Opuscula Math. 46, no. 4 (2026), 441–474
https://doi.org/10.7494/OpMath.202606241 Opuscula Mathematica

TOPOLOGICAL SENSITIVITY ANALYSIS FOR THE NARROW ESCAPE PROBLEM

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Communicated by Vicenţiu D. Rădulescu

Abstract. The topological sensitivity analysis method has been recognized as a promising, fast, and accurate approach for solving topology optimization and inverse problems. It is based on developing an asymptotic expansion of a design functional with respect to the creation of a small hole inside the computational domain. In this work, we extend this method to the narrow escape problem. The biological process is governed by a parabolic diffusion equation. We derive a sensitivity analysis for the parabolic problem solution with respect to the creation of a small absorbing boundary subset. We develop a rigorous mathematical framework that is valid in two- and three-dimensional space. It provides an asymptotic formula that describes the behavior of the perturbed solution with respect to the location and size of an arbitrary perturbed boundary subset. The Sobolev capacity notion has been employed to measure the smallness of the boundary subset and to describe the asymptotic behavior with respect to the perturbation size. The performed mathematical analysis is general and can be adapted for a large class of partial differential equations. The obtained asymptotic formula can serve as a useful tool to perform numerical algorithms for solving optimization and control problems.

Keywords: topological sensitivity analysis, boundary perturbation, parabolic equation, asymptotic expansion.

Mathematics Subject Classification: 65M80, 35C20, 35B40.

1. INTRODUCTION AND PROBLEM STATEMENT

During the last decades, the narrow escape theory has garnered significant attention due to its relevance in biophysics and molecular biology [18, 21]. The narrow escape problem, whose history goes back to H. von Helmholtz [19] (Physiological Optics, 1960) and Lord Rayleigh [31] (The Theory of Sound, 1945), is concerned with the calculation of the narrow escape time (NET) (also called the mean first passage time) of a Brownian particle that moves in a bounded domain before it escapes through a small absorbing boundary window [20]. The primary mathematical concern regarding the narrow-escape problem is connected with that of analyzing the sensitivity of a convection-diffusion model equation with respect to the creation of a small absorbing boundary subset [20, 33].

The classical topological sensitivity analysis method consists of studying the behavior of a given shape functional with respect to a small topological perturbation of the domain, typically by insertion of small holes, inclusions, cracks, or non-homogeneous regions. Theoretically, the topological sensitivity analysis has been established for various elliptic, parabolic, hyperbolic, and nonlocal operators; e.g. see [5, 6, 11, 13, 16, 17, 24, 29]. From a practical point of view, this method has been applied for solving various geometric inverse problems; topology optimization of turbine blade cooling system [12], tumor identification [8], determination of the initial density in a nonlocal diffusion problem [22], recovering of sources in a time-fractional diffusion equation [25], optimization of contact regions in semiconductor transistors [23], etc.

In all of these investigations, the topological sensitivity analysis has been derived under the assumption that the geometric perturbations are strictly included inside the computational domain. It is the purpose of this work to deal with a sensitivity analysis with respect to the creation of a small surface perturbation.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a simply connected domain with smooth boundary $\partial\Omega$ (of class C^2). Assume that $\partial\Omega$ is decomposed into two disjoint parts Σ^n and Σ^d such that $\overline{\partial\Omega} = \overline{\Sigma^n} \cup \overline{\Sigma^d}$ and $\Sigma^n \cap \Sigma^d = \emptyset$. In this study, the computational domain Ω is a biological media. The Brownian motion of particles suspended in Ω will be described by the following parabolic diffusion system

$$\begin{cases} \dfrac{\partial \Phi_0}{\partial t} - \operatorname{div}(\mu \nabla \Phi_0) = G & \text{in } \Omega \times (0, T), \\ \Phi_0 = 0 & \text{on } \Sigma^d \times (0, T), \\ \mu \nabla \Phi_0 \cdot \nu = 0 & \text{on } \Sigma^n \times (0, T), \\ \Phi_0(\cdot, 0) = 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

with $\Phi_0(x, t)$ is the density of the particles at location x and time t, μ represents the conductivity of the medium, G is a source term and ν denotes the unit outward normal vector to the boundary $\partial\Omega$. The conductivity μ is assumed to be sufficiently smooth ($\mu \in C^1(\overline{\Omega})$) and satisfies that there exist two positive constants $\mu_{\min}$ and $\mu_{\max}$ such that $0 < \mu_{\min} \leq \mu(x) \leq \mu_{\max}$ for all $x \in \Omega$.

The problem that we consider is to study the behavior of the parabolic problem solution with respect to the location and size of a Dirichlet small surface perturbation. More precisely, let $\gamma_{z,\varepsilon}$ be an open Lipschitz subset of the boundary part $\Sigma^n \subset \partial\Omega$ that is located around an arbitrary point $z \in \Sigma^n$ and defined as $\gamma_{z,\varepsilon} = B(z, \varepsilon) \cap \Sigma^n$, where $B(z, \varepsilon)$ is the ball in $\mathbb{R}^d$ of center z and radius ε, with $0 < \varepsilon << 1$ is a small parameter describing the size of the subset $\gamma_{z,\varepsilon}$. In this study, we assume that $\gamma_{z,\varepsilon}$ is far from the Dirichlet boundary region Σ^d in the sense that there exists $d_{\min} > 0$ such that

$$\operatorname{dist}(z, \Sigma^d) \geq d_{\min}. \tag{1.2}$$

In the presence of such surface perturbation, the parabolic diffusion problem consists of determining $\Phi_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ as the solution to the following initial boundary value problem

$$\begin{cases} \dfrac{\partial \Phi_\varepsilon}{\partial t} - \operatorname{div}(\mu \nabla \Phi_\varepsilon) = G & \text{in } \Omega \times (0,T), \\ \Phi_\varepsilon = 0 & \text{on } \Sigma^d \times (0,T), \\ \Phi_\varepsilon = 0 & \text{on } \gamma_{z,\varepsilon} \times (0,T), \\ \mu \nabla \Phi_\varepsilon \cdot \nu = 0 & \text{on } \Sigma^{n,\varepsilon} \times (0,T), \\ \Phi_\varepsilon(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $\Sigma^{n,\varepsilon} = \Sigma^n \backslash \overline{\gamma_{z,\varepsilon}}$ is the perturbed Neumann boundary region after imposing a homogeneous Dirichlet condition on the small subset $\gamma_{z,\varepsilon}$.

Our aim in this work is to evaluate the behavior of the parabolic problem solution Φ_ε with respect to the location and size of the created boundary type perturbation on the small subset $\gamma_{z,\varepsilon}$. This model problem is closely connected with the well-known narrow escape time (NET) problem which aims to calculate the mean first passage time of a Brownian particle that moves in a bounded domain Ω before it escapes through a small absorbing boundary window $\gamma_{z,\varepsilon}$.

The main contribution of this study is the development of a rigorous mathematical framework describing the behavior of the perturbed density field Φ_ε with respect to the location and size of the created window $\gamma_{z,\varepsilon}$. The leading term of the variation $\Phi_\varepsilon - \Phi_0$ has been expressed as a function involving a modified fundamental solution of the considered parabolic operator. The Sobolev capacity notion has been introduced to measure the smallness of the subset $\gamma_{z,\varepsilon}$.

The remainder of the paper is organized as follows. Section 2 is devoted to recalling the definition and some properties concerning the Sobolev capacity notion. Section 3 provides an energy estimate for the density field variation involving the Sobolev capacity of the boundary subset $\gamma_{z,\varepsilon}$. An improved L^2-norm estimate for the variation $\Phi_\varepsilon - \Phi_0$ is illustrated in Section 4. Section 5 provides an asymptotic formula that describes the behavior of the density field variation with respect to the location and size of an arbitrary perturbed boundary subset $\gamma_{z,\varepsilon}$. Section 6 presents an explicit asymptotic expansion of the density field variation in the particular case where the boundary condition perturbation is supported by a planar surface.

2. SOBOLEV CAPACITY

This section is concerned with the capacity notion. We begin by briefly recalling the definition of the Sobolev capacity of an arbitrary subset of $\mathbb{R}^d$. For more details on this notion, one can consult [15]. In Subsection 2.2, we establish explicit estimates for the Sobolev capacity of particular examples of small subsets that are supported by a planar surface. In Subsection 2.3, we consider a small open Lipschitz subset $\gamma_{z,\varepsilon}$ of the boundary $\partial\Omega$, and we derive an estimate linking the Sobolev capacity of $\gamma_{z,\varepsilon}$ and the H^1-energy norm.

2.1. SOBOLEV CAPACITY DEFINITION AND CHARACTERIZATIONS

The Sobolev capacity is a positive scalar quantity defined with the help of the H^1-Sobolev norm.

Definition 2.1. Let $\mathcal{S}$ be an arbitrary subset of $\mathbb{R}^d$. The Sobolev capacity of $\mathcal{S}$, denoted by $\operatorname{cap}(\mathcal{S})$, is defined by

$$\operatorname{cap}(\mathcal{S}) = \inf\left\{\|\phi\|^2_{H^1(\mathbb{R}^d)} :\ \phi \in \mathcal{H}^+_{\mathcal{S}}\right\},$$

where $\mathcal{H}^+_{\mathcal{S}} \subset H^1(\mathbb{R}^d)$ is given as

$$\mathcal{H}^+_{\mathcal{S}} = \left\{\phi \in H^1(\mathbb{R}^d)\ :\ \phi \geq 1 \text{ a.e. on an open neighborhood of } \mathcal{S}\right\}.$$

A more useful characterization for the capacity $\operatorname{cap}(\mathcal{S})$ is given by the following lemma. For more details, one can see [15].

Lemma 2.2 ([15])**.** *Let $\mathcal{S} \subset \mathbb{R}^d$ be an arbitrary subset. The capacity of $\mathcal{S}$ has the following characterization*

$$\operatorname{cap}(\mathcal{S}) = \inf\left\{\|\phi\|^2_{H^1(\mathbb{R}^d)} :\ \phi \in \mathcal{H}_{\mathcal{S}}\right\},$$

where $\mathcal{H}_{\mathcal{S}} = \left\{\phi \in H^1(\mathbb{R}^d)\ :\ \phi = 1 \text{ a.e. on an open neighborhood of } \mathcal{S}\right\}$.

The next part of this section concerns the Sobolev capacity of a boundary subset. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded and smooth domain and let γ be a Lipschitz open subset of $\partial\mathcal{O}$. The following proposition provides an estimate of $\operatorname{cap}(\gamma)$ involving the energy norm of an arbitrary function belonging to the set

$$\mathcal{H}_\gamma = \left\{\phi \in H^1(\mathbb{R}^d)\ :\ \phi = 1 \text{ on } \gamma \text{ in the trace sense } H^{1/2}(\gamma)\right\}.$$

Proposition 2.3. *Let γ be an open Lipschitz subset of the boundary $\partial\mathcal{O}$. Then the Sobolev capacity of γ satisfies the estimate*

$$\operatorname{cap}(\gamma) \leq \|\phi\|^2_{H^1(\mathbb{R}^d)}, \quad \forall \phi \in \mathcal{H}_\gamma.$$

Proof. For the proof and more details, see [1] or [27]. □

2.2. EXPLICIT ESTIMATES FOR PARTICULAR BOUNDARY SUBSET

Now, as an application of the previous proposition, we derive explicit estimates of the capacity for particular examples of small boundary subsets in two and three-dimensional cases. Let $\gamma^j_{z,\varepsilon} \subset \mathbb{R}^d$ be a small subset located around an arbitrary point $z \in \mathbb{R}^d$ defined as

$$\gamma^j_{z,\varepsilon} := \mathcal{P}^j_z \cap B(z,\varepsilon),$$

with $B(z,\varepsilon) = \left\{x \in \mathbb{R}^d : |x - z| < \varepsilon\right\}$ and $\mathcal{P}^j_z$ is the plane that passes through the point $z = (z_1, \ldots, z_d)$ and normal to the j^{th} coordinate axis:

$$\mathcal{P}^j_z := \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d\ :\ x_j = z_j\right\}, \quad 1 \leq j \leq d.$$

For instance, in the two-dimensional case, $\gamma^j_{z,\varepsilon}$ is defined by the vertical (if $j = 1$) and the horizontal (if $j = 2$) line segment of length 2ε: $\gamma^j_{z,\varepsilon} =]z_j - \varepsilon,\, z_j + \varepsilon[$. In the three-dimensional case, the subset $\gamma^j_{z,\varepsilon}$ is represented by the disk of center $z = (z_1, z_2, z_3)$ and radius ε:

$$\gamma^j_{z,\varepsilon} = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \; x_j = z_j, \; \sum_{\substack{k=1 \\ k \neq j}}^{3} (x_k - z_k)^2 < \varepsilon^2 \right\}, \quad 1 \leq j \leq 3.$$

From Proposition 2.3, we deduce the following corollary.

Corollary 2.4. *When ε is sufficiently small (i.e. $0 < \varepsilon << 1$), the Sobolev capacity of the subset $\gamma^j_{z,\varepsilon} \subset \mathbb{R}^d$ satisfies there exists $c > 0$, independent of ε, such that*

$$\operatorname{cap}(\gamma^j_{z,\varepsilon}) \leq \frac{c}{|\log \varepsilon|} \quad \text{if } d = 2. \tag{2.1}$$

and

$$\operatorname{cap}(\gamma^j_{z,\varepsilon}) \leq c\,\varepsilon \quad \text{if } d = 3. \tag{2.2}$$

Proof. By Proposition 2.3, we have $\operatorname{cap}(\gamma^j_{z,\varepsilon}) \leq \|\phi\|^2_{H^1(\mathbb{R}^d)}$ for all $\phi \in \mathcal{H}_{\gamma^j_{z,\varepsilon}}$, where

$$\mathcal{H}_{\gamma^j_{z,\varepsilon}} = \left\{ \phi \in H^1(\mathbb{R}^d) : \phi = 1 \text{ on } \gamma^j_{z,\varepsilon} \text{ in the trace sense } H^{1/2}(\gamma^j_{z,\varepsilon}) \right\}.$$

To prove the estimate (2.1), we consider the function $\phi_{z,\varepsilon}$ defined by

$$\phi_{z,\varepsilon}(x) = \begin{cases} 1, & \text{if } 0 \leq |x - z| < \varepsilon, \\ \dfrac{\log |x - z|}{\log \varepsilon}, & \text{if } \varepsilon \leq |x - z| \leq 1, \\ 0, & \text{if } |x - z| > 1. \end{cases}$$

Using polar coordinates, one can easily prove that there exists $c > 0$, independent of ε, such that

$$\|\phi_{z,\varepsilon}\|^2_{H^1(\mathbb{R}^2)} = \|\phi_{z,\varepsilon}\|^2_{H^1(B(z,1))} \leq \frac{c}{|\log \varepsilon|}.$$

Consequently, there exists $c > 0$, independent of ε, such that

$$\operatorname{cap}(\gamma^j_{z,\varepsilon}) \leq \frac{c}{|\log \varepsilon|}.$$

To prove (2.2), we deal with the function $\phi_{z,\varepsilon}$ defined by $\phi_{z,\varepsilon}(x) = 1$ if $|x - z| < \varepsilon$, $\phi_{z,\varepsilon}(x) = \frac{2\varepsilon}{|x-z|} - 1$ if $\varepsilon \leq |x - z| \leq 2\varepsilon$ and $\phi_{z,\varepsilon}(x) = 0$ if $|x - z| > 2\varepsilon$. Using spherical coordinates, one can easily prove that there exists $c > 0$, independent of ε, such that

$$\|\phi_{z,\varepsilon}\|^2_{H^1(\mathbb{R}^3)} \leq c\,\varepsilon,$$

which implies that there exists $c > 0$, independent of ε, such that

$$\operatorname{cap}(\gamma^j_{z,\varepsilon}) \leq c\,\varepsilon. \qquad \square$$

2.3. SOBOLEV CAPACITY AND ENERGY ESTIMATES

We derive, in this paragraph, an energy estimate for the Sobolev capacity of $\gamma_{z,\varepsilon}$. To this end, we introduce the following elliptic boundary value problem which of finding $\vartheta_\varepsilon \in H^1(\Omega)$ as the solution to the system

$$\begin{cases} -\Delta\vartheta_\varepsilon = 0 & \text{in } \Omega, \\ \vartheta_\varepsilon = 1 & \text{on } \gamma_{z,\varepsilon}, \\ \vartheta_\varepsilon = 0 & \text{on } \Sigma^d, \\ \nabla\vartheta_\varepsilon \cdot \nu = 0 & \text{on } \Sigma^{n,\varepsilon}, \end{cases} \tag{2.3}$$

where $\Sigma^{n,\varepsilon} = \Sigma^n \setminus \overline{\gamma_{z,\varepsilon}}$ and ν is the normal unit vector to the boundary Σ^n, outward to Ω. The established estimates are summarized by Proposition 2.5.

Proposition 2.5. *Let $\gamma_{z,\varepsilon}$ be an open small subset of Σ^n such that $\gamma_{z,\varepsilon} = B(z,\varepsilon)\cap\Sigma^n$. Then, under Assumption (1.2), there exist two constants $0 < c_1 \le c_2$, independent of ε, such that*

$$c_1 \,\mathrm{cap}(\gamma_{z,\varepsilon}) \le \|\vartheta_\varepsilon\|^2_{H^1(\Omega)} \le c_2 \,\mathrm{cap}(\gamma_{z,\varepsilon}), \tag{2.4}$$

with ϑ_ε is the unique solution to (2.3).

Proof. The first inequality can be established using Proposition 2.3 after extending the function ϑ_ε to the whole space $\mathbb{R}^d$. Therefore, from the fact that Ω is a smooth domain there exist $\widetilde{\vartheta}_\varepsilon \in H^1(\mathbb{R}^d)$ and a constant $c_0 > 0$, depending only on Ω, such that (see Appendix A in [10])

$$\|\widetilde{\vartheta}_\varepsilon\|_{H^1(\mathbb{R}^d)} \le c_0\|\vartheta_\varepsilon\|_{H^1(\Omega)}.$$

Since $\widetilde{\vartheta}_\varepsilon = 1$ on $\gamma_{z,\varepsilon}$, Proposition 2.3 implies the desired estimate

$$\mathrm{cap}(\gamma_{z,\varepsilon}) \le \|\widetilde{\vartheta}_\varepsilon\|^2_{H^1(\mathbb{R}^d)} \le c_0^2\,\|\vartheta_\varepsilon\|^2_{H^1(\Omega)}.$$

To prove the second inequality, we make use Assumption (1.2) which implies the existence of a function $\psi \in C^1(\overline{\Omega})$ such that $\psi(x) = 1$ on $\gamma_{z,\varepsilon}$, $\psi(x) = 0$ on Σ^d and $\|\psi\|_{C^1(\overline{\Omega})} \le c$, where c is a positive constant independent of ε. Thanks to the Green's formula, we have

$$\int\limits_\Omega |\nabla\vartheta_\varepsilon|^2\,dx = \int\limits_{\partial\Omega}[\nabla\vartheta_\varepsilon\cdot\nu]\,\vartheta_\varepsilon ds(x) = \int\limits_{\partial\Omega}[\nabla\vartheta_\varepsilon\cdot\nu]\,\psi\,\phi\,ds(x), \quad \forall\phi \in \mathcal{H}_{\gamma_{z,\varepsilon}},$$

where

$$\mathcal{H}_{\gamma_{z,\varepsilon}} = \left\{\phi \in H^1(\mathbb{R}^d)\ :\ \phi = 1 \text{ a.e. on an open neighborhood of } \gamma_{z,\varepsilon}\right\}.$$

Then, it follows that

$$\int\limits_\Omega |\nabla\vartheta_\varepsilon|^2\,dx = \int\limits_\Omega \nabla\vartheta_\varepsilon\cdot\nabla(\psi\,\phi)\,ds(x) \le c\|\phi\|_{H^1(\mathbb{R}^d)}\|\nabla\vartheta_\varepsilon\|_{L^2(\Omega)}.$$

Using the Poincaré inequality (since $\vartheta_\varepsilon = 0$ on Σ^d) one can deduce that there exists $c_2 > 0$, independent of ε, such that

$$\|\vartheta_\varepsilon\|^2_{H^1(\Omega)} \leq c_2 \, \|\phi\|^2_{H^1(\mathbb{R}^d)}, \quad \forall \phi \in \mathcal{H}_{\gamma_{z,\varepsilon}}.$$

Finally, we apply Lemma 2.2 which implies the desired estimate. □

Corollary 2.6. *Under the assumption of Proposition 2.5, there exists $c > 0$, independent of ε, such that*

$$\|\vartheta_\varepsilon\|_{L^2(\Omega)} \leq c \, |\mathrm{cap}(\gamma_{z,\varepsilon})|^{3/4}.$$

Proof. We proceed in two main steps. We start by recalling some regularities results. In the second step, we apply Proposition 2.5. Let us begin by introducing the following auxiliary problem, which consists of finding $\psi_\varepsilon \in H^1(\Omega)$ as the solution to

$$\begin{cases} -\Delta \psi_\varepsilon = \vartheta_\varepsilon & \text{in } \Omega, \\ \psi_\varepsilon = 0 & \text{on } \Sigma^d, \\ \nabla \psi_\varepsilon \cdot \nu = 0 & \text{on } \Sigma^n. \end{cases} \tag{2.5}$$

Since $\vartheta_\varepsilon \in H^1(\Omega)$, by standard elliptic regularity ψ_ε is sufficiently smooth ($\psi_\varepsilon \in H^3(\Omega)$) away from the interface between Σ^n and Σ^d. Thanks to Assumption (1.2), one can construct a cut-off function $\lambda \in C_c^\infty(\mathbb{R}^d)$ such that $\lambda(x) = 1$ on an open neighborhood of $\overline{\gamma_{z,\varepsilon}}$ and $\lambda(x) = 0$ on an open set $\mathcal{O} \subset \mathbb{R}^d$ with $\Sigma^d \Subset \mathcal{O}$. More precisely, by standard interior elliptic regularity results (see [7] or [14]) one can prove that there exists $c > 0$ such that

$$\|\lambda \psi_\varepsilon\|_{H^3(\Omega)} \leq c \|\vartheta_\varepsilon\|_{H^1(\Omega)}.$$

On the other hand, by the classical Sobolev embedding theorem in the two- and three-dimensional spaces (see e.g. [2] or [7] for more details), we have $H^3(\Omega) \subset C^1(\overline{\Omega})$ and there exists $c > 0$ such that

$$\|v\|_{C^1(\overline{\Omega})} \leq c \|v\|_{H^3(\Omega)}, \quad \forall v \in H^3(\Omega).$$

Then, from the combination of the previous inequalities one can check that there exists $c > 0$ such that

$$\|\lambda \psi_\varepsilon\|_{C^1(\overline{\Omega})} \leq c \|\vartheta_\varepsilon\|_{H^1(\Omega)}.$$

Now, in order to apply Proposition 2.5, we use the weak formulation of (2.5) which implies

$$\|\vartheta_\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega \nabla \psi_\varepsilon \cdot \nabla \vartheta_\varepsilon \, dx = \int_{\partial\Omega} \big[\nabla \vartheta_\varepsilon \cdot \nu\big] \psi_\varepsilon \, ds(x) = \int_\Omega \nabla \vartheta_\varepsilon \cdot \nabla\big(\lambda \psi_\varepsilon \vartheta_\varepsilon\big) \, dx.$$

The last equality follows from the fact that $\big[\nabla \vartheta_\varepsilon \cdot \nu\big] \psi_\varepsilon = \big[\nabla \vartheta_\varepsilon \cdot \nu\big] \big(\lambda \psi_\varepsilon \vartheta_\varepsilon\big)$ on $\partial\Omega$. Then, by the Cauchy–Schwartz inequality one can obtain

$$\|\vartheta_\varepsilon\|^2_{L^2(\Omega)} \leq 2 \|\vartheta_\varepsilon\|^2_{H^1(\Omega)} \|\lambda \psi_\varepsilon\|_{C^1(\overline{\Omega})} \leq c \|\vartheta_\varepsilon\|^3_{H^1(\Omega)}.$$

Finally, from Proposition 2.5, we deduce the desired estimate

$$\|\vartheta_\varepsilon\|_{L^2(\Omega)} \leq c \, |\mathrm{cap}(\gamma_{z,\varepsilon})|^{3/4}.$$

□

3. ESTIMATES OF THE PERTURBED SOLUTION

In this section, we consider the parabolic diffusion problem in the presence of a small perturbation in the type of boundary conditions. The aim is to evaluate the density field variation caused by imposing a zero-Dirichlet boundary condition on a small subset of the Neumann boundary Σ^n. The first result of this section is illustrated by the following theorem. It provides an energy estimate for the density field variation involving the Sobolev capacity of $\gamma_{z,\varepsilon}$.

Theorem 3.1. *Let $\gamma_{z,\varepsilon} \subset \Sigma^n$ be an open Lipschitz subset of small size $0 < \varepsilon << 1$. Under Assumption (1.2), there exists a constant $c > 0$, independent of ε, such that the perturbed density field satisfies the estimate*

$$\left\| \frac{\partial \Phi_\varepsilon}{\partial t} - \frac{\partial \Phi_0}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \|\Phi_\varepsilon - \Phi_0\|_{L^\infty(0,T;H^1(\Omega))} \leq c \left| \operatorname{cap}(\gamma_{z,\varepsilon}) \right|^{1/2}. \tag{3.1}$$

To prove the estimate (3.1), we first establish a preliminary result, which will be illustrated in the next section. The proof of Theorem 3.1 will be presented in Section 3.2.

3.1. PRELIMINARY ESTIMATES

From (1.1) and (1.3) one can easily check that the variation $\rho_\varepsilon = \Phi_\varepsilon - \Phi_0$ satisfies the system

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} - \operatorname{div}(\mu \nabla \rho_\varepsilon) = 0 & \text{in } \Omega \times (0,T), \\ \rho_\varepsilon = 0 & \text{on } \Sigma^d \times (0,T), \\ \rho_\varepsilon = -\Phi_0 & \text{on } \gamma_{z,\varepsilon} \times (0,T), \\ \mu \nabla \rho_\varepsilon \cdot \nu = 0 & \text{on } \Sigma^{n,\varepsilon} \times (0,T), \\ \rho_\varepsilon(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

Here, $\Sigma^{n,\varepsilon} = \Sigma^n \backslash \overline{\gamma_{z,\varepsilon}}$ represented the perturbed Neumann boundary region.

Thanks to Assumption (1.2), one can establish that there exist $r > 0$ and an open set $\mathcal{O} \subset \mathbb{R}^d$ such that

$$\gamma_{z,\varepsilon} \Subset B(z,r), \quad \Sigma^d \Subset \mathcal{O}, \text{ and } B(z,r) \cap \mathcal{O} = \emptyset.$$

Then, one can construct a function $\beta \in C_c^\infty(\mathbb{R}^d)$ such that

$$\beta(y) = \begin{cases} 1 & \text{in } B(z,r), \\ 0 & \text{in } \mathcal{O}. \end{cases} \tag{3.3}$$

Therefore, the system (3.2) can be rewritten, with a more regular function on the boundary $\gamma_{z,\varepsilon}$, as

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} - \operatorname{div}(\mu \nabla \rho_\varepsilon) = 0 & \text{in } \Omega \times (0,T), \\ \rho_\varepsilon = 0 & \text{on } \Sigma^d \times (0,T), \\ \rho_\varepsilon = \Phi_{0,\beta} & \text{on } \gamma_{z,\varepsilon} \times (0,T), \\ \mu \nabla \rho_\varepsilon \cdot \nu = 0 & \text{on } \Sigma^{n,\varepsilon} \times (0,T), \\ \rho_\varepsilon(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.4}$$

with $\Phi_{0,\beta}$ is defied in $\Omega \times (0,T)$ as

$$\Phi_{0,\beta}(x,t) = -\beta(x)\Phi_0(x,t), \quad \forall (x,t) \in \Omega \times (0,T).$$

Thanks to the smoothness of $\Phi_{0,\beta}$, we derive the following preliminary estimate.

Lemma 3.2. *Let $\gamma_{z,\varepsilon}$ be an open Lipschitz subset of Σ^n. Then, under Assumption* (1.2), *there exists a constant $c > 0$, independent of ε, such that*

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c \left|\operatorname{cap}(\gamma_{z,\varepsilon})\right|^{1/2}.$$

Proof. Using the variational formulation of system (3.4), we obtain

$$\int_\Omega \frac{\partial \rho_\varepsilon}{\partial t} \rho_\varepsilon \, dx + \int_\Omega \mu |\nabla \rho_\varepsilon|^2 \, dx = \int_{\gamma_{z,\varepsilon}} [\mu \nabla \rho_\varepsilon \cdot \nu] \, \Phi_{0,\beta} \, ds(x).$$

Since $\mu \nabla \rho_\varepsilon \cdot \nu = 0$ on $\Sigma^{n,\varepsilon}$, $\vartheta_\varepsilon = 0$ on Σ^d and $\vartheta_\varepsilon = 1$ on $\gamma_{z,\varepsilon}$, it follows that

$$\int_{\gamma_{z,\varepsilon}} [\mu \nabla \rho_\varepsilon \cdot \nu] \, \Phi_{0,\beta} \, ds(x) = \int_{\partial\Omega} [\mu \nabla \rho_\varepsilon \cdot \nu] \, \vartheta_\varepsilon(x) \Phi_{0,\beta}(x,t) \, ds(x),$$

where ϑ_ε is the unique solution to (2.3).

Rewriting the variational formulation of (3.4), using $(x,t) \mapsto \vartheta_\varepsilon(x)\Phi_{0,\beta}(x,t)$ as a test function, we derive

$$\int_\Omega \frac{\partial \rho_\varepsilon}{\partial t} \rho_\varepsilon dx + \int_\Omega \mu |\nabla \rho_\varepsilon|^2 dx = \int_\Omega \frac{\partial \rho_\varepsilon}{\partial t} \vartheta_\varepsilon \Phi_{0,\beta} \, dx + \int_\Omega \mu \nabla \rho_\varepsilon \cdot \nabla [\vartheta_\varepsilon \Phi_{0,\beta}] dx.$$

Let $\tau \in (0,T)$. Since $\rho_\varepsilon(\cdot,0)=0$ in Ω, integrating over the interval $[0,\tau]$ and using the identity $\frac{\partial\rho_\varepsilon}{\partial t}(x,t)\rho_\varepsilon(x,t)=\frac{1}{2}\frac{\partial}{\partial t}\left(\rho_\varepsilon(x,t)\right)^2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\rho_\varepsilon(x,\tau)|^2\,dx+\int_0^\tau\int_\Omega \mu|\nabla\rho_\varepsilon|^2 dxdt\\
&=\int_0^\tau\int_\Omega \frac{\partial\rho_\varepsilon}{\partial t}(x,t)\vartheta_\varepsilon(x)\Phi_{0,\beta}(x,t)dxdt \qquad (3.5)\\
&\quad+\int_0^\tau\int_\Omega \mu\nabla\rho_\varepsilon(x,t)\cdot\nabla[\vartheta_\varepsilon(x)\Phi_{0,\beta}(x,t)]dxdt.
\end{aligned}$$

By an integration by parts and the fact that $\rho_\varepsilon(\cdot,0)=0$, the first term in the right-hand side of (3.5) can be reformulated as

$$\begin{aligned}
\int_0^\tau\int_\Omega \frac{\partial\rho_\varepsilon}{\partial t}\,\vartheta_\varepsilon\Phi_{0,\beta}\,dx\,dt=&\int_\Omega \rho_\varepsilon(x,\tau)\vartheta_\varepsilon(x)\Phi_{0,\beta}(x,\tau)\,dx\\
&-\int_0^\tau\int_\Omega \frac{\partial\Phi_{0,\beta}}{\partial t}\,\vartheta_\varepsilon(x)\rho_\varepsilon\,dx\,dt.
\end{aligned}$$

Using the identity

$$\nabla\rho_\varepsilon\cdot\nabla[\vartheta_\varepsilon(x)\Phi_{0,\beta}]=\Phi_{0,\beta}\,\nabla\rho_\varepsilon\cdot\nabla\vartheta_\varepsilon(x)+\vartheta_\varepsilon(x)\nabla\rho_\varepsilon\cdot\nabla\Phi_{0,\beta},$$

the second integral term in the right-hand side of (3.5) can be rewritten as

$$\begin{aligned}
\int_0^\tau\int_\Omega \mu(x)\nabla\rho_\varepsilon\cdot\nabla\big[\vartheta_\varepsilon(x)\Phi_{0,\beta}\big]\,dx\,dt=&\int_0^\tau\int_\Omega \mu(x)\Phi_{0,\beta}\nabla\rho_\varepsilon\cdot\nabla\vartheta_\varepsilon(x)\,dx\,dt\\
&+\int_0^\tau\int_\Omega \mu(x)\vartheta_\varepsilon(x)\nabla\rho_\varepsilon\cdot\nabla\Phi_{0,\beta}(x,t)\,dx\,dt.
\end{aligned}$$

Making use of the Cauchy–Schwartz inequality, the fact that $\mu(x)\le\mu_{\max}$ for all $x\in\overline{\Omega}$, and the smoothness of the function $\Phi_{0,\beta}$, one can derive that there exists a constant $c_0>0$, independent of ε, such that

$$\frac{1}{2}\int_\Omega |\rho_\varepsilon(x,\tau)|^2\,dx+\int_0^\tau\int_\Omega \mu|\nabla\rho_\varepsilon|^2dx\,dt\le c_0\,\|\vartheta_\varepsilon\|_{H^1(\Omega)}\,\mathcal{N}_1(\rho_\varepsilon), \qquad (3.6)$$

with

$$\mathcal{N}_1(\rho_\varepsilon)=\|\rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}+\|\nabla\rho_\varepsilon\|_{L^2(0,T;L^2(\Omega))}.$$

Since (3.6) is valid for all $\tau \in [0,T]$, then it follows

$$\|\rho_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \le 2\, c_0\, \|\vartheta_\varepsilon\|_{H^1(\Omega)}\, \mathcal{N}_1(\rho_\varepsilon). \tag{3.7}$$

Using the fact that $\mu(x) \ge \mu_{\min}$ for all $x \in \overline{\Omega}$, from (3.6) we get

$$\|\nabla\rho_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \le \frac{c_0}{\mu_{\min}}\, \|\vartheta_\varepsilon\|_{H^1(\Omega)}\, \mathcal{N}_1(\rho_\varepsilon). \tag{3.8}$$

Integrating the inequality (3.6) with respect to t over $(0,T)$, we obtain

$$\|\rho_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \le 2T\, c_0\, \|\vartheta_\varepsilon\|_{H^1(\Omega)}\, \mathcal{N}_1(\rho_\varepsilon). \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), one can establish that there exists a constant $c > 0$, independent of ε, such that

$$\|\rho_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} \le c\, \|\vartheta_\varepsilon\|_{H^1(\Omega)}\, \mathcal{N}_1(\rho_\varepsilon). \tag{3.10}$$

Using the identity $(a+b)^2 \le 2(a^2+b^2)$, we obtain

$$\Big(\mathcal{N}_1(\rho_\varepsilon)\Big)^2 \le 2\, \Big[\|\rho_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))}\Big].$$

From the estimate (3.10), we get

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le 2\; c\; \|\vartheta_\varepsilon\|_{H^1(\Omega)}. \tag{3.11}$$

Furthermore, using Proposition 2.5 and the estimate (3.11), we conclude that there exists $c > 0$, independent of ε, such that

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c\, |\mathrm{cap}(\gamma_{z,\varepsilon})|^{1/2}. \qquad \square$$

3.2. PROOF OF THEOREM 3.1

In this paragraph, we provide a demonstration for the established energy-estimate in Theorem 3.1. We start with the variational formulation of system (3.4) using $\frac{\partial\rho_\varepsilon}{\partial t}$ as a test function, it follows

$$\int_\Omega \left|\frac{\partial\rho_\varepsilon}{\partial t}\right|^2\, dx + \int_\Omega \mu\nabla\rho_\varepsilon\cdot\frac{\partial}{\partial t}(\nabla\rho_\varepsilon)\, dx = \int_{\gamma_{z,\varepsilon}} [\mu\nabla\rho_\varepsilon\cdot\nu]\,\frac{\partial\Phi_{0,\beta}}{\partial t}\, ds(x).$$

It is important to recall here that $\frac{\partial\Phi_{0,\beta}}{\partial t}(x,t) = -\beta(x)\frac{\partial\Phi_0}{\partial t}(x,t)$ for all $(x,t) \in \Omega\times(0,T)$, which is a sufficient smooth function (belongs to $C^0(0,T;C^1(\overline{\Omega}))$, see [3] or [9]. Using the fact that $\mu\nabla\rho_\varepsilon\cdot\nu = 0$ on $\Sigma^{n,\varepsilon}$, $\vartheta_\varepsilon = 1$ on $\gamma_{z,\varepsilon}$ and $\vartheta_\varepsilon = 0$ on Σ^d, the last integral term can be written as

$$\int_{\gamma_{z,\varepsilon}} [\mu\nabla\rho_\varepsilon\cdot\nu]\,\frac{\partial\Phi_{0,\beta}}{\partial t}\, ds(x) = \int_{\partial\Omega} [\mu\nabla\rho_\varepsilon\cdot\nu]\,\vartheta_\varepsilon(x)\frac{\partial\Phi_{0,\beta}}{\partial t}\, ds(x),$$

with ϑ_ε is the unique solution to (2.3). Again, with the help of the variational formulation of (3.4), one can establish

$$\int_\Omega \left|\frac{\partial \rho_\varepsilon}{\partial t}\right|^2 dx + \int_\Omega \mu \,\nabla\rho_\varepsilon \cdot \frac{\partial}{\partial t}(\nabla \rho_\varepsilon)\, dx = \int_\Omega \frac{\partial \rho_\varepsilon}{\partial t}\,\vartheta_\varepsilon(x)\,\frac{\partial \Phi_{0,\beta}}{\partial t}\, dx + \int_\Omega \mu\,\nabla\rho_\varepsilon \cdot \nabla\left[\vartheta_\varepsilon(x)\frac{\partial \Phi_{0,\beta}}{\partial t}\right] dx.$$

Let $\tau \in (0,T)$. Integrating over the interval $[0,\tau]$ and using the fact that $\rho_\varepsilon(\cdot,0) = 0$ in Ω, we obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega \left|\frac{\partial \rho_\varepsilon}{\partial t}\right|^2 dx\,dt + \frac{1}{2}\int_\Omega \mu\left|\nabla\rho_\varepsilon(x,\tau)\right|^2 dx \\
&= \int_0^\tau \int_\Omega \vartheta_\varepsilon(x)\,\frac{\partial \rho_\varepsilon}{\partial t}(x,t)\,\frac{\partial \Phi_{0,\beta}}{\partial t}(x,t)\,dx\,dt \\
&\quad + \int_0^\tau \int_\Omega \mu\,\nabla\rho_\varepsilon(x,t)\cdot\nabla\left[\vartheta_\varepsilon(x)\frac{\partial \Phi_{0,\beta}}{\partial t}(x,t)\right] dx\,dt.
\end{aligned} \tag{3.12}$$

Using the identity $\nabla_x[\vartheta_\varepsilon \frac{\partial \Phi_{0,\beta}}{\partial t}] = \frac{\partial \Phi_{0,\beta}}{\partial t}(x,t)\nabla\vartheta_\varepsilon(x) + \vartheta_\varepsilon(x)\nabla_x(\frac{\partial \Phi_{0,\beta}}{\partial t})(x,t)$ and the fact that $\mu(x) \geq \mu_{\min}$ for all $x \in \overline{\Omega}$, from (3.12) it follows

$$\begin{aligned}
&\int_0^\tau \int_\Omega \left|\frac{\partial \rho_\varepsilon}{\partial t}\right|^2 dx\,dt + \frac{\mu_{\min}}{2}\int_\Omega \left|\nabla\rho_\varepsilon(x,\tau)\right|^2 dx \\
&\leq \int_0^\tau \int_\Omega \vartheta_\varepsilon(x)\,\frac{\partial \rho_\varepsilon}{\partial t}\,\frac{\partial \Phi_{0,\beta}}{\partial t}\,dx\,dt \\
&\quad + \int_0^\tau \int_\Omega \mu\,\frac{\partial \Phi_{0,\beta}}{\partial t}\,\nabla\rho_\varepsilon\cdot\nabla\vartheta_\varepsilon(x)\,dx\,dt \\
&\quad + \int_0^\tau \int_\Omega \mu\,\vartheta_\varepsilon(x)\,\nabla\rho_\varepsilon\cdot\nabla\left(\frac{\partial \Phi_{0,\beta}}{\partial t}\right) dx\,dt.
\end{aligned} \tag{3.13}$$

Due to the smoothness of the function $\frac{\partial \Phi_{0,\beta}}{\partial t}(x,t)$ and the Cauchy–Schwartz inequality, one can show that there exists $c_0 > 0$, independent of ε, such that

$$\int_0^\tau \int_\Omega \left|\frac{\partial \rho_\varepsilon}{\partial t}\right|^2 dx + \int_\Omega \left|\nabla\rho_\varepsilon(x,\tau)\right|^2 dx \leq c_0\,\|\vartheta_\varepsilon\|_{H^1(\Omega)}\,\mathcal{N}_2(\rho_\varepsilon) \tag{3.14}$$

with

$$\mathcal{N}_2(\rho_\varepsilon) = \left\| \frac{\partial \rho_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla \rho_\varepsilon\|_{L^2(0,T;L^2(\Omega))}.$$

Now, taking the supremum over the interval $[0,T]$ in (3.14), it follows

$$\left\| \frac{\partial \rho_\varepsilon}{\partial t} \right\|^2_{L^2(0,T;L^2(\Omega))} + \|\nabla \rho_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \leq c_0 \, \|\vartheta_\varepsilon\|_{H^1(\Omega)} \, \mathcal{N}_2(\rho_\varepsilon).$$

Since $\|\nabla \rho_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq \sqrt{T} \, \|\nabla \rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}$, then one can derive

$$\left\| \frac{\partial \rho_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla \rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq 2\, c_0 \big(1 + \sqrt{T}\big) \, \|\vartheta_\varepsilon\|_{H^1(\Omega)}.$$

Therefore, using Proposition 2.5 one can justify that there exists $c > 0$, independent of ε, such that

$$\left\| \frac{\partial \rho_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla \rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq c \, |\mathrm{cap}(\gamma_{z,\varepsilon})|^{1/2}. \tag{3.15}$$

According to Lemma 3.2, we have

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq c \; |\mathrm{cap}(\gamma_{z,\varepsilon})|^{1/2}. \tag{3.16}$$

Combining the relations (3.15) and (3.16), one can deduce

$$\left\| \frac{\partial \rho_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq c \; |\mathrm{cap}(\gamma_{z,\varepsilon})|^{1/2}.$$

Finally, using the fact that $\rho_\varepsilon = \Phi_\varepsilon - \Phi_0$, one can conclude that there exists a constant $c > 0$, independent of ε, such that

$$\left\| \frac{\partial \Phi_\varepsilon}{\partial t} - \frac{\partial \Phi_0}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \|\Phi_\varepsilon - \Phi_0\|_{L^\infty(0,T;H^1(\Omega))} \leq c \; |\mathrm{cap}(\gamma_{z,\varepsilon})|^{1/2},$$

which completes the proof of Theorem 3.1.

4. AN IMPROVED ESTIMATE OF THE L^2-NORM

In this section, we derive an improved estimate for the L^2-norm of the quantity $\Phi_\varepsilon - \Phi_0$, that describes the variation between the perturbed and unperturbed density fields. The established estimate is illustrated by the following theorem.

Theorem 4.1. *Let $\gamma_{z,\varepsilon} \subset \Sigma^n$ be an open Lipschitz subset of small size $0 < \varepsilon << 1$. Under Assumption (1.2), there exists a constant $c > 0$, independent of ε, such that the perturbed density field satisfies the estimate:*

$$\|\Phi_\varepsilon - \Phi_0\|_{L^2(0,T;L^2(\Omega))} \leq c \, |\mathrm{cap}(\gamma_{z,\varepsilon})|^{3/4}. \tag{4.1}$$

In order to prove this theorem, we introduce the following terminal-boundary value problem, which consists of finding $\chi_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ as the solution to the system

$$\begin{cases} -\dfrac{\partial \chi_\varepsilon}{\partial t} - \operatorname{div}(\mu \nabla \chi_\varepsilon) = \Phi_\varepsilon - \Phi_0 & \text{in } \Omega \times (0,T), \\ \chi_\varepsilon = 0 & \text{on } \Sigma^d \times (0,T), \\ \mu \nabla \chi_\varepsilon \cdot \nu = 0 & \text{on } \Sigma^n \times (0,T), \\ \chi_\varepsilon(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{4.2}$$

In the next section, we will derive an energy type estimate describing the behavior of the solution χ_ε with respect to the Sobolev capacity of the boundary subset $\gamma_{z,\varepsilon}$. The proof of Theorem 4.1 will be illustrated in Section 4.2.

4.1. PRELIMINARY RESULTS

In this paragraph, we establish two preliminary results. The first one is concerned with the smoothness of the function $x \mapsto \beta(x)\chi_\varepsilon(x,t)$, with $\beta \in C_c^\infty(\mathbb{R}^d)$ is a cut-off function such that $\beta(x) = 1$ on an open neighborhood of $\overline{\gamma_{z,\varepsilon}}$ and $\beta(x) = 0$ on an open set $\mathcal{O} \subset \mathbb{R}^d$ with $\Sigma^d \Subset \mathcal{O}$. The existence of β is guaranteed by Assumption (1.2). The second one provides an energy type estimate for the auxiliary problem solution.

Lemma 4.2. *Let $\gamma_{z,\varepsilon}$ be an open Lipschitz subset of Σ^n verifying Assumption* (1.2). *Then, there exists a constant $c > 0$, independent of ε, such that*

$$\|\beta \chi_\varepsilon\|_{L^2(0,T;C^1(\overline{\Omega}))} \leq c\, |\operatorname{cap}(\gamma_{z,\varepsilon})|^{1/2}.$$

Proof. Since the source term of the system (4.2) is regular enough ($\Phi_\varepsilon - \Phi_0 \in L^2(0,T;H^1(\Omega))$), by standard parabolic regularity results (e.g. see [32] or [7]) one can prove that the solution χ_ε has an improved regularity far from the boundary interface between Σ^d and Σ^n. More precisely, one can check that $\beta\chi_\varepsilon$ belongs to $L^2(0,T;H^3(\Omega))$ and there exists $c > 0$ such that

$$\|\beta \chi_\varepsilon\|_{L^2(0,T;H^3(\Omega))} \leq c\, \|\Phi_\varepsilon - \Phi_0\|_{L^2(0,T;H^1(\Omega))}.$$

On the other hand, the classical Sobolev embedding theorems in two and three space dimensions ensure that $H^3(\Omega) \subset C^1(\overline{\Omega})$ and for all $\varphi \in H^3(\Omega)$,

$$\|\varphi\|_{L^2(0,T;C^1(\overline{\Omega}))} \leq c \|\varphi\|_{L^2(0,T;H^3(\Omega))}.$$

Therefore, one can check that there exists $c > 0$, independent of ε, such that

$$\|\beta \chi_\varepsilon\|_{L^2(0,T;C^1(\overline{\Omega})} \leq c \|\Phi_\varepsilon - \Phi_0\|_{L^2(0,T;H^1(\Omega))}. \tag{4.3}$$

The desired estimate follows immediately from (4.3) and Theorem 3.1. □

Next, we provide an energy type estimate for the solution χ_ε of the auxiliary problem (4.2). The established result is illustrated by the following lemma.

Lemma 4.3. *Let $\gamma_{z,\varepsilon}$ be an open Lipschitz subset of Σ^n verifying Assumption* (1.2). *Then, there exists a constant $c > 0$, independent of ε, such that*

$$\|\chi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\chi_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c\ |\mathrm{cap}(\gamma_{z,\varepsilon})|^{1/2}.$$

Proof. Due to Green's formula, the variational formulation of (4.2) implies

$$-\int_\tau^T\int_\Omega \frac{\partial\chi_\varepsilon}{\partial t}\chi_\varepsilon\ dx + \int_\tau^T\int_\Omega \mu|\nabla\chi_\varepsilon|^2\ dx = \int_\tau^T\int_\Omega \big(\Phi_\varepsilon - \Phi_0\big)\chi_\varepsilon\ dx, \quad \forall\tau\in(0,T).$$

Using the Cauchy–Schwarz inequality and the fact that $\chi_\varepsilon(\cdot,T) = 0$, one can check $c > 0$, independent of ε, such that

$$\|\chi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\chi_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c\|\Phi_\varepsilon - \Phi_0\|_{L^2(0,T;L^2(\Omega))}. \tag{4.4}$$

Exploiting again the variational formulation of (4.2), with $(x,t)\mapsto -\frac{\partial\chi_\varepsilon}{\partial t}(x,t)$ as a test function, one can deduce that there exists a constant $c > 0$, independent of ε, such that

$$\left\|\frac{\partial\chi_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\chi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le c\|\Phi_\varepsilon - \Phi_0\|_{L^2(0,T;L^2(\Omega))}. \tag{4.5}$$

Combining the relations (4.4) and (4.5), we deduce

$$\left\|\frac{\partial\chi_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\chi_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le c\|\Phi_\varepsilon - \Phi_0\|_{L^2(0,T;L^2(\Omega))}. \tag{4.6}$$

Finally, according to Theorem 3.1 and relation (4.6) one can check that there exists $c > 0$, independent of ε, such that

$$\left\|\frac{\partial\chi_\varepsilon}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))} + \|\chi_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le c\,|\mathrm{cap}(\gamma_{z,\varepsilon})|^{1/2}. \qquad \square$$

4.2. PROOF OF THEOREM 4.1

In this paragraph, we provide a demonstration of the main results of this section. To establish the estimate (4.1), we begin by exploiting the weak formulation of (4.2) which gives

$$\begin{aligned}\int_0^T\int_\Omega |\Phi_\varepsilon - \Phi_0|^2\,dx\,dt = &-\int_0^T\int_\Omega \frac{\partial\chi_\varepsilon}{\partial t}\,(\Phi_\varepsilon - \Phi_0)\,dx\,dt\\ &+\int_0^T\int_\Omega \mu\nabla\chi_\varepsilon\cdot\nabla(\Phi_\varepsilon - \Phi_0)\,dx\,dt.\end{aligned} \tag{4.7}$$

Since $(\Phi_\varepsilon - \Phi_0)(\cdot,0) = \chi_\varepsilon(\cdot,T) = 0$, the integration by parts formula implies that the second integral term can be rewritten as

$$-\int_0^T\int_\Omega \frac{\partial\chi_\varepsilon}{\partial t}(\Phi_\varepsilon - \Phi_0)\ dx\ dt = \int_0^T\int_\Omega \frac{\partial(\Phi_\varepsilon - \Phi_0)}{\partial t}\chi_\varepsilon\ dx\ dt. \tag{4.8}$$

Using the fact that $\rho_\varepsilon = \Phi_\varepsilon - \Phi_0$ satisfies the system (3.2), then from (4.7) and (4.8) we deduce

$$\begin{aligned}\int_0^T\int_\Omega |\Phi_\varepsilon - \Phi_0|^2\,dx\,dt &= \int_0^T\int_\Omega \frac{\partial\rho_\varepsilon}{\partial t}\,\chi_\varepsilon\,dx\,dt + \int_0^T\int_\Omega \mu\nabla\rho_\varepsilon\cdot\nabla\chi_\varepsilon\,dx\,dt\\ &= \int_0^T\int_{\partial\Omega}(\mu\nabla\rho_\varepsilon\cdot\nu)\,\chi_\varepsilon\,ds(x)\,dt.\end{aligned}$$

In other hand, from the properties of the function ϑ_ε (solution to (2.3)) and the smooth function β (defined by (3.3)) one can observe that $\chi_\varepsilon = \beta\vartheta_\varepsilon\chi_\varepsilon$ on $\Sigma^d\cup\gamma_{z,\varepsilon}$. Then, the last identity can be reformulated as

$$\int_0^T\int_\Omega |\Phi_\varepsilon - \Phi_0|^2\,dx\,dt = \int_0^T\int_{\partial\Omega}(\mu\nabla\rho_\varepsilon\cdot\nu)\,\beta\,\vartheta_\varepsilon\,\chi_\varepsilon\,ds(x)\,dt.$$

According to the variational formulation of (3.2), it follows

$$\begin{aligned}\int_0^T\int_\Omega |\Phi_\varepsilon - \Phi_0|^2\,dx\,dt = &\int_0^T\int_\Omega \frac{\partial\rho_\varepsilon}{\partial t}\,\beta\,\vartheta_\varepsilon\,\chi_\varepsilon\,dx\,dt\\ &+\int_0^T\int_\Omega \mu\nabla\rho_\varepsilon\cdot\nabla\big(\beta\,\vartheta_\varepsilon\,\chi_\varepsilon\big)\,dx\,dt.\end{aligned}$$

It can be rewritten as

$$\begin{aligned}\int_0^T\int_\Omega |\Phi_\varepsilon - \Phi_0|^2\,dx\,dt = &\int_0^T\int_\Omega \frac{\partial\rho_\varepsilon}{\partial t}\,\vartheta_\varepsilon\,\beta\,\chi_\varepsilon\,dx\,dt\\ &+\int_0^T\int_\Omega \mu\,\vartheta_\varepsilon\,\nabla\rho_\varepsilon\cdot\nabla(\beta\chi_\varepsilon)\,dx\,dt\\ &+\int_0^T\int_\Omega \mu\,(\beta\chi_\varepsilon)\,\nabla\rho_\varepsilon\cdot\nabla\vartheta_\varepsilon\,dx\,dt.\end{aligned}$$

Thanks to the Cauchy–Schwarz inequality and the smoothness of the function $x \mapsto \beta(x)\chi_\varepsilon(x,t)$ (belongs to $C^1(\overline{\Omega})$), one can derive the following estimate

$$\int_0^T \int_\Omega |\Phi_\varepsilon - \Phi_0|^2 \, dx \, dt \le c_0 \|\vartheta_\varepsilon\|_{H^1(\Omega)} \|\beta\chi_\varepsilon\|_{L^2(0,T;C^1(\overline{\Omega}))} \mathcal{N}_3(\rho_\varepsilon), \tag{4.9}$$

with $c_0 = 1 + 2\mu_{\max}$, and

$$\mathcal{N}_3(\rho_\varepsilon) = \left\| \frac{\partial \rho_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|_{L^2(0,T;H^1(\Omega))}.$$

According to Proposition 2.5 (see (2.4)), we have

$$\|\vartheta_\varepsilon\|_{H^1(\Omega)} \le c \left|\mathrm{cap}(\gamma_{z,\varepsilon})\right|^{1/2}.$$

Since $\rho_\varepsilon = \Phi_\varepsilon - \Phi_0$, from Theorem 3.1 one deduce that

$$\left\| \frac{\partial \rho_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} + \|\rho_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c \left|\mathrm{cap}(\gamma_{z,\varepsilon})\right|^{1/2}.$$

From Lemma 4.2, it follows

$$\|\beta\chi_\varepsilon\|_{L^2(0,T;C^1(\overline{\Omega}))} \le c \left|\mathrm{cap}(\gamma_{z,\varepsilon})\right|^{1/2}.$$

Finally, inserting the previous estimates into (4.9) we obtain

$$\int_0^T \int_\Omega |\Phi_\varepsilon - \Phi_0|^2 \, dx \, dt \le c \left|\mathrm{cap}(\gamma_{z,\varepsilon})\right|^{3/2},$$

which implies the desired estimates.

5. ASYMPTOTIC FORMULA FOR THE PERTURBED DENSITY

In this section, we derive an asymptotic formula for the perturbed density field Φ_ε with respect to imposing Dirichlet boundary condition on a small subset $\gamma_{z,\varepsilon}$ of the Neumann boundary Σ^n. According to the previous notations, $\gamma_{z,\varepsilon}$ be an open subset of the boundary part $\Sigma^n \subset \partial\Omega$, located around an arbitrary point $z \in \Sigma^n$ and defined as $\gamma_{z,\varepsilon} = \Sigma^n \cap B(z,\varepsilon)$, with $0 < \varepsilon << 1$ is a small parameter describing the size of the subset $\gamma_{z,\varepsilon}$. The employed mathematical framework is based on a sensitivity analysis of the density field variation $\Phi_\varepsilon - \Phi_0$ with respect to the location and size of the perturbed subset. The leading term of the variation $\Phi_\varepsilon - \Phi_0$ will be calculated with the help of a modified fundamental solution related to the considered parabolic operator. The Sobolev capacity will be used to measure the "smallness" of the perturbed subset $\gamma_{z,\varepsilon}$.

To present the main result of this section, we first need to introduce some useful notations. For any point $(x,t) \in \Omega \times (0,T)$, we denote by $(y,\tau) \mapsto E_{x,t}(y,\tau)$, the modified fundamental solution that is defined as the unique solution to

$$\begin{cases} -\dfrac{\partial E_{x,t}}{\partial \tau} - \operatorname{div}_y\big(\mu(y)\nabla_y E_{x,t}\big) = \delta_{x,t} & \text{in } \Omega \times (0,T), \\ E_{x,t} = 0 & \text{on } \Sigma^d \times (0,T), \\ \mu(y)\nabla_y E_{x,t} \cdot \nu = 0 & \text{on } \Sigma^n \times (0,T), \\ E_{x,t}(\cdot, T) = 0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

where $\delta_{x,t}(y,\tau)$ is the Dirac distribution function, i.e. $\delta_{x,t}(x,t) = 1$ and $\delta_{x,t}(y,\tau) = 0$ if $(y,\tau) \neq (x,t)$.

In this study, we assume that the subset $\gamma_{z,\varepsilon}$ is not close to the Dirichlet boundary part Σ^d. Then, according to the hypothesis (1.2) one can easily construct a smooth cutoff function $\lambda \in C^\infty(\partial\Omega)$ satisfying

$$\lambda(y) = \begin{cases} 1 & \text{if } y \in \partial\Omega^+, \\ 0 & \text{if } y \in \partial\Omega^-, \end{cases} \tag{5.2}$$

where $\partial\Omega^+$ and $\partial\Omega^-$ are the following two subsets of the boundary $\partial\Omega$:

$$\partial\Omega^+ = \left\{ \xi \in \partial\Omega : \operatorname{dist}(\xi, \Sigma^d) > \frac{d_{\min}}{2} \right\},$$
$$\partial\Omega^- = \left\{ \xi \in \partial\Omega : \operatorname{dist}(\xi, \Sigma^d) < \frac{d_{\min}}{4} \right\}.$$

We are now ready to present the main result of this paper.

Theorem 5.1. *Let $\gamma_{z,\varepsilon} \subset \Sigma^n$ be an open Lipschitz subset, created around an arbitrary point $z \in \Sigma^n$. Assume that Assumption (1.2) is verified and the Sobolev capacity of $\gamma_{z,\varepsilon}$ tends to zero when ε tends to zero. Then the perturbed density field Φ_ε satisfies the following asymptotic formula*

$$\Phi_\varepsilon(x,t) = \Phi_0(x,t) - \operatorname{cap}(\gamma_{z,\varepsilon})\mu(z)\mathfrak{R}(\lambda) \int_0^T E_{x,t}(z,\tau)\Phi_0(z,\tau)d\tau + o\Big(\operatorname{cap}(\gamma_{z,\varepsilon})\Big),$$

where $\mathfrak{R}$ is a non-trivial non negative Radon measure on $\partial\Omega$ whose support is included in any compact $\mathfrak{C} \subset \partial\Omega$ containing the small subset $\gamma_{z,\varepsilon}$ for $0 < \varepsilon << 1$.

To prove this theorem, we first need to derive some auxiliary results. We start by considering the following distribution $\mathfrak{R}$, which is defined as

$$\mathfrak{R}(\phi) = \lim_{\varepsilon \to 0} \frac{1}{\operatorname{cap}(\gamma_{z,\varepsilon})} \int_{\partial\Omega} \left[\nabla\vartheta_\varepsilon \cdot \nu\right] \vartheta_\varepsilon\, \phi\, ds(y), \quad \forall \phi \in C^1(\partial\Omega), \tag{5.3}$$

where $\vartheta_\varepsilon \in H^1(\Omega)$ is the solution of (2.3).

Based on the Banach–Alaoglu theorem [4], we will establish in the next section that the distribution $\mathfrak{R}$ is a Radon measure on $\partial\Omega$. The proof of Theorem 5.1 will be illustrated in Section 5.3. Section 5.2 we will concerned with a useful preliminary estimate.

5.1. THE DISTRIBUTION $\mathfrak{R}$ IS A NON NEGATIVE RADON MEASURE

In this subsection, we examine the properties of the linear map $\mathfrak{R}$ that is introduced in (5.3). Based on Proposition 2.5 and Corollary 2.6, we establish the following result.

Proposition 5.2. *Under the assumptions of Theorem 5.1, $\mathfrak{R}$ is a non-trivial, non-negative Radon measure on $\partial\Omega$. Moreover, the support of $\mathfrak{R}$ is contained in any compact subset $\mathfrak{C} \subset \partial\Omega$ such that $\gamma_{z,\varepsilon} \subset \mathfrak{C}$ for $0 < \varepsilon << 1$.*

Proof. Let $\phi \in C^1(\partial\Omega)$ be an arbitrary function. Since $\partial\Omega$ is regular, one can construct a function $p_\phi \in C^1(\overline{\Omega})$ such that

$$p_\phi = \phi \text{ on } \partial\Omega, \text{ and } \|p_\phi\|_{C^1(\overline{\Omega})} \leq c_0 \|\phi\|_{C^1(\partial\Omega)}, \tag{5.4}$$

where $c_0 > 0$ is a positive constant which depends only on Ω, see [7] for more details. Since $\Delta\vartheta_\varepsilon = 0$ in Ω (see (2.3)), by Green's formula, one can derive

$$\int\limits_{\partial\Omega} \left[\nabla\vartheta_\varepsilon \cdot \nu\right] \vartheta_\varepsilon \, \phi \, ds(y) = \int\limits_{\Omega} (\nabla\vartheta_\varepsilon \cdot \nabla\vartheta_\varepsilon) \, p_\phi dy + \int\limits_{\Omega} (\nabla\vartheta_\varepsilon \cdot \nabla p_\phi) \, \vartheta_\varepsilon \, dy. \tag{5.5}$$

Using Proposition 2.5 and Corollary 2.6, we deduce

$$\left| \int\limits_{\Omega} (\nabla\vartheta_\varepsilon \cdot \nabla p_\phi)\vartheta_\varepsilon dy \right| \leq \|p_\phi\|_{C^1(\overline{\Omega})} \|\vartheta_\varepsilon\|_{L^2(\Omega)} \|\vartheta_\varepsilon\|_{H^1(\Omega)} \leq c \left|\mathrm{cap}(\gamma_{z,\varepsilon})\right|^{5/4}. \tag{5.6}$$

Therefore, combining (5.3), (5.5) and (5.6) we deduce

$$\mathfrak{R}(\phi) = \lim_{\varepsilon \to 0} \frac{1}{\mathrm{cap}(\gamma_{z,\varepsilon})} \int\limits_{\Omega} (\nabla\vartheta_\varepsilon \cdot \nabla\vartheta_\varepsilon) \, p_\phi \, dy.$$

for any function $p_\phi \in C^1(\overline{\Omega})$, verifying the conditions (5.4).

Next, we use this new expression to examine the properties of $\mathfrak{R}$. Thanks to Proposition 2.5 and the estimate (5.4), one can establish that there exists a constant $c_0 > 0$, independent of ε, such that

$$\left| \frac{1}{\mathrm{cap}(\gamma_{z,\varepsilon})} \int\limits_{\Omega} (\nabla\vartheta_\varepsilon \cdot \nabla\vartheta_\varepsilon) p_\phi \; dy \right| \leq c \; \|\phi\|_{C^1(\partial\Omega)}, \quad \forall \phi \in C^1(\partial\Omega).$$

With the help of the Banach–Alaoglu theorem, one can deduce the existence of a subsequence (still indexed by ε) and a non negative Radon measure η on $\overline{\Omega}$ such that

$$\frac{1}{\operatorname{cap}(\gamma_{z,\varepsilon})}\int\limits_{\Omega}(\nabla\vartheta_\varepsilon\cdot\nabla\vartheta_\varepsilon)p_\phi\;dy \xrightarrow[\varepsilon\to 0]{} \int\limits_{\Omega} p_\phi\;d\eta,\quad \forall\phi\in C^1(\partial\Omega).$$

Then, it follows that

$$\mathfrak{R}(\phi)=\int\limits_{\Omega} p_\phi\;d\eta,$$

for all $\phi\in C^1(\partial\Omega)$ with p_ϕ given by (5.4).

On the other hand, using the inequality (5.4), the integral term can be estimated as

$$\left|\int\limits_{\Omega} p_\phi\;d\nu\right|\le\tilde{c}\;\|p_\phi\|_{C^1(\overline{\Omega})}\le c\|\phi\|_{C^1(\partial\Omega)},$$

which implies

$$|\mathfrak{R}(\phi)|\le c\;\|\phi\|_{C^1(\partial\Omega)},\quad\forall\phi\in C^1(\partial\Omega).$$

This proves that $\mathfrak{R}$ is a Radon measure on $\partial\Omega$. Its non-negativity follows from that of η. Also, one can show that $\mathfrak{R}$ is a non-trivial measure. Such a property follows from the established estimates in Proposition 2.5 and Corollary 2.6:

$$\mathfrak{R}(1)=\lim_{\varepsilon\to 0}\frac{1}{\operatorname{cap}(\gamma_{z,\varepsilon})}\int\limits_{\Omega}|\nabla\vartheta_\varepsilon|^2\,dy\ge c>0.$$

Now, let us discuss the support of $\mathfrak{R}$. Consider $\mathfrak{C}\Subset\partial\Omega$ be a compact subset of $\partial\Omega$ such that $\gamma_{z,\varepsilon}\subset\mathfrak{C}$ for sufficiently small $0<\varepsilon<<1$. Let $\phi\in C^1(\partial\Omega)$ be an arbitrary function with support in $\mathcal{S}:=\partial\Omega\setminus\mathfrak{C}$. Then, the function $y\mapsto\vartheta_\varepsilon(x)\phi(x)$ belongs to $H^{1/2}(\partial\Omega)$ and satisfies $\vartheta_\varepsilon(x)\phi(x)=0$ if $x\in\Sigma^d\cup\gamma_{z,\varepsilon}$, which implies

$$\mathfrak{R}(\phi)=\lim_{\varepsilon\to 0}\frac{1}{\operatorname{cap}(\gamma_{z,\varepsilon})}\int\limits_{\partial\Omega}\left[\nabla\vartheta_\varepsilon\cdot\nu\right]\vartheta_\varepsilon\,\phi\;ds(y)=0.$$

Therefore, one can conclude that for any $\phi\in C^1(\partial\Omega)$ such that $\operatorname{supp}(\phi)\subset\partial\Omega\setminus\mathfrak{C}$, we have $\mathfrak{R}(\phi)=0$. Consequently, the support of $\mathfrak{R}$ is contained in any compact subset $\mathfrak{C}\subset\partial\Omega$ containing $\gamma_{z,\varepsilon}$. □

5.2. PRELIMINARY ESTIMATE

In this paragraph, we establish an estimate of the integral term

$$\int\limits_0^T\int\limits_{\partial\Omega}[\mu\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nu]\varphi\;ds(x)\,dt,$$

with $\varphi \in C^1(0,T;C^1(\partial\Omega))$ be an arbitrary function that vanishes on the set $\partial\Omega^- = \{\xi \in \partial\Omega : \mathrm{dist}(\xi,\Sigma^d) < d_{\min}/4\}$. Since Ω is smooth, it is easy to construct a function $\Psi \in C^1(0,T;C^1(\Omega))$ verifying

(i) $\Psi = \varphi$ on $\partial\Omega \times (0,T)$,
(ii) $\|\Psi\|_{C^1(0,T;C^1(\overline{\Omega}))} \leq c\, \|\varphi\|_{C^1(0,T;C^1(\partial\Omega))}$,

where the constant c depends only on Ω. One can refer to [7] for more details.

Lemma 5.3. *Under the assumptions of Theorem 5.1, we have*

$$\begin{aligned}&\int_0^T\int_{\partial\Omega}[\mu\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nu]\varphi\,ds(x)\,dt\\&=-\int_0^T\int_{\partial\Omega}\big[\nabla\vartheta_\varepsilon\cdot\nu\big]\vartheta_\varepsilon\mu\Phi_0\Psi\,ds(x)\,dt+o\Big(\,\mathrm{cap}(\gamma_{z,\varepsilon})\Big).\end{aligned}$$

Proof. From the definition of Ψ and the properties of ϑ_ε, one can observe that $\varphi=\vartheta_\varepsilon\Psi$ on $\Sigma^d\cup\gamma_{z,\varepsilon}$. Then, using that fact that $\mu\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nu=0$ on $\Sigma^{n,\varepsilon}=\Sigma^n\backslash\overline{\gamma_{z,\varepsilon}}$, we deduce

$$\int_0^T\int_{\partial\Omega}[\mu\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nu]\,\varphi\,ds(x)\,dt=\int_0^T\int_{\partial\Omega}[\mu\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nu]\,\vartheta_\varepsilon\Psi\,ds(x)\,dt.$$

Since $\rho_\varepsilon=\Phi_\varepsilon-\Phi_0$ is solution to (3.2), then by Green's formula one can obtain

$$\begin{aligned}&\int_0^T\int_{\partial\Omega}[\mu\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nu]\,\varphi\,ds(x)\,dt\\&=\int_0^T\int_{\Omega}\mu\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nabla(\vartheta_\varepsilon\Psi)\,dx\,dt+\int_0^T\int_{\Omega}\frac{\partial(\Phi_\varepsilon-\Phi_0)}{\partial t}\,\vartheta_\varepsilon\Psi\,dx\,dt.\end{aligned}$$

From the fact that $\nabla(\vartheta_\varepsilon\Psi)=\Psi\nabla\vartheta_\varepsilon+\vartheta_\varepsilon\nabla\Psi$, it follows that

$$\begin{aligned}&\int_0^T\int_{\partial\Omega}[\mu\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nu]\,\varphi\,ds(x)\,dt\\&=\int_0^T\int_{\Omega}\mu\Psi\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nabla\vartheta_\varepsilon\,dx\,dt+\int_0^T\int_{\Omega}\frac{\partial(\Phi_\varepsilon-\Phi_0)}{\partial t}\vartheta_\varepsilon\Psi\,dx\,dt\\&\quad+\int_0^T\int_{\Omega}\mu\vartheta_\varepsilon\nabla(\Phi_\varepsilon-\Phi_0)\cdot\nabla\Psi\,dx\,dt.\end{aligned}\tag{5.7}$$

Using the Cauchy–Schwarz inequality, Corollary 2.6, Theorem 3.1 and the fact that $\|\Psi\|_{L^2(0,T;C^1(\overline{\Omega}))}$ is uniformly bounded, one can deduce that there exists a constant $c > 0$, independent of ε, such that

$$\left| \int_0^T \int_\Omega \frac{\partial(\Phi_\varepsilon - \Phi_0)}{\partial t} \vartheta_\varepsilon \Psi \, dx \, dt + \int_0^T \int_\Omega \mu \vartheta_\varepsilon \nabla(\Phi_\varepsilon - \Phi_0) \cdot \nabla\Psi \, dx \, dt \right| \\ \leq c \left|\mathrm{cap}(\gamma_{z,\varepsilon})\right|^{5/4}.$$

Consequently, the relation (5.7) can be rewritten as

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} [\mu \nabla(\Phi_\varepsilon - \Phi_0) \cdot \nu] \, \varphi \, ds \, dt \\ &= \int_0^T \int_\Omega \mu \Psi \nabla(\Phi_\varepsilon - \Phi_0) \cdot \nabla\vartheta_\varepsilon \, dx \, dt + o(\mathrm{cap}(\gamma_{z,\varepsilon})). \end{aligned} \tag{5.8}$$

On the other hand, the second integral term in (5.8) can be formulated as

$$\begin{aligned} &\int_0^T \int_\Omega \mu \Psi \nabla(\Phi_\varepsilon - \Phi_0) \cdot \nabla\vartheta_\varepsilon \, dx \, dt \\ &= \int_0^T \int_\Omega \nabla\Big[\mu \Psi (\Phi_\varepsilon - \Phi_0)\Big] \cdot \nabla\vartheta_\varepsilon \, dx \, dt \\ &\quad - \int_0^T \int_\Omega (\Phi_\varepsilon - \Phi_0) \nabla(\mu\Psi) \cdot \nabla\vartheta_\varepsilon \, dx \, dt. \end{aligned} \tag{5.9}$$

According to Proposition 2.5 and Theorem 4.1, the last integral term satisfies

$$\left| \int_0^T \int_\Omega (\Phi_\varepsilon - \Phi_0) \nabla(\mu\Psi) \cdot \nabla\vartheta_\varepsilon \, dx \, dt \right| \leq c \left|\mathrm{cap}(\gamma_{z,\varepsilon})\right|^{5/4}. \tag{5.10}$$

Combining (5.8), (5.9) and (5.10), we obtain

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} [\mu \nabla(\Phi_\varepsilon - \Phi_0) \cdot \nu] \, \varphi \, ds \, dt \\ &= \int_0^T \int_\Omega \nabla\Big[\mu \Psi (\Phi_\varepsilon - \Phi_0)\Big] \cdot \nabla\vartheta_\varepsilon \, dx \, dt + o(\mathrm{cap}(\gamma_{z,\varepsilon})). \end{aligned}$$

The weak formulation of (2.3) implies

$$\int_0^T \int_{\partial\Omega} [\mu \nabla(\Phi_\varepsilon - \Phi_0) \cdot \nu] \, \varphi \, ds \, dt$$
$$= \int_0^T \int_{\partial\Omega} \big(\nabla \vartheta_\varepsilon \cdot \nu\big) \, \big[\mu \Psi(\Phi_\varepsilon - \Phi_0)\big] \, ds \, dt + o(\mathrm{cap}(\gamma_{z,\varepsilon}))\,.$$

Finally, since $[\nabla \vartheta_\varepsilon \cdot \nu](\Phi_\varepsilon - \Phi_0) = 0$ on $\Sigma^{n,\varepsilon} \cup \Sigma^d$ and $\Phi_\varepsilon - \Phi_0 = -\vartheta_\varepsilon \, \Phi_0$ on $\gamma_{z,\varepsilon}$, the last identity becomes

$$\int_0^T \int_{\partial\Omega} [\mu \nabla(\Phi_\varepsilon - \Phi_0) \cdot \nu] \, \varphi \, ds \, dt$$
$$= -\int_0^T \int_{\partial\Omega} \big(\nabla \vartheta_\varepsilon \cdot \nu\big) \, \vartheta_\varepsilon \mu \Psi \Phi_0 \, ds \, dt + o(\mathrm{cap}(\gamma_{z,\varepsilon}))\,. \qquad \square$$

5.3. PROOF OF THEOREM 5.1

We start this proof by the variational formulation of the system (5.1). For any $(x,t) \in \Omega \times (0,T)$, we obtain

$$\begin{aligned}\Phi_\varepsilon(x,t) - \Phi_0(x,t) = &-\int_0^T \int_\Omega \frac{\partial E_{x,t}}{\partial \tau}(y,\tau)\big[\Phi_\varepsilon - \Phi_0\big](y,\tau) \, dy \, d\tau \\ &+ \int_0^T \int_\Omega \mu \nabla E_{x,t}(y,\tau) \cdot \nabla\big[\Phi_\varepsilon - \Phi_0\big](y,\tau) \, dy \, d\tau \\ &- \int_0^T \int_{\partial\Omega} \mu(y) \, (\nabla E_{x,t} \cdot \nu)(y,\tau)\big[\Phi_\varepsilon - \Phi_0\big](y,\tau) \, ds(y) \, d\tau.\end{aligned}$$

Taking into account the boundary conditions, it follows that

$$\begin{aligned}\Phi_\varepsilon(x,t) - \Phi_0(x,t) = &-\int_0^T \int_\Omega \frac{\partial E_{x,t}}{\partial \tau}(y,\tau)\big[\Phi_\varepsilon - \Phi_0\big](y,\tau) \, dy \, d\tau \\ &+ \int_0^T \int_\Omega \mu(y) \nabla E_{x,t}(y,\tau) \cdot \nabla\big[\Phi_\varepsilon - \Phi_0\big](y,\tau) \, dy \, d\tau.\end{aligned}$$

Using the fact that $\rho_\varepsilon = \Phi_\varepsilon - \Phi_0$ satisfies the system (3.2), then from the weak formulation one can deduce

$$\begin{aligned}&\int_0^T \int_\Omega \mu \nabla E_{x,t}(y,\tau) \cdot \nabla\big[\Phi_\varepsilon - \Phi_0\big](y,\tau)\, dy\, d\tau \\ &= -\int_0^T \int_\Omega \frac{\partial(\Phi_\varepsilon - \Phi_0)}{\partial \tau}(y,\tau)\, E_{x,t}(y,\tau)\, dy\, d\tau \\ &\quad + \int_0^T \int_{\partial\Omega} [\mu \nabla(\Phi_\varepsilon - \Phi_0)\cdot \nu](y,\tau)\, E_{x,t}(y,\tau)\, ds(y)\, d\tau.\end{aligned} \tag{5.11}$$

Integrating with respect to the time variable and using the fact that $(\Phi_\varepsilon - \Phi_0)(\cdot, 0) = 0$ and $E_{x,t}(\cdot, T) = 0$, the first integral term in the right-hand side of (5.11) can be rewritten as

$$\begin{aligned}&\int_0^T \int_\Omega \frac{\partial(\Phi_\varepsilon - \Phi_0)}{\partial \tau}(y,\tau)\, E_{x,t}(y,\tau)\, dy\, d\tau \\ &= -\int_0^T \int_\Omega \frac{\partial E_{x,t}}{\partial \tau}(y,\tau)\, (\Phi_\varepsilon - \Phi_0)(y,\tau)\, dy\, d\tau.\end{aligned}$$

Consequently, we have

$$\Phi_\varepsilon(x,t) - \Phi_0(x,t) = \int_0^T \int_{\partial\Omega} [\mu \nabla(\Phi_\varepsilon - \Phi_0)(y,\tau)\cdot \nu(y)] E_{x,t}(y,\tau) ds(y) d\tau. \tag{5.12}$$

According to the definition of λ (see (5.2)), $y \longmapsto \lambda(y)\, E_{x,t}(y,\tau)$ is a C^∞ function on the boundary $\partial\Omega$. Moreover, one can easily check that $\lambda(\cdot)\, E_{x,t}(\cdot,\tau)$ coincides with $E_{x,t}(\cdot,\tau)$ on $(\gamma_{z,\varepsilon} \cup \Sigma^d) \times (0,T)$. Since $\mu\nabla(\Phi_\varepsilon - \Phi_0)(y,\tau)\cdot\nu = 0$ on $\Sigma^{n,\varepsilon}$, the relation (5.12) becomes

$$\Phi_\varepsilon(x,t) - \Phi_0(x,t) = \int_0^T \int_{\partial\Omega} [\mu \nabla(\Phi_\varepsilon - \Phi_0)(y,\tau)\cdot \nu] \lambda(y) E_{x,t}(y,\tau) ds(y) d\tau.$$

Applying Lemma 5.3 (with $\varphi(y,\tau) = \lambda(y)\, E_{x,t}(y,\tau)$), we deduce

$$\begin{aligned}&\Phi_\varepsilon(x,t) - \Phi_0(x,t) \\ &= -\int_0^T \int_{\partial\Omega} \big[\nabla \vartheta_\varepsilon(y)\cdot \nu\big]\, \vartheta_\varepsilon(y)\, \mu(y)\, \Phi_0(y,\tau)\, \lambda(y)\, E_{x,t}(y,\tau)\, ds(y)\, d\tau \\ &\quad + o\Big(\operatorname{cap}(\gamma_{z,\varepsilon})\Big).\end{aligned} \tag{5.13}$$

Since the function $(y,\tau) \longmapsto \mu(y)\Phi_0(y,\tau)\, E_{x,t}(y,\tau)$ is sufficiently smooth (of class C^1) on a neighborhood of $\gamma_{z,\varepsilon}$, in such away there exists $c > 0$, independent of ε such that for each small $\varepsilon > 0$

$$|\mu(y)\Phi_0(y,\tau)\, E_{x,t}(y,\tau) - \mu(z)\Phi_0(z,\tau)\, E_{x,t}(z,\tau)| \le c\,\varepsilon, \quad \forall y \in \overline{\gamma_{z,\varepsilon}}.$$

Therefore, the relation (5.13) can be reformulated as

$$\begin{aligned} &\Phi_\varepsilon(x,t) - \Phi_0(x,t) \\ &= -\mu(z)\int_0^T \Phi_0(z,\tau)\, E_{x,t}(z,\tau) \int_{\partial\Omega} \big[\nabla\vartheta_\varepsilon(y)\cdot\nu\big]\, \vartheta_\varepsilon(y)\, \lambda(y)\, ds(y)\, d\tau \\ &\quad + o\Big(\operatorname{cap}(\gamma_{z,\varepsilon})\Big). \end{aligned} \tag{5.14}$$

Finally, since $\lambda \in C^1(\partial\Omega)$, by the definition of $\mathfrak{R}$ (see Section 5.1), we have

$$\lim_{\varepsilon\to 0} \frac{1}{\operatorname{cap}(\gamma_{z,\varepsilon})} \int_{\partial\Omega} \big[\nabla\vartheta_\varepsilon(y)\cdot\nu\big]\, \vartheta_\varepsilon(y)\, \lambda(y)\; ds(y) = \mathfrak{R}(\lambda).$$

Then it follows that

$$\int_{\partial\Omega} \big[\nabla\vartheta_\varepsilon(y)\cdot\nu\big]\, \vartheta_\varepsilon(y)\, \lambda(y)\; ds(y) = \operatorname{cap}(\gamma_{z,\varepsilon})\, \mathfrak{R}(\lambda) + o\Big(\operatorname{cap}(\gamma_{z,\varepsilon})\Big). \tag{5.15}$$

Finally, inserting (5.15) into (5.14), one can conclude that the perturbed density field Φ_ε satisfies the following asymptotic expansion:

$$\Phi_\varepsilon(x,t) = \Phi_0(x,t) - \operatorname{cap}(\gamma_{z,\varepsilon})\, \mu(z)\mathfrak{R}(\lambda) \int_0^T \Phi_0(z,\tau)E_{x,t}(z,\tau)d\tau + o\Big(\operatorname{cap}(\gamma_{z,\varepsilon})\Big).$$

According to Proposition 5.2, $\mathfrak{R}$ is a non-trivial, non-negative Radon measure on $\partial\Omega$. Its support is included in any compact subset $\mathfrak{C} \subset \partial\Omega$ containing $\gamma_{z,\varepsilon}$ for small enough $0 < \varepsilon << 1$.

6. AN EXPLICIT ASYMPTOTIC FORMULA

In this section, we consider the particular case where the boundary condition perturbation is supported by a planar surface. More precisely, we assume that the subset $\gamma_{z,\varepsilon}$ satisfies the following conditions:

(i) the subset $\gamma_{z,\varepsilon}$ is far from Σ^d in the sense of Hypothesis (1.2),
(ii) there exists $r_0 > 0$ such that the subset $\Sigma^n_{z,r_0} = B(z,r_0) \cap \Sigma^n$ belongs to the tangent plane at z.

Let $\nu_z = (\alpha_k^z)_{1\le k\le d}$ be an outward normal vector to the boundary $\Sigma^n \subset \partial\Omega$ at the point z. Denoting by $\mathcal{P}_{\nu,z}$ the plane defined by the normal vector ν_z and the point z. In other words, $\mathcal{P}_{\nu,z}$ is the plane tangent to the surface $\partial\Omega$ at the point z. Based on the Cartesian coordinates, $\mathcal{P}_{\nu,z}$ can be characterized as

$$\mathcal{P}_{\nu,z} = \Big\{\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \,:\, \sum_{k=1}^{d} \alpha_k^z\, \xi_k + \mathbf{c}_\nu^z = 0\Big\}, \tag{6.1}$$

with $\mathbf{c}_\nu^z$ depends on z and the normal vector ν_z as $\mathbf{c}_\nu^z = \sum_{k=1}^d \alpha_k^z z_k$. According to this notations, the condition (ii) can be explained as

$$\gamma_{z,\varepsilon} \subset \Sigma_{z,r_0}^n \subset \Sigma^n \cap P_{\nu,z}.$$

Let $\mathcal{P}_{\nu,z}^\star$ be the half-space defined by

$$\mathcal{P}_{\nu,z}^\star = \Big\{\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \,:\, \sum_{k=1}^{d} \alpha_k^z\, \xi_k + \mathbf{c}_\nu^z < 0\Big\}.$$

In the context of this particular geometric configuration, one can easily construct a smooth domain Ω_{z,r_0} such that $\Omega_{z,r_0} \subset \Omega \cap \mathcal{P}_{\nu,z}^\star$ and $\Sigma_{z,r_0}^n \subset \partial\Omega_{z,r_0}$.

Next, we will prove, in the context of this particular configuration, that the perturbed density field admits an explicit asymptotic formula. The main results of this section are illustrated by the following theorem.

Theorem 6.1. *Assume that Assumptions* (i) *and* (ii) *hold. Then the perturbed density field Φ_ε satisfies an explicit asymptotic formula in both two- and three-dimensional cases, namely: in the two-dimensional case, we have*

$$\Phi_\varepsilon(x,t) = \Phi_0(x,t) - \frac{\pi}{\log\varepsilon}\mu(z) \int_0^T E_{x,t}(z,\tau)\,\Phi_0(z,\tau)\,d\tau + o\Big(\frac{1}{\log\varepsilon}\Big), \tag{6.2}$$

and in the three-dimensional case, we have

$$\Phi_\varepsilon(x,t) = \Phi_0(x,t) - 4\pi\varepsilon\,\mu(z) \int_0^T E_{x,t}(z,\tau)\,\Phi_0(z,\tau)\,d\tau + o(\varepsilon). \tag{6.3}$$

To prove the asymptotic formulas (6.2) and (6.3), we start by the construction of a particular Green's function that satisfying a homogeneous Neumann boundary condition on $\mathcal{P}_{\nu,z}$. In section 6.2, we will establish some preliminary useful estimates. The proof of Theorem 6.1 will be presented in Section 6.3.

6.1. CONSTRUCTION OF A MODIFIED GREEN'S FUNCTION

For a given $x \in \Omega$, we denote by $\mathfrak{F}_x$ a modified Green's function for the Laplace operator in $\mathbb{R}^d$, such that $y \longmapsto \mathfrak{F}_x(y)$ satisfies the following conditions:

$$-\Delta_y \mathfrak{F}_x(y) = \delta_x(y) \quad \text{in } \mathbb{R}^d, \quad \nabla_y \mathfrak{F}_x(y)\cdot\nu_z(y) = 0 \quad \text{on } \mathcal{P}_{\nu,z}, \tag{6.4}$$

where $\mathcal{P}_{\nu,z}$ is the hyperplane defined in (6.1), i.e. $\mathcal{P}_{\nu,z} = \{\xi \in \mathbb{R}^d : \langle \xi - z, \nu_z \rangle = 0\}$. Based on the classical method of images (see e.g. [26]), the solution $\mathfrak{F}_x$ can be calculated explicitly.

Lemma 6.2. *For each $x \in \Omega$, the function $\mathfrak{F}_x$ admits the explicit expression*

$$\mathfrak{F}_x(y) = \Gamma(y - x) + \Gamma(y^\star - x)$$

with Γ is the fundamental solution of the Laplace operator in $\mathbb{R}^d$ $(d = 2, 3)$ and $y^\star$ is a mirror image of the point y across the plane $\mathcal{P}_{\nu,z}$.

Proof. We will distinguish the cases $d = 2$ and $d = 3$, and we will prove, in both cases, that the function $\mathfrak{F}_x$ satisfies the system (6.4).

In the two-dimensional case, the function $\mathfrak{F}_x$ admits the expression

$$\mathfrak{F}_x(y) = -\frac{1}{4\pi}\log(|y - x|^2) - \frac{1}{4\pi}\log(|y^\star - x|^2),$$

where

$$|y^\star - x|^2 = \big(y_1 - x_1 - 2\alpha_1^z \aleph(y)\big)^2 + \big(y_2 - x_2 - 2\alpha_2^z \aleph(y)\big)^2,$$

with $\aleph(y) = \alpha_1^z\, y_1 + \alpha_2^z\, y_2 + \mathbf{c}_\nu^z$.

In the three-dimensional case, the function $\mathfrak{F}_x$ admits the expression

$$\mathfrak{F}_x(y) = \frac{1}{4\pi}\Big[\frac{1}{|y - x|} + \frac{1}{|y^\star - x|}\Big],$$

where

$$|y^\star - x|^2 = \Big(y_1 - x_1 - 2\alpha_1^z\, \aleph(y)\Big)^2 + \Big(y_2 - x_2 - 2\alpha_2^z\, \aleph(y)\Big)^2 + \Big(y_3 - x_3 - 2\alpha_3^z\, \aleph(y)\Big)^2$$

with $\aleph(y) = \alpha_1^z\, y_1 + \alpha_2^z\, y_2 + \alpha_3^z\, y_3 + \mathbf{c}_\nu^z$. □

6.2. PRELIMINARY ESTIMATES

In this paragraph, we present some preliminary estimates.

Proposition 6.3. *Let $\gamma_{z,\varepsilon}$ be an open Lipschitz small subset of Σ^n, verifying conditions* (i)–(ii)*. Then, if $x \in \gamma_{z,\varepsilon}$, we have the estimate*

$$\int\limits_{\gamma_{z,\varepsilon}} [\mu\nabla(\Phi_\varepsilon - \Phi_0)\cdot\nu](y,\tau)\mathfrak{F}_x(y)\, ds(y) = -\mu(x)\Phi_0(x,\tau) + \mathfrak{T}(\varepsilon),$$

where $\mathfrak{T}(\varepsilon)$ is a quantity going to zero as $\varepsilon \to 0^+$.

Proof. According to (6.4), for any $(x,\tau) \in \overline{\Omega} \times (0,T)$, we have

$$\mu(x)\big[\Phi_\varepsilon(x,\tau) - \Phi_0(x,\tau)\big] = -\int\limits_{\Omega} \mu(y)\big[\Phi_\varepsilon(y,\tau) - \Phi_0(y,\tau)\big]\Delta_y\mathfrak{F}_x(y)dy.$$

Using Green's formula, one can obtain

$$\begin{aligned}
&\mu(x)\big[\Phi_\varepsilon(x,\tau) - \Phi_0(x,\tau)\big] \\
&= \int\limits_{\Omega} \mu(y)\nabla\big(\Phi_\varepsilon - \Phi_0\big)(y,\tau)\cdot\nabla_y\mathfrak{F}_x(y)dy \\
&\quad + \int\limits_{\Omega} \mathfrak{F}_x(y)\nabla\big(\Phi_\varepsilon - \Phi_0\big)(y,\tau)\cdot\nabla\mu(y)dy \\
&\quad - \int\limits_{\partial\Omega} \nabla_y\mathfrak{F}_x(y)\cdot\nu(y)\mu(y)\big(\Phi_\varepsilon - \Phi_0\big)(y,\tau)ds(y).
\end{aligned} \tag{6.5}$$

Since $\Sigma^n_{z,r_0} \subset \Sigma^n$, then from the fact that $\Phi_\varepsilon - \Phi_0 = 0$ on Σ^d and $\nabla_y\mathfrak{F}_x(y)\cdot\nu(y) = 0$ on Σ^n_{z,r_0}, we deduce that the last integral term can be rewritten as

$$\begin{aligned}
&\int\limits_{\partial\Omega} \nabla_y\mathfrak{F}_x(y)\cdot\nu(y)\mu(y)\big(\Phi_\varepsilon - \Phi_0\big)(y,\tau)ds(y) \\
&= \int\limits_{\Sigma^n\setminus\overline{\Sigma^n_{z,r_0}}} \nabla_y\mathfrak{F}_x(y)\cdot\nu(y)\mu(y)\big(\Phi_\varepsilon - \Phi_0\big)(y,\tau)ds(y).
\end{aligned} \tag{6.6}$$

On other hand, from the weak formulation (3.2) (since $\rho_\varepsilon = \Phi_\varepsilon - \Phi_0$), we obtain

$$\begin{aligned}
&\int\limits_{\Omega} \mu(y)\nabla\big(\Phi_\varepsilon - \Phi_0\big)(y,\tau)\cdot\nabla_y\mathfrak{F}_x(y)dy \\
&= \int\limits_{\gamma_{z,\varepsilon}} [\mu\nabla(\Phi_\varepsilon - \Phi_0)\cdot\nu](y,\tau)\mathfrak{F}_x(y)ds(y) \\
&\quad + \int\limits_{\Sigma^d} [\mu\nabla(\Phi_\varepsilon - \Phi_0)\cdot\nu](y,\tau)\mathfrak{F}_x(y)ds(y) \\
&\quad - \int\limits_{\Omega} \frac{\partial(\Phi_\varepsilon - \Phi_0)}{\partial\tau}(y,\tau)\mathfrak{F}_x(y)dy.
\end{aligned} \tag{6.7}$$

Combining the relations (6.5), (6.6) and (6.7) and using the fact that $\Phi_\varepsilon = 0$ on $\gamma_{z,\varepsilon}\times(0,T)$, one can deduce that for each $x \in \gamma_{z,\varepsilon}$ we have

$$\int\limits_{\gamma_{z,\varepsilon}} [\mu\nabla(\Phi_\varepsilon - \Phi_0)\cdot\nu](y,\tau)\mathfrak{F}_x(y)\ ds(y) = -\mu(x)\Phi_0(x,\tau) + \mathfrak{T}(\varepsilon),$$

with $\mathfrak{T}(\varepsilon) = \sum_{k=1}^{4} \mathfrak{T}_k(\varepsilon)$, where the terms $\mathfrak{T}_k(\varepsilon)$ are defined by

$$\mathfrak{T}_1(\varepsilon) = -\int_{\Omega} \mathfrak{F}_x(y)\nabla\big(\Phi_\varepsilon - \Phi_0\big) \cdot \nabla\mu \, dy,$$

$$\mathfrak{T}_2(\varepsilon) = \int_{\Sigma^n \setminus \overline{\Sigma^n_{z,r_0}}} \mu\nabla_y \mathfrak{F}_x(y) \cdot \nu\big(\Phi_\varepsilon - \Phi_0\big) ds,$$

$$\mathfrak{T}_3(\varepsilon) = -\int_{\Sigma^d} [\mu\nabla(\Phi_\varepsilon - \Phi_0) \cdot \nu]\mathfrak{F}_x(y) ds,$$

$$\mathfrak{T}_4(\varepsilon) = \int_{\Omega} \frac{\partial(\Phi_\varepsilon - \Phi_0)}{\partial\tau}\mathfrak{F}_x(y)\, dy.$$

Finally, based on Theorem 3.1, one can show that for each $1 \leq k \leq 4$, $\mathfrak{T}_k(\varepsilon)$ goes to zero when $\varepsilon \to 0^+$. □

The following lemma provides a preliminary estimate describing the behavior of the variation $\Phi_\varepsilon - \Phi_0$ with respect to ε.

Lemma 6.4. *We have*

$$\Phi_\varepsilon(x,t) - \Phi_0(x,t) = \int_0^T \int_{\gamma_{z,\varepsilon}} \big[\mu\nabla(\Phi_\varepsilon - \Phi_0)(y,\tau) \cdot \nu\big] E_{x,t}(z,\tau)\, ds(y)\, d\tau + O\Big(\operatorname{cap}(\gamma_{z,\varepsilon})\Big).$$

Proof. According to (5.12) and using the fact that

$$\big[\mu\nabla(\Phi_\varepsilon - \Phi_0)(y,\tau) \cdot \nu\big] E_{x,t}(y,\tau) = 0 \quad \text{on } \Sigma^d \cup \Sigma^{n,\varepsilon} \times (0,T).$$

the variation $\Phi_\varepsilon - \Phi_0$ can be expressed as

$$\Phi_\varepsilon(x,t) - \Phi_0(x,t) = \int_0^T \int_{\gamma_{z,\varepsilon}} \big[\mu\nabla(\Phi_\varepsilon - \Phi_0)(y,\tau) \cdot \nu\big] E_{x,t}(y,\tau)\ ds(y)\ d\tau,$$

where $E_{x,t}$ is the solution to the system (5.1). Then, it follows that

$$\begin{aligned}
&\Phi_\varepsilon(x,t) - \Phi_0(x,t) \\
&= \int_0^T \int_{\gamma_{z,\varepsilon}} \big[\mu\nabla(\Phi_\varepsilon - \Phi_0)(y,\tau) \cdot \nu\big] E_{x,t}(z,\tau)\, ds(y)\, d\tau \\
&\quad + \int_0^T \int_{\gamma_{z,\varepsilon}} \big[\mu\nabla(\Phi_\varepsilon - \Phi_0)(y,\tau) \cdot \nu\big] \Big(E_{x,t}(y,\tau) - E_{x,t}(z,\tau)\Big)\, ds(y)\, d\tau.
\end{aligned}$$

Using Theorem 3.1 and Lemma 3.2, one can derive

$$\left| \int_0^T \int_{\gamma_{z,\varepsilon}} \left[\mu \nabla (\Phi_\varepsilon - \Phi_0)(y,\tau) \cdot \nu\right] \Big(E_{x,t}(y,\tau) - E_{x,t}(z,\tau) \Big)\, ds(y) d\tau \right|$$
$$\leq c \left| \mathrm{cap}(\gamma_{z,\varepsilon}) \right|,$$

which implies the desired estimate. □

6.3. PROOF OF THEOREM 6.1

According to Lemma 6.4, the variation $\Phi_\varepsilon - \Phi_0$ can be expressed as

$$\Phi_\varepsilon(x,t) - \Phi_0(x,t) = \int_0^T \int_{\gamma_{z,\varepsilon}} \left[\mu \nabla (\Phi_\varepsilon - \Phi_0)(y,\tau) \cdot \nu\right] E_{x,t}(z,\tau)\, ds(y)\, d\tau$$
$$+ O\Big(\mathrm{cap}(\gamma_{z,\varepsilon}) \Big).$$

By the change of variable $y = z + \varepsilon\xi$ and the fact that $\gamma_{z,\varepsilon} = z + \varepsilon\gamma$, we obtain

$$\Phi_\varepsilon(x,t) - \Phi_0(x,t) = \int_0^T E_{x,t}(z,\tau) \int_\gamma \Psi_{z,\varepsilon}(\xi,\tau)\ ds(\xi)\ d\tau,$$

with

$$\Psi_{z,\varepsilon}(\xi,\tau) = \varepsilon^{d-1} \mu(z+\varepsilon\xi) \Big[\nabla_y (\Phi_\varepsilon - \Phi_0)(z+\varepsilon\xi,\tau) \cdot \nu(z+\varepsilon\xi) \Big].$$

To evaluate the behavior of the last integral term with respect to ε, we apply Proposition 6.3 which provides an estimate of the variation $\Phi_\varepsilon - \Phi_0$ involving the explicit expression of $\mathfrak{F}_x$. It is described by the following formula:

$$\Phi_\varepsilon(x,\tau) - \Phi_0(x,\tau) = \frac{1}{\mu(x)} \int_{\gamma_{z,\varepsilon}} [\mu \nabla (\Phi_\varepsilon - \Phi_0) \cdot \nu](y,\tau) \mathfrak{F}_x(y)\ ds(y) + \mathfrak{T}(\varepsilon).$$

In the particular case when x belongs to $\gamma_{z,\varepsilon}$, the last identity becomes

$$-\mu(x)\Phi_0(x,\tau) = \int_{\gamma_{z,\varepsilon}} [\mu \nabla (\Phi_\varepsilon - \Phi_0) \cdot \nu](y,\tau) \mathfrak{F}_x(y)\ ds(y) + \mathfrak{T}(\varepsilon).$$

Using the change of variable $y = z + \varepsilon\xi$ and the fact that $\gamma_{z,\varepsilon} = z + \varepsilon\gamma$, we get

$$-\mu(z)\Phi_0(z,\tau) = \int_\gamma \Psi_{z,\varepsilon}(\xi,\tau) \mathfrak{F}_{z+\varepsilon x}(z+\varepsilon\xi)\ ds(\xi) + \widetilde{\mathfrak{T}}(\varepsilon), \tag{6.8}$$

with

$$\widetilde{\mathfrak{T}}(\varepsilon) = \mathfrak{T}(\varepsilon) + \mu(z+\varepsilon x)\Phi_0(z+\varepsilon x,\tau) - \mu(z)\Phi_0(z,\tau), \quad x \in \gamma,\, \tau \in (0,T).$$

In order to determine the behavior of the last integral term with respect to ε, next we will distinguish between the two- and three-dimensional cases.

(a) *The two-dimensional case.* For $x \in \gamma$, the term $\mathfrak{F}_{z+\varepsilon x}(z+\varepsilon\xi)$ can be formulated as

$$\mathfrak{F}_{z+\varepsilon x}(z+\varepsilon\xi) = 2\,\Gamma(\varepsilon(\xi - x)) = -\frac{1}{\pi}\log\varepsilon + 2\,\Gamma(\xi - x).$$

Using the smoothness of the functions Φ_0 and μ near the point z and Proposition 6.3, one can deduce that the term $\widetilde{\mathfrak{R}}(\varepsilon)$ goes to zero as $\varepsilon \to 0^+$. Consequently, relation (6.8) can be rewritten as

$$\int\limits_{\gamma} \Psi_{z,\varepsilon}(\xi,\tau)\, ds(\xi) = \frac{\pi}{\log\varepsilon}\mu(z)\Phi_0(z,\tau) + o\Big(\frac{1}{\log\varepsilon}\Big).$$

(b) *The three-dimensional case.* For $x \in \gamma$, the term $\mathfrak{F}_{z+\varepsilon x}(z+\varepsilon\xi)$ can be formulated as

$$\mathfrak{F}_{z+\varepsilon x}(z+\varepsilon\xi) = 2\,\Gamma(\varepsilon(\xi - x)) = \frac{2}{\varepsilon}\,\Gamma(\xi - x).$$

Then, in this case, relation (6.8) can be rewritten as

$$\int\limits_{\gamma} \Psi_{z,\varepsilon}(\xi,\tau)\Gamma(\xi - x)\, ds(\xi) = -\frac{\varepsilon}{2}\mu(z)\Phi_0(z,\tau) + o(\varepsilon).$$

7. CONCLUSION AND FURTHER COMMENTS

In this study, we have derived a topological sensitivity analysis for the considered parabolic diffusion problem with respect to the creation of a boundary condition-type perturbation on a small subset. We have developed a rigorous mathematical framework that is valid in two- and three-dimensional space. It provides an asymptotic formula that describes the behavior of the perturbed density field Φ_ε with respect to the location and size of the created boundary perturbation. The dominant term of the density field variation has been expressed in terms of a modified fundamental solution of the considered parabolic operator and the unperturbed density field. The Sobolev capacity of $\gamma_{z,\varepsilon}$ has been employed to measure the smallness of the boundary subset and to describe the asymptotic behavior of Φ_ε with respect to the perturbation size.

The performed mathematical analysis is general and can be adapted for a large class of partial differential equations. The obtained asymptotic formula can serve as a useful tool to perform numerical algorithms for solving optimization and control problems.

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Received: April 6, 2026.
Revised: June 23, 2026.
Accepted: June 24, 2026.
Published online: August 4, 2026.