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NORMALIZED GROUND STATES FOR A p-LAPLACIAN SYSTEM IN THE MASS SUPER-CRITICAL CASE

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Abstract. In this paper, we study the existence of positive normalized solutions to the following p-Laplacian system:

$$\begin{cases} -\Delta_p u + \lambda_1 u^{p-1} = \mu_1 u^{m_1-1} + \beta r_1 u^{r_1-1} v^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta_p v + \lambda_2 v^{p-1} = \mu_2 v^{m_2-1} + \beta r_2 u^{r_1} v^{r_2-1} & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p = a, \quad \int_{\mathbb{R}^N} |v|^p = b, \end{cases}$$

where $1 < p < N$, $\mu_1, \mu_2, \beta, a, b > 0$ are prescribed, $\lambda_1, \lambda_2 \in \mathbb{R}$ are known as the Lagrange multiplier, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator. We prove the existence of positive solutions for the coupled purely mass super-critical case (i.e., $\frac{p^2}{N} + p < m_1, m_2, r_1 + r_2 < p^*$) by a minimization argument based on a closed ball and the Pohozaev constraint.

Keywords: p-Laplacian system, positive normalized solution, coupled purely mass super-critical case.

Mathematics Subject Classification: 35J47, 35J62.

1. INTRODUCTION

In this paper, our objective is to prove the existence of solution to the following p-Laplacian system:

$$\begin{cases} -\Delta_p u + \lambda_1 u^{p-1} = \mu_1 u^{m_1-1} + \beta r_1 u^{r_1-1} v^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta_p v + \lambda_2 v^{p-1} = \mu_2 v^{m_2-1} + \beta r_2 u^{r_1} v^{r_2-1} & \text{in } \mathbb{R}^N, \\ 0 < u, v \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

with the L^p-norm constraint:

$$\int_{\mathbb{R}^N} |u|^p \mathrm{d}x = a, \quad \int_{\mathbb{R}^N} |v|^p \mathrm{d}x = b. \tag{1.2}$$

Here $1 < p < N, p > 1, a, b, \mu_1, \mu_2, \beta > 0$, $\frac{p^2}{N} + p < m_1, m_2, r_1 + r_2 < p^*$,

$$p^* := \begin{cases} +\infty, & \text{if } N \le p, \\ \frac{Np}{N-p}, & \text{if } N > p, \end{cases}$$

and $\lambda_1, \lambda_2 \in \mathbb{R}$ are Lagrange multiplier, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator.

The p-Laplacian operator indeed plays a significant role in various fluid dynamics models, see e.g. [11, 14, 24]. It has the capacity to account for complex nonlinear phenomena. For example, it can explain shear thickening and shear thinning in non-Newtonian fluids as well as nonlinear flow in porous media. In the past few years, many scholars have studied the existence of p-Laplacian equations, see e.g. [29, 30, 32]. In [12], Byeon, Jeanjean and Maris proved the existence of least energy solutions of the following system:

$$-\operatorname{div}(|\nabla u_i|^{p-2}\nabla u_i) = g_i(u), \quad i = 1, \dots, m,$$

where $u = (u_1, \dots, u_m) : \mathbb{R}^N \longrightarrow \mathbb{R}^m, 1 < p \le N$ $g_i(0) = 0$ and there exists $G \in C^1(\mathbb{R}^m \setminus \{0\}, \mathbb{R}) \cap C(\mathbb{R}^m, \mathbb{R})$ such that $g_i(u) = \frac{\partial G}{\partial u_i}(u)$ for $u \neq 0$. In [31], Wang studied the components symmetry property of the following γ-Laplacian systems:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{\gamma-2}\nabla u) = f(u, v) & \text{in } \mathbb{R}^n, \\ -\operatorname{div}(|\nabla v|^{\gamma-2}\nabla v) = g(u, v) & \text{in } \mathbb{R}^n. \end{cases}$$

Here $n > \gamma, \gamma > 1$, and under some monotonicity assumption

$$(X - Y)[f(X, Y) - g(X, Y)] \le 0, \quad X, Y \ge 0.$$

In [18], Guo, Perera and Zou considered the following critical p-Laplacian systems:

$$\begin{cases} -\Delta_p u - \dfrac{\lambda a}{p} |u|^{a-2} u \, |v|^b = \mu_1 |u|^{p^*-2} u + \dfrac{\alpha\gamma}{p^*} |u|^{\alpha-2} u \, |v|^\beta, & x \in \Omega, \\ -\Delta_p v - \dfrac{\lambda b}{p} |u|^a \, |v|^{b-2} v = \mu_2 |v|^{p^*-2} v + \dfrac{\beta\gamma}{p^*} |u|^\alpha \, |v|^{\beta-2} v, & x \in \Omega, \\ u, v \in D_0^{1,p}(\Omega), \end{cases}$$

where $N \ge 3$, $1 < p < N$, $\lambda, \mu_1, \mu_2 \ge 0$, $\gamma \neq 0$, $a, b, \alpha, \beta > 1$ satisfy $a + b = p$, $\alpha + \beta = p^* := \frac{Np}{N-p}$, $\Omega = \mathbb{R}^N$ or a bounded domain in $\mathbb{R}^N$. By variational methods, they obtained the existence, nonexistence results of a positive least energy solution and the multiplicity of the nontrivial nonnegative solutions of this problem.

In the case $p = 2$, problem (1.1) comes from the study of the following time-dependent systems of coupled nonlinear Schrödinger equations:

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + |\Phi_1|^{m_1-2}\Phi_1 + r_1|\Phi_1|^{r_1-2}|\Phi_2|^{r_2}\Phi_1, & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + |\Phi_2|^{m_2-2}\Phi_2 + r_2|\Phi_1|^{r_1}|\Phi_2|^{r_2-2}\Phi_2, & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \ j = 1, 2, \ N \ge 1, \end{cases} \tag{1.3}$$

The problem (1.3) appears more naturally in mathematical physics and used as model for various physical phenomena, see e.g. [2, 3]. So it is necessary to consider that for any solution $[\Phi_1, \Phi_2]$ of problem (1.3) with the preserves the L^2-mass, namely

$$\int_{\mathbb{R}^N} |\Phi_1(t,x)|^2 \mathrm{d}x = a, \int_{\mathbb{R}^N} |\Phi_2(t,x)|^2 \mathrm{d}x = b, \ \forall t \in (0, +\infty).$$

Obviously, the solitary wave is a solution of problem (1.3) with the form

$$[\Phi_1, \Phi_2] = [e^{-i\lambda_1 t} u(x), e^{-i\lambda_2 t} v(x)],$$

and satisfies the nonlinear elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u|^{m_1-2} u + \beta r_1 |u|^{r_1-2} |v|^{r_2} u & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{m_2-2} v + \beta r_2 |u|^{r_1} |v|^{r_2-2} v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

with the constraints

$$\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a, \quad \int_{\mathbb{R}^N} |v|^2 \mathrm{d}x = b. \tag{1.5}$$

The different cases of the problem specified by (1.4)–(1.5) have been researched by some mathematicians, which appear in [5–9, 16, 17, 21]. In [7], Bartsch and Soave consider the case $N = 3$, $\mu_1, \mu_2 > 0$, $m_1 = m_2 = 4$, $r_1 = r_2 = 2$ and $\beta < 0$, i.e.,

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 u^3 + \beta u v^2, & \text{in } \mathbb{R}^3, \\ -\Delta v - \lambda_2 v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2 \mathrm{d}x = a \quad \text{and} \quad \int_{\mathbb{R}^3} v^2 \mathrm{d}x = b. \end{cases}$$

They obtained the existence of positive normalized solutions by using the Pohozaev manifold constraint. Moreover, they also derived the multiplicity results presented in [8]. For the case $\mu_1, \mu_2 > 0$, $\beta < 0$ and $2 < m_1, m_2, r_1 + r_2 < \frac{4}{N} + 2$, Gou and Jeanjean in [16] proved the existence of normalized solutions by means of a minimization argument. They in [5] also obtained a multiplicity result when $m_1, m_2 < \frac{4}{N} + 2 < r_1 + r_2$ or $r_1 + r_2 < \frac{4}{N} + 2 < m_1, m_2$.

When $\beta > 0$, the existence results of equations (1.1)–(1.2) are different. In [5, 6], Bartsch, Jeanjean and Soave studied the existence of positive normalized solutions of the following problem:

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^3, \\ -\Delta v - \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2 = a, \ \int_{\mathbb{R}^3} v^2 = b. \end{cases}$$

They obtained the existence of normalized solution through the variational argument in two intervals of β depending on a, b, μ_1, μ_2. In [9], Bartsch, Zhong and Zou overcame the dependence of β on the masses a, b by a new approach based on bifurcation theory. They obtained the existence of normalized solution provided β is in a range for any $a, b > 0$. In [4], Bartsch, Li and Zou investigated the existence and asymptotic properties of normalized ground states of the following Sobolev critical Schrödinger system:

$$\begin{cases} -\Delta u + \lambda_1 u = |u|^{2^*-2}u + \beta r_1 |u|^{r_1-2}|v|^{r_2}u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = |v|^{2^*-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 = a, \quad \int_{\mathbb{R}^N} v^2 = b, \end{cases}$$

where $N = 3, 4$, $r_1, r_2 > 1$ and $2 < r_1 + r_2 < 2^*$. When $\beta > 0$, they obtained the existence and non-existence results in different cases. While when $\beta < 0$, they proved the ground state does not exist. Recently, Jeanjean, Zhang and Zhong derived a new range of β by combining Liouville type theorem with the closed balls of radius a, b in [21]. More precisely, they obtained the existence of positive normalized ground states of the following systems of coupled Schrödinger equations:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^{m_1-1} + \beta r_1 u^{r_1-1} v^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^{m_2-1} + \beta r_2 u^{r_1} v^{r_2-1} & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 = a, \quad \int_{\mathbb{R}^N} v^2 = b, \end{cases}$$

where $\mu_1, \mu_2, \beta > 0$ and $\frac{4}{N} + 2 < m_1, m_2 < 2^*$. In particular, if $N = 1, 2$ or $N = 3, 4$ with $r_1, r_2 \in (1, 2)$, they just need $\beta > 0$ to guarantee that the existence result holds for any $a, b > 0$.

However, up to now, there have been relatively few studies on the problems of this kind of p-Laplacian system (1.1) with the mass constraints (1.2). It is worthwhile to investigate the existence of the ground state solutions to (1.1)–(1.2).

Denote $\mathcal{W} := W^{1,p}(\mathbb{R}^N) \times W^{1,p}(\mathbb{R}^N)$, and we define the energy functional corresponding to (1.1) as follows:

$$J_\beta[u, v] = \frac{1}{p}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) - \frac{\mu_1}{m_1}\|u\|_{m_1}^{m_1} - \frac{\mu_2}{m_2}\|v\|_{m_2}^{m_2} - \beta \int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} \mathrm{d}x,$$

constrained to the $\mathcal{S}_a \times \mathcal{S}_b$, where

$$\mathcal{S}_a := \left\{u \in W^{1,p}(\mathbb{R}^N) : \|u\|_p^p = a\right\}, \quad \mathcal{S}_b := \left\{v \in W^{1,p}(\mathbb{R}^N) : \|v\|_p^p = b\right\}.$$

Note that since $\frac{p^2}{N} + p < m_1, m_2, r_1 + r_2 < p^*$, $J_\beta\big|_{\mathcal{S}_a \times \mathcal{S}_b}$ is unbounded from below, it is necessary to consider the so-called Pohozaev manifold

$$\mathcal{P}_\beta := \left\{[u, v] \in \mathcal{W} \setminus \{[0, 0]\} : P_\beta[u, v] = 0\right\},$$

and

$$\begin{aligned} P_\beta[u, v] = {}& \|\nabla u\|_p^p + \|\nabla v\|_p^p - \mu_1 \delta_{m_1} \|u\|_{m_1}^{m_1} - \mu_2 \delta_{m_2} \|v\|_{m_2}^{m_2} \\ & - (r_1 + r_2)\delta_{r_1+r_2} \beta \int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} \mathrm{d}x, \end{aligned}$$

where $\delta_m := \frac{N}{p} - \frac{N}{m}$.

Prompted by the above literature and methods, we use the similar way in [21]. So we define

$$\mathcal{D}_a := \left\{u \in W^{1,p}(\mathbb{R}^N) : \|u\|_p^p \le a\right\}, \quad \mathcal{D}_b := \left\{v \in W^{1,p}(\mathbb{R}^N) : \|v\|_p^p \le b\right\}.$$

We will prove the existence of ground state solution on $\mathcal{D}_a \times \mathcal{D}_b$. So, for any $a, b > 0$, we can define

$$\mathcal{P}_\beta^{(a,b)} := \mathcal{P}_\beta \cap (\mathcal{D}_a \times \mathcal{D}_b).$$

We denote

$$M_\beta(a,b) := \inf_{[u,v]\in\mathcal{P}_\beta^{(a,b)}} J_\beta[u,v]. \tag{1.6}$$

Next, we need to show that $[u,v] \in \mathcal{S}_a \times \mathcal{S}_b$ provided $\lambda_1, \lambda_2 > 0$. At this point, the Liouville type theorem plays a vital role.

Now, we can state our main results as follows.

Theorem 1.1. *Let $1 < p < N \le p^2$, $m_1, m_2, r_1 + r_2 \in \left(\frac{p^2}{N} + p, p^*\right)$ and $r_1, r_2 > 1$. There exist $b_{m_1,m_2,\mu_1,\mu_2,a}$ defined in (2.7), and $\beta_{m_1,\mu_1,a,N,r}, \beta_{m_2,\mu_2,b,N,r}$ defined in (2.4), such that the following hold:*

- (i) *For any $a > 0$ and $b \in [b_{m_1,m_2,\mu_1,\mu_2,a}, +\infty)$, there exists a ground state solution $(\lambda_1, \lambda_2, u, v)$ to the equations (1.1)–(1.2), provided that either $r_1 < p$ with $\beta > 0$ or $r_1 = p$ with $\beta > \beta_{m_2,\mu_2,b,N,r}$.*
- (ii) *For any $a > 0$ and $b \in (0, b_{m_1,m_2,\mu_1,\mu_2,a}]$, there exists a ground state solution $(\lambda_1, \lambda_2, u, v)$ to the equations (1.1)–(1.2), provided that either $r_2 < p$ with $\beta > 0$ or $r_2 = p$ with $\beta > \beta_{m_1,\mu_1,a,N,r}$.*

Remark 1.2. When $p = 2$, our theorem holds for $N = 3, 4$, which generalizes the result for Schrödinger system in [21] for the case $2 < N$.

Furthermore, we can study some asymptotic properties of the normalized solutions obtained in Theorem 1.1.

Theorem 1.3. *Under the assumption of Theorem 1.1. Let $m_{m_1,\mu_1,a}, m_{m_2,\mu_2,b}$ defined in (2.6) and $w_{m_1,\mu_1,a}, w_{m_2,\mu_2,b}$ defined in (2.3). Then the following results hold:*

- (i) *For any $a > 0$, if either $b \in [b_{m_1,m_2,\mu_1,\mu_2,a}, +\infty)$ and $r_1 < p$ or $b \in (0, b_{m_1,m_2,\mu_1,\mu_2,a}]$ and $r_2 < p$, then as $\beta \to 0^+$, we have*

$$M_\beta(a,b) \to \min\{m_{m_1,\mu_1,a}, m_{m_2,\mu_2,b}\}.$$

  In particular,

$$[u_\beta, v_\beta] \to \begin{cases} [0, w_{m_2,\mu_2,b}] & \text{if } b \in (b_{m_1,m_2,\mu_1,\mu_2,a}, +\infty) \text{ and } r_1 < p, \\ [w_{m_1,\mu_1,a}, 0] & \text{if } b \in (0, b_{m_1,m_2,\mu_1,\mu_2,a}) \text{ and } r_2 < p. \end{cases}$$

- (ii) *For any $a, b > 0$, $M_\beta(a,b) \to 0^+$ as $\beta \to +\infty$.*

2. PROPERTIES OF A p-LAPLACIAN PROBLEM

In this section, we introduce some results about p-Laplacian equations. Firstly, a couple of nonnegative solutions of (1.1) is semitrivial when one component is 0 while the other is not. In order to understand the properties of semitrivial solutions of (1.1), we need the ground state solution to the following problem:

$$\begin{cases} -\Delta_p u + u^{p-1} = u^{m-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N). \end{cases} \tag{2.1}$$

For $p < N$, $m \in \left(\frac{p^2+p}{N}, p^*\right)$, the uniqueness and existence of ground state of (2.1) are given by [27, Theorem 3]. Moreover, the ground state is positive, radially symmetric and decreasing.

Denote the ground solution of (2.1) by U_m. Then for any $a, \mu > 0$ fixed, by scaling U_m, we can obtain the ground state solution of the following problem:

$$\begin{cases} -\Delta_p u + \lambda u^{p-1} = \mu u^{m-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \|u\|_p^p = a, \quad u \in W^{1,p}(\mathbb{R}^N). \end{cases} \tag{2.2}$$

We denote the ground state of (2.2) by $w_{m,\mu,a}$. More precisely,

$$w_{m,\mu,a} := \left(\frac{\lambda}{\mu}\right)^{\frac{1}{m-p}} U_m\left(\lambda^{\frac{1}{p}} x\right), \tag{2.3}$$

where

$$\lambda := \mu^{-\frac{p^2}{N(m-p)-p^2}} \|U_m\|_p^{\frac{p^2(m-p)}{N(m-p)-p^2}} a^{-\frac{p(m-p)}{N(m-p)-p^2}}.$$

Then, we can define

$$\beta_{m,\mu,a,N,r} = \frac{1}{p} \inf_{h \in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla h|^p \mathrm{d}x}{\int_{\mathbb{R}^N} |w_{m,\mu,a}|^r |h|^p \mathrm{d}x}. \tag{2.4}$$

From [32], we can obtain the positive ground state solution of (2.1) as well. Moreover, some properties of the ground state solution are also presented in [32].

Define the energy functional associated to (2.2) as

$$J_{m,\mu}(u) := \frac{1}{p}\|\nabla u\|_p^p - \frac{\mu}{m}\|u\|_m^m, \tag{2.5}$$

and the corresponding Pohozaev identity $\mathcal{P}_{m,\mu,a}$ is defined as

$$\mathcal{P}_{m,\mu,a} := \left\{u \in W^{1,p}(\mathbb{R}^N) : \|\nabla u\|_p^p - \mu\delta_m\|u\|_m^m = 0\right\}.$$

For any $t \in \mathbb{R}^+$ and $u \in W^{1,p}(\mathbb{R}^N)$, define

$$(t \star u)(x) := t^{\frac{N}{p}} u(tx).$$

Then we have

$$[t \star u(x), t \star v(x)] = [t^{\frac{N}{p}} u(tx), t^{\frac{N}{p}} u(tx)] \in \mathcal{S}_{\|u\|_p^p} \times \mathcal{S}_{\|v\|_p^p}.$$

Lemma 2.1. *Let $N \geq 2$, $m \in (\frac{p^2+p}{N}, p^*)$. Then $w_{m,\mu,a} \in \mathcal{P}_{m,\mu,a}$ and*

$$m_{m,\mu,a} := J_{m,\mu}(w_{m,\mu,a}) = \inf_{u \in \mathcal{S}_a} \max_{t>0} J_{m,\mu}(t \star u) = \inf_{u \in \mathcal{P}_{m,\mu,a}} J_{m,\mu}(u),$$

where $w_{m,\mu,a}$ is the positive ground state solution to (2.2).

Proof. The proof can be found in [32, Section 5]. □

Now, we can derive from a direct calculation that

$$\begin{aligned} m_{m,\mu,a} &= J_{m,\mu}(w_{m,\mu,a}) = \frac{1}{p}\|\nabla w_{m,\mu,a}\|_p^p - \frac{\mu}{m}\|w_{m,\mu,a}\|_m^m \\ &= \left(\frac{1}{p} - \frac{1}{m\delta_m}\right)\|\nabla w_{m,\mu,a}\|_m^m \\ &= \left(\frac{1}{p} - \frac{1}{m\delta_m}\right)\|U_m\|_p^{\frac{p[mp-N(m-p)]}{N(m-p)-p^2}} \\ &\quad \cdot \|\nabla U_m\|_p^p \mu^{-\frac{p^2}{N(m-p)-p^2}} a^{\frac{N(m-p)-mp}{N(m-p)-p^2}} \\ &= -\frac{N(m-p)-p^2}{p[N(m-p)-mp]}\|U_m\|_p^{\frac{p^2(m-p)}{N(m-p)-p^2}} \mu^{-\frac{p^2}{N(m-p)-p^2}} a^{\frac{N(m-p)-mp}{N(m-p)-p^2}}. \end{aligned} \tag{2.6}$$

Then we define

$$\begin{aligned} b_{m_1,m_2,\mu_1,\mu_2,a} := &\left[\frac{N(m_1-p)-p^2}{N(m_1-p)-m_1p} \cdot \frac{N(m_2-p)-m_2p}{N(m_2-p)-p^2}\right]^{\frac{N(m_2-p)-p^2}{N(m_2-p)-m_2p}} \\ &\cdot \|U_m\|_p^{\frac{p^2(m_1-p)}{N(m_1-p)-p^2} \cdot \frac{N(m_2-p)-p^2}{N(m_2-p)-m_2p}} \|U_{m_2}\|_p^{-\frac{p^2(m_2-p)}{N(m_2-p)-m_2p}} \\ &\cdot \mu_1^{-\frac{p^2}{N(m_1-p)-p^2} \cdot \frac{N(m_2-p)-p^2}{N(m_2-p)-m_2p}} \mu_2^{\frac{p}{N(m_2-p)-m_2p}} \\ &\cdot a^{\frac{N(m_1-p)-m_1p}{N(m_1-p)-p^2} \cdot \frac{N(m_2-p)-p^2}{N(m_2-p)-m_2p}}. \end{aligned} \tag{2.7}$$

It is easy to check that the semi-trivial solutions of (1.1) satisfy

$$J_\beta[u,0]=\frac{1}{p}\|\nabla u\|_p^p-\frac{\mu_1}{m_1}\|u\|_{m_1}^{m_1}=J_{m_1,\mu_1}(u),$$

and

$$J_\beta[0,v]=\frac{1}{p}\|\nabla v\|_p^p-\frac{\mu_2}{m_2}\|v\|_{m_2}^{m_2}=J_{m_2,\mu_2}(v).$$

Remark 2.2. By the definition of $m_{m_1,\mu_1,a}, m_{m_2,\mu_2,a}$, we have

$$J_\beta[0,w_{m_2,\mu_2,b}]=m_{m_2,\mu_2,b}<\text{(resp. }=,>)\,m_{m_1,\mu_1,a}=J_\beta[w_{m_1,\mu_1,a},0]$$

if and only if

$$b>\text{(resp. }=,<)\,b_{m_1,m_2,\mu_1,\mu_2,a}.$$

3. PRELIMINARIES

In this section, we introduce some preliminary results. Firstly, let us recall the famous Gagliardo–Nirenberg inequality.

Lemma 3.1 ([1, 25])**.** *For every $m\in(p,p^*)$, there exists a sharp constant $C_{N,m}>0$ such that*

$$\|u\|_m\le C_{N,m}\|\nabla u\|_p^{\delta_m}\|u\|_p^{1-\delta_m},\quad \forall u\in W^{1,p}(\mathbb{R}^N),\tag{3.1}$$

where $\delta_m=\frac{N}{p}-\frac{N}{m}$.

Next, we introduce some properties of the Pohozaev manifold.

Lemma 3.2. *Let $[u,v]\in\mathcal{W}$ be a weak solution of* (1.1)*, then the following identity holds:*

$$\begin{aligned}&\frac{N-p}{p}\int\limits_{\mathbb{R}^N}(|\nabla u|^p+|\nabla v|^p)+\frac{N}{p}\int\limits_{\mathbb{R}^N}(\lambda_1|u|^p+\lambda_2|v|^p)\\&=N\int\limits_{\mathbb{R}^N}\left(\frac{\mu_1}{m_1}|u|^{m_1}+\frac{\mu_2}{m_2}|v|^{m_2}\right)+N\beta\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}.\end{aligned}\tag{3.2}$$

Proof. By the regularity of the elliptic equation (see, e.g., [10]), $u, v \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$. Let

$$\mathcal{L}_i(s,\xi) := \frac{1}{p}|\xi|^p + \frac{\lambda_i}{p}|s|^p - \frac{\mu_i}{m_i}|s|^{m_i}, \quad i = 1, 2,$$

and

$$f_1(x) := \beta r_1 |u(x)|^{r_1-2}u(x)|v(x)|^{r_2}, \quad f_2(x) := \beta r_2 |u(x)|^{r_1}|v(x)|^{r_2-2}v(x).$$

Then it follows from [13, Lemma 1] that, for any $h \in C^1_0(\mathbb{R}^N, \mathbb{R}^N)$, we have

$$\begin{aligned} &\sum_{i,j=1}^{N} \int_{\mathbb{R}^N} D_i h_j D_{\xi_i}\mathcal{L}_1(u,\nabla u) D_j u \mathrm{d}x - \int_{\mathbb{R}^N} (\mathrm{div} h)\mathcal{L}_1(u,\nabla u)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} (h\cdot\nabla u) f_1 \mathrm{d}x, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\sum_{i,j=1}^{N} \int_{\mathbb{R}^N} D_i h_j D_{\xi_i}\mathcal{L}_2(v,\nabla v) D_j v \mathrm{d}x - \int_{\mathbb{R}^N} (\mathrm{div} h)\mathcal{L}_2(v,\nabla v)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} (h\cdot\nabla v) f_2 \mathrm{d}x. \end{aligned} \tag{3.4}$$

Now we can choose $\phi \in C^1_0(\mathbb{R}^N, \mathbb{R})$ such that $\phi(x) \equiv 1$ for $|x| \le 1$, $\phi \equiv 0$ for $|x| \ge 2$ and $0 \le \phi(x) \le 1$. Define $\phi_k(x) := \phi(\frac{x}{k})$. Taking $h(x) = \phi_k(x)x$ respectively in (3.3) and (3.4), we have that

$$\begin{aligned} &\sum_{i,j=1}^{N} \int_{\mathbb{R}^N} D_i\phi\left(\frac{x}{k}\right)\frac{x_j}{k}|\nabla u|^{p-2}D_i u D_j u \mathrm{d}x + \int_{\mathbb{R}^N} \phi\left(\frac{x}{k}\right)|\nabla u|^p \mathrm{d}x \\ &- \int_{\mathbb{R}^N} \left[(\nabla\phi)\left(\frac{x}{k}\right)\cdot\frac{x}{k}\right]\left[|\nabla u|^p + \frac{\lambda_1}{p}|u|^p - \frac{\mu_1}{m_1}|u|^{m_1}\right]\mathrm{d}x \\ &- N\int_{\mathbb{R}^N} \phi\left(\frac{x}{k}\right)\left[\frac{1}{p}|\nabla u|^p + \frac{\lambda_1}{p}|u|^p - \frac{\mu_1}{m_1}|u|^{m_1}\right]\mathrm{d}x \\ &= \beta\int_{\mathbb{R}^N} \left[\phi\left(\frac{x}{k}\right)x\cdot\nabla u\right]\left[r_1|u|^{r_1-2}u|v|^{r_2}\right]\mathrm{d}x, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} D_i\phi\left(\frac{x}{k}\right)\frac{x_j}{k}|\nabla v|^{p-2}D_ivD_jv\mathrm{d}x+\int_{\mathbb{R}^N}\phi\left(\frac{x}{k}\right)|\nabla v|^p\mathrm{d}x\\
&-\int_{\mathbb{R}^N}\left[(\nabla\phi)\left(\frac{x}{k}\right)\cdot\frac{x}{k}\right]\left[\frac{1}{p}|\nabla v|^p+\frac{\lambda_2}{p}|v|^p-\frac{\mu_2}{m_2}|v|^{m_2}\right]\mathrm{d}x\\
&-N\int_{\mathbb{R}^N}\phi\left(\frac{x}{k}\right)\left[|\nabla v|^p+\frac{\lambda_2}{p}|v|^p-\frac{\mu_2}{m_2}|v|^{m_2}\right]\mathrm{d}x\\
&=\beta\int_{\mathbb{R}^N}\left[\phi\left(\frac{x}{k}\right)x\cdot\nabla v\right]\left[r_2|v|^{r_2-2}v|u|^{r_1}\right]\mathrm{d}x.
\end{aligned}\tag{3.6}$$

By the Lebesgue's Dominated Convergence theorem, as $k\to+\infty$, we can derive

$$\int_{\mathbb{R}^N} D_i\phi\left(\frac{x}{k}\right)\frac{x_j}{k}|\nabla z|^{p-2}D_izD_jz\mathrm{d}x\to 0,\quad z=\{u,v\},$$

$$\int_{\mathbb{R}^N}\phi\left(\frac{x}{k}\right)|\nabla z|^p\mathrm{d}x\to\int_{\mathbb{R}^N}|\nabla z|^p\mathrm{d}x,\quad z=\{u,v\},$$

$$\int_{\mathbb{R}^N}\left[(\nabla\phi)\left(\frac{x}{k}\right)\cdot\frac{x}{k}\right]\left[|\nabla z|^p+\frac{\lambda_i}{p}|z|^p-\frac{\mu_i}{m_i}|z|^{m_i}\right]\mathrm{d}x\to 0,\ (i,z)=\{(1,u),(2,v)\},$$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\phi\left(\frac{x}{k}\right)\left[\frac{1}{p}|\nabla z|^p+\frac{\lambda_i}{p}|z|^p-\frac{\mu_i}{m_i}|z|^{m_i}\right]\mathrm{d}x\\
&\qquad\to\int_{\mathbb{R}^N}\left[\frac{1}{p}|\nabla z|^p+\frac{\lambda_i}{p}|z|^p-\frac{\mu_i}{m_i}|z|^{m_i}\right]\mathrm{d}x,\ (i,z)=\{(1,u),(2,v)\},
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{N}\left[\int_{\mathbb{R}^N}\phi\left(\frac{x}{k}\right)x_iD_iu\left(r_1|u|^{r_1-2}u|v|^{r_2}\right)\mathrm{d}x+\int_{\mathbb{R}^N}\phi\left(\frac{x}{k}\right)x_iD_iv\left(r_2|u|^{r_1}|v|^{r_2-2}v\right)\mathrm{d}x\right]\\
&=\sum_{i=1}^{N}\int_{\mathbb{R}^N}\phi\left(\frac{x}{k}\right)\left(|u|^{r_1}|v|^{r_2}\right)_{x_i}x_i\mathrm{d}x\\
&=N\int_{\mathbb{R}^N}\phi\left(\frac{x}{k}\right)\left(|u|^{r_1}|v|^{r_2}\right)\mathrm{d}x+\sum_{i=1}^{N}\int_{\mathbb{R}^N}D_i\phi\left(\frac{x}{k}\right)\frac{x_i}{k}\left(|u|^{r_1}|v|^{r_2}\right)\mathrm{d}x\\
&\to N\int_{\mathbb{R}^N}\left(|u|^{r_1}|v|^{r_2}\right)\mathrm{d}x.
\end{aligned}$$

Finally, combining (3.5) and (3.6), one obtains

$$\frac{N-p}{p}\int\limits_{\mathbb{R}^N}(|\nabla u|^p+|\nabla v|^p)+\frac{N}{p}\int\limits_{\mathbb{R}^N}(\lambda_1|u|^p+\lambda_2|v|^p)$$
$$=N\int\limits_{\mathbb{R}^N}\left(\frac{\mu_1}{m_1}|u|^{m_1}+\frac{\mu_2}{m_2}|v|^{m_2}\right)+N\beta\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}. \qquad \square$$

Lemma 3.3. *If* $[u,v]$ *is a solution of* (1.1) *for some* $\lambda_1,\lambda_2\in\mathbb{R}$*, then* $[u,v]\in\mathcal{P}_\beta$*.*

Proof. Since $[u,v]$ is a solution of (1.1), it follows that

$$\int\limits_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x+\lambda_1\int\limits_{\mathbb{R}^N}|u|^p\mathrm{d}x=\mu_1\int\limits_{\mathbb{R}^N}|u|^{m_1}\mathrm{d}x+\beta r_1\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x,$$

and

$$\int\limits_{\mathbb{R}^N}|\nabla v|^p\mathrm{d}x+\lambda_2\int\limits_{\mathbb{R}^N}|v|^p\mathrm{d}x=\mu_2\int\limits_{\mathbb{R}^N}|v|^{m_2}\mathrm{d}x+\beta r_2\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x.$$

On the other hand, it follows from Lemma 3.2 that

$$\frac{N-p}{p}\int\limits_{\mathbb{R}^N}(|\nabla u|^p+|\nabla v|^p)\mathrm{d}x+\frac{N}{p}\int\limits_{\mathbb{R}^N}(\lambda_1|u|^p+\lambda_2|v|^p)\mathrm{d}x$$
$$=N\int\limits_{\mathbb{R}^N}\left(\frac{\mu_1}{m_1}|u|^{m_1}+\frac{\mu_2}{m_2}|v|^{m_2}\right)\mathrm{d}x+N\beta\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x.$$

Combining the above three formulas, one obtains

$$\begin{aligned}P_\beta[u,v]=&\int\limits_{\mathbb{R}^N}(|\nabla u|^p+|\nabla v|^p)\mathrm{d}x-\mu_1\delta_{m_1}\int\limits_{\mathbb{R}^N}|u|^{m_1}\mathrm{d}x-\mu_2\delta_{m_2}\int\limits_{\mathbb{R}^N}|v|^{m_2}\mathrm{d}x\\&-(r_1+r_2)\delta_{r_1+r_2}\beta\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x\\=&\,0.\end{aligned} \qquad \square$$

Now, we define $\Psi^\beta_{[u,v]}:\mathbb{R}^+\to\mathbb{R}$ by

$$\begin{aligned}\Psi^\beta_{[u,v]}(t):=&\,J_\beta[t\star u,t\star v]\\=&\,\frac{1}{p}[\|\nabla u\|_p^p+\|\nabla v\|_p^p]t^p-\frac{\mu_1}{m_1}\|u\|_{m_1}^{m_1}t^{m_1\delta_{m_1}}\\&-\frac{\mu_2}{m_2}\|v\|_{m_2}^{m_2}t^{m_2\delta_{m_2}}-\beta\left(\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x\right)t^{(r_1+r_2)\delta_{r_1+r_2}}.\end{aligned}$$

Lemma 3.4. *Let $[u,v] \in \mathcal{W} \setminus \{[0,0]\}$, then for any $t > 0$, $\Psi'_{[u,v]}(t) = 0$ if and only if $[t \star u, t \star v] \in \mathcal{P}$.*

Proof. By direct calculation, we can obtain

$$\begin{aligned}
(\Psi^{\beta}_{[u,v]})'(t) &= \frac{d}{dt} J_\beta[t\star u, t\star v] \\
&= \left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) t^{p-1} - \mu_1 \delta_{m_1} \|u\|_{m_1}^{m_1} t^{m_1\delta_{m_1}-1} \\
&\quad - \mu_2 \delta_{m_2} \|v\|_{m_2}^{m_2} t^{m_2\delta_{m_2}-1} \\
&\quad - (r_1+r_2)\delta_{r_1+r_2}\beta \left(\int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} \mathrm{d}x \right) t^{(r_1+r_2)\delta_{r_1+r_2}-1} \\
&= \frac{P_\beta[t\star u, t\star v]}{t}, \quad \forall t > 0.
\end{aligned} \tag{3.7}$$

Hence, we can obtain that $(\Psi^{\beta}_{[u,v]})'(t) = 0 \Leftrightarrow P_\beta[t\star u, t\star v] = 0$ for any $t > 0$. □

Lemma 3.5. *$\mathcal{P}_\beta$ is a C^1 manifold of codimension 1 in $\mathcal{W}$.*

Proof. For any $[u,v] \in \mathcal{P}_\beta$, suppose by contradiction that $P'_\beta[u,v] = 0$, then $[u,v]$ satisfies the following system:

$$\begin{cases}
-p\Delta_p u - \mu_1 m_1 \delta_{m_1} |u|^{m_1-2} u \\
\qquad - \beta r_1 (r_1+r_2)\delta_{r_1+r_2} \int\limits_{\mathbb{R}^N} |u|^{r_1-2}|v|^{r_2} u \,\mathrm{d}x = 0, \\
-p\Delta_p v - \mu_2 m_2 \delta_{m_2} |v|^{m_2-2} v \\
\qquad - \beta r_2 (r_1+r_2)\delta_{r_1+r_2} \int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2-2} v \,\mathrm{d}x = 0.
\end{cases} \tag{3.8}$$

Similar to Lemma 3.2, we get that

$$\begin{aligned}
\frac{p}{p^*} \int\limits_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) &= \int\limits_{\mathbb{R}^N} (\mu_1 \delta_{m_1} |u|^{m_1} + \mu_2 \delta_{m_2} |v|^{m_2}) \\
&\quad + \beta (r_1+r_2)\delta_{r_1+r_2} \int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}.
\end{aligned} \tag{3.9}$$

Combining (3.9) with $P_\beta[u,v] = 0$, we deduce that

$$\frac{p}{p^*} \int\limits_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p) = \int\limits_{\mathbb{R}^N} (|\nabla u|^p + |\nabla v|^p),$$

which implies that $\|\nabla u\|_p^p = 0$ and $\|\nabla v\|_p^p = 0$. Then $[u,v] = [0,0]$ in $\mathcal{W}$. This is a contradiction with $[0,0] \neq [u,v] \in \mathcal{P}_\beta$. □

Lemma 3.6. *Suppose that $[u,v]$ is a critical point of $J_\beta\big|_{\mathcal{P}_\beta^{(a,b)}}$, then there exist some $\lambda_1,\lambda_2\in\mathbb{R}$ such that*

$$J'_\beta[u,v]+\lambda_1[u,0]+\lambda_2[0,v]=0.$$

Proof. For any $[u,v]\in\mathcal{P}_\beta^{(a,b)}$, by direct calculation, we have

$$\begin{aligned}(\Psi^\beta_{[u,v]})''(1)&=(p-1)(\|\nabla u\|_p^p+\|\nabla v\|_p^p)\\&\quad-(m_1\delta_{m_1}-1)\mu_1\delta_{m_1}\|u\|_{m_1}^{m_1}-(m_2\delta_{m_2}-1)\mu_2\delta_{m_2}\|v\|_{m_2}^{m_2}\\&\quad-[(r_1+r_2)\delta_{r_1+r_2}-1](r_1+r_2)\delta_{r_1+r_2}\beta\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x.\end{aligned}\tag{3.10}$$

On the other hand, by $P_\beta[u,v]=0$, we have

$$\begin{aligned}\|\nabla u\|_p^p+\|\nabla v\|_p^p&=\mu_1\delta_{m_1}\|u\|_{m_1}^{m_1}+\mu_2\delta_{m_2}\|v\|_{m_2}^{m_2}\\&\quad+(r_1+r_2)\delta_{r_1+r_2}\beta\int\limits_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x.\end{aligned}\tag{3.11}$$

Combining (3.10) and (3.11), we can obtain that $(\Psi^\beta_{[u,v]})''(1)<0$.

From Lemma 3.5, we have that $P'_\beta[u,v]\neq 0$ and there exist $\lambda_1,\lambda_2,\nu\in\mathbb{R}$ such that

$$J'_\beta[u,v]+\lambda_1[u,0]+\lambda_2[0,v]+\nu P'_\beta[u,v]=0.\tag{3.12}$$

Now, we just need to show that $\nu=0$. The functional of equation (3.12) is defined as

$$\Phi_\beta[u,v]:=J_\beta[u,v]+\lambda_1\|u\|_p^p+\lambda_2\|v\|_p^p+\nu P_\beta[u,v].$$

Similar to Lemma 3.4, $[u,v]$ satisfies a Pohozaev identity which is in the form of $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\beta[t\star u,t\star v]\big|_{t=1}=0$. Moreover, through direct computation, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\beta[t\star u,t\star v]\big|_{t=1}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left[J_\beta[t\star u,t\star v]+\lambda_1\|u\|_p^p+\lambda_2\|v\|_p^p+\nu P_\beta[t\star u,t\star v]\right]\big|_{t=1}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\Psi^\beta_{[u,v]}(t)+\nu t(\Psi^\beta_{[u,v]})'(t)\right]\Big|_{t=1}\\&=(1+\nu)(\Psi^\beta_{[u,v]})'(1)+\nu(\Psi^\beta_{[u,v]})''(1).\end{aligned}$$

Note that $(\Psi^\beta_{[u,v]})''(1)<0$. Hence, we can deduce that $\nu=0$. □

Lemma 3.7. *Let $\frac{p^2}{N}+p<m_1,m_2,r_1+r_2<p^*$. For every $[0,0]\neq[u,v]\in\mathcal{D}_a\times\mathcal{D}_b$, there exists a unique $t=t_{[u,v]}>0$ such that $[t\star u,t\star v]\in\mathcal{P}_\beta^{(a,b)}$. Moreover, $t_{[u,v]}<$ (resp. $=,>$) 1 if and only if $P_\beta[u,v]<$ (resp. $=,>$) 0.*

Proof. For any $[u,v] \in \mathcal{D}_a \times \mathcal{D}_b$ and $t > 0$, we have that $\|t \star u\|_p^p = \|u\|_p^p \le a$ and $\|t \star v\|_p^p = \|v\|_p^p \le b$. So we just need to verify that there exists a unique t such that $P_\beta[t \star u, t \star v] = 0$. Since $\frac{p^2}{N} + p < m_1, m_2, r_1 + r_2 < p^*$, we have that

$$m_1\delta_{m_1},\ m_2\delta_{m_2},\ (r_1+r_2)\delta_{r_1+r_2} > p.$$

Then it follows from (3.7) that there exists just one point $t = t_{[u,v]}$ such that $(\Psi^\beta_{[u,v]})'(t) = 0$, and

$$(\Psi^\beta_{[u,v]})'(s) > 0,\ \forall s \in (0, t_{[u,v]}), \quad (\Psi^\beta_{[u,v]})'(s) < 0,\ \forall s \in (t_{[u,v]}, +\infty),$$

which means that $P_\beta[t \star u, t \star v] = 0$ by Lemma 3.4. And t is the maximum critical point of $\Psi^\beta_{[u,v]}(t)$.

Moreover, since $P_\beta[u,v] = (\Psi^\beta_{[u,v]})'(1)$, we can obtain that

$$\begin{aligned} P_\beta[u,v] < (\text{resp. } =, >)\, 0 &\Leftrightarrow (\Psi^\beta_{[u,v]})'(1) < (\text{resp. } =, >)\, 0 \\ &\Leftrightarrow t_{[u,v]} < (\text{resp. } =, >)\, 1. \end{aligned}$$

□

Denote the Schwartz symmetrization of u as u^*. We can derive the following result.

Lemma 3.8. *Let $\frac{p^2}{N} + p < m_1, m_2, r_1 + r_2 < p^*$. For any $[u,v] \in \mathcal{P}^{(a,b)}_\beta$, there exists a unique $t = t_{[u^*,v^*]} \in (0,1]$ such that $[t\star u^*, t\star v^*] \in \mathcal{P}^{(a,b)}_\beta$ and $J_\beta[t\star u^*, t\star v^*] \le J_\beta[u,v]$.*

Proof. For any $[u,v] \in \mathcal{P}^{(a,b)}_\beta$, we have $[u,v] \ne [0,0]$. So by $\|u^*\|_p^p = \|u\|_p^p$ and $\|v^*\|_p^p = \|v\|_p^p$, we see that $[u^*, v^*] \in \mathcal{D}_a \times \mathcal{D}_b \setminus \{[0,0]\}$. Then by Lemma 3.7, there exists a unique $t = t_{[u^*,v^*]} > 0$ such that $[t \star u^*, t \star v^*] \in \mathcal{P}^{(a,b)}_\beta$.

By the properties of rearrangement, we also have that

$$\|\nabla u^*\|_p^p \le \|\nabla u\|_p^p, \quad \|\nabla v^*\|_p^p \le \|\nabla v\|_p^p, \quad \int_{\mathbb{R}^N} |u^*|^{r_1}|v^*|^{r_2}\mathrm{d}x \ge \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}\mathrm{d}x.$$

Thus, we have $P_\beta[u^*,v^*] \le P_\beta[u,v] = 0$ and $J_\beta[u^*,v^*] \le J_\beta[u,v]$. By Lemma 3.7 again, we can get the fact that $t = t_{[u^*,v^*]} \le 1$.

Moreover, we can deduce that

$$\begin{aligned} \max_{s>0} J_\beta[s\star u^*, s\star v^*] &= J_\beta[t\star u^*, t\star v^*] = J_\beta[(t\star u)^*, (t\star v)^*] \\ &\le J_\beta[t\star u, t\star v] \le \max_{s>0} J_\beta[s\star u, s\star v] = J_\beta[u,v]. \end{aligned}$$

□

Lemma 3.9. *Let $1 < p < N$, $\frac{p^2}{N} + p < m_1, m_2, r_1 + r_2 < p^*$. Then for any $[u,v] \in \mathcal{P}_\beta$, there exists a constant $C_0 > 0$ such that*

$$J_\beta[u,v] \ge C_0\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right).$$

Proof. Since $p + \frac{p^2}{N} < m_1, m_2, r_1 + r_2 < p^*$, we have

$$m_1\delta_{m_1},\, m_2\delta_{m_2},\, (r_1+r_2)\delta_{r_1+r_2} > p,$$

Define

$$\kappa := \max\left\{\frac{1}{m_1\delta_{m_1}}, \frac{1}{m_2\delta_{m_2}}, \frac{1}{(r_1+r_2)\delta_{r_1+r_2}}\right\} < \frac{1}{p}.$$

Hence, for any $[u,v] \in \mathcal{P}_\beta$, we have

$$\begin{aligned}\kappa\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) &= \kappa\Bigg[\mu_1\delta_{m_1}\|u\|_{m_1}^{m_1} + \mu_2\delta_{m_2}\|v\|_{m_2}^{m_2} \\ &\qquad + \beta(r_1+r_2)\delta_{r_1+r_2}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x\Bigg] \\ &\geq \frac{\mu_1}{m_1}\|u\|_{m_1}^{m_1} + \frac{\mu_2}{m_2}\|v\|_{m_2}^{m_2} + \beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x.\end{aligned}$$

Thus, one can obtain that

$$\begin{aligned}J_\beta[u,v] &= \frac{1}{p}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) - \frac{\mu_1}{m_1}\|u\|_{m_1}^{m_1} - \frac{\mu_2}{m_2}\|v\|_{m_2}^{m_2} - \beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x \\ &\geq \left(\frac{1}{p} - \kappa\right)\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) =: C_0\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right).\end{aligned}$$ □

From Lemma 3.9 above, we can deduce the following result.

Corollary 3.10. *Let $1 < p < N$, $p + \frac{p^2}{N} < m_1, m_2, r_1 + r_2 < p^*$. For any $a, b > 0$, $J_\beta\big|_{\mathcal{P}_\beta^{(a,b)}}$ is coercive.*

Lemma 3.11. *Let $1 < p < N$, $p + \frac{p^2}{N} < m_1, m_2, r_1 + r_2 < p^*$. For any $(a,b) \neq (0,0)$, there exists a constant $R_0 > 0$ such that*

$$\inf_{[u,v]\in\mathcal{P}_\beta^{(a,b)}}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) \geq R_0.$$

Proof. Suppose by contradiction that for any $\varepsilon > 0$ small enough, there exists $[u_0, v_0] \in \mathcal{P}_\beta^{(a,b)}$ such that

$$\left(\|\nabla u_0\|_p^p + \|\nabla v_0\|_p^p\right) < \varepsilon.$$

By the Gagliardo–Nirenberg inequality (3.1), Hölder inequality and Young inequality, for any $[u,v] \in \mathcal{W}$, we have

$$\begin{aligned}\delta_{m_1}\mu_1\|u\|_{m_1}^{m_1} &\leq \delta_{m_1}\mu_1 C_{N,m_1}^{m_1} a^{m_1(1-\delta_{m_1})}\|\nabla u\|_p^{m_1\delta_{m_1}} \\ &\leq \delta_{m_1}\mu_1 C_{N,m_1}^{m_1} a^{m_1(1-\delta_{m_1})}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right)^{\frac{m_1\delta_{m_1}}{p}} \\ &=: R_1\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right)^{\frac{m_1\delta_{m_1}}{p}},\end{aligned}$$

$$\begin{aligned}\delta_{m_2}\mu_2\|v\|_{m_2}^{m_2} &\le \delta_{m_2}\mu_2 C_{N,m_2}^{m_2} b^{m_2(1-\delta_{m_2})}\|\nabla v\|_p^{m_2\delta_{m_2}} \\ &\le \delta_{m_2}\mu_2 C_{N,m_2}^{m_2} b^{m_2(1-\delta_{m_2})}\left(\|\nabla u\|_p^p+\|\nabla v\|_p^p\right)^{\frac{m_2\delta_{m_2}}{p}} \\ &=: R_2\left(\|\nabla u\|_p^p+\|\nabla v\|_p^p\right)^{\frac{m_2\delta_{m_2}}{p}},\end{aligned}$$

and

$$\begin{aligned}&\beta(r_1+r_2)\delta_{r_1+r_2}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}\mathrm{d}x \\ &\le \beta(r_1+r_2)\delta_{r_1+r_2}\|u\|_{r_1+r_2}^{r_1}\|v\|_{r_1+r_2}^{r_2} \\ &\le \beta(r_1+r_2)\delta_{r_1+r_2}C_{N,r_1+r_2}^{r_1+r_2}a^{r_1(1-\delta_{r_1+r_2})}b^{r_2(1-\delta_{r_1+r_2})} \\ &\quad\cdot\left(\|\nabla u\|_p^p+\|\nabla v\|_p^p\right)^{\frac{(r_1+r_2)\delta_{r_1+r_2}}{p}} \\ &=: R_3\left(\|\nabla u\|_p^p+\|\nabla v\|_p^p\right)^{\frac{(r_1+r_2)\delta_{r_1+r_2}}{p}}.\end{aligned}$$

Then we can obtain that

$$\begin{aligned}P_\beta[u_0,v_0] \ge (\|\nabla u_0\|_p^p+\|\nabla v_0\|_p^p) &- R_1\left(\|\nabla u_0\|_p^p+\|\nabla v_0\|_p^p\right)^{\frac{m_1\delta_{m_1}}{p}} \\ &- R_2\left(\|\nabla u_0\|_p^p+\|\nabla v_0\|_p^p\right)^{\frac{m_2\delta_{m_2}}{p}} \\ &- R_3\left(\|\nabla u_0\|_p^p+\|\nabla v_0\|_p^p\right)^{\frac{(r_1+r_2)\delta_{r_1+r_2}}{p}}.\end{aligned}$$

By $p+\frac{p^2}{N}<m_1,m_2,r_1+r_2<p^*$, we have

$$\frac{m_1\delta_{m_1}}{p},\ \frac{m_2\delta_{m_2}}{p},\ \frac{(r_1+r_2)\delta_{r_1+r_2}}{p}>1.$$

Since $\left(\|\nabla u_0\|_p^p+\|\nabla v_0\|_p^p\right)$ is small enough, we can derive that

$$P_\beta[u_0,v_0]>0.$$

This is a contradiction. □

Then from Lemma 3.11 and Lemma 3.9, we have the following result.

Corollary 3.12. *Let* $1<p<N$, $p+\frac{p^2}{N}<m_1,m_2,r_1+r_2<p^*$ *and* $M_\beta(a,b)$ *be defined in* (1.6). *Then* $M_\beta(a,b)>0$.

Define

$$\mathcal{W}_r := \left\{[u,v]\in\mathcal{W} : [u(x),v(x)] = [u(|x|),v(|x|)],\ x\in\mathbb{R}^N\right\},$$

and

$$M_\beta^{\mathrm{rad}}(a,b) := \inf_{[u,v]\in\mathcal{P}_\beta^{(a,b)}\cap\mathcal{W}_r} J_\beta[u,v].$$

Lemma 3.13. $M_\beta^{\mathrm{rad}}(a,b) = M_\beta(a,b)$.

Proof. It is clear that $M_\beta^{\mathrm{rad}}(a,b) \geq M_\beta(a,b)$ by the fact that $\mathcal{P}_\beta^{(a,b)} \cap \mathcal{W}_r \subset \mathcal{P}_\beta^{(a,b)}$.

On the other hand, for any $[u,v] \in \mathcal{P}_\beta^{(a,b)}$, we can deduce from Lemma 3.8 that there exists $t = t_{[u^*,v^*]}$ such that $[t \star u^*, t \star v^*] \in \mathcal{P}_\beta^{(a,b)}$ and satisfies

$$J_\beta[t \star u^*, t \star v^*] \leq J_\beta[u,v],$$

which implies that

$$M_\beta^{\mathrm{rad}}(a,b) \leq \inf_{[u,v]\in\mathcal{P}_\beta^{(a,b)}} J_\beta[t \star u^*, t \star v^*] \leq \inf_{[u,v]\in\mathcal{P}_\beta^{(a,b)}} J_\beta[u,v] = M_\beta(a,b). \qquad \square$$

4. PROOF OF THEOREM 1.1

In this section, we will prove the existence of normalized solution to (1.1)–(1.2). Firstly, we need to construct some conditions to rule out the semi-trivial solutions mentioned in Section 2.

Lemma 4.1. *Let* $1 < p < N$, $\mu > 0$, $a > 0$, $r > 0$ *and* $\beta_{p,\mu,a,N,r}$ *be defined in* (2.4). *Then it holds that* $\beta_{p,\mu,a,N,r} > 0$.

Proof. Let $w = w_{p,\mu,a}$ be defined in (2.2). Since $1 < p < N$, we can derive that $w \in L^\infty(\mathbb{R}^N)$ by the Moser iteration technique (see, e.g., [15, 22]). Then it follows from Hölder's inequality and the critical Sobolev inequality that

$$\int_{\mathbb{R}^N} |w|^r |h|^p \mathrm{d}x \leq \|w\|^r_{\frac{Nr}{p}} \|h\|^p_{p^*} \leq S_N^{-1} \|w\|^r_{\frac{Nr}{p}} \|\nabla h\|^p_p, \quad \forall h \in W^{1,p}(\mathbb{R}^N),$$

which implies that

$$\beta_{p,\mu,a,N,r} = \frac{1}{p} \inf_{h\in W^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla h|^p \mathrm{d}x}{\int_{\mathbb{R}^N} |w|^r |h|^p \mathrm{d}x} \geq S_N \|w\|^{-r}_{\frac{Nr}{p}} > 0. \qquad \square$$

Lemma 4.2. *Let* $1 < p < N$ *and* $p + \frac{p^2}{N} < m_1, m_2, r_1 + r_2 < p^*$.

(i) *If* $1 < r_2 < p$ *or* $r_2 = p$ *with* $\beta > \beta_{m_1,\mu_1,a,N,r_1}$, *then* $M_\beta(a,b) < m_{m_1,\mu_1,a}$,
(ii) *If* $1 < r_1 < p$ *or* $r_1 = p$ *with* $\beta > \beta_{m_2,\mu_2,b,N,r_2}$, *then* $M_\beta(a,b) < m_{m_2,\mu_2,b}$.

Proof. This proof is similar to the proof in [21, Lemma 7.3]. We only need to show (i), and the proof of (ii) is completely analogous. For the sake of convenience, we denote $w_{m_1,\mu_1,a}$ defined in (2.2) as w here.

For any $\bar{h} \in W^{1,p}(\mathbb{R}^N) \setminus \{0\}$, let $h := \frac{\bar{h}}{\|\bar{h}\|_p}$. So we have that $[w, sh] \in \mathcal{D}_a \times \mathcal{D}_b$ for any $|s| \leq b^{\frac{1}{p}}$. By Lemma 3.7, there exists a unique $t = t(s) > 0$ for which $[t(s) \star w, t(s) \star sh] \in \mathcal{P}_\beta^{(a,b)}$, and $t(s)$ satisfies

$$\begin{aligned}
&\|\nabla w\|_p^p + |s|^p \|\nabla h\|_p^p \\
&= \delta_{m_1}\mu_1 \|w\|_{m_1}^{m_1} t^{m_1\delta_{m_1}-p} + \delta_{m_2}\mu_2 \|h\|_{m_2}^{m_2} |s|^{m_2} t^{m_2\delta_{m_2}-p} \\
&\quad + (r_1+r_2)\delta_{r_1+r_2}\beta \left(\int_{\mathbb{R}^N} |w|^{r_1}|h|^{r_2} \mathrm{d}x \right) |s|^{r_2} t^{(r_1+r_2)\delta_{r_1+r_2}-p},
\end{aligned} \tag{4.1}$$

which implies that $t(0) = 1$ by the definition of w. Then it follows from the implicit function theorem and (4.1) that $t(s) \in C^1$ locally around $s = 0$ and

$$t'(s) = \frac{P_h(s)}{Q_h(s)},$$

where

$$\begin{aligned}
P_h(s) &:= p|s|^{p-2}s\|\nabla h\|_p^p - m_2\delta_{m_2}\mu_2 \|h\|_{m_2}^{m_2} |s|^{m_2-2} s t^{m_2\delta_{m_2}-p} \\
&\quad - r_2(r_1+r_2)\delta_{r_1+r_2}\beta \left(\int_{\mathbb{R}^N} |w|^{r_1}|h|^{r_2} \mathrm{d}x \right) |s|^{r_2-2} s t^{(r_1+r_2)\delta_{r_1+r_2}-p},
\end{aligned}$$

$$\begin{aligned}
Q_h(s) &:= (m_1\delta_{m_1}-p)\delta_{m_1}\mu_1 \|w\|_{m_1}^{m_1} t^{m_1\delta_{m_1}-p-1} \\
&\quad + (m_2\delta_{m_2}-p)\delta_{m_2}\mu_2 \|h\|_{m_2}^{m_2} |s|^{m_2} t^{m_2\delta_{m_2}-p-1} \\
&\quad + \big[(r_1+r_2)\delta_{r_1+r_2}-p\big](r_1+r_2)\delta_{r_1+r_2}\beta \\
&\qquad \cdot \left(\int_{\mathbb{R}^N} |w|^{r_1}|h|^{r_2} \mathrm{d}x \right) |s|^{r_2} t^{(r_1+r_2)\delta_{r_1+r_2}-p-1}.
\end{aligned}$$

Case 1. $1 < r_2 < p$. For $|s|$ small enough, we have that $t(s) = 1 + o(1)$,

$$P_h(s) = -r_2(r_1+r_2)\delta_{r_1+r_2}\beta \left(\int_{\mathbb{R}^N} |w|^{r_1}|h|^{r_2} \mathrm{d}x \right) |s|^{r_2-2} s(1+o(1)),$$

and

$$Q_h(s) = (m_1\delta_{m_1}-p)\delta_{m_1}\mu_1 \|w\|_{m_1}^{m_1}(1+o(1)) = (m_1\delta_{m_1}-p)\|\nabla w\|_p^p(1+o(1)).$$

Define

$$C_h := \frac{(r_1+r_2)\delta_{r_1+r_2}\beta \left(\int_{\mathbb{R}^N} |w|^{r_1}|h|^{r_2} \mathrm{d}x \right)}{(m_1\delta_{m_1}-p)\|\nabla w\|_p^p}.$$

So we can obtain

$$t'(s) = -r_2 C_h |s|^{r_2-2} s(1+o(1)).$$

Then by integrating the function $t'(\cdot)$ from 0 to s, we can obtain

$$t(s) = 1 - C_h |s|^{r_2}(1+o(1)),$$

and for any $\tau > 0$,

$$t^\tau(s) = 1 - \tau C_h |s|^{r_2}(1+o(1)).$$

Hence, combining $w \in \mathcal{P}_{m_1,\mu_1,a}$ and $r_2 < p < m_1, m_2$, we have

$$\begin{aligned}
&J_\beta[t(s)\star w, t(s)\star(sh)] - J_\beta[w,0] \\
&= \frac{1}{p}\|\nabla w\|_p^p (t^p(s)-1) + \frac{1}{p}\|\nabla h\|_p^p |s|^p t^p(s) - \frac{\mu_1}{m_1}\|w\|_{m_1}^{m_1}(t^{m_1\delta_{m_1}}-1) \\
&\quad - \frac{\mu_2}{m_2}\|h\|_{m_2}^{m_2}|s|^{m_2} t^{m_2\delta_{m_2}} - \beta\left(\int\limits_{\mathbb{R}^N} |w|^{r_1}|h|^{r_2}\mathrm{d}x\right)|s|^{r_2} t^{(r_1+r_2)\delta_{r_1+r_2}} \\
&= \left[\|\nabla w\|_p^p - \mu_1\delta_{m_1}\|w\|_{m_1}^{m_1}\right] C_h|s|^{r_2}(1+o(1)) + \frac{1}{p}\|\nabla h\|_p^p|s|^p t^p(s) \\
&\quad - \frac{\mu_2}{m_2}\|h\|_{m_2}^{m_2}|s|^{m_2} t^{m_2\delta_{m_2}} - \beta\left(\int\limits_{\mathbb{R}^N} |w|^{r_1}|h|^{r_2}\mathrm{d}x\right)|s|^{r_2} t^{(r_1+r_2)\delta_{r_1+r_2}} \\
&\le -\beta\left(\int\limits_{\mathbb{R}^N} |w|^{r_1}|h|^{r_2}\mathrm{d}x\right)|s|^{r_2}(1+o(1)) < 0,
\end{aligned}$$

which implies that for any $|s| < b^{\frac{1}{p}}$ small enough, we have

$$M_\beta(a,b) \le J_\beta[t(s)\star w, t(s)\star(sh)] < J_\beta[w,0] = m_{m_1,\mu_1,a}.$$

Case 2. $r_2 = p$. For $|s|$ small enough, we have that

$$P_h(s) = \left(p\|\nabla h\|_p^p - p\beta \int\limits_{\mathbb{R}^N} |w|^{r_1}|h|^p \mathrm{d}x\right)|s|^{p-2}s(1+o(1)),$$

and

$$Q_h(s) = (m_1\delta_{m_1} - p)\delta_{m_1}\mu_1\|w\|_{m_1}^{m_1}(1+o(1)) = (m_1\delta_{m_1}-p)\|\nabla w\|_p^p(1+o(1)).$$

Define

$$\bar{C}_h := \frac{p\|\nabla h\|_p^p - p\beta\int_{\mathbb{R}^N}|w|^{r_1}|h|^p\mathrm{d}x}{(m_1\delta_{m_1}-p)\|\nabla w\|_p^p}.$$

Similar to the analysis in Case 1, we have

$$t'(s) = p\bar{C}_h|s|^{p-2}s(1+o(1)), \quad t(s) = 1 + \bar{C}_h|s|^p(1+o(1)),$$

and

$$t^\tau(s) = 1 + \tau\bar{C}_h|s|^p(1+o(1)).$$

Hence, it follows from $\beta > \beta_{m_1,\mu_1,a,N,r_1}$, $w \in \mathcal{P}_{m_1,\mu_1,a}$ and $r_2 = p < m_1, m_2$ that

$$\begin{aligned}
&J_\beta[t(s) \star w, t(s) \star (sh)] - J_\beta[w,0] \\
&= \frac{1}{p}\|\nabla w\|_p^p(t^p(s)-1) + \frac{1}{p}\|\nabla h\|_p^p|s|^p t^p(s) - \frac{\mu_1}{m_1}\|w\|_{m_1}^{m_1}(t^{m_1\delta_{m_1}}-1) \\
&\quad - \frac{\mu_2}{m_2}\|h\|_{m_2}^{m_2}|s|^{m_2}t^{m_2\delta_{m_2}} - \beta\left(\int_{\mathbb{R}^N}|w|^{r_1}|h|^p\mathrm{d}x\right)|s|^p t^{(r_1+r_2)\delta_{r_1+r_2}} \\
&= \left[\|\nabla w\|_p^p - \mu_1\delta_{m_1}\|w\|_{m_1}^{m_1}\right]\bar{C}_h|s|^p(1+o(1)) + \frac{1}{p}\|\nabla h\|_p^p|s|^p(1+o(1)) \\
&\quad - \frac{\mu_2}{m_2}\|h\|_{m_2}^{m_2}|s|^{m_2}t^{m_2\delta_{m_2}} - \beta\left(\int_{\mathbb{R}^N}|w|^{r_1}|h|^p\mathrm{d}x\right)|s|^p(1+o(1)) \\
&\le \left[\frac{1}{p}\|\nabla h\|_p^p - \beta\int_{\mathbb{R}^N}|w|^{r_1}|h|^p\mathrm{d}x\right]|s|^p(1+o(1)) < 0,
\end{aligned}$$

which implies that for any $|s| < b^{\frac{1}{p}}$ small enough, we have

$$M_\beta(a,b) \le J_\beta[t(s)\star w, t(s)\star(sh)] < J_\beta[w,0] = m_{m_1,\mu_1,a}. \qquad \square$$

Next, we prove the following existence result on $\mathcal{D}_a \times \mathcal{D}_b$.

Lemma 4.3. *Let* $1 < p < N$, $r_1, r_2 > 1$, $p + \frac{p^2}{N} < m_1, m_2, r_1 + r_2 < p^*$ *and let* $M_\beta(a,b)$ *be defined in* (1.6). *Then for any* $a, b > 0$, $M_\beta(a,b)$ *is achieved by some nonnegative Schwarz symmetric function* $[u,v]$.

Proof. By Lemma 3.13 and Corollary 3.12, we can take a minimizing sequence $[u_n, v_n] \in \mathcal{P}_\beta^{(a,b)}$ which is Schwarz symmetric such that

$$[u_n,v_n] = [|u_n|,|v_n|] = [u_n^*, v_n^*],\ 0 < \|u_n\|_p^p \le a,\ 0 < \|v_n\|_p^p \le b,\ P_\beta[u_n,v_n] = 0,$$

and

$$J_\beta[u_n,v_n] \to M_\beta(a,b) \quad \text{as } n \to +\infty.$$

It follows from Corollary 3.10 that J_β is coercive. So we can derive that $\{\|\nabla u_n\|_p\}$ and $\{\|\nabla v_n\|_p\}$ are bounded, i.e., $\{[u_n,v_n]\}$ is bounded in $\mathcal{W}$.

By the fact that the embedding $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is compact for any $s \in (p,p^*)$, up to a subsequence if necessary, there exists $[u,v] \in \mathcal{W}$ such that

$$\begin{aligned}
&u_n \rightharpoonup u \text{ in } W_r^{1,p}(\mathbb{R}^N), \quad u_n \to u \text{ in } L^s(\mathbb{R}^N), \quad u_n \to u \text{ a.e. in } \mathbb{R}^N, \\
&v_n \rightharpoonup v \text{ in } W_r^{1,p}(\mathbb{R}^N), \quad v_n \to v \text{ in } L^s(\mathbb{R}^N), \quad v_n \to v \text{ a.e. in } \mathbb{R}^N.
\end{aligned} \tag{4.2}$$

Since $r_1, r_2 > 1$, we can derive from the Mean Value Theorem and Hölder inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2}\mathrm{d}x - \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |u_n|^{r_1}(|v_n|^{r_2} - |v|^{r_2})\mathrm{d}x + \int_{\mathbb{R}^N} (|u_n|^{r_1} - |u|^{r_1})|v|^{r_2}\mathrm{d}x \\
&\leq r_2 \int_{\mathbb{R}^N} |u_n|^{r_1}(|v_n|^{r_2-1} + |v|^{r_2-1})|v_n - v|\mathrm{d}x \\
&\quad + r_1 \int_{\mathbb{R}^N} (|u_n|^{r_1-1} + |u|^{r_1-1})|v|^{r_2}|u_n - u|\mathrm{d}x \\
&\leq r_2\|u_n\|_{r_1+r_2}^{r_1}\big(\|v_n\|_{r_1+r_2}^{r_2-1} + \|v\|_{r_1+r_2}^{r_2-1}\big)\|v_n - v\|_{r_1+r_2} \\
&\quad + r_1\big(\|u_n\|_{r_1+r_2}^{r_1-1} + \|u\|_{r_1+r_2}^{r_1-1}\big)\|v\|_{r_1+r_2}^{r_2}\|u_n - u\|_{r_1+r_2} \\
&\to 0, \quad \text{as } n \to +\infty.
\end{aligned} \tag{4.3}$$

We claim that $u \not\equiv 0$ and $v \not\equiv 0$. Suppose by contradiction that $u \equiv 0$ or $v \equiv 0$. By $[u_n, v_n] \in \mathcal{P}_\beta^{(a,b)}$, (4.2) and (4.3), we have that

$$\begin{aligned}
\|\nabla u_n\|_p^p + \|\nabla v_n\|_p^p &= \mu_1\delta_{m_1}\|u_n\|_{m_1}^{m_1} + \mu_2\delta_{m_2}\|v_n\|_{m_2}^{m_2} \\
&\quad + \beta(r_1 + r_2)\delta_{r_1+r_2}\int_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2}\mathrm{d}x \\
&= o(1),
\end{aligned}$$

which implies $\|\nabla u_n\|_p^p = o(1)$ and $\|\nabla v_n\|_p^p = o(1)$. This is a contradiction with Lemma 3.11.

Moreover, by the weak lower semicontinuity, we have

$$\|\nabla u\|_p \leq \liminf_{n\to+\infty} \|\nabla u_n\|_p, \quad \|\nabla v\|_p \leq \liminf_{n\to+\infty} \|\nabla v_n\|_p. \tag{4.4}$$

Now, we have that

$$P_\beta[u, v] \leq \liminf_{n\to+\infty} P_\beta[u_n, v_n] = 0.$$

Then by Lemma 3.7, there exists a unique $t = t_{[u,v]} \in (0, 1]$ such that $[t\star u, t\star v] \in \mathcal{P}_\beta^{(a,b)}$. Hence, we have

$$\begin{aligned}
&J_\beta[t\star u, t\star v] \\
&= \frac{1}{p}\left[\|\nabla(t\star u)\|_p^p + \|\nabla(t\star v)\|_p^p\right] - \frac{\mu_1}{m_1}\|t\star u\|_{m_1}^{m_1} - \frac{\mu_2}{m_2}\|t\star v\|_{m_2}^{m_2} \\
&\quad - \beta\int\limits_{\mathbb{R}^N} |t\star u|^{r_1}|t\star v|^{r_2}\mathrm{d}x \\
&= \mu_1\left(\frac{\delta_{m_1}}{p} - \frac{1}{m_1}\right)\|t\star u\|_{m_1}^{m_1} + \mu_2\left(\frac{\delta_{m_2}}{p} - \frac{1}{m_2}\right)\|t\star v\|_{m_2}^{m_2} \\
&\quad + \beta\left(\frac{(r_1+r_2)\delta_{r_1+r_2}}{p} - 1\right)\int\limits_{\mathbb{R}^N} |t\star u|^{r_1}|t\star v|^{r_2}\mathrm{d}x \\
&= \mu_1\left(\frac{\delta_{m_1}}{p} - \frac{1}{m_1}\right)\|u\|_{m_1}^{m_1}t^{m_1\delta_{m_1}} + \mu_2\left(\frac{\delta_{m_2}}{p} - \frac{1}{m_2}\right)\|v\|_{m_2}^{m_2}t^{m_2\delta_{m_2}} \\
&\quad + \beta\left(\frac{(r_1+r_2)\delta_{r_1+r_2}}{p} - 1\right)\left(\int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}\mathrm{d}x\right)t^{(r_1+r_2)\delta_{r_1+r_2}}.
\end{aligned} \tag{4.5}$$

Furthermore, combining with $p + \frac{p^2}{N} < m_1, m_2, r_1 + r_2 < p^*$, (4.2), (4.3) and (4.5), we can obtain that

$$\begin{aligned}
M_\beta(a,b) &= \inf_{[\bar{u},\bar{v}]\in\mathcal{P}_\beta^{(a,b)}} J_\beta[\bar{u},\bar{v}] \le J_\beta[t\star u, t\star v] \\
&= \mu_1\left(\frac{\delta_{m_1}}{p} - \frac{1}{m_1}\right)\|u\|_{m_1}^{m_1}t^{m_1\delta_{m_1}} + \mu_2\left(\frac{\delta_{m_2}}{p} - \frac{1}{m_2}\right)\|v\|_{m_2}^{m_2}t^{m_2\delta_{m_2}} \\
&\quad + \beta\left(\frac{(r_1+r_2)\delta_{r_1+r_2}}{p} - 1\right)\left(\int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}\mathrm{d}x\right)t^{(r_1+r_2)\delta_{r_1+r_2}} \\
&\le \mu_1\left(\frac{\delta_{m_1}}{p} - \frac{1}{m_1}\right)\|u\|_{m_1}^{m_1} + \mu_2\left(\frac{\delta_{m_2}}{p} - \frac{1}{m_2}\right)\|v\|_{m_2}^{m_2} \\
&\quad + \beta\left(\frac{(r_1+r_2)\delta_{r_1+r_2}}{p} - 1\right)\int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}\mathrm{d}x \\
&\le \lim_{n\to+\infty}\mu_1\left(\frac{\delta_{m_1}}{p} - \frac{1}{m_1}\right)\|u_n\|_{m_1}^{m_1} + \lim_{n\to+\infty}\mu_2\left(\frac{\delta_{m_2}}{p} - \frac{1}{m_2}\right)\|v_n\|_{m_2}^{m_2} \\
&\quad + \lim_{n\to+\infty}\beta\left(\frac{(r_1+r_2)\delta_{r_1+r_2}}{p} - 1\right)\int\limits_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2}\mathrm{d}x \\
&= \lim_{n\to+\infty} J_\beta[u_n, v_n] = M_\beta(a,b),
\end{aligned}$$

which implies that $t = 1$ and

$$J_\beta[u, v] = \lim_{n\to+\infty} J_\beta[u_n, v_n] = M_\beta(a, b).$$

So, it follows from (4.2), (4.3) and $\lim_{n\to+\infty} J_\beta[u_n, v_n] = J_\beta[u, v]$ that

$$\frac{1}{p}\big(\|\nabla u_n\|_p^p + \|\nabla v_n\|_p^p\big) + o(1) = \frac{1}{p}\big(\|\nabla u\|_p^p + \|\nabla v\|_p^p\big),$$

and combining with (4.4), we can derive that

$$\|\nabla u_n\|_p^p \to \|\nabla u\|_p^p, \quad \|\nabla v_n\|_p^p \to \|\nabla v\|_p^p \quad \text{as } n \to +\infty$$

and $P_\beta[u, v] = 0$.

We proved that $M_\beta(a, b)$ is achieved by $[u, v] \in \mathcal{P}_\beta^{(a,b)}$. □

To show that a minimizer $[u, v] \in \mathcal{P}_\beta^{(a,b)}$ satisfies $\|u\|_p^p = a$ and $\|v\|_p^p = b$, we need the following Liouville type result for p-Laplacian, which is similar to [20, Lemma A.2] and [23, Lemma 2.7].

Lemma 4.4. *Suppose* $1 < p < N$, $q \in \big(0, \frac{N(p-1)}{N-p}\big]$. *If* $u \in L^q(\mathbb{R}^N)$ *is a nonnegative function and satisfies the following inequality:*

$$-\Delta_p u \geq 0 \quad \text{in } \mathbb{R}^N,$$

then $u \equiv 0$.

Proof. Suppose by contradiction that $u \not\equiv 0$. Then by the strong maximum principle (see [26, Theorem 1]), we have $u > 0$. Hence, it follows from [28, Lemma 2.3] that

$$u(x) \geq C|x|^{-\frac{N-p}{p-1}}, \quad |x| > 2,$$

where C is a positive constant depending on N, p, u. For any $q \in \big(0, \frac{N(p-1)}{N-p}\big]$, we have

$$\int\limits_{\mathbb{R}^N} u^q \mathrm{d}x \geq C^q \int\limits_{|x|>2} \left(\frac{1}{|x|^{\frac{N-p}{p-1}}}\right)^q \mathrm{d}x \geq C^q \int\limits_{|x|>2} \frac{1}{|x|^N} \mathrm{d}x = +\infty.$$

This contradicts the assumption that $u \in L^q(\mathbb{R}^N)$. □

Remark 4.5. Assume $1 < p < N$. If $u \in W^{1,p}(\mathbb{R}^N)$, we have $u \in L^q(\mathbb{R}^N)$ for $q \in [p, p^*]$ by the Sobolev embedding inequality. In this situation, we can obtain that $q \in \big(0, \frac{N(p-1)}{N-p}\big]$ if $N \leq p^2$.

Lemma 4.6. *Let* $1 < p < N \leq p^2$. *Assume that*

$$M_\beta(a, b) < \min\{m_{m_1,\mu_1,a},\, m_{m_2,\mu_2,b}\}.$$

Then there exists a ground state solution $[u, v]$ *of equations* (1.1)–(1.2).

Proof. By Lemma 4.3, we can assume that $[u,v] \neq [0,0]$ is a minimizer on $\mathcal{D}_a \times \mathcal{D}_b$. So there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{cases} -\Delta_p u + \lambda_1 u^{p-1} = \mu_1 u^{m_1-1} + \beta r_1 u^{r_1-1} v^{r_2}, \\ -\Delta_p v + \lambda_2 v^{p-1} = \mu_2 v^{m_2-1} + \beta r_2 u^{r_1} v^{r_2-1}. \end{cases} \tag{4.6}$$

Step 1. $u \not\equiv 0$, $v \not\equiv 0$. If $u \not\equiv 0$ and $v \equiv 0$, one obtains

$$m_{m_1,\mu_1,a} = \inf_{u \in \mathcal{P}_{m_1,\mu_1,a}} J_\beta[u,0] \leq J_\beta[u,0] = M_\beta(a,b),$$

which contradicts the assumption

$$M_\beta(a,b) < \min\big\{m_{m_1,\mu_1,a},\, m_{m_2,\mu_2,b}\big\}.$$

If $u \equiv 0$ and $v \not\equiv 0$, we can derive a similar contradiction in the same way.

Step 2. $\lambda_1, \lambda_2 > 0$. If not, we assume that $\lambda_1 \leq 0$ or $\lambda_2 \leq 0$. Then we can obtain that

$$-\Delta_p u \geq \mu_1 u^{m_1-1} \quad \text{or} \quad -\Delta_p v \geq \mu_2 v^{m_2-1}.$$

Since $1 < p < N \leq p^2$, by Lemma 4.4, one obtains $u \equiv 0$ or $v \equiv 0$, which is impossible.

Step 3. $[u,v] \in \mathcal{S}_a \times \mathcal{S}_b$. In this step, the proof is similar to [21, Lemma 8.2]. If $\lambda_1 > 0$, we claim that $u \in \mathcal{S}_a$. Suppose by contradiction that $\delta := \|u\|_p^p \in (0,a)$. Then for any $s \in \big(0, \big(\frac{a}{\delta}\big)^{\frac{1}{p}}\big]$, we have

$$[su, v] \in \mathcal{D}_a \times \mathcal{D}_b \setminus \{[0,0]\}.$$

So, there exists a unique $t = t(s) > 0$ such that $[t \star (su), t \star v] \in \mathcal{P}_\beta^{(a,b)}$ by Lemma 3.7. Precisely, $t = t(s)$ is determined by

$$\begin{aligned} &\|\nabla u\|_p^p s^p + \|\nabla v\|_p^p \\ &= \delta_{m_1}\mu_1 \|u\|_{m_1}^{m_1} s^{m_1} t^{m_1\delta_{m_1}-p} + \delta_{m_2}\mu_2 \|v\|_{m_2}^{m_2} t^{m_2\delta_{m_2}-p} \\ &\quad + (r_1+r_2)\delta_{r_1+r_2}\beta \left(\int\limits_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} \mathrm{d}x \right) s^{r_1} t^{(r_1+r_2)\delta_{r_1+r_2}-p}. \end{aligned} \tag{4.7}$$

Then we can derive from (4.7) and the implicit function theorem that $t(s) \in C^1$ locally around $s = 1$. Then we have

$$
\begin{aligned}
&\frac{d}{ds} J_\beta[t(s)\star(su), t(s)\star v] \\
&= \|\nabla u\|_p^p s^{p-1} t^p - \mu_1 \|u\|_{m_1}^{m_1} s^{m_1-1} t^{m_1\delta_{m_1}} \\
&\quad - \beta r_1 \left(\int_{\mathbb{R}^N} u^{r_1} v^{r_2} \mathrm{d}x \right) s^{r_1-1} t^{(r_1+r_2)\delta_{r_1+r_2}} + \Bigg[\|\nabla u\|_p^p s^p t^{p-1} + \|\nabla v\|_p^p t^{p-1} \\
&\quad - \delta_{m_1}\mu_1 \|u\|_{m_1}^{m_1} s^{m_1} t^{m_1\delta_{m_1}-1} - \delta_{m_2}\mu_2 \|v\|_{m_2}^{m_2} t^{m_2\delta_{m_2}-1} \\
&\quad - \beta(r_1+r_2)\delta_{r_1+r_2} \left(\int_{\mathbb{R}^N} u^{r_1} v^{r_2} \mathrm{d}x \right) s^{r_1} t^{(r_1+r_2)\delta_{r_1+r_2}-1} \Bigg] t'(s).
\end{aligned}
$$

Noting that $t(1) = 1$ and $P_\beta[u,v] = 0$, then we have

$$
\begin{aligned}
&\frac{d}{ds} J_\beta[t(s)\star(su), t(s)\star v]\Big|_{s=1} \\
&= \|\nabla u\|_p^p - \mu_1\|u\|_{m_1}^{m_1} - \beta r_1 \int_{\mathbb{R}^N} u^{r_1} v^{r_2} \mathrm{d}x + P_\beta[u,v] t'(1) \\
&= -\lambda_1 \|u\|_p^p < 0,
\end{aligned}
$$

which means that for any s near $s = 1$,

$$
M_\beta(a,b) \le J_\beta[t(s)\star(su), t(s)\star v] < J_\beta[u,v] = M_\beta(a,b),
$$

a contradiction.

So the claim that $u \in \mathcal{S}_a$ is guaranteed by $\lambda_1 > 0$. Similarly, $\lambda_2 > 0$ implies that $v \in \mathcal{S}_b$. □

Proof of Theorem 1.1. It follows from the definition of $b_{m_1,m_2,\mu_1,\mu_2,a}$ that

$$
m_{m_2,\mu_2,b} \le m_{m_1,\mu_1,a}, \quad \forall\, b \in [b_{m_1,m_2,\mu_1,\mu_2,a}, +\infty),
$$

and

$$
m_{m_2,\mu_2,b} \ge m_{m_1,\mu_1,a}, \quad \forall\, b \in (0, b_{m_1,m_2,\mu_1,\mu_2,a}].
$$

Since $N \ge 2$, $p \in (\sqrt{N}, N)$, $\frac{p^2}{N} + p < m_1, m_2, r_1 + r_2 < p^*$ and $r_1, r_2 > 1$.

(i) If $r_1 < p$, $\beta > 0$, or $r_1 = p$, $\beta > \beta_{m_2,\mu_2,b,N,r_1}$, then since $1 < p < N$, $\frac{p^2}{N} + p < m_1, m_2, r_1 + r_2 < p^*$, and $r_1, r_2 > 1$, we can derive from Lemma 4.2(ii) that for any $b \in [b_{m_1,m_2,\mu_1,\mu_2,a}, +\infty)$

$$
M_\beta(a,b) < m_{m_2,\mu_2,b} \le m_{m_1,\mu_1,a}.
$$

Now, since $1 < p < N \le p^2$, by Remark 4.5, the requirement of the Liouville-type lemma is satisfied. Hence, Lemma 4.3 holds, which implies that there exists a minimizer on $\mathcal{D}_a \times \mathcal{D}_b$. Then by Lemma 4.6, we can prove that there exists $(\lambda_1, \lambda_2, u, v) \in \mathbb{R}^2 \times \mathcal{W}$ which is a ground state solution of (1.1)–(1.2). Moreover, by the strong maximum principle (see [26, Theorem 1]), u and v are positive.

(ii) The proof is similar to (i), it suffices to use Lemma 4.2 (i), Lemma 4.3 and Lemma 4.6. □

5. PROOF OF THEOREM 1.3

In this section, we show some properties of normalized ground states of (1.1)–(1.2) obtained in Theorem 1.1.

Lemma 5.1. *$M_\beta(a,b)$ is non-increasing in β.*

Proof. By Theorem 1.1, for any $\beta_1 > 0$, there exists $[u_{\beta_1}, v_{\beta_1}] \in \mathcal{S}_a \times \mathcal{S}_b$ such that

$$J_{\beta_1}[u_{\beta_1}, v_{\beta_1}] = M_{\beta_1}(a,b).$$

For any $0 < \beta_1 < \beta_2$, we have

$$\begin{aligned} M_{\beta_2}(a,b) &\leq \max_{t>0} J_{\beta_2}[t \star u_{\beta_1}, t \star v_{\beta_1}] \\ &\leq \max_{t>0} J_{\beta_1}[t \star u_{\beta_1}, t \star v_{\beta_1}] = J_{\beta_1}[u_{\beta_1}, v_{\beta_1}] = M_{\beta_1}(a,b). \end{aligned}$$ □

Lemma 5.2. *Let $1 < p < N$ and $m_1, m_2, r_1 + r_2 \in \left(\frac{p^2}{N} + p, p^*\right)$. Then $M_\beta(a,b)$ is uniformly bounded with respect to β.*

Proof. Let $w_1 = w_{m_1,\mu_1,a}$ and $w_2 = w_{m_2,\mu_2,b}$ be defined in (2.3). For any $\beta > 0$, we have

$$\begin{aligned} M_\beta(a,b) &\leq \max_{t>0} J_\beta[t \star w_1, t \star w_2] \\ &= \max_{t>0} \Bigg[\frac{1}{p}\|\nabla(t \star w_1)\|_p^p + \frac{1}{p}\|\nabla(t \star w_2)\|_p^p - \frac{\mu_1}{m_1}\|t \star w_1\|_{m_1}^{m_1} \\ &\quad - \frac{\mu_2}{m_2}\|t \star w_2\|_{m_2}^{m_2} - \beta \int\limits_{\mathbb{R}^N} |t \star w_1|^{r_1} |t \star w_2|^{r_2} \mathrm{d}x\Bigg] \\ &\leq \max_{t>0} \left[\frac{1}{p}\|\nabla(t \star w_1)\|_p^p - \frac{\mu_1}{m_1}\|t \star w_1\|_{m_1}^{m_1}\right] \\ &\quad + \max_{t>0} \left[\frac{1}{p}\|\nabla(t \star w_2)\|_p^p - \frac{\mu_2}{m_2}\|t \star w_2\|_{m_2}^{m_2}\right] \\ &= m_{m_1,\mu_1,a} + m_{m_2,\mu_2,b}, \end{aligned}$$

where $m_{m_1,\mu_1,a}, m_{m_2,\mu_2,b}$ are defined in (2.6), which do not depend on β. □

Proof of Theorem 1.3. (i) By Theorem 1.1, for any $\beta > 0$, $M_\beta(a,b)$ is achieved by some $[u_\beta, v_\beta] \in \mathcal{S}_a \times \mathcal{S}_b$, where u_β, v_β are positive radial functions.

Since

$$\min\{m_{m_1,\mu_1,a},\, m_{m_2,\mu_2,b}\} > M_\beta(a,b) = J_\beta[u_\beta, v_\beta],$$

we obtain the boundedness of $\{[u_\beta, v_\beta]\}$ in $\mathcal{W}$ by a similar argument as in Lemma 3.9. Then up to a subsequence as $\beta \to 0^+$, there exists $[\bar{u}, \bar{v}] \in \mathcal{W}$ such that

$$\begin{aligned} &u_\beta \rightharpoonup \bar{u}, \quad v_\beta \rightharpoonup \bar{v} \quad \text{in } W^{1,p}(\mathbb{R}^N), \\ &u_\beta \to \bar{u}, \quad v_\beta \to \bar{v} \quad \text{in } L^s(\mathbb{R}^N), \end{aligned} \tag{5.1}$$

where $s \in (p, p^*)$ and $\bar{u}, \bar{v} \geq 0$.

Note that $[u_\beta, v_\beta]$ is a normalized solution of the system

$$\begin{cases} -\Delta_p u_\beta + \lambda_{1,\beta}|u_\beta|^{p-2}u_\beta = \mu_1|u_\beta|^{m_1-2}u_\beta + \beta r_1|u_\beta|^{r_1-2}u_\beta|v_\beta|^{r_2}, \\ -\Delta_p v_\beta + \lambda_{2,\beta}|v_\beta|^{p-2}v_\beta = \mu_2|v_\beta|^{m_2-2}v_\beta + \beta r_2|u_\beta|^{r_1}|v_\beta|^{r_2-2}v_\beta, \end{cases} \tag{5.2}$$

and satisfies the Pohozaev identity in Lemma 3.3. We deduce that

$$\lambda_{1,\beta}a + \lambda_{2,\beta}b = \mu_1(1-\delta_{m_1})\|u_\beta\|_{m_1}^{m_1} + \mu_2(1-\delta_{m_2})\|v_\beta\|_{m_2}^{m_2}. \tag{5.3}$$

Hence, $\{\lambda_{1,\beta}\}$ and $\{\lambda_{2,\beta}\}$ are bounded. Combining $\lambda_{1,\beta}, \lambda_{2,\beta} > 0$, up to a subsequence,

$$\lambda_{1,\beta} \to \overline{\lambda}_1 \geq 0, \quad \lambda_{2,\beta} \to \overline{\lambda}_2 \geq 0.$$

We consider the following cases.

Case 1. $\overline{\lambda}_1 = 0$, $\overline{\lambda}_2 = 0$. From (5.3) and Lemma 5.1, for any fixed $\beta_0 > \beta$,

$$\begin{aligned} 0 &= \mu_1\left(\frac{\delta_{m_1}}{p} - \frac{1}{m_1}\right)\|\bar{u}\|_{m_1}^{m_1} + \mu_2\left(\frac{\delta_{m_2}}{p} - \frac{1}{m_2}\right)\|\bar{v}\|_{m_2}^{m_2} \\ &= \lim_{\beta\to 0^+} J_\beta[u_\beta, v_\beta] \geq M_{\beta_0}(a,b) > 0, \end{aligned}$$

a contradiction.

Case 2. $\overline{\lambda}_1 > 0$, $\overline{\lambda}_2 > 0$. By (5.1), $(\bar{u}, \bar{v})$ is a weak solution of

$$\begin{aligned} -\Delta_p\bar{u} + \overline{\lambda}_1|\bar{u}|^{p-2}\bar{u} = \mu_1|\bar{u}|^{m_1-2}\bar{u}, \\ -\Delta_p\bar{v} + \overline{\lambda}_2|\bar{v}|^{p-2}\bar{v} = \mu_2|\bar{v}|^{m_2-2}\bar{v}. \end{aligned} \tag{5.4}$$

Testing the first equations in (5.2) and (5.4) with $u_\beta - \bar{u}$ and arguing as in [19, Lemma 3.6], we obtain

$$\|\nabla(u_\beta - \bar{u})\|_p^p + \overline{\lambda}_1\|u_\beta - \bar{u}\|_p^p \to 0, \quad \beta \to 0^+.$$

Thus, $u_\beta \to \bar{u}$ in $W^{1,p}(\mathbb{R}^N)$. Similarly, $v_\beta \to \bar{v}$ in $W^{1,p}(\mathbb{R}^N)$. Moreover,

$$\|\bar{u}\|_p^p = a, \quad \|\bar{v}\|_p^p = b, \quad \bar{u}, \bar{v} > 0.$$

Since $[\bar{u}, \bar{v}]$ are normalized solutions of (5.4),

$$\min\bigl\{m_{m_1,\mu_1,a},\, m_{m_2,\mu_2,b}\bigr\} \geq \lim_{\beta\to 0^+} J_\beta[u_\beta, v_\beta] \geq m_{m_1,\mu_1,a} + m_{m_2,\mu_2,b},$$

a contradiction.

Case 3. $\overline{\lambda}_1 = 0$, $\overline{\lambda}_2 > 0$. By Lemma 4.4, $\bar{u} \equiv 0$, and from (5.3), $\bar{v} > 0$. By the same argument as in Case 2,

$$\min\bigl\{m_{m_1,\mu_1,a},\, m_{m_2,\mu_2,b}\bigr\} \geq \lim_{\beta\to 0^+} J_\beta[u_\beta, v_\beta] = m_{m_2,\mu_2,b}.$$

If $b \in [b_{m_1,m_2,\mu_1,\mu_2,a}, +\infty)$, then

$$M_\beta(a,b) \to m_{m_2,\mu_2,b}, \quad [u_\beta, v_\beta] \to [0, w_{m_2,\mu_2,b}] \quad \text{as } \beta \to 0^+.$$

If $b \in (0, b_{m_1,m_2,\mu_1,\mu_2,a})$, this case cannot occur.

Case 4. $\overline{\lambda}_1 > 0$, $\overline{\lambda}_2 = 0$. This case is similar to Case 3. If $b \in (0, b_{m_1,m_2,\mu_1,\mu_2,a}]$, then

$$M_\beta(a,b) \to m_{m_1,\mu_1,a}, \quad [u_\beta, v_\beta] \to [w_{m_1,\mu_1,a}, 0] \quad \text{as } \beta \to 0^+.$$

If $b \in (b_{m_1,m_2,\mu_1,\mu_2,a}, +\infty)$, this case cannot occur.

(ii) We now prove that

$$M_\beta(a,b) \to 0^+ \quad \text{as } \beta \to +\infty.$$

Without loss of generality, assume $a \leq b$. Let $w = w_{r_1+r_2,\beta/2,a}$ be defined in (2.3). By the definition of $M_\beta(a,b)$,

$$\begin{aligned}
0 < M_\beta(a,b) &\leq \max_{t>0} J_\beta[t \star w, t \star w] \\
&= \max_{t>0} \left[\frac{2}{p} \|\nabla(t \star w)\|_p^p - \frac{\mu_1}{m_1} \|t \star w\|_{m_1}^{m_1} - \frac{\mu_2}{m_2} \|t \star w\|_{m_2}^{m_2} - \beta \|t \star w\|_{r_1+r_2}^{r_1+r_2} \right] \\
&\leq \max_{t>0} 2 \left(\frac{1}{p} \|\nabla(t \star w)\|_p^p - \frac{\beta}{2(r_1+r_2)} \|t \star w\|_{r_1+r_2}^{r_1+r_2} \right) \\
&= 2m_{r_1+r_2,\beta/2,a},
\end{aligned}$$

where $m_{r_1+r_2,\beta/2,a}$ is defined in (2.6). This implies $M_\beta(a,b) \to 0^+$ as $\beta \to +\infty$. □

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