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# A NOTE ON THE DEVELOPMENT OF SINGULARITIES ON SOLUTIONS TO THE NAVIER–STOKES EQUATIONS UNDER SUPERCRITICAL FORCING TERMS

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**Abstract.** Recently, Qi S. Zhang provided examples of solutions to the Navier–Stokes equations which, under suitable hypotheses, blow-up in finite time. He considers axially symmetric solutions in a cylinder $D$ under appropriate boundary conditions and under the effect of supercritical external forces $f$. In its main result Zhang exhibits, for each $q < \infty$, a blow-up solution with suitable $f \in L^q(0,T;L^1(D))$. Following Zhang, we construct blow-up solutions with forcing terms in the space $L^q(0,T;L^p(D))$, for suitable pairs $(q,p)$. In particular our results contain Zhang's result and provide blow-up examples along the supercritical curve

$$\frac{2}{q} + \frac{3}{p} = \frac{7}{2}, \quad 1 \le p < 2,$$

thereby approaching the classical endpoint $f \in L^1(0,T;L^2(D))$. A particularly significant case is the existence of external forces $f \in L^q(0,T;L^p(D))$, for every $p < 2$ and some well determined $q(p) > 1$, for which singularities occur in a finite time. The significant case $f \in L^1(0,T;L^2(D))$, which corresponds to the classical definition of weak solution, remains open. A particularly significant feature of our approach is that external forces, and solutions, are smooth in $[0,T)$. Blow-up occurs only as $t \to T$.

**Keywords:** Navier–Stokes equations, self-similar solutions under forcing terms, development of singularities, blow-up solutions.

**Mathematics Subject Classification:** 35Q30, 76D03, 35A01.

## 1. INTRODUCTION

In this paper, we are concerned with the Navier–Stokes equations

$$\begin{cases} \Delta v - v \cdot \nabla v - \nabla P - \partial_t v = f & \text{in } D \times (0,\infty), \\ \nabla \cdot v = 0 & \text{in } D \times (0,\infty), \\ v(x,0) = v_0 \end{cases} \tag{1.1}$$

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under suitable boundary conditions, where $v$ is the velocity, $P$ is the pressure, and $f$ is a given forcing term. The well-known classical spaces $L^q(0,T;L^p(D))$ are simply denoted by $L^q_t L^p_x$.

J. Leray [21] proved that for divergence-free $v_0 \in L^2$ and $f \in L^1_t L^2_x$, there exists a global weak solution to the above problem, now known as Leray-Hopf solutions (see [15] for the contribution of Hopf in bounded domains). Existence of global regular solutions and uniqueness remain the most important open problem in the theory of the Navier–Stokes equations, at least from the theoretical point of view. Below, by looking for blow-up solutions, we are indirectly concerned with the above first problem (however we give some information on the second problem).

After Leray and Hopf's works many results on the above subjects appeared. An important progress is given by the so-called Ladyzhenskaya–Prodi–Serrin sufficient condition for regularity: If $u \in L^p(0,T;L^q(\Omega))$ with

$$\frac{2}{q} + \frac{3}{p} = 1, \quad 3 < p \leq +\infty, \tag{1.2}$$

then $u$ is regular. See [3, 13] and [14], where completely different and independent proofs are shown (concerning [3] we refer to Section 7 of reference [5]). Assumption (1.2) was previously considered by other authors in studying the uniqueness problem. See Galdi's reference [12] for a very interesting treatment of the classical results on energy equality, uniqueness, and regularity. Further, the first author of this paper extended the above result to the case of assumptions on the vorticity $\omega = \nabla \times u$, see [4]. It is proved that if $\omega \in L^q(0,T;L^p(\Omega))$ with

$$\frac{2}{q} + \frac{3}{p} = 2, \quad \frac{3}{2} \leq p \leq +\infty,$$

then $u$ is regular. The much harder case $p = 3$ in equation (1.2) was reached by L. Escauriaza, G.A. Seregin, V. Šverák, see [11].

Concerning non-uniqueness of (1.1), not treated below, a fundamental contribution was given in O.A. Ladyzhenskaya reference [18], following a previous idea due to V. Scheffer. However, the spatial domain depends on time (roughly, like a revolution cone with the time direction as rotation axis, and the vertex at time zero). Nevertheless, the result is significant, since for regular data the solution is unique. In [7] non-uniqueness was proven for some solutions in the space $L^\infty_t L^2_x$, for $f = 0$. On the other hand, D. Albritton, E. Brué and M. Colombo established, in reference [1], the non-uniqueness of (1.1) in the energy space for a forcing term in the super-critical space $L^1_t L^2_x$. Recall that a forcing term in local $L^q_t L^p_x$ spaces is super-critical, critical, or sub-critical if $\frac{2}{q} + \frac{3}{p}$ is larger, equal, or smaller than 2, respectively.

## 2. THE BLOW-UP PROBLEM

### 2.1. LERAY'S PROPOSAL

In a famous 1934 work (see [21, Ch. III, para. 20, p. 224]) Leray proposed the construction of possible self-similar solutions of the Navier–Stokes equations developing a singularity at a certain time $t = T$. Leray looked for self-similar solutions of the following form:

$$u = \frac{1}{\sqrt{2(T-t)}} U\left(\frac{x}{\sqrt{2(T-t)}}\right), \quad P = \frac{1}{2(T-t)} \Pi\left(\frac{x}{\sqrt{(T-t)}}\right),$$

where $(U(y), \Pi(y))$ should satisfy

$$\begin{aligned} -\Delta U + U + (y \cdot \nabla)U + (U \cdot \nabla)U + \nabla\Pi = 0, \\ \operatorname{div} U = 0. \end{aligned}$$

If $U \neq 0$ then $u$ develops a singularity at $T$. A main difficulty in Leray's approach is due to avoiding external forces. In fact, in the absence of external forces, even in the most general case, the non-existence of Leray's blow-up self-similar solutions was proved in 1996, after 62 years of attempts, by J. Nečas, M. Růžička, and V. Šverák in the historical reference [23].

However, if we introduce a self-similar external force $f$, namely

$$f = \frac{1}{[2(T-t)]^{3/2}} F\left(\frac{x}{\sqrt{2(T-t)}}\right),$$

the situation becomes much simpler. In this case, $(U(y), \Pi(y))$ should satisfy

$$\begin{aligned} -\Delta U + U + (y \cdot \nabla)U + (U \cdot \nabla)U + \nabla\Pi = F, \\ \operatorname{div} U = 0. \end{aligned}$$

If $U = 0$, then $\nabla\Pi = F$ which implies that $\nabla \times F = 0$. The following conclusion holds: If $\nabla \times F \neq 0$, then $U \neq 0$, hence, the external force will lead to a singularity.

## 2.2. ZHANG'S APPROACH

One may find the very basic approach underlying Zhang's article in the above famous Leray's 1934 work. In comparison to Leray's proposal, Zhang's setting is extremely simplified. On the other hand, these restrictions make the actual construction of blow-up solutions more difficult. However, by admitting suitable external forces, the existence of significant blow-up solutions holds. It is worth noting that, as far as we know, there is no proof of blow-up under bounded external forces (similarly to non-uniqueness). We may merely quote the 1985 reference [25] where V. Scheffer considered weak solutions to the Navier–Stokes equations with an external force that always acts against the direction of the flow, and proved that there exists a solution with an internal singularity. However, as the author remarks, $f$ is very singular.

Let us now give some useful insight into the proof of Zhang's main theorem, which establishes the following result, see [27, Theorem 1.1(2)]: For any $q > 1$ there is a regular solution of (1.1) in $[0, T[\times D$, with a forcing term in the super-critical space $L^q_t L^1_x$, which blows up at the final time $T$. The author starts from the following axially symmetric ($\partial_\theta v = 0$) Navier–Stokes equations under the Navier (total) slip boundary condition:

$$\begin{cases} \left(\Delta - \frac{1}{r^2}\right) v_r - (v_r\partial_r + v_3\partial_{x_3})v_r + \frac{v_\theta^2}{r} - \partial_r P - \partial_t v_r = f_r & \text{in } D \times (0,\infty), \\ \left(\Delta - \frac{1}{r^2}\right) v_\theta - (v_r\partial_r + v_3\partial_{x_3})v_\theta + \frac{v_\theta v_r}{r} - \partial_t v_\theta = f_\theta & \text{in } D \times (0,\infty), \\ \Delta v_3 - (v_r\partial_r + v_3\partial_{x_3})v_3 - \partial_{x_3} P - \partial_t v_{x_3} = f_3 & \text{in } D \times (0,\infty), \\ \frac{1}{r}\partial_r(rv_r) + \partial_{x_3} v_3 = 0 & \text{in } D \times (0,\infty), \\ \partial_{x_3} v_r = \partial_{x_3} v_\theta = 0, v_3 = 0 & \text{on } \partial^H D, \\ \partial_{x_3} v_\theta = \frac{v_\theta}{r},\ \partial_r v_{x_3} = 0, v_r = 0 & \text{on } \partial^V D, \end{cases} \tag{2.1}$$

where

$$v_r = v \cdot e_r, \quad v_\theta = v \cdot e_\theta, \quad v_3 = v \cdot e_3,$$

and

$$f_r = v \cdot f_r, \quad f_\theta = f \cdot e_\theta, \quad f_3 = f \cdot e_3.$$

$\partial^H D$ and $\partial^V D$ denote respectively the horizontal and vertical parts of the boundary $\partial D$. Concerning results on the axially symmetric Navier–Stokes equations, classical references are, among others, [17] and [26]. It is known that if $v_\theta = 0$ and $f$ is regular, then blow-up does not occur in $\mathbb{R}^3$ for any time $t > 0$. Hence, it looks suitable to start by the more stringent case, namely by assuming in (2.1) not merely that $v_\theta \neq 0$, but also that $v_r = v_3 = 0$, $f_r = f_3 = 0$, $f_\theta = f_\theta(r,t)$. The problem reduces to solving the following system:

$$\begin{cases} \frac{v_\theta^2}{r} - \partial_r P = 0, \\ \left(\Delta - \frac{1}{r^2}\right) v_\theta - \partial_t v_\theta = f_\theta. \end{cases} \tag{2.2}$$

Let $\phi$ be a solution of the equation

$$\begin{cases} \left(\partial_r^2 - \frac{1}{r^2}\right) \phi - \partial_t \phi = h \quad \text{in } D \times [0,T], \\ \phi(0,t) = 0, \end{cases} \tag{2.3}$$

where $v_\theta = \phi$, $h = f_\theta$, and

$$P = \int_0^r \frac{v_\theta^2}{l}\, dl.$$

Then the couple $v_\theta, P$ satisfy (2.2).

Next Zhang looked for self-similar solution of (2.3) in the form of

$$\phi = \frac{1}{\sqrt{2(T-t)}}\phi_0\left(\frac{r}{\sqrt{2(T-t)}}\right), \quad h = \frac{1}{[2(T-t)]^{\frac{3}{2}}} k\left(\frac{r}{\sqrt{2(T-t)}}\right).$$

Then (2.3) reduces to the following ODE:

$$\begin{cases} \phi_0'' + \frac{1}{r}\phi_0' - \frac{1}{r^2}\phi_0 - \phi_0 - r\phi_0' = k(r), \\ \phi_0(0) = 0, \end{cases} \tag{2.4}$$

where $k = k(r)$, $r \in [0,\infty)$, is a smooth function supported in the unit interval $[0,1]$.

Zhang obtained an explicit solution of (2.4), namely

$$\phi_0 = -\frac{1}{r}\int\limits_0^r s e^{\frac{s^2}{2}} \int\limits_0^\infty e^{-\frac{1}{2}l^2} k(l)\,dl\,ds.$$

Starting from this explicit solution, by setting

$$\phi = \frac{1}{\sqrt{2(T-t)}}\phi_0\left(\frac{r}{\sqrt{2(T-t)}}\right),$$

Zhang constructed the desired blow-up solution by taking

$$v = [\ln(\phi(r,t)+1) - \ln(\phi(1,t)+1)r]\, e_\theta,$$

which is no longer self-similar.

**Remark 2.1.** It is worth noting that the assumption $v_r = v_3 = 0$ implies that $v \cdot \nabla v = (v_r \partial_r + v_3 \partial_{x_3})v = 0$. Hence, the nonlinear term in the Navier–Stokes equation has essentially no effect. It follows that proofs and results also apply to the Stokes evolution problem. It follows that under bounded external forces blow-up is here not possible. However, this fact is not decisive by itself since, as far as we know, better results obtained with the help of the nonlinear term are not available. If so, then the proofs below look more interesting since they are obtained without the above help. Furthermore, the results are obtained under the minimal assumptions, more precisely, a unique non-zero velocity component, and dependence on a unique spatial variable. This looks to us very positive in itself, since it leads us to believe that the addition of another component and another spatial variable, even in a minimally invasive form, could improve the results. A clarifying feature of our blow-up result is that external forces and solutions are smooth in $D \times [0,T)$. blow-up occurs only as $t \to T$. To end, note that the situation for the Stokes evolution problem is a generalization of that for real functions: Blow-up of $u'(t)$ does not imply blow-up of $u(t)$.

Concerning related recent results, for the reader's convenience we repeat the references quoted in [27], namely [8–10, 16, 22, 24]. In [22], the authors constructed a finite time blow-up example for a special cusp type domain.

## 3. STRATEGY, MAIN RESULTS, AND REMARKS

### 3.1. STRATEGY

We appeal to Zhang's basic ideas, by starting from his explicit solution $\phi$. But we obtain a supplementary degree of freedom $\alpha$ essentially by multiplication of the previous solutions by $r^\alpha$. More precisely, we consider solutions of the form $v = [r^\alpha \phi(r,t) - \phi(1,t)r]e_\theta$. Compared with Zhang's paper [27], our advantage is that we may adjust $\alpha$ to obtain a distinct, much larger, set of blow-up solutions.

Clearly, we lose the self-similar structure of the very basic solutions. Note that our parameter $\alpha$ is not related to the $\alpha$ in Zhang's reference, introduced to satisfy the boundary condition on the lateral surface of the cylinder.

### 3.2. MAIN RESULTS

Full results and complete proofs have previously appeared in reference [6]. In the sequel $D$ denotes the cylinder $B_2(0,1) \times [0,1]$ in $\mathbb{R}^3$, where $B_2(0,1)$ is the unit ball in $\mathbb{R}^2$ centered at the origin. Everywhere we assume that $p, q \geq 1$ and $\alpha \geq 0$.

**Theorem 3.1.** *Let $\alpha$ and a couple of exponents $q, p$ be given, such that $k-1 \leq \alpha < k$ for some positive integer $k$, and such that the constraint*

$$[(3-\alpha)p - 2]q < 2 \tag{3.1}$$

*holds. Then there exists a force $f \in L^q_t L^p_x$ and a corresponding, unique, solution $v \in L^\infty_t L^2_x \cap L^2_t H^1_x$ of the Navier–Stokes equations*

$$\Delta v - v \cdot \nabla v - \nabla p - \partial_t v = f \tag{3.2}$$

*in $D \times [0,T)$ which satisfies the non-slip boundary condition $v = 0$ on the vertical boundary of $D$ and the Navier total slip boundary condition on the horizontal boundary of $D$. In addition, $v$ satisfies the point-wise blow-up relation*

$$\|\nabla^{k-1} v(x,t)\|_{L^\infty_x} \to +\infty \quad as \quad t \to T.$$

*Moreover, $v \in L^{\tilde{q}}_t L^{\tilde{p}}_x$ for all pairs $\tilde{p}, \tilde{q}$ satisfying the inequality*

$$[(1-\alpha)\tilde{p} - 2]\tilde{q} < 2. \tag{3.3}$$

*External forces $f$ and solutions $v$ are smooth in $D \times [0,T)$. If $k \geq 2$ then one also has $v \in C^{k-2}[0,T]$.*

Let us put in evidence the case $k = 1$.

**Corollary 3.2.** *For any $0 \leq \alpha < 1$, and any couple $q, p$ satisfying the constraint* (3.1)*, there exists a force $f \in L^q_t L^p_x$ such that the corresponding, unique, solution $v$ of the Navier–Stokes equations* (3.2) *satisfying the no slip boundary condition $v = 0$ on the vertical boundary of $D$ and the Navier total slip boundary condition on the horizontal boundary of $D$, satisfies the blow-up condition*

$$\lim_{t \to T} \|v(x,t)\|_{L^\infty_x} = +\infty. \tag{3.4}$$

*Moreover, $v \in L^{\tilde{q}}_t L^{\tilde{p}}_x$ for all couple $\tilde{p}, \tilde{q}$ such that $[(1-\alpha)\tilde{p} - 2]\tilde{q} < 2$. External forces $f$ and solutions $v$ are smooth in $D \times [0,T)$. blow-up occurs only at time $T$.*

**Corollary 3.3.** *Let $1 \leq p < 2$, and let $q$ be determined by*

$$\frac{2}{q} + \frac{3}{p} = \frac{7}{2}. \tag{3.5}$$

*Hence, $q$, given by*

$$q = \frac{4p}{7p-6}$$

*is larger than* 1*. Then there exists a force*

$$f \in L^q(0,T; L^p(D))$$

*such that the corresponding unique solution $v$ of the Navier–Stokes equations* (3.2)*, satisfying the no-slip boundary condition on the vertical boundary of $D$ and the Navier total slip boundary condition on the horizontal boundary of $D$, satisfies the blow-up condition* (3.4)*. Moreover, the external force $f$ and the solution $v$ are smooth in $D \times [0,T)$, and the blow-up occurs only at time $T$.*

*Proof.* By Corollary 3.2, it is enough to find $0 \le \alpha < 1$ such that

$$\big((3-\alpha)p - 2\big)q < 2. \tag{3.6}$$

Condition (3.6) is equivalent to

$$\alpha > 3 - \frac{2}{p} - \frac{2}{pq}.$$

Using (3.5), we have

$$\frac{1}{q} = \frac{7}{4} - \frac{3}{2p}.$$

Therefore,

$$3 - \frac{2}{p} - \frac{2}{pq} = 3 - \frac{2}{p} - \frac{2}{p}\left(\frac{7}{4} - \frac{3}{2p}\right) = \frac{6p^2 - 11p + 6}{2p^2}.$$

Hence, it remains to check that this lower bound is strictly smaller than 1. Indeed,

$$\frac{6p^2 - 11p + 6}{2p^2} < 1$$

is equivalent to

$$4p^2 - 11p + 6 < 0.$$

Since

$$4p^2 - 11p + 6 = (4p-3)(p-2),$$

the above inequality holds precisely for

$$\frac{3}{4} < p < 2.$$

In particular, it holds for every $1 \le p < 2$. Thus, we may choose

$$\max\left\{0, \frac{6p^2 - 11p + 6}{2p^2}\right\} < \alpha < 1.$$

For this choice of $\alpha$, condition (3.6) holds. Therefore, the hypotheses of Corollary 3.2 are satisfied, and the conclusion follows. By the way, note that the above fraction is positive for all values assumed here by $p$. □

The uniqueness claim follows from the smoothness for $t < T$.

**Remark 3.4.** For any arbitrarily large $q$, by taking $p = 1$ and $\alpha > 1 - \frac{2}{q}$, we obtain [27, Theorem 1.1, item (2)].

**Remark 3.5.** The endpoint $p = 2$, $q = 1$, corresponding to $f \in L^1(0, T; L^2(D))$, is not covered by Corollary 3.3. Indeed, when $p = 2$ and $q = 1$, condition

$$\big((3 - \alpha)p - 2\big)q < 2$$

reduces to

$$4 - 2\alpha < 2,$$

which requires $\alpha > 1$. This is incompatible with the restriction $0 \leq \alpha < 1$ used in Corollary 3.2 to obtain the blow-up of $\|v(\cdot, t)\|_{L^\infty_x}$. Thus, the curve

$$\frac{2}{q} + \frac{3}{p} = \frac{7}{2}$$

approaches the classical endpoint $L^1_t L^2_x$, but does not reach it.

Note that there are external forces $f \in L^1_t L^p_x$, for arbitrarily large $p$ for which

$$\|\nabla v(x, t)\|_{L^\infty_x} \to \infty, \quad \text{as } t \to T, \tag{3.7}$$

holds. On the other hand, for $p = 2$, there exist external forces $f \in L^q_t L^2_x$, for arbitrarily large $q$, for which (3.7) holds.

It could be of some interest to consider the case $q = p$. The above corollary shows that blow-up holds if $f \in L^q_t L^p_x$, with $q = p < \frac{1+\sqrt{5}}{2}$.

Open Problem: We conclude this section with an interesting and challenging open problem, the classical case of the existence or non-existence of an external force in $L^1(L^2)$ that gives rise to a blowup.

## 4. PROOF OF THEOREM 3.1

First, by Zhang's paper [27, (1.11)], it follows that

$$\phi = \frac{1}{\sqrt{2(T-t)}} \phi_0 \left( \frac{r}{\sqrt{2(T-t)}} \right)$$

solves the equation

$$\left( \Delta - \frac{1}{r^2} \right) \phi - \partial_t \phi = h,$$

where $r = |x|$,

$$\phi_0 = \phi_0(r) = -\frac{1}{r} \int_0^r s e^{\frac{s^2}{2}} \int_0^\infty e^{-\frac{1}{2} l^2} k(l) dl ds,$$

$$h = \frac{1}{[2(T-t)]^{\frac{3}{2}}} k\left(\frac{r}{\sqrt{2(T-t)}}\right),$$

and $k = k(r)$ is a fixed smooth function, defined for all $r > 0$,but supported in the unit interval $[0, 1]$.

Set $\tilde{\phi} = r^{\alpha}\phi$. One has

$$\left(\Delta - \frac{1}{r^2}\right)\tilde{\phi} - \partial_t\tilde{\phi} = r^{\alpha}h + \alpha^2 r^{\alpha-2}\phi + 2\alpha r^{\alpha-1}\partial_r\phi := h_1.$$

We remark that $\tilde{\phi}$ and $h_1$ will be, respectively, the solution $v$ and the data $f$ in equation (3.2). It is easy to check that (see Zhang's paper [27, (1.16)])

$$|\phi| \leq \frac{Cr}{r^2+T-t}, \quad |\partial_r\phi| \leq \frac{C}{r^2+T-t},$$

so we have that

$$|\alpha^2 r^{\alpha-2}\phi + 2\alpha r^{\alpha-1}\partial_r\phi| \leq C\frac{r^{\alpha-1}}{r^2+T-t}.$$

**Lemma 4.1.** *For all $1 + p\alpha - p > -1$, that is $\alpha > 1 - \frac{2}{p}$, we have that*

$$\int\limits_D |h_1|^p dx \leq \begin{cases} \frac{C}{(T-t)^{\frac{3p-2-p\alpha}{2}}} & \text{if } \frac{3p-p\alpha}{2} \neq 1, \\ C|\ln(T-t)| & \text{if } \frac{3p-p\alpha}{2} = 1. \end{cases}$$

*Proof.* First, we have that

$$\int\limits_D |\alpha^2 r^{\alpha-2}\phi + 2\alpha r^{\alpha-1}\partial_r\phi|^p dx \leq C\int\limits_0^1 \frac{r^{1+p\alpha-p}}{(r^2+T-t)^p}dr.$$

When $1 - \frac{2}{p} < \alpha \leq 1$, we split the interval $[0, 1]$ into two parts: $[0, \sqrt{T-t}]$ and $[\sqrt{T-t}, 1]$.

In $[0, \sqrt{T-t}]$, we have that

$$\int\limits_0^{\sqrt{T-t}} \frac{r^{1+p\alpha-p}}{(r^2+T-t)^p}dr \leq \frac{1}{(T-t)^p}\int\limits_0^{\sqrt{T-t}} r^{1+p\alpha-p}dr \leq \frac{C}{(T-t)^{\frac{3p-2-p\alpha}{2}}}.$$

In $[\sqrt{T-t}, 1]$, since $p - p\alpha \geq 0$, we have that

$$\frac{(r^2+T-t)^{\frac{p-p\alpha}{2}}}{r^{p-p\alpha}} = \left(\frac{r^2+T-t}{r^2}\right)^{\frac{p-p\alpha}{2}} \leq C$$

for $r \in [\sqrt{T-t}, 1]$. Thus, we have

$$\begin{aligned}
&\int_{\sqrt{T-t}}^{1} \frac{r^{1+p\alpha-p}}{(r^2+T-t)^p} dr \\
&= \int_{\sqrt{T-t}}^{1} \frac{r}{(r^2+T-t)^{\frac{3p-p\alpha}{2}}} \frac{(r^2+T-t)^{\frac{p-p\alpha}{2}}}{r^{p-p\alpha}} dr \\
&\leq C \int_{\sqrt{T-t}}^{1} \frac{r}{(r^2+T-t)^{\frac{3p-p\alpha}{2}}} dr \\
&= \frac{C}{2} \int_{\sqrt{T-t}}^{1} \frac{1}{(r^2+T-t)^{\frac{3p-p\alpha}{2}}} d(r^2+T-t) \\
&= C \begin{cases} \frac{1}{3p-2-p\alpha} \left[ \frac{1}{[2(T-t)]^{\frac{3p-2-p\alpha}{2}}} - \frac{1}{(1+T-t)^{\frac{3p-2-p\alpha}{2}}} \right], & \frac{3p-p\alpha}{2} \neq 1, \\ \frac{1}{2}[\ln(1+T-t) - \ln(T-t)], & \frac{3p-p\alpha}{2} = 1, \end{cases} \\
&= \begin{cases} \frac{C}{(T-t)^{\frac{3p-2-p\alpha}{2}}}, & \frac{3p-p\alpha}{2} \neq 1, \\ C|\ln(T-t)|, & \frac{3p-p\alpha}{2} = 1. \end{cases}
\end{aligned}$$

When $\alpha > 1$, since $p\alpha - p > 0$, it follows that

$$\frac{r^{p\alpha-p}}{(r^2+T-t)^{\frac{p\alpha-p}{2}}} = \left( \frac{r^2}{r^2+T-t} \right)^{\frac{p\alpha-p}{2}} \leq 1,$$

for $r \in [0, 1]$, which gives

$$\begin{aligned}
\int_D |\alpha^2 r^{\alpha-2}\phi + 2\alpha r^{\alpha-1}\partial_r \phi|^p dx &\leq C \int_0^1 \frac{r^{1+p\alpha-p}}{(r^2+T-t)^p} dr \\
&= C \int_0^1 \frac{r}{(r^2+T-t)^{\frac{3p-p\alpha}{2}}} \frac{r^{p\alpha-p}}{(r^2+T-t)^{\frac{p\alpha-p}{2}}} dr \\
&\leq C \int_0^1 \frac{r}{(r^2+T-t)^{\frac{3p-p\alpha}{2}}} dr \\
&\leq \begin{cases} \frac{C}{(T-t)^{\frac{3p-2-p\alpha}{2}}}, & \frac{3p-p\alpha}{2} \neq 1, \\ C|\ln(T-t)|, & \frac{3p-p\alpha}{2} = 1. \end{cases}
\end{aligned}$$

The proof of Lemma 4.1 is accomplished. □

Now let us move to the proof of Theorem 3.1.

From $(3p-2-p\alpha)q<2$ we obtain

$$p\alpha - p + 1 > 2p - \frac{2}{q} - 1 \geq -1.$$

Thus, one shows that $(3p-2-p\alpha)q<2$ implies $p\alpha-p+1>-1$. Then, from Lemma 4.1, it follows that

$$h_1 \in L_t^q L_x^p$$

for all $p$, $q$, and $\alpha$ such that

$$(3p-2-p\alpha)q<2.$$

Next we study the solution $v=\tilde{\phi}$. One has

$$|\tilde{\phi}| \leq C\frac{r^{\alpha+1}}{r^2+T-t}, \quad |\partial_r\tilde{\phi}| \leq C\frac{r^{\alpha}}{r^2+T-t}.$$

Hence,

$$\int_D |\tilde{\phi}|^2 dx \leq C\int_0^1 \frac{r^{2\alpha+3}}{(r^2+T-t)^2}dr \leq C,$$

and also

$$\begin{aligned}
\int_D |\partial_r\tilde{\phi}|^2 dx &\leq C\int_0^1 \frac{r^{2\alpha+1}}{(r^2+T-t)^2}dr \\
&\leq C\int_0^1 \frac{r}{(r^2+T-t)^{2-\alpha}}\frac{r^{2\alpha}}{(r^2+T-t)^{\alpha}}dr \\
&\leq C\int_0^1 \frac{r}{(r^2+T-t)^{2-\alpha}}dr \\
&\leq \begin{cases} \frac{C}{(T-t)^{1-\alpha}}, & \alpha<1, \\ C|\ln(T-t)|, & \alpha=1, \\ C(1+T-t)^{\alpha-1}, & \alpha>1. \end{cases}
\end{aligned}$$

If $\alpha>0$, then $\frac{C}{(T-t)^{1-\alpha}} \in L_t^1$, so $\tilde{\phi} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$ for $\alpha>0$.

Recall that

$$\tilde{\phi} = r^\alpha \phi = \frac{r^\alpha}{\sqrt{2(T-t)}}\phi_0\left(\frac{r}{\sqrt{2(T-t)}}\right).$$

One has

$$\begin{aligned}
\partial^k\tilde{\phi} &= \sum_{i=0}^k C_k^i (\partial^i r^\alpha)\partial^{k-i}\phi \\
&\sim \frac{r^\alpha}{(\sqrt{2(T-t)})^{k+1}}(\partial^k\phi_0)\left(\frac{r}{\sqrt{2(T-t)}}\right) + \frac{C(\alpha,k)r^{\alpha-k}}{\sqrt{2(T-t)}}\phi_0\left(\frac{r}{\sqrt{2(T-t)}}\right).
\end{aligned}$$

Note that $\phi_0$ has support in $[0,1]$, hence we may assume that

$$\frac{r}{\sqrt{2(T-t)}} \leq 1.$$

So if $k-1 \leq \alpha < k$, one has

$$\frac{r^\alpha}{(\sqrt{2(T-t)})^{(k-2)+1}} = \left(\frac{r}{\sqrt{2(T-t)}}\right)^{k-1} r^{\alpha-(k-1)} \leq 1,$$

and

$$\frac{r^{\alpha-(k-2)}}{\sqrt{2(T-t)}} = \frac{r}{\sqrt{2(T-t)}} r^{\alpha-(k-1)} \leq 1,$$

which gives that

$$\tilde{\phi} \in C^{k-2}[0,T].$$

On the other hand, if $k-1 \leq \alpha < k$, one has

$$\frac{r^\alpha}{(\sqrt{2(T-t)})^{(k-1)+1}} = \left(\frac{r}{\sqrt{2(T-t)}}\right)^{\alpha} \left(\frac{1}{\sqrt{2(T-t)}}\right)^{k-\alpha} \to \infty, \quad \text{as } t \to T,$$

which shows that

$$\partial_r^{k-1}\tilde{\phi}(x,t) \to \infty, \quad \text{as } t \to T.$$

One has

$$\begin{aligned}
\int_D |\tilde{\phi}|^{\tilde{p}} dx &\leq C \int_0^1 \frac{r^{\tilde{p}\alpha+\tilde{p}+1}}{(r^2+T-t)^{\tilde{p}}} dr \\
&= C \int_0^1 \frac{r}{(r^2+T-t)^{\tilde{p}-\frac{\tilde{p}\alpha+\tilde{p}}{2}}} \frac{r^{\tilde{p}\alpha+\tilde{p}}}{(r^2+T-t)^{\frac{\tilde{p}\alpha+\tilde{p}}{2}}} dr \\
&\leq C \int_0^1 \frac{r}{(r^2+T-t)^{\tilde{p}-\frac{\tilde{p}\alpha+\tilde{p}}{2}}} dr \\
&\leq \begin{cases} \frac{C}{(T-t)^{\frac{\tilde{p}-2-\tilde{p}\alpha}{2}}}, & \frac{\tilde{p}-\tilde{p}\alpha}{2} \neq 1, \\ C|\ln(T-t)|, & \frac{\tilde{p}-\tilde{p}\alpha}{2} = 1. \end{cases}
\end{aligned} \tag{4.1}$$

Hence, $\tilde{\phi} \in L_t^{\tilde{q}} L_x^{\tilde{p}}$ provided that $[(1-\alpha)\tilde{p}-2]\tilde{q} < 2$.

Finally, to verify that the boundary conditions are satisfied, set

$$v = (\tilde{\phi} + \beta r)e_\theta,$$

where

$$\beta = \int_0^1 s e^{\frac{s^2}{2}} \int_s^\infty e^{-\frac{l^2}{2}} k(l) dl ds.$$

Actually, the Navier total slip boundary condition on the horizontal boundary of $D$ is satisfied since $v$ is independent of the vertical variable $x_3$ and $v_r = v_3 = 0$. On the other hand, it follows from Zhang's equation (1.20) that

$$\tilde{\phi}(1,t) = -\int_0^1 s e^{\frac{s^2}{2}} \int_s^\infty e^{-\frac{l^2}{2}} k(l)\,dl\,ds = -\beta.$$

So the no slip boundary condition is satisfied on the vertical boundary. The proof of Theorem 3.1 is accomplished.

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