $\Omega$ with the property that $\Omega \setminus K$ is connected. If $f$ is holomorphic on $\Omega \setminus K$ then there is a holomorphic $F$ on $\Omega$ such that $F|_{\Omega\setminus K} = f$.

It is worthwhile to have a unifying idea that explains this difference between dimension 1 and dimension $n$, $n > 1$. That is what we present in this paper.

It is classical to define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It is easy to check that

$$\frac{\partial}{\partial z}z = 1, \quad \frac{\partial}{\partial z}\overline{z} = 0, \quad \frac{\partial}{\partial \overline{z}}z = 0, \quad \frac{\partial}{\partial \overline{z}}\overline{z} = 1.$$

Now let $\phi$ be a continuously differentiable function on a domain $\Omega$ in $\mathbb{C}$ and taking values in $\mathbb{C}$. Then $\phi$ is holomorphic if and only if $(\partial/\partial\overline{z})\phi(z) = 0$. For if we write $\phi = \phi_1 + i\phi_2$ – the decomposition of $\phi$ into its real and imaginary parts – then

$$\begin{aligned}\frac{\partial\phi}{\partial\overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(\phi_1 + i\phi_2)\\ &= \frac{1}{2}\left(\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x}\right).\end{aligned}$$

Thus, the condition that $\partial/\partial\overline{z}\phi(\zeta) = 0$ corresponds to the simultaneous vanishing of the real and imaginary parts of the displayed equations. But that is just the classical Cauchy–Riemann equations for $\phi$.

It will be useful for us to have a general version of the Cauchy integral formula. The idea here is that we want a formula that is not true only for holomorphic functions, but rather for virtually "all" functions.

**Proposition 3.1.** *Let $\Omega$ be a domain in $\mathbb{C}$ with $C^1$ boundary and let $f$ be a function that is $C^1$ on $\overline{\Omega}$. Then, for $z \in \Omega$,*

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f/\partial\overline{\zeta}}{\zeta - z} \, d\overline{\zeta} \wedge d\zeta.$$

*Proof.* We apply Stokes's theorem to the function

$$\frac{f(\zeta)}{\zeta - z}$$

on the domain

$$\Omega_\epsilon \equiv \Omega \setminus \overline{D}(z, \epsilon)$$

for $\epsilon > 0$ small. Then we have

$$\oint_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \oint_{\partial D(0,\epsilon)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \iint_{\Omega_\epsilon} \frac{\partial f/\partial\overline{\zeta}}{\zeta - z} \, d\overline{\zeta} \wedge d\zeta.$$

Observe that the left side is simply the integral over the boundary of $\Omega_\epsilon$.

It is easy to see that

$$\lim_{\epsilon \to 0} \iint_{\Omega_\epsilon} \frac{\partial f/\partial\overline{\zeta}}{\zeta - z} \, d\overline{\zeta} \wedge d\zeta = \iint_{\Omega} \frac{\partial f/\partial\overline{\zeta}}{\zeta - z} \, d\overline{\zeta} \wedge d\zeta.$$

Also,

$$\begin{aligned} \lim_{\epsilon \to 0} \oint_{\partial D(0,\epsilon)} \frac{f(\zeta)}{\zeta - z} \, d\zeta &= \lim_{\epsilon \to 0} \int_0^{2\pi} \frac{f(z + \epsilon e^{i\theta})}{(z + \epsilon e^{i\theta}) - z} \, \epsilon i \; e^{i\theta} \, d\theta \\ &= \lim_{\epsilon \to 0} \int_0^{2\pi} f(z + \epsilon e^{i\theta}) \, i \, d\theta = 2\pi i \; f(z). \end{aligned}$$

Altogether then, we find that

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial f/\partial\overline{\zeta}}{\zeta - z} \, d\overline{\zeta} \wedge d\zeta. \qquad \square$$

It is of interest to be able to solve the inhomogeneous equation

$$\frac{\partial u}{\partial\bar{\zeta}} = \psi$$

when $\psi$ is a given function on $\mathbb{C}$ that is, say, $C^1$ with compact support. It turns out (as we shall prove below) that the formula

$$u(\zeta) = -\frac{1}{\pi} \int \frac{\psi(\xi)}{(\xi - \zeta)} dA(\xi)$$

always supplies a solution to this problem. However, *even though $\psi$ is compactly supported*, no solution of the equation $\partial u/\partial\bar{\zeta} = \psi$ can be compactly supported if $\int \psi(\zeta)d\zeta \neq 0$. For if there were a compactly supported $u$, say $u(\zeta) = 0$ when $|\zeta| \geq R$, that satisfied the equation, then

$$0 = \oint_{|\xi|=R} u(\xi)d\xi \overset{\text{(Stokes)}}{=} \int_{|\xi|<R} \frac{\partial u}{\partial\bar{\xi}}(\xi)d\bar{\xi} \wedge d\xi = 2i \int_{|\xi|<R} \psi(\xi)dA(\xi) \neq 0$$

which is a contradiction.

Now the analogous problem in several complex variables is as follows: Let $\psi_1, \ldots, \psi_n$ be given functions that are in $C^1_c(\mathbb{C}^n)$ (that is, $C^1$ with compact support in $\mathbb{C}^n$). We seek a $C^2$ function $u$ such that

$$\frac{\partial u}{\partial\bar{z}_j} = \psi_j, \quad j = 1, \ldots, n. \tag{3.1}$$

Since

$$\frac{\partial^2 u}{\partial\bar{z}_j\partial\bar{z}_k} = \frac{\partial^2 u}{\partial\bar{z}_k\partial\bar{z}_j},$$

it is clearly necessary, for (3.1) to have a solution, that

$$\frac{\partial\psi_j}{\partial\bar{z}_k} = \frac{\partial\psi_k}{\partial\bar{z}_j}, \quad j, k = 1, \ldots, n.$$

Subject to this compatibility condition, we shall see below that the system is always solvable, and a solution is given by

$$u(z) = -\frac{1}{\pi}\int \frac{\psi_j(z_1, \ldots, z_{j-1}, \xi, z_{j+1}, \ldots, z_n)}{(\xi - z_j)} dV(\xi)\,,$$

for any choice of $j = 1, \ldots, n$. This solution is *always* compactly supported when the dimension $n > 1$. Suppose for simplicity that $j = 1$. Now $\partial u/\partial\bar{z}_\ell = \psi_\ell = 0$, for all $\ell$,when $z$ is large. Thus, $u$ is holomorphic in each variable separately, hence holomorphic, for $z$ large – say when $|z| > R$. But, looking at the definition of $u$,we see that $u(z)$ itself must be zero when $z$ is large. By analytic continuation, $u(z) = 0$ on the unbounded component of ${}^c(\cup_j \text{supp}\psi_j)$ In other words, $u$ is compactly supported.

For later use, we record here the notation

$$\overline{\partial} f = \frac{\partial f}{\partial\overline{z}_1}d\overline{z}_1 + \frac{\partial f}{\partial\overline{z}_2}d\overline{z}_2 + \cdots \frac{\partial f}{\partial\overline{z}_n}d\overline{z}_n$$

for a $C^1$ function $f$.

## 4. THE HARTOGS PHENOMENON

Now the main point of this paper is to give a proof of the following theorem that depends on the Cauchy–Riemann equations – particularly that in several complex variables it holds that compactly supported data yields compactly supported solutions.

We next derive the explicit integral solution formula for the inhomogeneous Cauchy–Riemann equations that we discussed earlier. We will see that the holomorphicity of the one-dimensional Cauchy kernel will play an important role in the derivation of this formula.

**Theorem 4.1.** *Let $\psi \in C^1_c(\mathbb{C})$. The function defined by*

$$u(\zeta) = -\frac{1}{2\pi i}\int \frac{\psi(\xi)}{\xi - \zeta} d\bar{\xi} \wedge d\xi = -\frac{1}{\pi}\int \frac{\psi(\xi)}{\xi - \zeta} dA(\xi)$$

*satisfies*

$$\overline{\partial} u(\zeta) = \frac{\partial u}{\partial \bar{\zeta}}(\zeta) d\bar{\zeta} = \psi(\zeta) d\bar{\zeta}.$$

*Proof.* Let $D(0,R)$ be a large disc that contains the support of $\psi$. Then

$$\begin{aligned} \frac{\partial u}{\partial \bar{\zeta}}(\zeta) &= -\frac{1}{2\pi i}\frac{\partial}{\partial \bar{\zeta}} \int\limits_{\mathbb{C}} \frac{\psi(\xi)}{\xi - \zeta} d\bar{\xi} \wedge d\xi = -\frac{1}{2\pi i}\frac{\partial}{\partial \bar{\zeta}} \int\limits_{\mathbb{C}} \frac{\psi(\xi + \zeta)}{\xi} d\bar{\xi} \wedge d\xi \\ &= -\frac{1}{2\pi i}\int\limits_{\mathbb{C}} \frac{\frac{\partial \psi}{\partial \bar{\xi}}(\xi + \zeta)}{\xi} \delta\bar{\xi} \wedge d\xi = -\frac{1}{2\pi i}\int\limits_{D(0,R)} \frac{\frac{\partial \psi}{\partial \bar{\xi}}(\xi)}{\xi - \zeta} \delta\bar{\xi} \wedge d\xi. \end{aligned}$$

By Proposition 3.1 this last equals

$$\psi(\zeta) - \frac{1}{2\pi i}\int\limits_{\partial D(0,R)} \frac{\psi(\xi)}{\xi - \zeta} d\xi = \psi(\zeta).$$

Here we have used the support condition on $\psi$. This is the result that we wish to prove. ☐

Let us just say that there is a similar formula for the solution of the $\overline{\partial}$ problem (that is, the problem $\overline{\partial} u = f$) in several complex variables, proved in just the same way. We have enunciated this formula above. We omit the details here, but refer to the reader to [9, Section 1.1] for the full story.

**Theorem 4.2** (The Hartogs phenomenon)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain, $n > 1$. Let $K$ be a compact subset of $\Omega$ with the property that $\Omega \setminus K$ is connected. If $f$ is holomorphic on $\Omega \setminus K$ then there is a holomorphic $F$ on $\Omega$ such that $F|_{\Omega \setminus K} = f$.*

*Proof.* Let $\phi$ be a function in $C^\infty_c(\Omega)$ that is identically 1 on a neighborhood of $K$. Define

$$\tilde{f}(z) = \begin{cases} (1 - \phi(z))\, f(z), & \text{if } z \in \Omega \setminus K, \\ 0, & \text{if } z \in K. \end{cases}$$

Then $\tilde{f} \in C^\infty(\Omega)$. Finally, set

$$\psi(z) = \overline{\partial} \tilde{f}(z).$$

Then $\psi$ satisfies the following crucial properties:

1. $\psi$ has $C^\infty$ coefficients,
2. $\bar{\partial}\psi = \bar{\partial}^2 \tilde{f} \equiv 0$,
3. $\operatorname{supp}\psi$ is a *compact* subset $K_0$ of $\Omega$.

The first two of these properties are obvious and the last follows since $\tilde{f}$ is holomorphic in $\Omega \cap$ (neighborhood of $\partial\Omega$).

By our earlier discussion, there is a function $u \in C_c^\infty(\mathbb{C}^n)$, with support compact in $\Omega$, so that $\bar{\partial}u = \psi$. In particular the function $u$ is identically 0 in a neighborhood $U$ of $\partial\Omega$. We define $F = \tilde{f} - u$. Then

$$\bar{\partial}F = \bar{\partial}\tilde{f} - \bar{\partial}u = \psi - \psi = 0,$$

so $F$ is holomorphic on $\Omega$. Also, shrinking $U$ if necessary, we obtain

$$F|_U = (\tilde{f} - u)|_U = \tilde{f}|_U = f|_U.$$

Therefore, $F$ agrees with $f$ near $\partial\Omega$. Since $\Omega \setminus K$ is connected, we may conclude by the uniqueness of analytic continuation that $F = f$ on $\Omega \setminus K$. The proof is complete. □

**Remark 4.3.** Let us examine this proof and see why it works. The idea is that $\tilde{f}$ is a $C^\infty$ extension of $f$ to $\Omega$. It could not in general be holomorphic since it is identically 0 on $K$ (which could have interior). What we hope is that we can subtract a function $u$ from $\tilde{f}$ to make it holomorphic. That is, we want $\bar{\partial}(\tilde{f} - u) = 0$ or $\bar{\partial}u = \psi$.

Now the equation $\bar{\partial}u = \psi$ has many solutions because $\bar{\partial}$ is a linear operator with a large null space (all holomorphic functions). One solution is the function $\tilde{f}$ itself, but this would be a poor choice since, then $\tilde{f} - u$ would be identically zero. This is why it is so important that we be able to find a solution $u$ with support compact in $\Omega$. For since $\tilde{f}$ has support near $\partial\Omega$, then $\tilde{f} - u$ is not the zero function and agrees with the original function $f$ near $\partial\Omega$.

## 5. CONCLUDING REMARKS

The dialectic between the function theory of one complex variables and that of several complex variables is a subtle one. It is explored in some detail in the book [3]. This paper explores one particular aspect of that duality, and does so from the point of view of differential equations.

It is possible to examine the matter using analytic discs, or plurisubharmonic exhaustion functions. But the point of view taken here is more direct and more explicit. We hope that the reader will find it useful.

REFERENCES

[1] G. Bratti, *Su di un teorema di Hartogs*, Rend. Sem. Mat. Univ. Padova **79** (1988), 59–70.

[2] A. Brown, *On certain analytic continuations and analytic homeomorphisms*, Duke Math. J. **2** (1936), 20–28.

[3] P.V. Dovbush, S.G. Krantz, *One Complex Variable from the Several Variable Point of View*, Taylor & Francis, Boston, 2025.

[4] L. Ehrenpreis, *A new proof and an extension of Hartogs's theorem*, Bull. Amer. Math. Soc. **67** (1961), 507–509.

[5] G. Fichera, *Caratterizzazione della traccia, sulla frontiera di un campo, di una funzione analitica di più variabili complesse*, Rend. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Ser. VIII **22** (1957), 706–715.

[6] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. **68** (1958), 460–472.

[7] F. Hartogs, *Einige Folgerungen aus der Cauchyschen Integralformel bei Funktionen mehrerer Veränderlichen*, Sitzungsber. Königl. Bayer. Akad. Wiss. Math.-Phys. Kl. **36** (1906), 223–242.

[8] A. Kaneko, *On continuation of regular solutions of partial differential equations with constant coefficients*, Proc. Japan Acad. **49** (1973), 17–19.

[9] S.G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., Amer. Math. Soc., Providence, RI, 2001.

[10] K. Oka, *Sur les fonctions analytiques de plusieurs variables. IV. Domaines d'holomorphie et domaines rationnellement convexes*, Japan. J. Math. **17** (1941), 517–521.

[11] K. Oka, *Sur les fonctions analytiques de plusieurs variables. V. L'intégrale de Cauchy*, Japan. J. Math. **17** (1941), 523–531.

[12] K. Oka, *Sur les domaines pseudoconvexes*, Proc. Imp. Acad. Tokyo **17** (1941), 7–10.

[13] K. Oka, *Sur les fonctions analytiques de plusieurs variables. VI. Domaines pseudoconvexes*, Tôhoku Math. J. **49** (1942), 15–52.

[14] W.F. Osgood, *Lehrbuch der Funktionentheorie II*, 2nd ed., B. G. Teubner, Leipzig, 1929.

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