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AN ELLIPTIC SYSTEM INVOLVING p-LAPLACIAN AND A CHEMOTACTIC TERM

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Abstract. We study the nonlinear elliptic system:

$$\begin{cases} u \in W_0^{1,p}(\Omega) : - \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + u = - \operatorname{div}(a(x)u|\nabla\psi|^{p-2}\nabla\psi) + f(x), \\ \psi \in W_0^{1,p}(\Omega) : - \operatorname{div}(a(x)|\nabla\psi|^{p-2}\nabla\psi) = u^\theta \end{cases}$$

in a bounded, open subset of $\mathbb{R}^N$ for $N > 2$ and $2 < p < N$, where f satisfies:

$$0 \le f, \quad f \in L^{(p^*)'}(\Omega), \quad p^* = \frac{Np}{N-p},$$

$a \in L^\infty(\Omega)$ is a given function such that there exist $\alpha, \beta \in \mathbb{R}$ satisfying

$$0 < \alpha \le a(x) \le \beta, \quad x \in \Omega.$$

We prove the existence of weak solutions in $[W_0^{1,p}(\Omega)]^2$ under the assumption

$$0 < \theta < 1 - \frac{2}{p^*}.$$

Keywords: nonlinear elliptic systems, weak solutions, p-Laplacian, chemotaxis term.

Mathematics Subject Classification: 35J47, 35J60, 35D30.

1. INTRODUCTION

In this article, we are interested in the existence of weak solutions of the nonlinear elliptic system

$$\begin{cases} 0 \le u \in W_0^{1,p}(\Omega) : \\ - \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + u = - \operatorname{div}(a(x)u|\nabla\psi|^{p-2}\nabla\psi) + f(x), \\ 0 \le \psi \in W_0^{1,p}(\Omega) : \quad - \operatorname{div}(a(x)|\nabla\psi|^{p-2}\nabla\psi) = u^\theta, \end{cases} \tag{1.1}$$

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where Ω is a bounded, open subset of class C^1 of $\mathbb{R}^N$ and $N > 2$, under assumptions:

$$2 < p < N,$$

$$0 \leq f(x) \in L^{(p^*)'}(\Omega),$$

and

$$\theta \in \left(0, 1 - \frac{2}{p^*}\right), \tag{1.2}$$

for p^* and $(p^*)'$ given by

$$p^* = \frac{Np}{N-p}, \quad (p^*)' = \frac{Np}{Np - N + p}.$$

There exist two positive constants α and β such that

$$0 < \alpha \leq a(x) \leq \beta.$$

We note that, since $2 < p < N$ we have $p^* > 2$ and

$$\theta < 1 - \frac{2}{p^*} = \frac{Np - 2(N-p)}{Np} = \frac{N(p-2) + 2p}{Np} = \frac{p-2}{p} + \frac{2}{N}.$$

Remark 1.1. In the case $p = 2$, the assumption (1.2) reads $\theta < \frac{2}{N}$, already given in [5].

The problem is connected with chemotaxis systems of PDEs, where two parabolic equations describe the evolution of a biological species and a chemoattractant that orients its movement. The parabolic system has been widely studied from the early 1970s, after the pioneering works of Keller and Segel [13] (see, for instance [1, 11, 12, 17] and references therein for more details). In [13], the diffusion is linear and the chemoattractant coefficient is constant, whereas we consider nonlinear diffusion given by the p-Laplacian and non-constant chemotactic coefficient. Related problems have also been studied in [10] and [19].

Parabolic chemotaxis system with p-Laplacian have been recently studied in [15, 20] and [18], among others.

In [5], the system is analyzed for $p = 2$, $\theta < \frac{2}{N}$ and $f \in L^m(\Omega)$ for $m \geq 1$:

- If $m = 1$, $u \in W_0^{1,q}(\Omega) \cap L^r(\Omega)$, for $1 < q < \frac{N}{N-1}$ and $1 < r < \frac{N}{N-2}$.
- If $1 < m < \frac{2N}{N-2}$, $u \in W_0^{1,m_*}(\Omega)$ for $m_* = \frac{mN}{N+m}$.
- If $\frac{2N}{N+2} \leq m < \frac{N}{2}$, $u \in W_0^{1,2}(\Omega)$.
- If $m > \frac{N}{2}$, $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

After [5], the elliptic system with linear diffusion has been studied in several works; see, for instance [6–9, 16].

Since the data of the problem belong to Lebesgue spaces, we study the problem in terms of weak solutions.

Definition 1.2 (Weak formulation of the problem)**.** $(u,\psi) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ is a weak solution of (1.1) if

$$u|\nabla\psi|^{p-1} \in L^{p'}(\Omega)$$

and

$$\begin{aligned}
&\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega uv = \int_\Omega a(x)u|\nabla\psi|^{p-2}\nabla\psi\nabla v + \int_\Omega fv,\\
&\int_\Omega a(x)|\nabla\psi|^{p-2}\nabla\psi\nabla\varphi = \int_\Omega u^\theta\varphi,
\end{aligned}$$

for all $(v,\varphi) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

Remark 1.3. Notice that, since $v,\varphi \in W^{1,p}(\Omega)$ and $p < N$, we have $v,\varphi \in L^{p^*}(\Omega)$, and

$$\int_\Omega |u|^\theta\varphi \le \|\varphi\|_{L^{p^*}(\Omega)} \left[\int_\Omega |u|^{\theta(p^*)'}\right]^{\frac{1}{(p^*)'}}.$$

Notice that

$$(p^*)'\theta = \frac{1}{1-\frac{1}{p^*}}\theta = \frac{p^*}{p^*-1}\theta < 2\theta < 2 < p^*,$$

which implies $W_0^{1,p}(\Omega) \subset L^{\theta(p^*)'}(\Omega)$ to deduce that

$$\int_\Omega |u|^\theta\varphi$$

is well defined.

The article is organized as follows. In Section 2 we approximate the functions f, u^θ and u in the chemotactic term by $\frac{f}{1+\frac{1}{n}f}$, $\frac{u_n^\theta}{1+\frac{1}{n}u_n^\theta}$ and $\frac{u_n}{1+\frac{1}{n}u_n}$ to obtain the boundedness of the sequences $\{u_n\}$ and $\{\psi_n\}$ in the appropriate spaces. In Section 3 we obtain the weak convergence of the approximate solutions to the weak solution of (1.1) in the sense of Definition 1.2. The result is stated in Theorem 3.6.

2. APPROXIMATED PROBLEM

Let $f_n(x) = \frac{f(x)}{1+\frac{1}{n}f(x)}$. In this section we consider the approximated problem:

$$\begin{cases}
-\operatorname{div}(a(x)|\nabla u_n|^{p-2}\nabla u_n) + u_n = -\operatorname{div}\left(\dfrac{a(x)u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^{p-2}\nabla\psi_n\right) + f_n(x),\\
-\operatorname{div}\left(a(x)|\nabla\psi_n|^{p-2}\nabla\psi_n\right) = \dfrac{u_n^\theta}{1+\frac{1}{n}u_n^\theta}.
\end{cases} \tag{2.1}$$

Weak formulation of the approximated problem reads as follows:

Definition 2.1. $(u_n, \psi_n) \in [W_0^{1,p}(\Omega)]^2$ is a weak solution of (2.1) if it satisfies

$$\begin{aligned}\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v + \int_\Omega u_n v = \int_\Omega \frac{a(x)u_n}{1+\frac{1}{n}u_n}|\nabla \psi_n|^{p-2}\nabla \psi_n \cdot \nabla v \\ + \int_\Omega f_n(x) v,\end{aligned} \tag{2.2}$$

$$\int_\Omega a(x)|\nabla \psi_n|^{p-2}\nabla \psi_n \nabla \varphi = \int_\Omega \frac{u_n^\theta}{1+\frac{1}{n}u_n^\theta}\varphi \tag{2.3}$$

for all $(v, \varphi) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

Since $f(x) \geq 0$, we have

$$\frac{f(x)}{1+\frac{1}{n}f(x)} = \frac{nf(x)}{n+f(x)} \leq n.$$

We use the solution of the approximated problem and its weak formulation to obtain the appropriate a priori estimates of the approximated solution to pass to the limit. We first prove the existence of weak solutions to (2.1).

Lemma 2.2. *There exists a solution* $(u_n, \psi_n) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ *to* (2.1).

Proof. We define the operator $S_n : L^p(\Omega) \to L^p(\Omega)$ as the composition $\mathcal{T}_n \circ \mathcal{F}_n$ where $\mathcal{F}_n(\Gamma) = \xi$ such that ξ is the solution of

$$-\operatorname{div}(a(x)|\nabla \xi|^{p-2}\nabla \xi) = \frac{|\Gamma|^\theta}{1+\frac{1}{n}|\Gamma|^\theta} \tag{2.4}$$

and $\mathcal{T}_n(\xi) = Z$ such that

$$\begin{aligned}Z \in W_0^{1,p}(\Omega): \quad & -\operatorname{div}(a(x)|\nabla Z|^{p-2}\nabla Z) + Z \\ & = -\operatorname{div}\left(\tfrac{a(x)Z}{1+\frac{1}{n}|Z|}|\nabla \xi|^{p-2}\nabla \xi\right) + \tfrac{f(x)}{1+\frac{1}{n}f(x)}.\end{aligned} \tag{2.5}$$

Since $\frac{|\Gamma|^\theta}{1+\frac{1}{n}|\Gamma|^\theta} \in L^\infty(\Omega)$, there exists a unique $\xi \in W_0^{1,p}(\Omega)$ which proves that $\mathcal{F}_n$ is well defined. We note that, using ξ as test function in (2.4), we get

$$\alpha\|\nabla \xi\|^p_{L^p(\Omega)} \leq n\|\xi\|_{L^1(\Omega)} \leq nC_{1,p}\|\nabla \xi\|_{L^p(\Omega)}$$

where $C_{1,p}$ is the constant of the embedding $W^{1,p}(\Omega) \subset L^1(\Omega)$. We have

$$\|\nabla \xi\|_{L^p(\Omega)} \leq \left[\frac{nC_{1,p}}{\alpha}\right]^{\frac{1}{p-1}}. \tag{2.6}$$

To prove that $\mathcal{T}_n$ is well defined we note that a fixed point argument gives the existence of solutions of (2.5) by using the following application:

$$R : L^p(\Omega) \to W_0^{1,p}(\Omega)$$

where $R(V) = \tilde{Z}$ and $\tilde{Z}$ is the solution of the problem

$$\begin{aligned} - \operatorname{div}\big(a(x)|\nabla \tilde{Z}|^{p-2}\nabla \tilde{Z}\big) + \tilde{Z} = - \operatorname{div}\left(\frac{a(x)V}{1+\frac{1}{n}|V|}|\nabla\xi|^{p-2}\nabla\xi\right) \\ + \frac{f(x)}{1+\frac{1}{n}f(x)}. \end{aligned} \tag{2.7}$$

Since

$$\frac{V}{1+\frac{1}{n}|V|}|\nabla\xi|^{p-2}\nabla\xi \in [L^{p'}(\Omega)]^N$$

and

$$\frac{f(x)}{1+\frac{1}{n}f(x)} \in L^\infty(\Omega)$$

we have that there exists a unique $\tilde{Z} \in W_0^{1,p}(\Omega)$ such that $R(V) = \tilde{Z}$. We use $\tilde{Z}$ as test function in (2.7) to obtain, after some computations:

$$\alpha \int_\Omega |\nabla \tilde{Z}|^p + \int_\Omega \tilde{Z}^2 \leq n\beta \int_\Omega |\nabla\xi|^{p-1}|\nabla \tilde{Z}| + \int_\Omega \frac{f(x)}{1+\frac{1}{n}f(x)}\tilde{Z}\,,$$

then, thanks to Hölder's inequality we obtain

$$n\beta \int_\Omega |\nabla\xi|^{p-1}|\nabla \tilde{Z}| \leq n\beta \|\nabla\xi\|^{p-1}_{L^p(\Omega)}\|\nabla \tilde{Z}\|_{L^p(\Omega)}$$

and then by Young's inequality

$$\alpha\left(1-\frac{1}{p}\right)\int_\Omega |\nabla \tilde{Z}|^p + \int_\Omega \tilde{Z}^2 \leq \frac{(n\beta)^{p'}}{p'}\alpha^{-\frac{p'}{p}}\int_\Omega |\nabla\xi|^p + \int_\Omega \frac{f(x)}{1+\frac{1}{n}f(x)}\tilde{Z}.$$

Notice also that

$$\int_\Omega \frac{f(x)}{1+\frac{1}{n}f(x)}\tilde{Z} \leq \frac{1}{2}n^2|\Omega| + \frac{1}{2}\int_\Omega \tilde{Z}^2,$$

and then we claim that

$$\frac{\alpha}{p'}\int_\Omega |\nabla \tilde{Z}|^p + \frac{1}{2}\int_\Omega \tilde{Z}^2 \leq \alpha^{-\frac{p'}{p}}\frac{(n\beta)^{p'}}{p'}\int_\Omega |\nabla\xi|^p + \frac{1}{2}n^2|\Omega|.$$

Thanks to estimate (2.6), we obtain

$$\frac{\alpha}{p'}\int_\Omega |\nabla \tilde{Z}|^p + \frac{1}{2}\int_\Omega \tilde{Z}^2 \leq \alpha^{-\frac{p'}{p}}\frac{(n\beta)^{p'}}{p'}\left[\frac{nC_{1,p}}{\alpha}\right]^{\frac{p}{p-1}} + \frac{1}{2}n^2|\Omega|.$$

We apply the Schauder fixed point theorem to obtain the existence of a solution of (2.5) that we denote by Z. To see that $\mathcal{T}_n$ is well posed, we have to prove the uniqueness of (2.5). We denote by $\mathcal{A}$ the differential operator

$$\mathcal{A}(Z) := -\ \mathrm{div}(a(x)|\nabla Z|^{p-2}\nabla Z) + Z$$

and assume there exist two solutions Z_1 and Z_2 such that

$$\mathcal{A}(Z_1) - \mathcal{A}(Z_2) = -\ \mathrm{div}\left(a(x)\left[\frac{Z_1}{1+\frac{1}{n}|Z_1|} - \frac{Z_2}{1+\frac{1}{n}|Z_2|}\right]|\nabla\xi|^{p-2}\nabla\xi\right). \tag{2.8}$$

For $s \in \mathbb{R}$, we define

$$T_k(s) = \max(-k, \min(s,k)). \tag{2.9}$$

We use $T_h(Z_1 - Z_2)$ as test function in (2.8) to obtain after some computations:

$$\begin{aligned}
&\alpha \int\limits_{|Z_1-Z_2|<h} |\nabla(Z_1 - Z_2)|^p + \int\limits_{\Omega} (Z_1 - Z_2)T_h(Z_1 - Z_2) \\
&\leq \int\limits_{\Omega} a(x){T'}_h\left[\frac{Z_1}{1+\frac{1}{n}|Z_1|} - \frac{Z_2}{1+\frac{1}{n}|Z_2|}\right]|\nabla\xi|^{p-1}|\nabla(Z_1 - Z_2)| \\
&\leq \int\limits_{|Z_1-Z_2|<h} a(x)\left[\frac{Z_1}{1+\frac{1}{n}|Z_1|} - \frac{Z_2}{1+\frac{1}{n}|Z_2|}\right]|\nabla\xi|^{p-1}|\nabla(Z_1 - Z_2)| \\
&\leq h\beta \int\limits_{|Z_1-Z_2|<h} |\nabla\xi|^{p-1}|\nabla(Z_1 - Z_2)| \\
&\leq \alpha^{-\frac{p'}{p}}\frac{(h\beta)^{\frac{p}{p-1}}}{p'} \int\limits_{|Z_1-Z_2|<h} |\nabla\xi|^p + \frac{\alpha}{p}\int\limits_{|Z_1-Z_2|<h} |\nabla(Z_1 - Z_2)|^p.
\end{aligned}$$

We subtract the last term in the right-hand side to obtain

$$\begin{aligned}
&\frac{\alpha}{p'} \int\limits_{\{|Z_1-Z_2|<h\}} |\nabla(Z_1 - Z_2)|^p + \int\limits_{\Omega} (Z_1 - Z_2)T_h(Z_1 - Z_2) \\
&\leq \alpha^{-\frac{p'}{p}}\frac{(\beta h)^{\frac{p}{p-1}}}{p'} \int\limits_{\{|Z_1-Z_2|<h\}} |\nabla\xi|^p.
\end{aligned}$$

Dividing by h and eliminating the term

$$\frac{\alpha}{p'} \int\limits_{|Z_1-Z_2|<h} |\nabla(Z_1 - Z_2)|^p,$$

we deduce

$$\frac{1}{h}\int\limits_{\Omega} (Z_1 - Z_2)T_h(Z_1 - Z_2) \leq \alpha^{-\frac{p'}{p}}\frac{h^{\frac{1}{p-1}}\beta^{\frac{p}{p-1}}}{p'} \int\limits_{|Z_1-Z_2|<h} |\nabla\xi|^p.$$

We take the limits as $h \to 0$ to obtain

$$\lim_{h\to 0} \frac{1}{h} \int_\Omega (Z_1 - Z_2) T_h(Z_1 - Z_2) = 0.$$

Since

$$\lim_{h\to 0} \frac{1}{h} s T_h(s) = |s|,$$

we obtain

$$\int_\Omega |Z_1 - Z_2| = 0,$$

which proves the uniqueness of solution of (2.5) and that $\mathcal{T}_n$ is well defined.

We use Z as test function in (2.5):

$$\int_\Omega a(x)|\nabla Z|^p + \int_\Omega Z^2 = \int_\Omega a(x) \frac{Z}{1+\frac{1}{n}|Z|} |\nabla \xi|^{p-2} \nabla \xi \nabla Z + \int_\Omega \frac{f(x)}{1+\frac{1}{n}f(x)} Z.$$

Then

$$\alpha \int_\Omega |\nabla Z|^p + \int_\Omega Z^2 \leq n\beta \int_\Omega |\nabla \xi|^{p-1} |\nabla Z| + \int_\Omega \frac{f(x)}{1+\frac{1}{n}f(x)} Z. \tag{2.10}$$

We apply Hölder's inequality to the term

$$n\beta \int_\Omega |\nabla \xi|^{p-1} |\nabla Z|$$

to obtain

$$n\beta \int_\Omega |\nabla \xi|^{p-1} |\nabla Z| \leq n\beta \|\nabla \xi\|_{L^p(\Omega)}^{p-1} \|\nabla Z\|_{L^p(\Omega)}$$

and replace

$$\|\nabla \xi\|_{L^p(\Omega)}^{p-1}$$

by (2.6) to deduce

$$n\beta \int_\Omega |\nabla \xi|^{p-1} |\nabla Z| \leq n\beta \left[\frac{nC_{1,p}}{\alpha}\right]^{\frac{p-1}{p-1}} \|\nabla Z\|_{L^p(\Omega)}.$$

Notice also that

$$\int_\Omega \frac{f(x)}{1+\frac{1}{n}f(x)} Z \leq \frac{1}{2} n^2 |\Omega| + \frac{1}{2} \int_\Omega Z^2.$$

We replace into (2.10):

$$\alpha \int_\Omega |\nabla Z|^p + \frac{1}{2} \int_\Omega Z^2 \leq n^2 \beta C_{1,p} \left[\int_\Omega |\nabla Z|^p\right]^{\frac{1}{p}} + \frac{1}{2} n^2 |\Omega|.$$

We apply Young's inequality to the term

$$\frac{n^2\beta C_{1,p}}{\alpha}\left[\int_\Omega |\nabla Z|^p\right]^{\frac{1}{p}}$$

as follows:

$$\frac{n^2\beta C_{1,p}}{\alpha}\left[\int_\Omega |\nabla Z|^p\right]^{\frac{1}{p}} \le \alpha^{-\frac{p'}{p}}\frac{1}{p'}\left[\frac{n^2\beta C_{1,p}}{\alpha}\right]^{p'} + \frac{\alpha}{p}\int_\Omega |\nabla Z|^p.$$

Then

$$\frac{\alpha}{p'}\int_\Omega |\nabla Z|^p + \frac{1}{2}\int_\Omega Z^2 \le \alpha^{-\frac{p'}{p}}\frac{1}{p'}\left[\frac{n^2\beta C_{1,p}}{\alpha}\right]^{p'} + \frac{1}{2}n^2|\Omega|.$$

Therefore, $S_n : L^p(\Omega) \to L^p(\Omega)$ satisfies

$$S_n(L^p(\Omega)) \subset B_\rho(0),$$

where $B_\rho(0)$ is the ball in $W_0^{1,p}(\Omega)$ centered at 0 with radius ρ such that

$$\rho^p = \frac{p'}{\alpha}\left[\frac{\alpha^{-\frac{p'}{p}}}{p'}\left[\frac{n^2\beta C_{1,p}}{\alpha}\right]^{p'} + \frac{1}{2}n^2|\Omega|\right].$$

Since the embedding $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact, we have that $S_n(L^p(\Omega))$ is a relatively compact set of $L^p(\Omega)$ and therefore there exists at least a fixed point of S_n. From the estimates we also have that $Z \in W_0^{1,p}(\Omega)$ and there exists $\xi \in W_0^{1,p}(\Omega)$ such that (Z, ξ) is the solution of (2.1). This completes the proof. □

The following estimate plays a fundamental role in our approach, as in [3].

Lemma 2.3. *Let* (u_n, ψ_n) *be the solution to* (2.1) *given in Lemma 2.2. Then*

$$\int_\Omega |u_n| \le \int_\Omega f. \tag{2.11}$$

Proof. We follow [3], and we use $T_h(u_n)$ as test function in (2.2) to obtain after some straightforward computations:

$$\begin{aligned}
&\int\limits_{|u_n|<h} a(x)|\nabla u_n|^p + \int\limits_{\Omega} u_n T_h(u_n) \\
&\le \beta \int\limits_{\Omega} \frac{|u_n|}{1+\frac{1}{n}|u_n|} |\nabla \psi_n|^{p-1} |\nabla T_h(u_n)| + \int\limits_{\Omega} \frac{f}{1+\frac{1}{n}f} T_h(u_n) \\
&\le \beta \int\limits_{|u_n|<h} \frac{|u_n|}{1+\frac{1}{n}|u_n|} |\nabla \psi_n|^{p-1} |\nabla u_n| + \int\limits_{\Omega} \frac{f}{1+\frac{1}{n}f} T_h(u_n) \\
&\le h\beta \int\limits_{|u_n|<h} |\nabla \psi_n|^{p-1} |\nabla u_n| + \int\limits_{\Omega} \frac{f}{1+\frac{1}{n}f} T_h(u_n) \\
&\le \alpha^{-\frac{p'}{p}} \frac{h^{\frac{p}{p-1}}}{p'} \beta^{\frac{p}{p-1}} \int\limits_{|u_n|<h} |\nabla \psi_n|^p + \frac{\alpha}{p} \int\limits_{|u_n|<h} |\nabla u_n|^p + \int\limits_{\Omega} \frac{f}{1+\frac{1}{n}f} T_h(u_n).
\end{aligned}$$

We subtract the term $\frac{\alpha}{p}\int_{|u_n|<h} |\nabla u_n|^p$ from both sides of the inequality to obtain

$$\begin{aligned}
\frac{\alpha}{p'} \int\limits_{|u_n|<h} |\nabla u_n|^p + \int\limits_{\Omega} u_n T_h(u_n) \le \alpha^{-\frac{p'}{p}} \frac{h^{\frac{p}{p-1}}}{p'} \beta^{\frac{p}{p-1}} \int\limits_{|u_n|<h} |\nabla \psi_n|^p \\
+ \int\limits_{\Omega} \frac{f}{1+\frac{1}{n}f} T_h(u_n).
\end{aligned}$$

We divide by h and eliminate the positive term

$$\frac{\alpha}{p'} \int\limits_{|u_n|<h} |\nabla u_n|^p$$

to obtain

$$\frac{1}{h} \int\limits_{\Omega} u_n T_h(u_n) \le \alpha^{-\frac{p'}{p}} \frac{h^{\frac{1}{p-1}}}{p'} \beta^{\frac{p}{p-1}} \int\limits_{\Omega} |\nabla \psi_n|^p + \frac{1}{h} \int\limits_{\Omega} f_n\, T_h(u_n).$$

Since

$$\lim_{h\to 0} \frac{1}{h} s T_h(s) = |s| \quad \text{and} \quad \frac{1}{h}|T_h(s)| \le 1,$$

taking the limit as $h \to 0$, we get

$$\lim_{h\to 0} \frac{1}{h} \int\limits_{\Omega} u_n T_h(u_n) = \int\limits_{\Omega} |u_n| \le \int\limits_{\Omega} \frac{f}{1+\frac{1}{n}f} \le \int\limits_{\Omega} f.$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let* (u_n, ψ_n) *be the solution to* (2.1). *Then* $u_n \geq 0$.

Proof. Here we follow [2]. Let $h > 0$ and T_h as defined in (2.9) and $(\cdot)_-$ the negative part function defined as $(s)_- = -s$ if $s < 0$ and 0 otherwise.

We use $T_h(-(u_n)_-)$ as test function in (2.1) to obtain, after some computations:

$$\alpha \int\limits_{-h<u_n<0} |\nabla u_n|^p + \int\limits_{-h<u_n<0} u_n T_h(-(u_n)_-) \leq \beta \int\limits_{-h<u_n<0} |u_n||\nabla \psi_n|^{p-1}|\nabla u_n|.$$

Since

$$\beta \int\limits_{-h<u_n<0} u_n |\nabla \psi_n|^{p-1}|\nabla u_n| \leq \frac{\alpha}{p} \int\limits_{-h<u_n<0} |\nabla u_n|^p + \frac{\beta^{p'} h^{p'}}{p' \alpha^{\frac{p'}{p}}} \|\psi_n\|^p_{W_0^{1,p}(\Omega)},$$

we have, in view of $u_n, \psi_n \in W_0^{1,p}(\Omega)$, that

$$\frac{\alpha}{p'} \int\limits_{-h<u_n<0} |\nabla u_n|^p + \int\limits_{-h<u_n<0} u_n T_h(-(u_n)_-) \leq \frac{\beta^{p'} h^{p'}}{p' \alpha^{\frac{p'}{p}}} \|\psi_n\|^p_{W_0^{1,p}(\Omega)}.$$

Therefore, in view of the positivity of $\int_\Omega |\nabla u_n|^p$, we have

$$\int\limits_{-h<u_n<0} u_n T_h(-(u_n)_-) \leq \frac{\beta^{p'} h^{p'}}{p' \alpha^{\frac{p'}{p}}} \|\psi_n\|^p_{W_0^{1,p}(\Omega)}.$$

We divide by h and take the limit as $h \to 0$. Since $p' > 1$ and

$$\lim_{h \to 0} \frac{u_n T_h(-(u_n)_-)}{h} = (u_n)_-,$$

we have

$$\int\limits_\Omega (u_n)_- \leq 0$$

which ends the proof. □

Lemma 2.5. *The sequence* $\{\psi_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

Proof. We use ψ_n as test function in (2.3) to obtain, after some computations:

$$\alpha \int\limits_\Omega |\nabla \psi_n|^p \leq \int\limits_\Omega u_n^\theta \psi_n.$$

Thanks to Hölder's inequality we have

$$\int\limits_\Omega u_n^\theta \psi_n \leq \|u_n\|^\theta_{L^1(\Omega)} \|\psi_n\|_{L^{\frac{1}{1-\theta}}(\Omega)}$$

and, since $\theta < 1 - \frac{2}{p^*}$, we have that

$$\frac{1}{1-\theta} \leq \frac{p^*}{2}.$$

Therefore, using Lemma 2.3,

$$\alpha \int_\Omega |\nabla \psi_n|^p \leq \|u_n\|^\theta_{L^1(\Omega)} \|\psi_n\|_{L^{\frac{1}{1-\theta}}(\Omega)} \leq \|f\|^\theta_{L^1(\Omega)} C_p \|\psi_n\|_{W_0^{1,p}(\Omega)},$$

which implies the result. □

Lemma 2.6. *Let* (u_n, ψ_n) *be a weak solution to* (2.1)*,* $G : \mathbb{R} \to \mathbb{R}$ *a Lipschitz function such that* $G(0) = 0$ *and* $H_n : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$H_n{}'(s) = \frac{s}{1 + \frac{1}{n}s} G'(s), \quad H_n(0) = 0.$$

Then

$$\begin{aligned} \int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla G(u_n) + \int_\Omega u_n\, G(u_n) = \int_\Omega \frac{u_n^\theta}{1 + \frac{1}{n}u_n^\theta} H_n(u_n) \\ + \int_\Omega \frac{f}{1+\frac{1}{n}f} G(u_n). \end{aligned} \tag{2.12}$$

Proof. We use $G(u_n)$ as test function in the first equation to obtain

$$\begin{aligned} &\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \nabla G(u_n) + \int_\Omega u_n\, G(u_n) \\ &= \int_\Omega \frac{a(x)u_n}{1+\frac{1}{n}u_n} |\nabla \psi_n|^{p-2}\nabla\psi_n \nabla G(u_n) + \int_\Omega \frac{f}{1+\frac{1}{n}f} G(u_n). \end{aligned}$$

In the same fashion, we use $H_n(u_n)$ as test function in the second equation, and then

$$\int_\Omega a(x)|\nabla \psi_n|^{p-2}\nabla\psi_n \nabla H_n(u_n) = \int_\Omega \frac{u_n^\theta}{1+\frac{1}{n}u_n^\theta} H_n(u_n).$$

Then, in view of

$$\int_\Omega a(x)|\nabla \psi_n|^{p-2}\nabla\psi_n \nabla H_n(u_n) = \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n} |\nabla \psi_n|^{p-2}\nabla\psi_n \nabla G(u_n),$$

we have the equality (2.12). □

Lemma 2.7. *There exists a positive constant $C_1(f)$ such that*

$$\|u_n\|_{W_0^{1,p}(\Omega)} \le C_1(f).$$

Moreover, there exist $u \in W_0^{1,p}(\Omega)$ and a subsequence, still denoted by $\{u_n\}$, such that

$$\{u_n\} \text{ converges weakly to } u \text{ in } W_0^{1,p}(\Omega),$$

$$\{u_n\} \text{ converges strongly to } u \text{ in } L^q(\Omega) \text{ for } q < \frac{Np}{N-p}$$

and

$$\{u_n\} \text{ converges a.e. to } u.$$

Proof. Let $G : \mathbb{R} \to \mathbb{R}$ be given by $G(s) = s$ and H_n as defined in Lemma 2.6:

$$H_n(0) = 0; \quad {H_n}' = \frac{s}{1+\frac{1}{n}s} G'(s) = \frac{s}{1+\frac{1}{n}s}.$$

Thanks to (2.12), we have

$$\int_\Omega a(x)|\nabla u_n|^p + \int_\Omega u_n^2 = \int_\Omega \frac{u_n^\theta}{1+\frac{1}{n}u_n^\theta}\, H_n(u_n) + \int_\Omega \frac{f(x)}{1+\frac{1}{n}f(x)}\, u_n.$$

Thus, in view of

$$H_n(t) = \int_0^t \frac{s}{1+\frac{1}{n}\,s}\, ds \le \int_0^t s\, ds \le \frac{1}{2}t^2,$$

we have that

$$\alpha \int_\Omega |\nabla u_n|^p + \int_\Omega u_n^2 \le \frac{1}{2} \int_\Omega \frac{u_n^{\theta+2}}{1+\frac{1}{n}u_n^\theta} + \int_\Omega \frac{f(x)}{1+\frac{1}{n}f(x)}\, u_n.$$

Dropping the positive term on the left hand side of the inequality and in view of $f \in L^{(p^*)'}(\Omega)$, it follows that

$$\alpha \int_\Omega |\nabla u_n|^p \le \frac{1}{2} \int_\Omega u_n^{\theta+2} + \|f\|_{(p^*)'} \|u_n\|_{p^*}. \tag{2.13}$$

Then, we have

$$\int_\Omega u_n^{\theta+2} = \int_\Omega u_n^\theta u_n^2 \le \|u_n\|^\theta_{L^1(\Omega)} \|u_n\|^2_{L^{\frac{2}{1-\theta}}(\Omega)}.$$

Notice that

$$\frac{2}{1-\theta} < p^* \Longleftrightarrow \theta < 1 - \frac{2}{p^*}.$$

Therefore, thanks to Lemma 2.3, we have

$$\int_\Omega u_n^{\theta+2} \le \|u_n\|^\theta_{L^1(\Omega)} \|u_n\|^2_{L^{p^*}(\Omega)} \le \|f\|^\theta_{L^1(\Omega)} \|u_n\|^2_{L^{p^*}(\Omega)}.$$

We substitute the previous inequality into (2.13) to deduce

$$\alpha \|u_n\|^p_{W_0^{1,p}(\Omega)} \le \frac{1}{2} \|f\|^\theta_{L^1(\Omega)} \|u_n\|^2_{L^{p^*}(\Omega)} + \|f\|_{(p^*)'} \|u_n\|_{p^*},$$

which implies, in view of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, that

$$\alpha \|u_n\|^p_{W_0^{1,p}(\Omega)} \le S^2 \frac{1}{2} \|f\|^\theta_{L^1(\Omega)} \|u_n\|^2_{W_0^{1,p}(\Omega)} + S\|f\|_{(p^*)'} \|u_n\|_{W_0^{1,p}(\Omega)},$$

where S denotes the embedding constant. Since $p > 2$, it follows that

$$\|u_n\|_{W_0^{1,p}(\Omega)} \le C_1(f).$$

To end the proof, we use the compact embedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ for $q < \frac{Np}{N-p}$ and the fact that every sequence convergent in $L^1(\Omega)$ admits a subsequence that converges almost everywhere. □

3. EXISTENCE OF SOLUTIONS

In this section, we prove the existence of weak solutions of (1.1) by passing to the limit in the weak formulation of the approximated problem. We first notice that, thanks to Lemma 2.7, there exists a function $u \in L^q(\Omega)$ (for every $q < p^*$) and a subsequence of $\{u_n\}$ which converges strongly in $L^q(\Omega)$ to u. In the next lemma, we prove the convergence of a subsequence of $\{\frac{u_n}{1+\frac{1}{n}u_n}\}$ to u in $L^p(\Omega)$.

Lemma 3.1. *Let $n \in \mathbb{N}$ and (u_n, ψ_n) be the solution to (2.1). Then there exists a subsequence of $\frac{u_n}{1+\frac{1}{n}u_n}$ which converges strongly to u in $L^p(\Omega)$.*

Proof. Thanks to Lemma 2.7 we have that $\{u_n\}$ converges to u a.e. Then, we have that $\{\frac{u_n}{1+\frac{1}{n}u_n}\}$ converges to u a.e. $x \in \Omega$ (up to subsequences). Moreover, $\{u_n\}$ converges to u strongly in $L^1(\Omega)$. Now we apply the generalized Lebesgue theorem. Since

$$0 \le \frac{u_n}{1+\frac{1}{n}u_n} \le u_n$$

and

$$\lim_{n\to+\infty} \int_\Omega u_n = \int_\Omega u,$$

we conclude the proof of the lemma. □

Lemma 3.2. $\left\{\frac{u_n^\theta}{1+\frac{1}{n}u_n^\theta}\right\}$ *converges strongly to* u^θ*, up to subsequences, in* $L^q(\Omega)$ *for any* $q<p^*$*.*

Proof. As a consequence of the a.e. convergence of $\{u_n\}$ to u (see Lemma 2.7), we have

$$\lim_{n\to+\infty}\int_\Omega u_n^\theta=\int_\Omega u^\theta.$$

Moreover, since $\{u_n^\theta\}$ converges a.e. to u^θ,

$$\frac{u_n^\theta}{1+\frac{1}{n}u_n^\theta}<u_n^\theta$$

and thanks to general Lebesgue's dominated convergence theorem we obtain the convergence of $\left\{\frac{u_n^\theta}{1+\frac{1}{n}u_n^\theta}\right\}$ to u^θ in $L^1(\Omega)$. □

Lemma 3.3. *There exists* $\psi\in W_0^{1,p}(\Omega)$ *and a suitable subsequence of* $\{\psi_n\}$*, still denoted by* $\{\psi_n\}$*, such that*

$$\{\psi_n\}\ \textit{converges strongly to}\ \psi\ \textit{in}\ W_0^{1,p}(\Omega).$$

Proof. Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, there exists a subsequence, denoted by $\{u_n\}$, such that converges strongly in $L^q(\Omega)$ for $q<p^*$. In the same way as in Lemma 3.2, we have that $\{u_n^\theta\}$ converges strongly in $L^{(p^*)'}(\Omega)$.

For simplicity, we still denote by $\{u_n^\theta\}$ the cited subsequence. Since

$$|a-b|^p\le 2^{p-2}\langle|a|^{p-2}a-|b|^{p-2}b,a-b\rangle\quad\text{if } p\ge 2,$$

we have

$$\begin{aligned}0&\le\alpha 2^{2-p}\int|\nabla\psi_k-\nabla\psi_m|^p\\&\le 2^{2-p}\int_\Omega a(x)(|\nabla\psi_k|^{p-2}\nabla\psi_k-|\nabla\psi_m|^{p-2}\nabla\psi_m)(\nabla\psi_k-\nabla\psi_m)\\&=2^{2-p}\int_\Omega(u_k^\theta-u_m^\theta)(\psi_k-\psi_m)\\&\le c\|u_k^\theta-u_m^\theta\|_{L^{(p^*)'}(\Omega)}\|\psi_k-\psi_m\|_{W^{1,p}(\Omega)}\\&\le c\|u_k^\theta-u_m^\theta\|_{L^{(p^*)'}(\Omega)},\end{aligned}$$

so that the sequence $\{\psi_n\}$ converges strongly in $W_0^{1,p}(\Omega)$ (up to subsequences) to some ψ provided $\theta<1-\frac{1}{p^*}$. □

As consequence of the previous lemma, we state the following theorem.

Theorem 3.4. (u,ψ) *satisfies*

$$\int_\Omega a(x)|\nabla\psi|^{p-2}\nabla\psi\nabla\varphi = \int_\Omega u^\theta\varphi,$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

Proof. Letting $n\to\infty$ in (2.3), and thanks to Lemmas 3.2 and 3.3, the result follows. □

Lemma 3.5. *There exists a subsequence of* $\{u_n\}$ *which converges to u strongly in* $W_0^{1,q}(\Omega)$ *for any* $q\in[1,p)$. *Moreover,*

$$\nabla u_n(x) \text{ conv. (up to a subsequence) a.e. to } \nabla u(x). \tag{3.1}$$

Proof. We follow the proof of the Appendix of [4]. Thanks to Lemma 2.7, we have that $\|u_n\|_{W_0^{1,p}(\Omega)} \le C$.

Let $0<\gamma<1$ and consider

$$\begin{aligned} I_{\Omega,n} &= \int_\Omega \{a(x)[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u]\nabla(u_n-u)\}^\gamma \\ &= \int_{B_k} \{a(x)[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u]\nabla(u_n-u)\}^\gamma \\ &\quad+ \int_{A_k} \{a(x)[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u]\nabla(u_n-u)\}^\gamma = I_{B_k,n}+I_{A_k,n}, \end{aligned}$$

where $B_k=\{x:|u(x)|\le k\}$ and $A_k=\{x:|u(x)|>k\}$. We can write $I_{B_k,n}$ as

$$\begin{aligned} I_{B_k,n} &= \int_{B_k} \{a(x)[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)]\nabla(u_n-T_k(u))\}^\gamma, \\ &\le \int_\Omega \{a(x)[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)]\nabla(u_n-T_k(u))\}^\gamma = J_{\Omega,n} \end{aligned}$$

since the integrand is positive. It follows that

$$I_{\Omega,n} = I_{B_k,n}+I_{A_k,n} \le J_{\Omega,n}+I_{A_k,n}.$$

Then the use of Hölder's inequality (with exponents $\frac{1}{\gamma}$ and $\frac{1}{1-\gamma}$) in $I_{A_k,n}$ implies that

$$\begin{aligned} I_{\Omega,n} = I_{B_k,n}+I_{A_k,n} &\le J_{\Omega,n} + C_1\left[\int_{A_k}(|\nabla u_n|+|\nabla u|)^p\right]^\gamma \mu(A_k)^{1-\gamma} \\ &\le J_{\Omega,n} + C_2\,\mu(A_k)^{1-\gamma} = J_{\Omega,n}+\omega_1(k), \end{aligned}$$

where $\mu(E)$ denotes the Lebesgue measure of a measurable subset $E \subset \Omega$ and $\omega_i(k)$ (for $i = 1, \ldots, 4$) denote quantities such that $\lim_{k\to\infty} \omega_i(k) = 0$. Notice that, thanks to (2.11), $\lim_{k\to\infty} \mu(A_k) = 0$.

Now we study the behaviour of $J_{\Omega,n}$. We take $h \in \mathbb{R}^+$ and split $J_{\Omega,n}$ as follows:

$$\begin{aligned} J_{\Omega,n} = &\int\limits_{\{|u_n(x)-T_k(u)|\le h\}} \Big\{a(x)\big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big] \cdot \nabla[u_n - T_k(u)]\Big\}^\gamma \\ &+ \int\limits_{\{|u_n(x)-T_k(u)|> h\}} \Big\{a(x)\big[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\big] \cdot \nabla[u_n - T_k(u)]\Big\}^\gamma. \end{aligned}$$

The first integral can be written as

$$\int\limits_\Omega \{a(x)[|\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u)]\nabla T_h[u_n - T_k(u)]\}^\gamma.$$

Then we use Hölder's inequality in both terms (with exponents $\frac{1}{\gamma}$ and $\frac{1}{1-\gamma}$) and the estimate $\|u_n\|_{W_0^{1,q}(\Omega)} \le C_q$ to have that

$$\begin{aligned} J_{\Omega,n} \le \Bigg(\int\limits_\Omega &a(x)[|\nabla u_n|^{p-2}\nabla u_n \\ &- |\nabla T_k(u)|^{p-2}\nabla T_k(u)]\nabla T_h[u_n - T_k(u)]\Bigg)^\gamma \mu(\Omega)^{1-\gamma} \\ &+ C_3\, \mu\{x : |u_n(x) - T_k(u(x))| > h\}^{1-\gamma}. \end{aligned}$$

Thus, the use of $T_h[u_n - T_k(u)]$ in (2.2) implies that

$$\begin{aligned} J_{\Omega,n} \le C_4\Bigg(&\int\limits_\Omega (f_n - u_n)\, T_h[u_n - T_k(u)] \\ &+ \int\limits_\Omega \frac{a(x)u_n}{1 + \frac{1}{n}u_n} |\nabla\psi_n|^{p-2}\nabla\psi_n(x)\, \nabla T_h[u_n - T_k(u)] \\ &- \int\limits_\Omega \{a(x)|\nabla T_k(u)|^{p-2}\nabla T_k(u)\}\nabla T_h[u_n - T_k(u)]\Bigg)^\gamma \mu(\Omega)^{1-\gamma} \\ &+ C_3\, \mu\{x : |u_n(x) - T_k(u(x))| > h\}^{1-\gamma}. \end{aligned}$$

We remark that

$$\lim_{n\to\infty} \int\limits_\Omega (f_n - u_n)\, T_h[u_n - T_k(u)] = \int\limits_\Omega (f - u)\, T_h[u - T_k(u)] = \omega_2(k).$$

We have

$$\lim_{n\to\infty}\int_\Omega a(x)|\nabla T_k(u)|^{p-2}\nabla T_k(u)\nabla T_h[u_n - T_k(u)]$$
$$= \int_\Omega a(x)|\nabla T_k(u)|^{p-2}\nabla T_k(u)\cdot\nabla T_h[u - T_k(u)] = 0$$

and

$$\limsup_{n\to\infty}\mu\big(\{|u_n(x) - T_k(u(x))| > h\}\big)^{1-\gamma}$$
$$\le \mu\big(\{|u(x) - T_k(u(x))| \ge h\}\big)^{1-\gamma} = \omega_3(k).$$

The crucial point in the proof of the lemma is the use of ψ as a solution of the second equation (Theorem 3.4) and the argument used in Lemma 2.6 to obtain the convergence of the term

$$\lim_{n\to\infty}\int_\Omega \frac{a(x)u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^{p-2}\nabla\psi_n(x)\,\nabla T_h[u_n - T_k(u)]$$

to 0 which we detail below.

Let $G_k(u) = u - T_k(u)$. Then, thanks to the generalized Lebesgue theorem, we have

$$\lim_{n\to\infty}\int_\Omega \frac{a(x)u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^{p-2}\nabla\psi_n(x)\,\nabla T_h[u_n - T_k(u)]$$
$$= \int_\Omega u\; a(x)|\nabla\psi|^{p-2}\nabla\psi\,\nabla T_h[G_k(u)].$$

Let $h \in (0,1)$ and $Q_{k,h}$ by defined by

$$Q_{k,h}(t) = \begin{cases} 0, & \text{if } 0 \le t < k,\\ \frac{1}{2}t^2 - \frac{1}{2}k^2, & \text{if } k \le t < k+h,\\ h\,k + \frac{1}{2}h^2, & \text{if } k+h \le t, \end{cases}$$

and we note that

$$\begin{cases} 0 \le Q_{k,h}(t) \le k + \frac{1}{2} \le u + \frac{1}{2},\\ \nabla T_h[G_k(u)]\,u = \nabla Q_{k,h}(u), \end{cases}$$

so that (thanks to Theorem 3.4)

$$\int_\Omega u\; a(x)|\nabla\psi|^{p-2}\nabla\psi\;\nabla T_h[G_k(u)] = \int_\Omega a(x)|\nabla\psi|^{p-2}\nabla\psi\;\nabla Q_{k,h}(u)$$
$$= \int_\Omega u^\theta\, Q_{k,h}(u).$$

Thus,

$$\lim_{n\to\infty}\int\limits_{\Omega}\frac{a(x)u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^{p-2}\nabla\psi_n(x)\,\nabla T_h[u_n-T_k(u)]=\int\limits_{\Omega}u^{\theta}\,Q_{k,h}(u)=\omega_4(k,h),$$

with $\lim_{k\to\infty}\omega_4(k,h)=0$ (for $h\in(0,1)$, thanks to the Lebesgue theorem, since $u^{\theta}\,Q_{k,h}(u)$ converges to zero a.e. and $u^{\theta}\,Q_{k,h}(u)\le u^{\theta+1}$. Observe that $u^{\theta+1}\in L^1(\Omega)$ by $\theta+1<2-\frac{2}{p^*}\le 2<p^*$ (see (1.2)).

Thus,

$$\lim_{n\to\infty}J_{\Omega,n}\le C_1[\,\omega_2(k)^{\gamma}+\omega_4(k,h)]+C_3\omega_3(k).$$

Hence, we have proved that

$$\lim_{n\to\infty}I_{\Omega,n}\le\omega_1(k)+C_1[\,\omega_2(k)^{\gamma}+\omega_4(k,h)]+C_3\omega_3(k).$$

Let $\epsilon\in\mathbb{R}^+$. Now we fix k^* such that $\omega_1(k^*)+C_1\omega_2(k^*)^{\gamma}+C_3\omega_3(k^*)<\epsilon$ and then h_0 such that $C_1\omega_4(k^*,h_0)<\epsilon$. At that point, we have $\lim_{n\to\infty}I_{\Omega,n}\le 2\epsilon$, that is,

$$\|\{[|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u]\nabla(u_n-u)\}^{\gamma}\|_{L^1(\Omega)}\to\ 0,$$

which implies, in view of

$$2^{2-p}|\nabla u_n-\nabla u|^p\le[|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u][\nabla u_n-\nabla u],$$

that $\{u_n\}$ strongly converges to u (up to subsequences) in $W_0^{1,q}(\Omega)$ for any $q\in[1,p)$.

Moreover, for a suitable subsequence (still denoted by $\{u_n\}$) we have

$$\{[|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u]\nabla(u_n-u)\}^{\gamma}\ \longrightarrow 0\quad\text{almost everywhere},$$

and also (since γ is strictly positive)

$$\{[|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u]\nabla(u_n-u)\}\ \longrightarrow 0\quad\text{almost everywhere.}$$

Here we use a result proved in [14]: under our assumptions, the previous limit implies that $\{\nabla u_n(x)\}\to\nabla u(x)$ almost everywhere, that is the claim (3.1). □

Theorem 3.6. *There exists at least a weak solution $(u,\psi)\in[W_0^{1,p}(\Omega)]^2$ to (1.1), in the sense of Definition 1.2.*

Proof. We note that for any $n\in\mathbb{N}$, (u_n,ψ_n) satisfies

$$\int\limits_{\Omega}a(x)|\nabla u_n|^{p-2}\nabla u_n\nabla v+\int\limits_{\Omega}u_nv=\int\limits_{\Omega}\frac{a(x)u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^{p-2}\nabla\psi_n\nabla v+\int\limits_{\Omega}\frac{f(x)v}{1+\frac{f(x)}{n}}$$

$$\int\limits_{\Omega}a(x)|\nabla\psi_n|^{p-2}\nabla\psi_n\nabla\varphi=\int\limits_{\Omega}\frac{u_n^{\theta}}{1+\frac{1}{n}u_n^{\theta}}\varphi$$

for all $(v,\varphi) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$. Thanks to the previous lemmas, we have, up to subsequences,

$$u_n \longrightarrow u \text{ strongly in } W_0^{1,q}(\Omega) \text{ for any } q < p,$$

$$\psi_n \longrightarrow \psi \text{ strongly in } W_0^{1,p}(\Omega).$$

Therefore,

$$\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \nabla v + \int_\Omega u_n v \longrightarrow \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega uv. \tag{3.2}$$

and

$$\int_\Omega a(x)|\nabla \psi_n|^{p-2}\nabla \psi_n \nabla \varphi \longrightarrow \int_\Omega a(x)|\nabla \psi|^{p-2}\nabla \psi \nabla \varphi. \tag{3.3}$$

Moreover, Lemma 3.1 shows that

$$\frac{a(x)u_n}{1+\frac{1}{n}u_n} \longrightarrow a(x)u \text{ strongly in } L^p(\Omega).$$

Therefore, we have

$$\frac{a(x)u_n}{1+\frac{1}{n}u_n}|\nabla \psi_n|^{p-2}\nabla \psi_n \longrightarrow a(x)u|\nabla \psi|^{p-2}\nabla \psi, \text{ strongly in } L^1(\Omega),$$

which implies

$$\int_\Omega \frac{a(x)u_n}{1+\frac{1}{n}u_n}|\nabla \psi_n|^{p-2}\nabla \psi_n \nabla v \longrightarrow \int_\Omega a(x)|\nabla \psi|^{p-2}\nabla \psi \nabla v. \tag{3.4}$$

Lemma 3.2 proves that

$$\int_\Omega \frac{u_n^\theta}{1+\frac{1}{n}u_n^\theta}\varphi \longrightarrow \int_\Omega u^\theta \varphi. \tag{3.5}$$

Finally,

$$\int_\Omega f_n v \longrightarrow \int_\Omega fv. \tag{3.6}$$

We take limits in the weak formulation of (2.1) to obtain, thanks to (3.2)–(3.6) that

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega uv = \int_\Omega a(x)u|\nabla \psi|^{p-2}\nabla \psi \nabla v + \int_\Omega fv,$$

$$\int_\Omega a(x)|\nabla \psi|^{p-2}\nabla \psi \nabla \varphi = \int_\Omega u^\theta \varphi,$$

for all $(v,\varphi) \in [C_c^\infty(\Omega)]^2$.

□

Remark 3.7. We have proved the existence of a solution (u, ψ), with $u \geq 0$ and $\psi \geq 0$. Open problem: Is it true that $\mu\{x : u(x) = 0\} = 0$ and that $\mu\{x : \psi(x) = 0\} = 0$?

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