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NORMALIZED SOLUTIONS FOR PLANAR SCHRÖDINGER–POISSON SYSTEM WITH CRITICAL EXPONENTIAL GROWTH AND NONLOCAL INTERACTION

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Abstract. This paper focuses on the following planar Schrödinger–Poisson system with critical exponential growth and nonlocal interaction

$$\begin{cases} -\Delta u + \lambda u + \mu(\log|\cdot| * u^2)u = \gamma\left(I_\alpha * |u|^q\right)|u|^{q-2}u + \left(e^{u^2} - 1 - u^2\right)u, & x \in \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} u^2 \mathrm{d}x = c, \end{cases}$$

where $c > 0$, $\mu, \gamma > 0$, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier, $\alpha \in (0,2)$, $1 + \frac{\alpha}{2} \leq q < +\infty$, $I_\alpha : \mathbb{R}^2 \to \mathbb{R}$ denotes the Riesz potential and $1 + \frac{\alpha}{2}$ is the lower critical exponent with respect to the Hardy–Littlewood–Sobolev inequality. Through delicate energy estimates, under explicit conditions on c, we prove the existence of two normalized solutions: one is a local minimizer and the other is of mountain-pass type. The presence of the logarithmic kernel and the competition between the two nonlocal terms necessitates the development of new tools to address the loss of compactness caused by the critical exponential growth, for which the variational techniques developed for the local problem are no longer applicable. Our work not only generalizes the special case $\gamma = 0$, but also provides an analytical approach that is applicable to more L^2-constrained problems with competing nonlocal terms modelling long-range attraction in particle physics.

Keywords: normalized solution, logarithmic convolution potential, nonlocal interaction, critical exponential growth, Trudinger–Moser inequality.

Mathematics Subject Classification: 35J20, 35J62, 35Q55.

1. INTRODUCTION

This paper investigates the existence of multiple normalized solutions to the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \lambda u + \mu(\log|\cdot| * u^2)u = \gamma\left(I_\alpha * |u|^q\right)|u|^{q-2}u + \left(e^{u^2} - 1 - u^2\right)u, & x \in \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} u^2 \mathrm{d}x = c, \end{cases} \tag{1.1}$$

where $c, \mu, \gamma > 0$, $\lambda \in \mathbb{R}$ arises as a Lagrange parameter and is part of the unknowns, $1 + \frac{\alpha}{2} \leq q < +\infty$, $I_\alpha : \mathbb{R}^2 \to \mathbb{R}$ is the Riesz potential with order $\alpha \in (0, 2)$ defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi |x|^{2-\alpha}} := \frac{\mathcal{K}_\alpha}{|x|^{2-\alpha}}, \quad x \in \mathbb{R}^2 \setminus \{0\},$$

and the nonlinearity exhibits critical exponential growth of Trudinger–Moser type. As defined in [2, 17], a function f has *critical exponential growth* if $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies

$$\lim_{|t| \to +\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0, \end{cases} \quad \text{for some constant } \alpha_0 > 0, \tag{1.2}$$

which is motivated by the Trudinger–Moser inequality developed in [5] (see Lemma 2.8).

1.1. RESEARCH MOTIVATION AND MAIN DIFFICULTY

The study of (1.1) stems from the very active recent research on Brezis–Nirenberg type problems for the Choquard equation:

$$-\Delta u + \lambda u = \gamma(I_\alpha * |u|^q)|u|^{q-2}u + \kappa|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $N \geq 2$,

$$2_\alpha := \frac{N+\alpha}{N} \leq q \leq 2^\alpha := \begin{cases} \frac{N+\alpha}{N-2}, & N \geq 3, \\ +\infty, & N = 2, \end{cases} \quad 2 \leq p \leq 2^* := \begin{cases} \frac{2N}{N-2}, & N \geq 3, \\ +\infty, & N = 2, \end{cases}$$

$\gamma, \kappa \in \mathbb{R}$, I_α is the Riesz potential with order $\alpha \in (0, N)$ and 2^* denotes the Sobolev critical exponent. The constants 2_α and 2^α are the lower and upper critical exponents, respectively, for Choquard-type problems corresponding to the Hardy–Littlewood–Sobolev inequality given in Lemma 2.2. This problem naturally emerges from various fields within mathematical physics. In physics, it was originally introduced in 1954 by Pekar [30] to describe resting polarons. Subsequently, Lieb [24] employed (1.3) to characterize trapped electrons in plasma physics, and it was later employed as a model for self-gravitating matter in [28]. From a mathematical perspective, an extensive body of literature has been dedicated to exploring the existence, multiplicity, and qualitative characteristics of solutions associated with Choquard type equations. To the best of our knowledge, the pioneering mathematical analyses of equation (1.3) are due to Lieb [24] and Lions [26]. There are two possible choices to deal with (1.3) according to the role of λ. One can fix λ as an assigned parameter that has attracted considerable attention. We invite the reader to see [29, 35, 40] and the references therein.

From another perspective, equation (1.3) can be investigated under a prescribed L^2-norm constraint. When restricted to this mass constraint, λ is no longer a fixed parameter, but rather treated as an unknown Lagrange multiplier. Solutions with a prescribed L^2-norm are commonly referred to in the literature as *normalized solutions*. These solutions hold significant appeal in the physics community since the L^2-norm remains invariant during time evolution. Furthermore, their underlying variational structure offers an effective theoretical tool for investigating orbital stability properties (see [7, 32]). For further details on related developments in the study of normalized solutions, we refer the readers to [10, 14, 19, 37, 38].

We note that both multiplicity and qualitative behavior of normalized solutions to (1.3) are highly influenced by additional assumptions imposed on γ, κ and the powers q, p. For example, when $\gamma = 1, \kappa = 0$ and $N \geq 3, q = \alpha = 2$, the existence and uniqueness of normalized solutions for (1.3) were established by Lieb [24], while Lions [26] investigated their existence and stability properties. Later, Li and Ye [20] deduced the existence of a mountain-pass type normalized solution when $N \geq 3$ and $\frac{N+\alpha+2}{N} < q < 2_\alpha$. Here, $\frac{N+\alpha+2}{N}$ acts as the L^2-critical threshold that dictates whether the energy functional for (1.3) with $\kappa = 0$ remains bounded from below on the mass constraint $\bar{\mathcal{S}}_c := \{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c\}$. Thus, the problem can be categorized into the cases of L^2-subcritical ($2_\alpha < q < \frac{N+\alpha+2}{N}$), L^2-critical ($q = \frac{N+\alpha+2}{N}$) and L^2-supercritical ($\frac{N+\alpha+2}{N} < q < 2^\alpha$).

When $\gamma, \kappa \neq 0$, problem (1.3) exhibits a key feature that the inclusion of multiple power terms destroys the scaling invariance inherent in the homogeneous equation, thus classifying it as a mixed problem. Such mixed problems are currently an active area of investigation; see for instance [19, 21, 23, 39] and the references therein. In particular, Zhou and Zhang [39] considered (1.3) when the local power term is replaced by a more general Sobolev-subcritical nonlinearity, and the Choquard term is treated as a perturbation. By using a more general minimax principle on the manifold and establishing different variational geometries for the equation, the authors showed that for $N \geq 3$, (1.3) admits two solutions in the L^2-subcritical case: one is a local minimizer with negative energy, and the other is of mountain-pass type. For the L^2-critical and L^2-supercritical cases, they obtained that (1.3) has a ground state solution. Recently, Jin–Yang–Zhou [19] investigated normalized solutions to equation (1.3) for the Sobolev critical case when $N = 3$. By employing some new analytical approaches, the authors obtained multiple existence results. For L^2-critical and L^2-supercritical perturbation cases, they introduced a new test function and derived key energy estimates to overcome the lack of compactness. It should be noted that this work extends the existence results of (1.3) concerning the Sobolev subcritical case to the Sobolev critical case $p = 2^*$ in the higher dimensions $N \geq 3$. To the best of our knowledge, whether such an extension holds in the critical case when $N = 2$ remains unknown, and this fact constitutes a primary motivation for the present study.

Another motivation for this paper comes from recent works dealing with normalized solutions to the planar Schrödinger–Poisson system

$$-\Delta u + \mu(\log|\cdot| * u^2)u + \lambda u = \gamma|u|^{q-2}u + \omega\left(e^{u^2} - 1 - u^2\right)u, \quad x \in \mathbb{R}^2, \tag{1.4}$$

which can be seen as the limiting equation of (1.1) as $\alpha \to 0$. Such a system originates naturally from various physical areas, including quantum theory and the modeling of semiconductors. We refer the interested reader to [3, 27] for detailed background information.

From a variational point of view, the presence of the logarithmic convolution term, i.e., $(\log|\cdot| * u^2)u$, implies that the standard Sobolev space $H^1(\mathbb{R}^2)$ is not an appropriate framework for this problem. This is primarily due to the sign-changing and unbounded nature of the nonlocal logarithmic convolution kernel, which renders the corresponding energy functional not well-defined on $H^1(\mathbb{R}^2)$. Cingolani and Weth [9] overcame this obstacle by developing a variational framework within X, which is defined as

$$X := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(2 + |x|)u^2 \mathrm{d}x < \infty \right\}. \tag{1.5}$$

They proved that the unconstrained problem (1.4) admits solutions when $\omega = 0$ and $q > 4$ with λ being a fixed parameter. To the best of our knowledge, Cingolani and Jeanjean [8] made the first contribution to investigating normalized solutions for (1.4) within X when $\omega = 0$. By analyzing the various cases on parameters $\mu, \gamma \in \mathbb{R}$, they obtained a ground state solution when $\mu > 0$ and one of the following holds: (a) $\gamma \leq 0$ and $q > 2$, (b) $\gamma > 0$ and $q < 4$, (c) $\gamma > 0$, $q = 4$ and $c < \frac{2}{\gamma \mathcal{C}_4}$, where $\mathcal{C}_4 > 0$ is the constant from Lemma 2.1. In addition, when $\mu, \gamma > 0$ and $q > 4$, they demonstrated the existence of two normalized solutions for $c > 0$ sufficiently small. Notably, the exponent 4 corresponds to the L^2-critical exponent for (1.4). Subsequently, Chen–Rădulescu–Tang [15] developed novel approaches to study the case of $\gamma = 0$ and $\omega = 1$. Instead of working in X, the authors established a symmetric variational setting within E_{as} as defined by (1.10), originally developed in [11]. By imposing explicit conditions on c, they proved that (1.4) admits two normalized solutions. Particularly, to obtain the mountain-pass type solution, they demonstrated that the corresponding minimax level lies strictly below the compactness threshold, thereby overcoming the loss of compactness caused by the critical exponential term. For equation (1.4) with $\gamma > 0$, $\omega = 1$ and the extra term $V(x)u$, Song and Wang [33] applied variational techniques to obtain similar results in the L^2-subcritical and L^2-critical cases. It should be noted that their method cannot be applied to the L^2-supercritical case, and the scenario associated with the lower critical exponent $q = 2_\alpha$ remains untreated.

Recently, Song–Wang–Zhang [34] considered the L^2-supercritical case for the planar Schrödinger–Poisson system with mixed nonlinearities

$$\begin{cases} -\Delta u + \mu(\log|\cdot| * u^2)u + \lambda u = \gamma|u|^{q-2}u + \varsigma(u), & x \in \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} u^2 \mathrm{d}x = c, \end{cases} \tag{1.6}$$

where $\gamma > 0$, and ς satisfies the critical exponential growth (1.2) with $\alpha_0 = 4\pi$, $\mathrm{sgn}(t)\varsigma(t) \geq \nu|t|^{q_0-2}t$ for $t \in \mathbb{R}$ ($\nu > 0$, $q_0 > 4$), and some other assumptions. The authors first obtained a local minimizer when μ, c and γ are sufficiently small. Based on this solution, they proved that (1.6) admits a mountain-pass type solution for sufficiently large ν. Note that this perturbative character of this result appears in two ways:

(i) γ is sufficiently small so that the influence of the power term in (1.6) is negligible,
(ii) ν is sufficiently large such that the obtained mountain-pass level is small enough to allow recovering the compactness in the same way as in [8].

Clearly, their approach cannot be applied to study the case

$$\varsigma(u) = \left(e^{u^2} - 1 - u^2\right)u,$$

and effectively avoids the difficulties arising from the power term and critical exponential growth nonlinearity. As a consequence, no results are available in the existing literature on this research gap for (1.4).

Inspired by the aforementioned works, especially [15, 19, 33, 39], in this paper, we concentrate on the existence of multiple normalized solutions for planar Schrödinger–Poisson system (1.1) with critical exponential growth and Choquard type perturbations. The study of (1.1) presents greater challenges than those encountered in (1.3) and (1.6). For instance:

(i) Unlike the Choquard equation (1.3), the methods used in [19, 39] are not directly applicable to (1.1) because the logarithmic convolution term $\mu(\log|\cdot| * u^2)u$ is sign-changing and unbounded, rendering the standard variational framework inapplicable.
(ii) Compared to the previous work [15] on (1.4) (i.e., the case $\gamma = 0$ in (1.1)), studying normalized solutions for (1.1) introduces further difficulties. Indeed, the added Choquard term behaves differently across various ranges of the parameter q, thereby complicating the choice of strategies for seeking constrained critical points of the associated energy functional.
(iii) Establishing compactness for critical growth problems relies on sharp upper bound estimates for minimax levels. The logarithmic kernel and competition between the two nonlocal terms necessitate new tools to address the loss of compactness caused by critical exponential growth, for which local variational techniques are no longer applicable. To the best of our knowledge, no analogous estimate has been established for (1.1).

1.2. STATEMENT OF THE MAIN RESULTS

As we all know, normalized solutions of (1.1) correspond to critical points of the $\mathcal{C}^1$-functional $\Phi : X \to \mathbb{R}$ defined by

$$\begin{aligned}\Phi(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{\mu}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y - \frac{\gamma}{2q}\int_{\mathbb{R}^2}\left(I_\alpha * |u|^q\right)|u|^q \mathrm{d}x \\ - \frac{1}{2}\int_{\mathbb{R}^2}\left(e^{u^2} - 1 - u^2 - \frac{u^4}{2}\right)\mathrm{d}x\end{aligned} \tag{1.7}$$

under the constraint $\mathcal{S}_c := \left\{u \in X : \|u\|_2^2 = c > 0\right\}$. Note that Φ has no lower bound on $\mathcal{S}_c$, owing to the property that $\Phi(tu_t(x)) \to -\infty$ as $t \to \infty$, where

$$tu_t(x) := tu(tx), \quad \forall x \in \mathbb{R}^2,\ t > 0 \tag{1.8}$$

is a dilation preserving the L^2-norm, that is $\|tu_t\|_2 = \|u\|_2$ for $t > 0$. In addition, the L^2-Pohozaev functional $\mathcal{P} : X \to \mathbb{R}$ is defined by

$$\begin{aligned}\mathcal{P}(u) = \|\nabla u\|_2^2 - \frac{\mu c^2}{4} - \gamma\frac{2q-(2+\alpha)}{2q}\int_{\mathbb{R}^2}\left(I_\alpha * |u|^q\right)|u|^q \mathrm{d}x \\ - \int_{\mathbb{R}^2}\left[(u^2-1)e^{u^2} + 1 - \frac{u^4}{2}\right]\mathrm{d}x.\end{aligned} \tag{1.9}$$

As will be shown in Lemma 2.6, any solution to (1.1) satisfies the L^2-Pohozaev identity $\mathcal{P}(u) = 0$. In particular, one has $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(tu_t)\big|_{t=1} = \mathcal{P}(u)$.

Before presenting our results, we set the axially symmetric space as follows:

$$E_{as} := X \cap \left\{u \in H^1(\mathbb{R}^2) : u(x) := u(x_1, x_2) = u(|x_1|, |x_2|),\ \forall x \in \mathbb{R}^2\right\}, \tag{1.10}$$

which is equipped with the norm

$$\|u\|_{E_{as}} := \left(\|\nabla u\|_2^2 + \|u\|_*^2\right)^{1/2}, \text{ where } \|u\|_*^2 = \int_{\mathbb{R}^2}\log(2+|x|)u^2(x)\mathrm{d}x. \tag{1.11}$$

Accordingly, we restrict our problem to the constraint manifold

$$\hat{\mathcal{S}}_c := E_{as} \cap \mathcal{S}_c = \left\{u \in E_{as} : \|u\|_2^2 = c\right\}.$$

We establish a variational framework within E_{as}, which is beneficial to the proof of the L^2-convergence of a minimizing sequence and a Palais–Smale (PS) sequence.

In this paper, we will develop some new analytical tricks and approaches to overcome the aforementioned challenge and prove the existence of two distinct normalized solutions to (1.1) for $q \in [1 + \frac{\alpha}{2}, +\infty)$: one being a local minimizer and the other of mountain-pass type. Our results are stated as follows.

Theorem 1.1. *Let $\mu, \gamma > 0$ and $1 + \frac{\alpha}{2} \le q < +\infty$. Then there exists $c_1 > 0$ such that, for any $c \in (0, c_1)$, (1.1) has a pair of solutions $(u_c, \lambda_c) \in \mathcal{S}_c \times \mathbb{R}$ such that*

$$\Phi(u_c) = m(c) := \inf\left\{\Phi(u) : u \in \hat{\mathcal{S}}_c, \quad \|\nabla u\|_2^2 < \frac{\pi}{3}\right\}.$$

Theorem 1.2. *Let $\mu, \gamma > 0$ and $1 + \frac{\alpha}{2} \leq q < +\infty$. Then* (1.1) *has a second pair of solutions $(\bar{u}, \bar{\lambda}_c) \in \mathcal{S}_c \times \mathbb{R}$ such that*

$$m(c) < \Phi(\bar{u}) < m(c) + 2\pi,$$

if one of the following three conditions holds:

(i) $0 < \gamma$, $0 < c < c_1$ *if* $1 + \frac{\alpha}{2} \leq q < 2 + \frac{\alpha}{2}$ *or* $\frac{5}{2} + \frac{\alpha}{2} \leq q < +\infty$,
(ii) $0 < \gamma < \frac{q}{C_0(\alpha,q)c^{(\alpha+2)/2}}$, $0 < c < c_1$ *if* $q = 2 + \frac{\alpha}{2}$,
(iii) $0 < \gamma$, $0 < c < \min\{c_1, c_2\}$ *if* $2 + \frac{\alpha}{2} < q < \frac{5}{2} + \frac{\alpha}{2}$.

Remark 1.3. In this paper, we prove the multiplicity of normalized solutions to (1.1) across the L^2-subcritical, L^2-critical, L^2-supercritical cases and even the lower critical case $q = 1 + \frac{\alpha}{2}$. Note that $q = 1 + \frac{\alpha}{2}$ is the lower critical exponent $2_\alpha := \frac{N+\alpha}{N}$ for Choquard-type problems in the planar case. Therefore, system (1.1) includes a doubly critical scenario where the lower critical exponent and critical exponential growth in the sense of Trudinger–Moser coexist.

Remark 1.4. (i) It should be emphasized that the unfavorable properties of the non-local Choquard term prevent the direct application of the geometric structure analysis method for $g_u(t) := \Phi(tu_t)$ from [15]. We develop some new and simpler mathematical techniques (see Lemma 5.1), which are also adaptable to the setting in [15].

(ii) As is well known, obtaining sharp energy estimates for minimax levels is a crucial step in the compactness analysis of critical growth problems. In this paper, we introduce some refined estimates for the Choquard term to control the energy growth and thus establish sharp estimates for the energy levels. For further details, see Lemma 6.1.

This paper is organized as follows. Section 2 introduces the variational framework and presents several fundamental lemmas required for the subsequent analysis. In Section 3, we study the geometry of local minima on $E_{as} \cap \mathcal{S}_c$. In Section 4, we introduce several critical point theorems on manifolds and provide the proof of Theorem 1.1. Section 5 investigates the mountain-pass geometry for Φ on $E_{as} \cap \mathcal{S}_c$. Finally, Section 6 provides a precise energy estimate and recovers the compactness of the (PS) sequences obtained in Section 5. Then we prove the existence of the mountain pass solution and complete the proof of Theorem 1.2.

Throughout the paper, we adopt the following notation:

– Let $H^1(\mathbb{R}^2)$ denote the standard Sobolev space with the inner product and norm

$$(u, v) = \int\limits_{\mathbb{R}^2} (\nabla u \cdot \nabla v + uv)\mathrm{d}x, \quad \|u\| = (u, u)^{1/2}, \quad \forall u, v \in H^1(\mathbb{R}^2).$$

– $H^1_r(\mathbb{R}^2)$ denotes the subspace of $H^1(\mathbb{R}^2)$ consisting of radially symmetric functions:

$$H^1_r(\mathbb{R}^2) := \{v \in H^1(\mathbb{R}^2) \ : \ v(x) = v(|x|) \quad \text{a.e. in } \mathbb{R}^2\}.$$

– $L^{\mathcal{Q}}(\mathbb{R}^2)$ $(1 \leq \mathcal{Q} < \infty)$ denotes the Lebesgue space equipped with the norm $\|u\|_{\mathcal{Q}} = \left(\int_{\mathbb{R}^2} |u|^{\mathcal{Q}} \mathrm{d}x\right)^{1/\mathcal{Q}}$.

– For any $v \in H^1(\mathbb{R}^2) \setminus \{0\}$, we set $v_t(x) := v(tx)$ for $t > 0$.
– For any $x \in \mathbb{R}^2$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^2 : |y - x| < r\}$ and $B_r = B_r(0)$.
– $C_1, C_2, \ldots$ represent positive constants that may vary from line to line and depend on the parameter $c > 0$.

2. VARIATIONAL FRAMEWORK AND PRELIMINARIES

2.1. VARIATIONAL FRAMEWORK

Following [11], we introduce the symmetric bilinear forms

$$(u, v) \mapsto A_1(u, v) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(2 + |x - y|)\, u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.1}$$

$$(u, v) \mapsto A_2(u, v) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{2}{|x - y|}\right) u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.2}$$

$$(u, v) \mapsto A_0(u, v) := A_1(u, v) - A_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x - y| u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.3}$$

where the definition is restricted, in each case, to measurable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that the corresponding double integral is meaningful in the Lebesgue sense. Since for $r \geq 0$, the inequality $0 \leq \log(1 + r) \leq r$ for $r \geq 0$ holds, together with Hardy–Littlewood–Sobolev inequality (see Lemma 2.2), one has

$$|A_2(u, v)| \leq 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x - y|}|u(x)v(y)|\mathrm{d}x\mathrm{d}y \leq \bar{\mathcal{C}}_1\|u\|_{4/3}\|v\|_{4/3}, \tag{2.4}$$

where $\bar{\mathcal{C}}_1 > 0$ is a constant. According to (2.1), (2.2) and (2.3), we introduce the following functionals:

$$I_1 : H^1(\mathbb{R}^2) \to [0, \infty], \quad I_1(u) := A_1(u^2, u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(2 + |x - y|)\, u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y,$$

$$\begin{aligned} I_2 : L^{8/3}(\mathbb{R}^2) \to [0, \infty), \quad I_2(u) &:= A_2(u^2, u^2) \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{2}{|x - y|}\right) u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y, \end{aligned}$$

$$I_0 : H^1(\mathbb{R}^2) \to \mathbb{R} \cup \{\infty\}, \quad I_0(u) := A_0(u^2, u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x - y| u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y.$$

Note the functional I_2 is finite only on $L^{8/3}(\mathbb{R}^2)$. Indeed, (2.4) implies

$$|I_2(u)| \leq \bar{\mathcal{C}}_1 \|u\|_{8/3}^4, \quad \forall u \in L^{8/3}(\mathbb{R}^2). \tag{2.5}$$

Note that $\|\cdot\|_*$ is given by (1.11). The norm on the space X (see (1.5)) is then defined by

$$\|u\|_X := (\|\nabla u\|_2^2 + \|u\|_*^2)^{1/2}.$$

Using Rellich's Theorem [31], the embedding $X \hookrightarrow L^s(\mathbb{R}^2)$ is compact for $s \in [2, \infty)$. Consequently, the same holds for the embedding $E_{as} \hookrightarrow L^s(\mathbb{R}^2)$. For all $x, y \in \mathbb{R}^2$, it follows from the inequality $\log(2 + |x - y|) \leq \log(2 + |x|) + \log(2 + |y|)$ that

$$\begin{aligned} &|A_1(u_1v_1, u_2v_2)| \leq \|u_1\|_*\|v_1\|_*\|u_2\|_2\|v_2\|_2 + \|u_1\|_2\|v_1\|_2\|u_2\|_*\|v_2\|_*, \\ &\forall u_1, v_1, u_2, v_2 \in X. \end{aligned} \tag{2.6}$$

By [9, Lemma 2.2], for all $u, v \in X$ and $i = 0, 1, 2$, we can get that $I_0, I_1, I_2 \in \mathcal{C}^1(X)$, $I_0 = I_1 - I_2$ and $\langle I_i'(u), v\rangle = 4A_i(u^2, uv)$.

For any $u \in \hat{\mathcal{S}}_c$, let the function $g_u : (0, +\infty) \to \mathbb{R}$ be defined as follows:

$$\begin{aligned} g_u(t) := &\frac{t^2}{2}\|\nabla u\|_2^2 + \frac{\mu}{4} I_0(u) - \frac{\mu c^2}{4}\log t - \frac{\gamma}{2q} t^{2q-(2+\alpha)} \int_{\mathbb{R}^2} (I_\alpha * |u|^q)\, |u|^q \mathrm{d}x \\ &- \frac{1}{2t^2} \int_{\mathbb{R}^2} \left(e^{t^2u^2} - 1 - t^2u^2 - \frac{t^4u^4}{2}\right) \mathrm{d}x. \end{aligned} \tag{2.7}$$

Then we get that g_u is $\mathcal{C}^2$ on $(0, +\infty)$ and

$$\begin{aligned} g_u'(t) = &\frac{1}{t}\left\{ t^2\|\nabla u\|_2^2 - \frac{\mu c^2}{4} - \gamma \frac{2q - (2+\alpha)}{2q} t^{2q-(2+\alpha)} \int_{\mathbb{R}^2} (I_\alpha * |u|^q)\, |u|^q \mathrm{d}x \right. \\ &\left. - \frac{1}{t^2}\int_{\mathbb{R}^2} \left[(t^2u^2 - 1)e^{t^2u^2} + 1 - \frac{t^4u^4}{2}\right] \mathrm{d}x \right\} = \frac{1}{t}\mathcal{P}(tu_t), \quad \forall t > 0. \end{aligned} \tag{2.8}$$

2.2. PRELIMINARIES

The following preliminary results will be employed later in this paper.

Lemma 2.1 ([36], Gagliardo–Nirenberg inequality). *Let $s > 2$ and $u \in H^1(\mathbb{R}^2)$. Then there exists a sharp constant $\mathcal{C}_s > 0$ such that*

$$\|u\|_s \leq \mathcal{C}_s \|u\|_2^{\frac{2}{s}} \|\nabla u\|_2^{\frac{s-2}{s}}. \tag{2.9}$$

Lemma 2.2 ([25], Hardy–Littlewood–Sobolev inequality). *For $N \geq 1$, $t, r > 1$ and $0 < \alpha < N$ with $\frac{1}{t} + \frac{N-\alpha}{N} + \frac{1}{r} = 2$, $a \in L^t(\mathbb{R}^N)$ and $b \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(N, \alpha, t, r)$, independent of a and b, such that*

$$\left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{a(x)b(y)}{|x - y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y \right| \leq C(N, \alpha, t, r)\|a\|_t\|b\|_r. \tag{2.10}$$

Remark 2.3. Let $N = 2$, $t = r$ and $a = b = |u|^q$ in (2.10), we have that for any $q \in [1 + \frac{\alpha}{2}, \infty)$,

$$\int_{\mathbb{R}^2} (I_\alpha * |u|^q)\, |u|^q \mathrm{d}x \le \mathcal{K}_\alpha C(\alpha, q) \|u\|_{\frac{4q}{\alpha+2}}^{2q}, \quad \forall u \in H^1(\mathbb{R}^2). \tag{2.11}$$

Furthermore, using Lemma 2.1, we obtain that for any $q \in (1 + \frac{\alpha}{2}, \infty)$,

$$\begin{aligned}\int_{\mathbb{R}^2} (I_\alpha * |u|^q)\, |u|^q \mathrm{d}x &\le \mathcal{K}_\alpha C(\alpha, q) \mathcal{C}_{\frac{4q}{\alpha+2}}^{2q} c^{\frac{\alpha+2}{2}} \|\nabla u\|_2^{2q-\alpha-2} \\ &:= C_0(\alpha, q) c^{\frac{\alpha+2}{2}} \|\nabla u\|_2^{2q-\alpha-2}, \quad \forall u \in H^1(\mathbb{R}^2),\end{aligned} \tag{2.12}$$

where $C_0(\alpha, q) := \mathcal{K}_\alpha C(\alpha, q) \mathcal{C}_{\frac{4q}{\alpha+2}}^{2q}$.

Lemma 2.4 ([1, 5, 6])**.** (i) *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1 \right) \mathrm{d}x < \infty.$$

(ii) *If $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \le 1$, $\|u\|_2 \le \beta < \infty$ and $\alpha < 4\pi$, then there exists a constant $\bar{C}(\alpha, \beta)$, which depends only on α and β, such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1 \right) \mathrm{d}x \le \bar{C}(\alpha, \beta).$$

Lemma 2.5 ([11, Lemma 2.4])**.** *If v is a critical point of Φ restricted to E_{as}, then v is a critical point of Φ on X.*

Lemma 2.6. *Let $\mu, \gamma > 0$ and $1 + \frac{\alpha}{2} \le q < +\infty$. If there exist $u \in E_{as}$ and $\lambda \in \mathbb{R}$ satisfying* (1.1)*, then $\mathcal{P}(u) = 0$, where $\mathcal{P}$ is defined by* (1.9)*.*

Proof. Similar to [16, Theorem 1.4], the following Pohozaev identity

$$\begin{aligned}&\mu \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y + \frac{\mu}{4}\|u\|_2^4 + \lambda \|u\|_2^2 \\ &- \gamma \frac{2+\alpha}{2q} \int_{\mathbb{R}^2} (I_\alpha * |u|^q)\, |u|^q \mathrm{d}x - \int_{\mathbb{R}^2} \left(e^{u^2} - 1 - u^2 - \frac{u^4}{2} \right) \mathrm{d}x = 0\end{aligned} \tag{2.13}$$

holds. In addition, combining (2.13) with the following Nehari identity

$$\begin{aligned}&\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \lambda \|u\|_2^2 + \mu \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u^2(x) u^2(y) \mathrm{d}x\mathrm{d}y \\ &- \gamma \int_{\mathbb{R}^2} (I_\alpha * |u|^q)\, |u|^q \mathrm{d}x - \int_{\mathbb{R}^2} \left(e^{u^2} - 1 - u^2 \right) u^2 \mathrm{d}x = 0,\end{aligned} \tag{2.14}$$

it is easy to obtain that $\mathcal{P}(u) = 0$ and the proof is completed. □

Lemma 2.7 ([11, Lemma 2.2]). *For all $u_1, u_2 \in E_{as}$, there holds*

$$A_1(u_1^2, u_2^2) \geq \frac{1}{4}\|u_1\|_2^2\|u_2\|_*^2.$$

In particular, we have

$$I_1(u_1) \geq \frac{1}{4}\|u_1\|_2^2\|u_1\|_*^2, \quad \forall u_1 \in E_{as}. \tag{2.15}$$

Lemma 2.8 ([9, Lemma 2.6]). *Let $\{u_n\}$, $\{w_n\}$ and $\{v_n\}$ be bounded sequences with $u_n \rightharpoonup \bar{u} \in X$. Then for any $\varphi \in X$, we have $A_1(v_n w_n, \varphi(u_n - \bar{u})) \to 0$.*

Lemma 2.9 ([12, Lemma 2.4]). *If $u \in H^1(\mathbb{R}^2)$, then*

$$\int\limits_{\mathbb{R}^2} |u|^{2k}\mathrm{d}x \leq \frac{2 + 2^{2k-1}(k-2)}{(k-2)\pi^{k-1}}\|u\|_2^k\|\nabla u\|_2^k + \frac{k!}{2\pi^{k-1}}\|\nabla u\|_2^{2k}, \quad \forall k \in \mathbb{N}. \tag{2.16}$$

Furthermore, if $u \in \mathcal{S}_c$ and $\|\nabla u\|_2^2 < \pi$, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2} \left(e^{u^2} - 1 - u^2 - \frac{u^4}{2}\right)\mathrm{d}x \\
&\leq \frac{\|\nabla u\|_2^6}{2\pi(\pi - \|\nabla u\|_2^2)} + \frac{2c^{\frac{3}{2}}\|\nabla u\|_2^3}{\pi^2}\sum_{k=0}^{\infty}\frac{4^{k+2}(k+1)+1}{(k+1)(k+3)!}\left(\frac{\|u\|_2\|\nabla u\|_2}{\pi}\right)^k,
\end{aligned}$$

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2} \left[\left(u^2 - 1\right)e^{u^2} + 1 - \frac{u^4}{2}\right]\mathrm{d}x \\
&\leq \frac{\left(2\pi - \|\nabla u\|_2^2\right)\|\nabla u\|_2^6}{2\pi\left(\pi - \|\nabla u\|_2^2\right)^2} \\
&\quad + \frac{2c^{\frac{3}{2}}\|\nabla u\|_2^3}{\pi^2}\sum_{k=0}^{\infty}\frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!}\left(\frac{\|u\|_2\|\nabla u\|_2}{\pi}\right)^k.
\end{aligned}$$

3. LOCAL MINIMUM GEOMETRY ON $E_{as} \cap \mathcal{S}_c$

In this section we analyze the local minimum geometry of Φ restricted to $\hat{\mathcal{S}}_c = E_{as} \cap \mathcal{S}_c$. For any $\rho > 0$, set $A_\rho := \left\{u \in E_{as} : \|\nabla u\|_2^2 < \rho\right\}$ and $m(c) := \inf_{\hat{\mathcal{S}}_c \cap A_{\pi/3}} \Phi$. Let $c_1 = c_1(\gamma, \mu) > 0$ be the unique solution of the equation with respect to ϑ as follows:

$$\begin{aligned}
\frac{19\pi}{72} &= \frac{\mu\vartheta^2}{4} + \gamma\frac{2q - (2+\alpha)}{2q}C_0(\alpha, q)\vartheta^{\frac{\alpha+2}{2}}\left(\frac{\pi}{3}\right)^{\frac{2q-\alpha-2}{2}} \\
&\quad + \frac{2\vartheta^{3/2}}{3\sqrt{3\pi}}\sum_{k=0}^{\infty}\frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!}\left(\sqrt{\frac{\vartheta}{3\pi}}\right)^k,
\end{aligned} \tag{3.1}$$

where $q \in [1 + \frac{\alpha}{2}, \infty)$ and the definition of $C_0(\alpha, q)$ can be seen in Remark 2.3. Then we have the following lemma.

Lemma 3.1. *Assume $\mu, \gamma > 0$ and $1 + \frac{\alpha}{2} \leq q < +\infty$. The following properties hold.*

(i) *If $\|u\|_2^2 = c$, $\|\nabla u\|_2^2 = \frac{\pi}{3}$ and $\mathcal{P}(u) \leq 0$, then $c \geq c_1$.*

(ii) *If $u \in \hat{\mathcal{S}}_c \cap \partial A_{\pi/3}$ and $c \in (0, c_1)$, then $\mathcal{P}(u) > 0$. In particular, there exists $t_u \in (0,1)$ such that $\mathcal{P}(t_u u_{t_u}) = 0$.*

Proof. (i) By $\|u\|_2^2 = c$, $\|\nabla u\|_2^2 = \frac{\pi}{3}$, $\mathcal{P}(u) \leq 0$ and (1.9), we have

$$\frac{\pi}{3} = \|\nabla u\|_2^2 \leq \frac{\mu c^2}{4} + \int_{\mathbb{R}^2} \left[(u^2-1)e^{u^2} + 1 - \frac{u^4}{2}\right] \mathrm{d}x + \gamma \frac{2q-(2+\alpha)}{2q} \int_{\mathbb{R}^2} (I_\alpha * |u|^q)\, |u|^q \mathrm{d}x.$$

Next, we distinguish the following two cases.

Case 1. $q = 1 + \frac{\alpha}{2}$. Using Lemma 2.9, it is easy to deduce that

$$\begin{aligned}
\frac{\pi}{3} &\leq \frac{\mu c^2}{4} + \frac{(2\pi - \|\nabla u\|_2^2)\|\nabla u\|_2^6}{2\pi\left(\pi - \|\nabla u\|_2^2\right)^2} \\
&\quad + \frac{2c^{3/2}\|\nabla u\|_2^3}{\pi^2} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\frac{\sqrt{c}\|\nabla u\|_2}{\pi}\right)^k \\
&= \frac{\mu c^2}{4} + \frac{5\pi}{72} + \frac{2c^{3/2}}{3\sqrt{3\pi}} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\sqrt{\frac{c}{3\pi}}\right)^k.
\end{aligned}$$

Case 2. $q \in (1 + \frac{\alpha}{2}, \infty)$. It follows from (2.12) and Lemma 2.9 that

$$\begin{aligned}
\frac{\pi}{3} &\leq \frac{\mu c^2}{4} + \gamma \frac{2q-(2+\alpha)}{2q} C_0(\alpha,q) c^{\frac{\alpha+2}{2}} \|\nabla u\|_2^{2q-\alpha-2} + \frac{(2\pi - \|\nabla u\|_2^2)\|\nabla u\|_2^6}{2\pi\left(\pi - \|\nabla u\|_2^2\right)^2} \\
&\quad + \frac{2c^{3/2}\|\nabla u\|_2^3}{\pi^2} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\frac{\sqrt{c}\|\nabla u\|_2}{\pi}\right)^k \\
&= \frac{\mu c^2}{4} + \frac{5\pi}{72} + \gamma \frac{2q-(2+\alpha)}{2q} C_0(\alpha,q) c^{\frac{\alpha+2}{2}} \left(\frac{\pi}{3}\right)^{\frac{2q-\alpha-2}{2}} \\
&\quad + \frac{2c^{3/2}}{3\sqrt{3\pi}} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\sqrt{\frac{c}{3\pi}}\right)^k.
\end{aligned}$$

Combining Cases 1 and 2, we have

$$\begin{aligned}
\frac{19\pi}{72} &\leq \frac{\mu c^2}{4} + \gamma \frac{2q-(2+\alpha)}{2q} C_0(\alpha,q) c^{\frac{\alpha+2}{2}} \left(\frac{\pi}{3}\right)^{\frac{2q-\alpha-2}{2}} \\
&\quad + \frac{2c^{3/2}}{3\sqrt{3\pi}} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\sqrt{\frac{c}{3\pi}}\right)^k.
\end{aligned} \tag{3.2}$$

Then from (3.1) and (3.2), we get that $c \geq c_1$.

(ii) According to (i), it is easy to obtain $\mathcal{P}(u) > 0$. Furthermore, since $u \in \hat{\mathcal{S}}_c \cap \partial A_{\pi/3}$, we have $g_u'(1) > 0$. By (2.8), one has $t g_u'(t) \to -\frac{\mu c^2}{4} < 0$ as $t \to 0^+$. Thus, there exists $t_u \in (0,1)$ such that $g_u'(t_u) = 0$, that is $\mathcal{P}(t_u u_{t_u}) = 0$. This completes the proof of the lemma. □

4. PROOF OF THEOREM 1.1: EXISTENCE OF LOCAL MINIMUM

This section is devoted to proving Theorem 1.1 by applying the critical point theory on manifolds developed in [13].

Lemma 4.1 ([13])**.** *Let $\varphi \in \mathcal{C}^1(E,\mathbb{R})$ and $K \subset E$. If there exists $\tilde{\rho} > 0$ satisfying*

$$\tilde{a} := \inf_{v \in M \cap K} \varphi(v) < \tilde{b} := \inf_{v \in M \cap (K_{\tilde{\rho}} \setminus K)} \varphi(v),$$

where $K_{\tilde{\rho}} := \{v \in E : \|u - v\| < \tilde{\rho}, \forall u \in K\}$, then for any $\varepsilon \in (0, (\tilde{b} - \tilde{a})/2)$, $\delta \in (0, \tilde{\rho}/2)$ and $w \in M \cap A$ such that $\varphi(w) \le \tilde{a} + \varepsilon$, there exists $u \in M$ such that

(i) $\tilde{a} - 2\varepsilon \le \varphi(u) \le \tilde{a} + 2\varepsilon$,
(ii) $\|u - w\|_E \le 2\delta$,
(iii) $\|\varphi|'_M(u)\| \le \frac{8\varepsilon}{\delta}$.

Corollary 4.2. *Let $\varphi \in \mathcal{C}^1(E,\mathbb{R})$ and $K \subset E$. If there exist $\rho > 0$ and $\bar{u} \in M \cap K$ such that*

$$\varphi(\bar{u}) = \inf_{v \in M \cap K} \varphi(v) < \inf_{v \in M \cap (K_\rho \setminus K)} \varphi(v),$$

then $\varphi|'_M(\bar{u}) = 0$.

Lemma 4.3 ([13])**.** *Assume that $\tilde{\theta} \in \mathbb{R}$, $\tilde{\varphi} \in \mathcal{C}^1(E \times \mathbb{R}, \mathbb{R})$ and $\tilde{\Upsilon} \subset M \times \mathbb{R}$ is a closed set. Let*

$$\tilde{\Gamma} := \left\{ \tilde{\gamma} \in \mathcal{C}([0,1], M \times \mathbb{R}) : \tilde{\gamma}(0) \in \tilde{\Upsilon}, \tilde{\varphi}(\tilde{\gamma}(1)) < \tilde{\theta} \right\}.$$

If $\tilde{\varphi}$ satisfies

$$\tilde{a} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{\varphi}(\tilde{\gamma}(t)) > \tilde{b} := \sup_{\tilde{\gamma} \in \tilde{\Gamma}} \max \left\{ \tilde{\varphi}(\tilde{\gamma}(0)), \tilde{\varphi}(\tilde{\gamma}(1)) \right\},$$

then, for every $\varepsilon \in (0, (\tilde{a} - \tilde{b})/2)$, $\delta > 0$ and $\tilde{\gamma}_* \in \tilde{\Gamma}$ such that $\sup_{t \in [0,1]} \tilde{\varphi}(\tilde{\gamma}_*(t)) \le \tilde{a} + \varepsilon$, there exists $(v, \tau) \in M \times \mathbb{R}$ such that

(i) $\tilde{a} - 2\varepsilon \le \tilde{\varphi}(v, \tau) \le \tilde{a} + 2\varepsilon$,
(ii) $\min_{t \in [0,1]} \|(v, \tau) - \tilde{\gamma}_*(t)\|_{E \times \mathbb{R}} \le 2\delta$,
(iii) $\left\|\tilde{\varphi}|'_{M \times \mathbb{R}}(v, \tau)\right\| \le \frac{8\varepsilon}{\delta}$.

Lemma 4.4. *Assume that $u_n \rightharpoonup u_c$ in E_{as} with $\|u_n\|_2^2 = c$ and $\|\nabla u_n\|_2^2 \le \frac{\pi}{3}$. Then, up to a subsequence, there holds*

$$\Phi(u_n) = \Phi(u_c) + \Phi(v_n) + \frac{\mu}{2} A_0(u_c^2, v_n^2) + o(1). \tag{4.1}$$

Proof. Set $v_n := u_n - u_c$. Then, passing to a subsequence if necessary, we have $v_n \rightharpoonup 0$ in E_{as}, $\|v_n\|_s \to 0$ for $s \in [2, +\infty)$ and $v_n \to 0$ a.e. $x \in \mathbb{R}^2$. It follows from (2.1)–(2.4), Lemma 2.4 and [29, Lemma 2.4] that

$$\|\nabla v_n\|_2^2 = \|\nabla u_n\|_2^2 - \|\nabla u_c\|_2^2 + o(1), \tag{4.2}$$

$$I_0(u_n) = I_0(u_c) + I_0(v_n) + 2A_0(u_c^2, v_n^2) + o(1), \tag{4.3}$$

$$\int_{\mathbb{R}^2} (I_\alpha * |u_n|^q) |u_n|^q \mathrm{d}x - \int_{\mathbb{R}^2} (I_\alpha * |v_n|^q) |v_n|^q \mathrm{d}x = \int_{\mathbb{R}^2} (I_\alpha * |u_c|^q) |u_c|^q \mathrm{d}x + o(1) \tag{4.4}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left| \left(e^{u_n^2}-1\right) - \left(e^{v_n^2}-1\right) - \left(e^{u_c^2}-1\right) \right| \mathrm{d}x \\
&\le \int_{\mathbb{R}^2} (|u_n|+|u_c|)|v_n| e^{u_n^2+u_c^2} \mathrm{d}x + \int_{\mathbb{R}^2} e^{v_n^2}|v_n|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^2} (|u_n|+|u_c|)|v_n| \left(e^{u_n^2+u_c^2}-1\right) \mathrm{d}x + \int_{\mathbb{R}^2} (|u_n|+|u_c|)|v_n| \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2} \left(e^{v_n^2}-1\right) |v_n|^2 \mathrm{d}x + \|v_n\|_2^2 \\
&\le \left[\int_{\mathbb{R}^2} \left(e^{2u_n^2+2u_c^2}-1\right) \mathrm{d}x \right]^{\frac{1}{2}} (\|u_n\|_4 + \|u_c\|_4)\|v_n\|_4 \\
&\quad + \left[\int_{\mathbb{R}^2} \left(e^{2v_n^2}-1\right) \mathrm{d}x \right]^{\frac{1}{2}} \|v_n\|_4^2 + o(1) \\
&= o(1).
\end{aligned} \tag{4.5}$$

Hence, it follows from (1.7), (4.2)–(4.5) and the Brezis–Lieb lemma that (4.1) holds. □

Proof of Theorem 1.1. Let $\{u_n\} \subset \hat{\mathcal{S}}_c \cap A_{\pi/3}$ be a minimizing sequence of Φ for $m(c)$. Observe that $\{|u_n|\} \subset \hat{\mathcal{S}}_c \cap A_{\pi/3}$ is also a minimizing sequence for $m(c)$. We may assume, without loss of generality, that $u_n \ge 0$. Consequently,

$$\|u_n\|_2^2 = c, \quad \|\nabla u_n\|_2^2 \le \frac{\pi}{3}, \quad \Phi(u_n) = m(c) + o(1). \tag{4.6}$$

From (1.7), (2.5), (2.12), (4.6) and Lemma 2.9, we obtain that $\{I_1(u_n)\}$ is bounded. So by (2.15), one has that $\{\|u_n\|_*\}$ is bounded, and $\{u_n\}$ is bounded in E_{as}. Hence, up to a subsequence, we may assume that

$$u_n \rightharpoonup u_c \text{ in } E_{as}, \quad u_n \to u_c \text{ in } L^s(\mathbb{R}^2), \quad s \in [2,\infty) \quad \text{and} \quad u_n \to u_c \text{ a.e. on } \mathbb{R}^2. \tag{4.7}$$

Furthermore, we get $\|u_c\|_2^2 = \|u_n\|_2^2 = c, \|\nabla u_c\|_2^2 \le \frac{\pi}{3}$. Set $v_n := u_n - u_c$. By (4.6) and Lemma 4.4, we have that

$$\begin{aligned}
m(c) + o(1) &= \Phi(u_c) + \Phi(v_n) + \frac{\mu}{2} A_0(u_c^2, v_n^2) + o(1) \\
&\ge m(c) + \Phi(v_n) + \frac{\mu}{2} A_1(u_c^2, v_n^2) + o(1).
\end{aligned} \tag{4.8}$$

This means $\Phi(v_n) \leq o(1)$. It follows from (1.7), (2.5), (2.9), Remark 2.3 and Lemma 2.4 that

$$\begin{aligned}
&\|\nabla v_n\|_2^2 + \frac{\mu}{2} I_1(v_n) \\
&\leq \frac{\mu}{2} I_2(v_n) + \int_{\mathbb{R}^2} \left(e^{v_n^2} - 1 - v_n^2 - \frac{v_n^4}{2} \right) \mathrm{d}x + \frac{\gamma}{q} \int_{\mathbb{R}^2} \left(I_\alpha * |v_n|^q \right) |v_n|^q \mathrm{d}x + o(1) \\
&\leq \frac{\mu}{2} \bar{\mathcal{C}}_1 \|v_n\|_{8/3}^4 + \int_{\mathbb{R}^2} \left(e^{v_n^2} - 1 - v_n^2 - \frac{v_n^4}{2} \right) \mathrm{d}x + o(1) \\
&\leq \frac{\mu}{2} \bar{\mathcal{C}}_1 \mathcal{C}_{8/3}^4 \|\nabla v_n\|_2 \|v_n\|_2^3 + \int_{\mathbb{R}^2} \left(e^{v_n^2} - 1 \right) v_n^2 \mathrm{d}x + o(1) \\
&\leq \frac{\mu}{2} \bar{\mathcal{C}}_1 \mathcal{C}_{8/3}^4 \|\nabla v_n\|_2 \|v_n\|_2^3 + \mathcal{C}_4^2 \left[\int_{\mathbb{R}^2} \left(e^{2v_n^2} - 1 \right) \mathrm{d}x \right]^{\frac{1}{2}} \|\nabla v_n\|_2 \|v_n\|_2 + o(1) \\
&= o(1).
\end{aligned} \tag{4.9}$$

From (4.9), we obtain $\|\nabla v_n\|_2^2 = o(1)$ and $I_1(v_n) = o(1)$, therefore $\Phi(v_n) = o(1)$. In view of (4.8), we find $A_1(u_c^2, v_n^2) = o(1)$, and the Lemma 2.7 implies $v_n \to 0$ in E_{as}, i.e. $u_n \to u_c$ in E_{as}. Hence, $u_c \in A_{\pi/3}$ and $\Phi(u_c) = m(c)$. Note that $u_n \geq 0$, we also have $u_c \geq 0$.

We now show that $\|\nabla u_c\|_2^2 < \frac{\pi}{3}$. Let us assume by contradiction that $\|\nabla u_c\|_2^2 = \frac{\pi}{3}$. Then Lemma 3.1 immediately yields $\mathcal{P}(u_c) > 0$. Choose $t_0 < 1$ sufficiently close to 1. Using (2.8), one has $t_0 u_c(t_0 x) \in A_{\pi/3}$ and $\Phi(t_0 u_c(t_0 x)) < \Phi(u_c) = m(c)$, which is a contradiction. Therefore, it follows from Corollary 4.2 that $\Phi\big|_{\hat{\mathcal{S}}_c}'(u_c) = 0$, and thus there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$ such that $\langle \Phi'(u_c) + \lambda_c u_c, \phi \rangle = 0$ for any $\phi \in E_{as}$. By Lemma 2.5, for any $\phi \in X$, we have that $\langle \Phi'(u_c) + \lambda_c u_c, \phi \rangle = 0$, that is

$$-\Delta u_c + \lambda_c u_c + \mu(\log|x| * u_c^2) u_c - \gamma(I_\alpha * |u_c|^q)|u_c|^{q-2} u_c - (e^{u_c^2} - 1 - u_c^2) u_c = 0, \ x \in \mathbb{R}^2.$$

This completes the proof. □

5. THE MOUNTAIN PASS STRUCTURE AND (PS) SEQUENCES

In this section, based on the local minimum obtained in Theorem 1.1, we establish a mountain pass structure. Furthermore, by applying critical point theories on a manifold, we obtain a Palais–Smale sequence $\{u_n\}$ satisfying the additional property $\mathcal{P}(u_n) \to 0$. This property allows us to prove the boundedness of $\{\|\nabla u_n\|_2\}$ and $\{I_1(u_n)\}$. Throughout this section, we assume that $\mu, \gamma > 0$ and $q \in \left[1 + \frac{\alpha}{2}, +\infty\right)$.

Let $c_2 = c_2(\gamma, \mu) > 0$ denote the unique root of the equation below.

$$c = \sqrt{\frac{4(2q-\alpha-4)}{\mu(2q-\alpha-2)}\eta(c,\gamma)}, \tag{5.1}$$

where $q \in (2+\frac{\alpha}{2}, \frac{5}{2}+\frac{\alpha}{2})$ and $\eta(c,\gamma) > 0$ denotes the unique root of the following equation with respect to ϑ:

$$\begin{aligned}
&\gamma(2q-\alpha-2)^2 \frac{\vartheta^{2q-\alpha-4} C_0(\alpha,q) c^{(\alpha+2)/2}}{4q} \\
&+ \frac{2c}{\pi} \sum_{k=3}^{\infty} \frac{(k-1)^2[4^{k-1}(k-2)+1]}{(k-2)k!} \left(\frac{\vartheta^2\sqrt{c}}{\pi}\right)^{k-2} \\
&+ \frac{\vartheta\left(4\pi^2 - 3\pi\vartheta^2 + \vartheta^4\right)}{2\pi(\pi-\vartheta^2)^3} = 1.
\end{aligned} \tag{5.2}$$

Here the constant $C_0(\alpha, q)$ is defined in Remark 2.3. Let c_1 be defined by (3.1). Then we have the following result.

Lemma 5.1. *Let $c \in (0, c_1)$ if $q \in [1+\frac{\alpha}{2}, 2+\frac{\alpha}{2}] \cup [\frac{5}{2}+\frac{\alpha}{2}, +\infty)$ and $c \in (0, c_2)$ if $q \in (2+\frac{\alpha}{2}, \frac{5}{2}+\frac{\alpha}{2})$. Then for any $u \in \hat{\mathcal{S}}_c$, there exists a unique $s_u^+ > 0$ which is a strict local minimum point for g_u and a unique $s_u^- > 0$ which is a strict local maximum point for g_u, where g_u be given by (2.7).*

Proof. For any $u \in \hat{\mathcal{S}}_c$, set $\tau := 1/\|\nabla u\|_2$ and $\hat{u} := \tau u_\tau$. Then $\|\nabla \hat{u}\|_2^2 = 1$ and $t\hat{u}_t = (t\tau)u_{t\tau}$ for all $t > 0$. Thus, it suffices to establish this lemma under the specific assumption that $u \in \hat{\mathcal{S}}_c$ with $\|\nabla u\|_2^2 = 1$.

Fix $u \in \hat{\mathcal{S}}_c$ with $\|\nabla u\|_2^2 = 1$, we have

$$g_u'(t) = \frac{1}{t}\mathcal{P}(tu_t), \quad \forall t > 0. \tag{5.3}$$

The proof of this lemma is divided into the following two cases.

Case (i). $q \in [1+\frac{\alpha}{2}, 2+\frac{\alpha}{2}] \cup [\frac{5}{2}+\frac{\alpha}{2}, +\infty)$.

From (5.3), we define the function $\zeta_u : (0, +\infty) \to \mathbb{R}$ by

$$\begin{aligned}
\zeta_u(t) := \frac{g_u'(t)}{t} = \frac{1}{t^2}\mathcal{P}(tu_t) = 1 - \frac{\mu c^2}{4t^2} - \gamma \frac{2q-(\alpha+2)}{2q} t^{2q-(\alpha+4)} \int_{\mathbb{R}^2} \left(I_\alpha * |u|^q\right)|u|^q \mathrm{d}x \\
- \sum_{k=3}^{\infty} \frac{(k-1)t^{2k-4}}{k!} \|u\|_{2k}^{2k}, \quad \forall t > 0.
\end{aligned} \tag{5.4}$$

Then we can deduce that

$$\begin{aligned}\zeta_u'(t) = \frac{\mu c^2}{2t^3} - \gamma\frac{(2q-\alpha-2)(2q-\alpha-4)}{2q}t^{2q-(\alpha+5)}\int\limits_{\mathbb{R}^2}(I_\alpha * |u|^q)\,|u|^q \mathrm{d}x \\ -\sum_{k=3}^{\infty}\frac{2(k-1)(k-2)t^{2k-5}}{k!}\|u\|_{2k}^{2k}, \quad \forall t>0.\end{aligned} \tag{5.5}$$

It is straightforward to get that $\zeta_u'(t)$ is strictly decreasing when $q \in [1+\frac{\alpha}{2}, 2+\frac{\alpha}{2}] \cup [\frac{5}{2}+\frac{\alpha}{2}, +\infty)$, and

$$\lim_{t\to 0^+}\zeta_u'(t) = +\infty, \qquad \lim_{t\to+\infty}\zeta_u'(t) = -\infty.$$

This means $\zeta_u'(t)$ has only a zero point on $(0,+\infty)$. Furthermore, we can find $t_0 > 0$ such that $t_0 u_{t_0} \in \hat{\mathcal{S}}_c \cap \partial A_{\pi/3}$. By Lemma 3.1(ii), we have $\mathcal{P}(t_0 u_{t_0}) > 0$. Thus, we have that

$$\zeta_u(t_0) = \frac{g_u'(t)}{t_0} = \frac{1}{t_0^2}\mathcal{P}(t_0 u_{t_0}) > 0.$$

Consequently, $\zeta_u(t)$ has exactly two roots in $(0,+\infty)$. The first zero $s_u^+ > 0$ of $g_u'(t) = 0$ yields a strict local minimum of g_u, while the second zero $s_u^- > 0$ yields a strict local maximum.

Case (ii). $q \in (2+\frac{\alpha}{2}, \frac{5}{2}+\frac{\alpha}{2})$.

Let $t_* > 0$ such that

$$\begin{aligned}\gamma(2q-\alpha-2)^2\frac{t_*^{2q-\alpha-4}}{4q}\int\limits_{\mathbb{R}^2}(I_\alpha * |u|^q)\,|u|^q \mathrm{d}x \\ + t_*^{-4}\int\limits_{\mathbb{R}^2}\left[\left(1-t_*^2u^2+t_*^4u^4\right)e^{t_*^2u^2} - 1 - \frac{t_*^4u^4}{2}\right]\mathrm{d}x = 1.\end{aligned} \tag{5.6}$$

It follows that

$$\begin{aligned}t^2 > \gamma(2q-\alpha-2)^2\frac{t^{2q-\alpha-2}}{4q}\int\limits_{\mathbb{R}^2}(I_\alpha * |u|^q)\,|u|^q \mathrm{d}x \\ + t^{-2}\int\limits_{\mathbb{R}^2}\left[\left(1-t^2u^2+t^4u^4\right)e^{t^2u^2} - 1 - \frac{t^4u^4}{2}\right]\mathrm{d}x, \quad 0<t<t_*,\end{aligned} \tag{5.7}$$

$$\begin{aligned}t^2 < \gamma(2q-\alpha-2)^2\frac{t^{2q-\alpha-2}}{4q}\int\limits_{\mathbb{R}^2}(I_\alpha * |u|^q)\,|u|^q \mathrm{d}x \\ + t^{-2}\int\limits_{\mathbb{R}^2}\left[\left(1-t^2u^2+t^4u^4\right)e^{t^2u^2} - 1 - \frac{t^4u^4}{2}\right]\mathrm{d}x, \quad t_*<t<+\infty.\end{aligned} \tag{5.8}$$

According to (2.8) and (5.7), we get that for any $0 < t \leq t_*$,

$$\begin{aligned} g'_u(t) &= \frac{1}{t}\left\{t^2 - \frac{\mu c^2}{4} - \gamma\frac{2q-(\alpha+2)}{2q}t^{2q-(\alpha+2)}\int_{\mathbb{R}^2}(I_\alpha * |u|^q)|u|^q \mathrm{d}x \right.\\ &\quad \left. -t^{-2}\int_{\mathbb{R}^2}\left[(t^2u^2-1)e^{t^2u^2}+1-\frac{t^4u^4}{2}\right]\mathrm{d}x\right\}\\ &\geq \frac{1}{t}\left\{t^2 - \frac{\mu c^2}{4} - \gamma\frac{2q-\alpha-2}{2q}t^{2q-(\alpha+2)}\int_{\mathbb{R}^2}(I_\alpha * |u|^q)|u|^q \mathrm{d}x \right.\\ &\quad \left. -\frac{t^{-2}}{2}\int_{\mathbb{R}^2}\left[\left(1-t^2u^2+t^4u^4\right)e^{t^2u^2}-1-\frac{t^4u^4}{2}\right]\mathrm{d}x\right\} \\ &\geq \frac{1}{t}\left\{t^2 - \frac{2t^2}{2q-\alpha-2} + \frac{2t^{-2}}{2q-\alpha-2}\int_{\mathbb{R}^2}\left[\left(1-t^2u^2+t^4u^4\right)e^{t^2u^2}-1-\frac{t^4u^4}{2}\right]\mathrm{d}x \right.\\ &\quad \left. -\frac{\mu c^2}{4} - \frac{t^{-2}}{2}\int_{\mathbb{R}^2}\left[\left(1-t^2u^2+t^4u^4\right)e^{t^2u^2}-1-\frac{t^4u^4}{2}\right]\mathrm{d}x\right\}\\ &\geq \frac{1}{t}\left[\left(1-\frac{2}{2q-\alpha-2}\right)t^2 - \frac{\mu c^2}{4}\right]. \end{aligned} \tag{5.9}$$

If $t_* < \pi$, then from (2.12), (2.16) and (5.6), we have

$$\begin{aligned} 1 &= \gamma(2q-\alpha-2)^2\frac{t_*^{2q-\alpha-4}}{4q}\int_{\mathbb{R}^2}(I_\alpha * |u|^q)|u|^q \mathrm{d}x \\ &\quad + t_*^{-4}\int_{\mathbb{R}^2}\left[(1-t_*^2u^2+t_*^4u^4)e^{t_*^2u^2}-1-\frac{t_*^4u^4}{2}\right]\mathrm{d}x\\ &= \gamma(2q-\alpha-2)^2\frac{t_*^{2q-\alpha-4}}{4q}\int_{\mathbb{R}^2}(I_\alpha * |u|^q)|u|^q \mathrm{d}x + \sum_{k=3}^{\infty}\frac{(k-1)^2}{k!}\|u\|_{2k}^{2k}t_*^{2(k-2)} \\ &\leq \gamma(2q-\alpha-2)^2\frac{t_*^{2q-\alpha-4}C_0(\alpha,q)c^{\frac{\alpha+2}{2}}}{4q} \\ &\quad + \frac{2c}{\pi}\sum_{k=3}^{\infty}\frac{(k-1)^2[4^{k-1}(k-2)+1]}{(k-2)k!}\left(\frac{t_*^2\sqrt{c}}{\pi}\right)^{k-2}\\ &\quad + \frac{t_*\left(4\pi^2-3\pi t_*^2+t_*^4\right)}{2\pi(\pi-t_*^2)^3}. \end{aligned} \tag{5.10}$$

Combining (5.2) with (5.10), we get that $\eta(c,\gamma) \leq t_*$. Using (5.2) and the fact that $\eta(c,\gamma)$ is decreasing on $c > 0$. Therefore, by (5.1), one has

$$\frac{\mu(2q-\alpha-2)c^2}{4(2q-\alpha-4)} < \frac{\mu(2q-\alpha-2)c_2^2}{4(2q-\alpha-4)} = \eta^2(c_2,\gamma) < \eta^2(c,\gamma) \leq t_*^2, \quad \forall c \in (0,c_2). \tag{5.11}$$

Hence, (5.11) implies that there exists $\delta > 0$ such that $(1 - \frac{2}{2q-\alpha-2})t^2 - \frac{\mu c^2}{4} > 0$ for any $t \in (t_* - \delta, t_*]$. Using (5.9), we can deduce that $g_u'(t) > 0$ for any $t \in (t_* - \delta, t_*]$. Hence, $g_u(t)$ is increasing in $(t_* - \delta, t_*]$.

Noting that $g_u(t) \to +\infty$ as $t \to 0^+$ and $g_u(t) \to -\infty$ as $t \to +\infty$, we deduce the existence of at least one local minimum point $s_u^+ < t_*$ and one local maximum point $s_u^- > t_*$ for g_u. Since $s_u^- > t_*$, by (5.8), we obtain

$$\begin{aligned}(s_u^-)^2 &< \gamma(2q-\alpha-2)^2 \frac{(s_u^-)^{2q-\alpha-2}}{4q} \int\limits_{\mathbb{R}^2} (I_\alpha * |u|^q)\,|u|^q \mathrm{d}x \\ &\quad + (s_u^-)^{-2} \int\limits_{\mathbb{R}^2} \left[(1 - (s_u^-)^2 u^2 + (s_u^-)^4 u^4) e^{(s_u^-)^2 u^2} - 1 - \frac{(s_u^-)^4 u^4}{2} \right] \mathrm{d}x.\end{aligned} \tag{5.12}$$

Note that s_u^- is a local maximum point of g_u, so $g_u'(s_u^-) = 0$, i.e.,

$$\begin{aligned}&s_u^- - \frac{\mu c^2}{4 s_u^-} - \gamma \frac{2q-(2+\alpha)}{2q} (s_u^-)^{2q-(\alpha+3)} \int\limits_{\mathbb{R}^2} (I_\alpha * |u|^q)\,|u|^q \mathrm{d}x \\ &- (s_u^-)^{-3} \int\limits_{\mathbb{R}^2} \left[\left((s_u^-)^2 u^2 - 1 \right) e^{(s_u^-)^2 u^2} + 1 - \frac{(s_u^-)^4 u^4}{2} \right] \mathrm{d}x = 0.\end{aligned} \tag{5.13}$$

Then, from (5.12) and (5.13), we have

$$\begin{aligned}g_u''(s_u^-) &= 1 + \frac{\mu c^2}{4(s_u^-)^2} - \gamma \frac{2q-(2+\alpha)}{2q} \left[2q-(\alpha+3)\right] (s_u^-)^{2q-(\alpha+4)} \int\limits_{\mathbb{R}^2} (I_\alpha * |u|^q)\,|u|^q \mathrm{d}x \\ &\quad - \sum_{k=3}^{\infty} \frac{(k-1)(2k-3)}{k!} \|u\|_{2k}^{2k} (s_u^-)^{2k-4} \\ &= 2 - \gamma \frac{[2q-(2+\alpha)]^2}{2q} (s_u^-)^{2q-(\alpha+4)} \int\limits_{\mathbb{R}^2} (I_\alpha * |u|^q)\,|u|^q \mathrm{d}x \\ &\quad - \frac{2}{(s_u^-)^2} \sum_{k=3}^{\infty} \frac{(k-1)^2}{k!} \|u\|_{2k}^{2k} (s_u^-)^{2(k-1)} \\ &< 0.\end{aligned}$$

Therefore, g_u attains a strict local maximum at s_u^-.

We now establish the uniqueness of s_u^-. By contradiction, suppose that g_u admits another local maximum point $\hat{s}_u^- > 0$. First, we note that if $0 < \hat{s}_u^- < t_*$, it follows from $g_u'(\hat{s}_u^-) = 0$ and (5.7) that

$$g_u''(\hat{s}_u^-) = \frac{2}{(\hat{s}_u^-)^2}\Bigg\{(\hat{s}_u^-)^2 - \gamma\frac{[2q-(2+\alpha)]^2}{4q}(\hat{s}_u^-)^{2q-(\alpha+2)}\int_{\mathbb{R}^2}(I_\alpha * |u|^q)\,|u|^q\mathrm{d}x - \sum_{k=3}^{\infty}\frac{(k-1)^2}{k!}\|u\|_{2k}^{2k}(\hat{s}_u^-)^{2(k-1)}\Bigg\} > 0, \tag{5.14}$$

which is impossible. Hence, $\hat{s}_u^- > t_*$, and the same reasoning as before yields $g_u''(\hat{s}_u^-) < 0$. This implies the existence of an additional critical point: $\theta_u \in (\hat{s}_u^-, s_u^-)$ or $\theta_u \in (s_u^-, \hat{s}_u^-)$, which is a local minimum for g_u. In view of (5.8), we again deduce $g_u''(\theta_u) < 0$, a contradiction. Therefore, s_u^- is unique.

By an analogous argument, one can show that s_u^+ is the unique local minimum point of g_u.

Combining Cases (i) and (ii), the proof of this lemma is thus completed. □

We omit the proofs of the following two lemmas, as they can be derived by a straightforward adaptation of the arguments in [15, Section 4].

Lemma 5.2. *Assume $1 + \frac{\alpha}{2} \le q < +\infty$. For any $c \in (0, c_1)$, there exists $\kappa_c > 0$ such that*

$$M(c) := \inf_{\gamma\in\Gamma_c}\max_{t\in[0,1]}\Phi(\gamma(t)) \ge \kappa_c > \sup_{\gamma\in\Gamma_c}\{\Phi(\gamma(0)), \Phi(\gamma(1))\} \tag{5.15}$$

where

$$\Gamma_c = \Big\{\gamma \in \mathcal{C}([0,1], \hat{\mathcal{S}}_c) : \gamma(0) = u_c, \Phi(\gamma(1)) < m(c) - 1\Big\} \tag{5.16}$$

and u_c is given by Theorem 1.1.

Lemma 5.3. *Let $1 + \frac{\alpha}{2} \le q < +\infty$. Then for any $c \in (0, c_1)$, there exists $\{u_n\} \subset \hat{\mathcal{S}}_c$ such that*

$$\Phi(u_n) \to M(c) > m(c), \quad \Phi|'_{\hat{\mathcal{S}}_c}(u_n) \to 0 \quad \text{and} \quad \mathcal{P}(u_n) \to 0. \tag{5.17}$$

6. PROOF OF THEOREM 1.2: EXISTENCE OF MOUNTAIN PASS SOLUTION

This section is devoted to proving Theorem 1.2. To this end, we first establish the compactness of the Palais–Smale sequences from Lemma 5.3.

We begin by deriving a sharp energy estimate for the minimax level $M(c)$ introduced in Lemma 5.2. This estimate is crucial for recovering compactness. Let us introduce the Moser-type functions $\tilde{w}_n(x)$ supported in $B_1(0)$, defined by

$$\tilde{w}_n(x) = \frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log n}, & 0 \le |x| \le 1/n,\\ \frac{\log(1/|x|)}{\sqrt{\log n}}, & 1/n \le |x| \le 1,\\ 0, & |x| \ge 1.\end{cases} \tag{6.1}$$

A straightforward calculation yields

$$\|\nabla \tilde{w}_n\|_2^2 = 1, \quad \|\tilde{w}_n\|_2^2 = \frac{1}{4\log n} - \frac{1}{4n^2\log n} - \frac{1}{2n^2}, \tag{6.2}$$

$$\|\tilde{w}_n\|_{8/3}^{8/3} = O\left(\frac{1}{\log^{4/3} n}\right), \quad \|\tilde{w}_n\|_*^2 = \int_{\mathbb{R}^2} \log(2+|x|)|\tilde{w}_n|^2 \mathrm{d}x = O\left(\frac{1}{\log n}\right), \quad n \to \infty, \tag{6.3}$$

and

$$|I_0(\tilde{w}_n)| = \left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| \tilde{w}_n^2(x)\tilde{w}_n^2(y)\mathrm{d}x\mathrm{d}y \right| \leq O\left(\frac{1}{\log^2 n}\right), \ n \to \infty. \tag{6.4}$$

With the estimates (6.2)–(6.4), we now establish the following lemma.

Lemma 6.1. *If* $1 + \frac{\alpha}{2} \leq q < +\infty$ *and* $c \in (0, c_1)$*, then*

$$M(c) < m(c) + 2\pi. \tag{6.5}$$

Proof. Let u_c be given by Theorem 1.1. Then, together with Lemma 2.6, we have

$$\|u_c\|_2^2 = c, \quad \Phi(u_c) = m(c), \quad u_c(x) \geq 0, \, \forall x \in \mathbb{R}^2, \tag{6.6}$$

$$\begin{aligned} \int_{\mathbb{R}^2} \nabla u_c \cdot \nabla \tilde{w}_n \mathrm{d}x = \int_{\mathbb{R}^2} \Bigg[&- \mu \int_{\mathbb{R}^2} \log|x-y| u_c^2(y)\mathrm{d}y + \gamma\left(I_\alpha * |u_c|^q\right)|u_c|^{q-2} \\ &+ \left(e^{u_c^2} - 1 - u_c^2\right) - \lambda_c \Bigg] u_c \tilde{w}_n \mathrm{d}x \end{aligned} \tag{6.7}$$

and

$$-\lambda_c c = \frac{\mu c^2}{4} + \mu I_0(u_c) - \frac{\gamma(2+\alpha)}{2q} \int_{\mathbb{R}^2} \left(I_\alpha * |u_c|^q\right)|u_c|^q \mathrm{d}x - \int_{\mathbb{R}^2} \left(e^{u_c^2} - 1 - u_c^2 - \frac{u_c^4}{2}\right)\mathrm{d}x. \tag{6.8}$$

In view of (6.1)–(6.4) and the fact that $u_c \in E_{as}$, we can deduce, as $n \to \infty$,

$$\left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u_c(x)\tilde{w}_n(x)u_c(y)\tilde{w}_n(y)\mathrm{d}x\mathrm{d}y \right| = O\left(\frac{1}{\log n}\right), \tag{6.9}$$

$$\left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u_c^2(x)\tilde{w}_n^2(y)\mathrm{d}x\mathrm{d}y \right| = O\left(\frac{1}{\log n}\right), \tag{6.10}$$

$$\left| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| u_c(x)\tilde{w}_n(x)\tilde{w}_n^2(y)\mathrm{d}x\mathrm{d}y \right| = O\left(\frac{1}{\log^{3/2} n}\right) \tag{6.11}$$

and

$$\int_{\mathbb{R}^2} u_c \tilde{w}_n \mathrm{d}x = O\left(\frac{1}{\sqrt{\log n}}\right), \quad \int_{\mathbb{R}^2} \left(e^{u_c^2} - 1 - u_c^2\right) u_c \tilde{w}_n \mathrm{d}x = O\left(\frac{1}{\sqrt{\log n}}\right). \tag{6.12}$$

Applying Lemma 2.2 together with the Hölder inequality yields

$$\begin{aligned}
&\int_{\mathbb{R}^2} (I_\alpha * |u_c|^q)\, |u_c|^{q-1}\tilde{w}_n \mathrm{d}x \\
&\leq \mathcal{K}_\alpha C(\alpha,q) \left[\int_{\mathbb{R}^2} (|u_c|^q)^{\frac{4}{\alpha+2}}\,\mathrm{d}x\right]^{\frac{\alpha+2}{4}} \left[\int_{\mathbb{R}^2} \left(|u_c|^{q-1}\tilde{w}_n\right)^{\frac{4}{\alpha+2}}\,\mathrm{d}x\right]^{\frac{\alpha+2}{4}} \\
&\leq \mathcal{K}_\alpha C(\alpha,q) \|u_c\|^q_{\frac{4q}{\alpha+2}} \left[\left(\int_{\mathbb{R}^2} u_c^{\frac{4(q-1)}{\alpha+2}\cdot\frac{\alpha+2}{\alpha}}\mathrm{d}x\right)^{\frac{\alpha}{\alpha+2}} \left(\int_{\mathbb{R}^2} \tilde{w}_n^{\frac{4}{\alpha+2}\cdot\frac{\alpha+2}{2}}\mathrm{d}x\right)^{\frac{2}{\alpha+2}}\right]^{\frac{\alpha+2}{4}} \\
&= O\left(\frac{1}{\sqrt{\log n}}\right), \quad n\to\infty.
\end{aligned}$$

From (6.1) and (6.6), we deduce that

$$\|u_c+t\tilde{w}_n\|_2^2 = c+t^2\|\tilde{w}_n\|_2^2+2t\int_{\mathbb{R}^2} u_c\tilde{w}_n\mathrm{d}x = c+2t\int_{\mathbb{R}^2} u_c\tilde{w}_n\mathrm{d}x+t^2\left[O\left(\frac{1}{\log n}\right)\right], \quad n\to\infty.$$

Define $\tau := \|u_c+t\tilde{w}_n\|_2/\sqrt{c}$. Using (6.12), we get, as $n\to\infty$

$$\tau^2 = 1+\frac{2t}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n\mathrm{d}x + t^2\left[O\left(\frac{1}{\log n}\right)\right], \tag{6.13}$$

$$\tau^{-2p} = 1-\frac{2pt}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n\mathrm{d}x + t^2\left[O\left(\frac{1}{\log n}\right)\right], \quad \forall p\geq 1. \tag{6.14}$$

Set

$$W_{n,t}(x) := u_c(\tau x)+t\tilde{w}_n(\tau x). \tag{6.15}$$

Then $W_{n,t}\in\mathcal{S}_c$ for all $t>0$ due to

$$\|W_{n,t}\|_2^2 = \tau^{-2}\|u_c+t\tilde{w}_n\|_2^2 = c, \tag{6.16}$$

moreover, we can easily check that $\|\nabla W_{n,t}\|_2^2 = \|\nabla(u_c+t\tilde{w}_n)\|_2^2$,

$$\begin{aligned}
I_0(W_{n,t}) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|\left[u_c(\tau x)+t\tilde{w}_n(\tau x)\right]^2\left[u_c(\tau y)+t\tilde{w}_n(\tau y)\right]^2\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{\tau^4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|\left[u_c(x)+t\tilde{w}_n(x)\right]^2\left[u_c(y)+t\tilde{w}_n(y)\right]^2\mathrm{d}x\mathrm{d}y - c^2\log\tau
\end{aligned} \tag{6.17}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(e^{W_{n,t}^2} - 1 - W_{n,t}^2 - \frac{W_{n,t}^4}{2} \right) \mathrm{d}x \\
&= \frac{1}{\tau^2} \int_{\mathbb{R}^2} \left[e^{(u_c + t\tilde{w}_n)^2} - 1 - (u_c + t\tilde{w}_n)^2 - \frac{(u_c + t\tilde{w}_n)^4}{2} \right] \mathrm{d}x.
\end{aligned} \tag{6.18}$$

We can easily establish the following inequality

$$(m+k)^Q \geq \begin{cases} m^Q + Qm^{Q-1}k, & \text{if } Q > 1,\ m,k > 0, \\ m^Q + Qm^{Q-1}k + k^Q, & \text{if } Q > 2,\ m,k > 0, \\ m^Q + Qm^{Q-1}k + Qmk^{Q-1} + k^Q, & \text{if } Q > 3,\ m,k > 0. \end{cases} \tag{6.19}$$

Thus, we can get that for any $q \in [1 + \frac{\alpha}{2}, +\infty)$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} (I_\alpha * |W_{n,t}|^q) |W_{n,t}|^q \mathrm{d}x \\
&= \int_{\mathbb{R}^2} (I_\alpha * |u_c(\tau x) + t\tilde{w}_n(\tau x)|^q) |u_c(\tau x) + t\tilde{w}_n(\tau x)|^q \mathrm{d}x \\
&= \tau^{-2-\alpha} \int_{\mathbb{R}^2} (I_\alpha * |u_c + t\tilde{w}_n|^q) |u_c + t\tilde{w}_n|^q \mathrm{d}x \\
&\geq \tau^{-2-\alpha} \int_{\mathbb{R}^2} \left\{ I_\alpha * \left[|u_c|^q + qt|u_c|^{q-1}\tilde{w}_n \right] \right\} \times \left[|u_c|^q + qt|u_c|^{q-1}\tilde{w}_n \right] \mathrm{d}x \\
&\geq \tau^{-2-\alpha} \left[\int_{\mathbb{R}^2} (I_\alpha * |u_c|^q) |u_c|^q \mathrm{d}x + 2qt \int_{\mathbb{R}^2} (I_\alpha * |u_c|^q) |u_c|^{q-1}\tilde{w}_n \mathrm{d}x \right].
\end{aligned} \tag{6.20}$$

We shall construct a suitable path belonging to Γ_c with the help of $W_{n,t}$ to derive (6.5). For this, we now give an upper estimate of $\max_{t\geq 0} \Phi(W_{n,t})$. From (6.13), one has

$$\tau^2 = 1 + \frac{2t}{c} \int_{\mathbb{R}^2} u_c \tilde{w}_n \mathrm{d}x + t^2 \left[O\left(\frac{1}{\log n} \right) \right] \leq 1 + t + t^2, \quad \text{for large } n \in \mathbb{N}. \tag{6.21}$$

Let us define the following function:

$$\Psi_n(t) = \frac{t^2}{2} - \frac{1}{2\tau^2} \int_{\mathbb{R}^2} \left(e^{t^2\tilde{w}_n^2} - 1 - t^2\tilde{w}_n^2 - \frac{1}{2}t^4\tilde{w}_n^4 \right) \mathrm{d}x, \quad \forall t > 0. \tag{6.22}$$

By (6.16)–(6.18), (6.20) and (6.22), we have

$$
\begin{aligned}
\Phi(W_{n,t}) &= \frac{1}{2}\|\nabla W_{n,t}\|_2^2 + \frac{\mu}{4} I_0(W_{n,t}) - \frac{\gamma}{2q}\int_{\mathbb{R}^2} (I_\alpha * |W_{n,t}|^q)\,|W_{n,t}|^q \mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}\left(e^{W_{n,t}^2} - 1 - W_{n,t}^2 - \frac{W_{n,t}^4}{2}\right)\mathrm{d}x \\
&= \frac{1}{2}\|\nabla(u_c + t\tilde{w}_n)\|_2^2 + \frac{\mu}{4\tau^4} I_0(u_c + t\tilde{w}_n) - \frac{\mu c^2}{4}\log\tau \\
&\quad - \frac{\gamma}{2q}\int_{\mathbb{R}^2} \left[I_\alpha * |u_c(\tau x) + t\tilde{w}_n(\tau x)|^q\right] |u_c(\tau x) + t\tilde{w}_n(\tau x)|^q \mathrm{d}x \\
&\quad - \frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left[e^{(u_c+t\tilde{w}_n)^2} - 1 - (u_c + t\tilde{w}_n)^2 - \frac{(u_c+t\tilde{w}_n)^4}{2}\right]\mathrm{d}x \\
&\le \frac{1}{2}\|\nabla u_c\|_2^2 + \frac{\mu\tau^{-4}}{4} I_0(u_c) - \frac{\mu c^2}{4}\log\tau - \frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left(e^{u_c^2} - 1 - u_c^2 - \frac{u_c^4}{2}\right)\mathrm{d}x \\
&\quad + \frac{t^2}{2}\|\nabla\tilde{w}_n\|_2^2 - \frac{1}{2\tau^2}\int_{\mathbb{R}^2}\left(e^{t^2\tilde{w}_n^2} - 1 - t^2\tilde{w}_n^2 - \frac{t^4\tilde{w}_n^4}{2}\right)\mathrm{d}x + t\int_{\mathbb{R}^2}\nabla u_c\cdot\nabla\tilde{w}_n \mathrm{d}x \\
&\quad + \mu\tau^{-4}t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)\tilde{w}_n(y)\mathrm{d}x\mathrm{d}y \\
&\quad - \tau^{-2}t\int_{\mathbb{R}^2}\left(e^{u_c^2} - 1 - u_c^2\right)u_c\tilde{w}_n \mathrm{d}x \\
&\quad - \frac{\gamma\tau^{-2-\alpha}}{2q}\int_{\mathbb{R}^2}(I_\alpha * |u_c|^q)\,|u_c|^q\mathrm{d}x - \tau^{-2-\alpha}\gamma t\int_{\mathbb{R}^2}(I_\alpha * |u_c|^q)\,|u_c|^{q-1}\tilde{w}_n\mathrm{d}x \\
&\quad + (t^2 + t^4)\left[O\left(\frac{1}{\log n}\right)\right] \\
&= \Phi(u_c) + \Psi_n(t) + \langle\Phi'(u_c), t\tilde{w}_n\rangle - \frac{\mu(1-\tau^{-4})}{4} I_0(u_c) - \frac{\mu c^2}{4}\log\tau \\
&\quad + \frac{1-\tau^{-2}}{2}\int_{\mathbb{R}^2}\left(e^{u_c^2} - 1 - u_c^2 - \frac{u_c^4}{2}\right)\mathrm{d}x + (1-\tau^{-2})t\int_{\mathbb{R}^2}\left(e^{u_c^2} - 1 - u_c^2\right)u_c\tilde{w}_n\mathrm{d}x \\
&\quad + \frac{\gamma}{2q}(1-\tau^{-2-\alpha})\int_{\mathbb{R}^2}(I_\alpha * |u_c|^q)\,|u_c|^q\mathrm{d}x \\
&\quad + (1-\tau^{-2-\alpha})\gamma t\int_{\mathbb{R}^2}(I_\alpha * |u_c|^q)\,|u_c|^{q-1}\tilde{w}_n\mathrm{d}x \\
&\quad - \mu(1-\tau^{-4})t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)\tilde{w}_n(y)\mathrm{d}x\mathrm{d}y + (t^2+t^4)\left[O\left(\frac{1}{\log n}\right)\right]
\end{aligned}
\tag{6.23}
$$

$$
\begin{aligned}
&\le m(c)+\Psi_n(t)-\lambda_c t\int_{\mathbb{R}^2} u_c\tilde{w}_n \mathrm{d}x-\frac{\mu c^2}{4}\left[\frac{t}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n \mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]\\
&\quad-\mu I_0(u_c)\left[\frac{t}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n \mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]\\
&\quad+\left[\frac{t}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n \mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2-\frac{u_c^4}{2}\right)\mathrm{d}x\\
&\quad-\mu\left[\frac{4t}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n \mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)\tilde{w}_n(y)\mathrm{d}x\mathrm{d}y\\
&\quad+\left[\frac{2t}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n \mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]t\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2\right)u_c\tilde{w}_n\mathrm{d}x\\
&\quad+\frac{\gamma}{2q}\left[\frac{(2+\alpha)t}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n \mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]\int_{\mathbb{R}^2}\left(I_\alpha*|u_c|^q\right)|u_c|^q\mathrm{d}x\\
&\quad+\left[\frac{(2+\alpha)t}{c}\int_{\mathbb{R}^2} u_c\tilde{w}_n \mathrm{d}x+t^2\left(O\left(\frac{1}{\log n}\right)\right)\right]\gamma t\int_{\mathbb{R}^2}\left(I_\alpha*|u_c|^q\right)|u_c|^{q-1}\tilde{w}_n\mathrm{d}x\\
&\quad+(t^2+t^4)\left[O\left(\frac{1}{\log n}\right)\right]\\
&\le m(c)+\Psi_n(t)+(t^2+t^4)\left[O\left(\frac{1}{\log n}\right)\right]\\
&\quad+\frac{2t^2}{c}\left[\int_{\mathbb{R}^2}\left(e^{u_c^2}-1-u_c^2\right)u_c\tilde{w}_n\mathrm{d}x\right]\left(\int_{\mathbb{R}^2}u_c\tilde{w}_n\mathrm{d}x\right)\\
&\quad-\frac{4\mu t^2}{c}\left(\int_{\mathbb{R}^2}u_c\tilde{w}_n\mathrm{d}x\right)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log|x-y|u_c^2(x)u_c(y)\tilde{w}_n(y)\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{\gamma(2+\alpha)t^2}{c}\left(\int_{\mathbb{R}^2}u_c\tilde{w}_n\mathrm{d}x\right)\int_{\mathbb{R}^2}\left(I_\alpha*|u_c|^q\right)|u_c|^{q-1}\tilde{w}_n\mathrm{d}x\\
&\le m(c)+\Psi_n(t)+(t^2+t^4)\left[O\left(\frac{1}{\log n}\right)\right],\quad \forall t>0.
\end{aligned}
$$

In the sequel, we agree that all inequalities hold for large $n\in\mathbb{N}$ without mentioning. We claim that

$$
\sup_{t>0}\left[\Psi_n(t)+(t^2+t^4)\left(O\left(\frac{1}{\log n}\right)\right)\right]\le 2\pi-\frac{\pi}{\log n}\log\frac{\log n}{32\pi}<2\pi. \tag{6.24}
$$

In order to establish this claim, we proceed by analyzing the following three cases.
Case (i). $t \in \left[0, \sqrt{2\pi}\right]$. From (6.22), we obtain

$$\Psi_n(t) + (t^2 + t^4)\left[O\left(\frac{1}{\log n}\right)\right] \leq \frac{t^2}{2} + O\left(\frac{1}{\log n}\right) \leq \frac{3}{2}\pi. \tag{6.25}$$

Case (ii). $t \in \left[\sqrt{2\pi}, \sqrt{6\pi}\right)$. By (6.1) and (6.13), we have

$$\frac{1}{\tau^2}\int\limits_{\mathbb{R}^2}\left(e^{t^2\tilde{w}_n^2} - 1 - t^2\tilde{w}_n^2 - \frac{1}{2}t^4\tilde{w}_n^4\right)\mathrm{d}x \geq \frac{1}{2\tau^2}\int\limits_{B_{1/n}} e^{t^2\tilde{w}_n^2}\mathrm{d}x \geq \frac{1}{16n^2}e^{(2\pi)^{-1}t^2\log n}, \tag{6.26}$$

which, combined with (6.24), gives

$$\Psi_n(t) \leq \frac{t^2}{2} - \frac{1}{32n^2}e^{(2\pi)^{-1}t^2\log n} := \varphi_n(t). \tag{6.27}$$

Choose $t_n > 0$ such that $\varphi_n'(t_n) = 0$, we can deduce that $1 = \frac{\log n}{32\pi n^2}e^{(2\pi)^{-1}t_n^2\log n}$, which gives

$$t_n^2 = 4\pi\left[1 + \frac{\log(32\pi) - \log(\log n)}{2\log n}\right] \tag{6.28}$$

and

$$\varphi_n(t) \leq \varphi_n(t_n) = \frac{t_n^2}{2} - \frac{\pi}{\log n}, \quad \forall t \geq 0. \tag{6.29}$$

Consequently, (6.27), (6.28) and (6.29) imply

$$\Psi_n(t) \leq \varphi_n(t) \leq \frac{t_n^2}{2} - \frac{\pi}{\log n} = 2\pi - \frac{\pi}{\log n}\log\frac{e\log n}{32\pi},$$

and for any $t \in \left[\sqrt{2\pi}, \sqrt{6\pi}\right)$, we have

$$\Psi_n(t) + (t^2 + t^4)\left[O\left(\frac{1}{\log n}\right)\right] \leq 2\pi - \frac{\pi}{\log n}\log\frac{\log n}{32\pi}.$$

Case (iii). $t \in \left(\sqrt{6\pi}, +\infty\right)$. Then it follows from (6.1) and (6.21) that

$$\begin{aligned}\Psi_n(t) + (t^2 + t^4)\left[O\left(\frac{1}{\log n}\right)\right] &\leq \frac{t^4}{8\pi} - \frac{1}{2\tau^2}\int\limits_{\mathbb{R}^2}\left(e^{t^2\tilde{w}_n^2} - 1 - t^2\tilde{w}_n^2 - \frac{1}{2}t^4\tilde{w}_n^4\right)\mathrm{d}x \\ &\leq \frac{t^4}{8\pi} - \frac{\pi}{4n^2\tau^2}e^{(2\pi)^{-1}t^2\log n} \\ &\leq \frac{t^4}{8\pi} - \frac{\pi}{4n^2(1+t+t^2)}e^{(2\pi)^{-1}t^2\log n} \\ &\leq \frac{9\pi}{2} - \frac{\pi}{4n^2(1+\sqrt{6\pi}+6\pi)}e^{3\log n} \leq \frac{3}{2}\pi,\end{aligned} \tag{6.30}$$

due to the fact that $\phi_n(t) := \frac{t^4}{8\pi} - \frac{\pi}{4n^2(1+t+t^2)} e^{(2\pi)^{-1}t^2 \log n}$ is decreasing on the interval $t \in \left(\sqrt{6\pi}, +\infty\right)$ for large n. Indeed, $\phi_n'(t) = \frac{t^3}{2\pi} - \frac{\left(1+t+t^2\right)t \log n - (1+2t)\pi}{4n^2(1+t+t^2)^2} e^{(2\pi)^{-1}t^2 \log n}$. Assume that $s_n > 0$ such that $\phi_n'(s_n) = 0$ for large n. Then we have

$$2s_n^3 \left(1 + s_n + s_n^2\right)^2 = \frac{\pi \left(1 + s_n + s_n^2\right) s_n \log n - (1 + 2s_n)\pi^2}{n^2} e^{(2\pi)^{-1} s_n^2 \log n},$$

which leads to

$$s_n^2 = 4\pi \left\{ 1 + \frac{\log \left[2s_n^3 \left(1 + s_n + s_n^2\right)^2\right] - \log \left[\pi \left(1 + s_n + s_n^2\right) s_n \log n - (1 + 2s_n)\pi^2\right]}{2 \log n} \right\}.$$

This means $\lim_{n\to\infty} s_n^2 = 4\pi$. So $\phi_n(t)$ is decreasing on $t \in \left(\sqrt{6\pi}, +\infty\right)$ when n is sufficiently large.

Combining Cases (i)–(iii) yields (6.24). Therefore, one can find a sufficiently large $\bar{n} \in \mathbb{N}$ such that

$$\sup_{t>0} \Phi(W_{\bar{n},t}) < m(c) + 2\pi. \tag{6.31}$$

In addition, from (6.13), (6.15), (6.16) and (6.23), one gets $W_{\bar{n},t} \in \mathcal{S}_c$ for all $t > 0$, $W_{\bar{n},0} = u_c$ and $\Phi(W_{\bar{n},t}) < 2m(c)$ for large $t > 0$. Hence, there exists a number $\hat{t} > 0$ satisfying

$$\Phi(W_{\bar{n},\hat{t}}) < 2m(c). \tag{6.32}$$

Define $\gamma_{\bar{n}}(t) := W_{\bar{n},t\hat{t}}$. Then $\gamma_{\bar{n}} \in \Gamma_c$, where Γ_c is defined by (5.16). Combining (6.31) and (6.32), we obtain (6.5), which completes the proof. □

In the rest of this section we set

$$\begin{cases} \gamma(q) = +\infty & \text{if } 1 + \frac{\alpha}{2} \le q < 2 + \frac{\alpha}{2} \text{ or } 2 + \frac{\alpha}{2} < q < +\infty, \\ \gamma(q) = \frac{q}{C_0(\alpha,q)c^{(\alpha+2)/2}} & \text{if } q = 2 + \frac{\alpha}{2}. \end{cases}$$

Lemma 6.2. *Assume $\mu > 0$, $1 + \frac{\alpha}{2} \le q < +\infty$ and $0 < \gamma < \gamma(q)$. Let $\{u_n\} \subset \hat{\mathcal{S}}_c$ be a* (PS) *sequence satisfying* (5.17)*, then $\{\|\nabla u_n\|_2\}$ and $\{I_1(u_n)\}$ are bounded. Moreover, $\{u_n\}$ is bounded in E_{as}. Then, there exists $\bar{u} \neq 0$ such that, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in E_{as} as $n \to \infty$.*

Proof. From (5.17) together with Lemma 6.1, we obtain

$$\Phi(u_n) \to M(c) < 2\pi + m(c), \quad \Phi\big|_{\hat{\mathcal{S}}_c}'(u_n) \to 0 \quad \text{and} \quad \mathcal{P}(u_n) \to 0.$$

Using (1.7) and (1.9), one has

$$\begin{aligned}&\frac{1}{2}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_0(u_n)-\frac{\gamma}{2q}\int\limits_{\mathbb{R}^2}\left(I_\alpha*|u_n|^q\right)|u_n|^q\mathrm{d}x\\&-\frac{1}{2}\int\limits_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2-\frac{u_n^4}{2}\right)\mathrm{d}x=M(c)+o(1),\end{aligned}\tag{6.33}$$

$$\begin{aligned}&\|\nabla u_n\|_2^2-\frac{\mu c^2}{4}-\frac{\gamma(2q-2-\alpha)}{2q}\int\limits_{\mathbb{R}^2}\left(I_\alpha*|u_n|^q\right)|u_n|^q\mathrm{d}x\\&-\int\limits_{\mathbb{R}^2}\left[(u_n^2-1)e^{u_n^2}+1-\frac{u_n^4}{2}\right]\mathrm{d}x=o(1).\end{aligned}\tag{6.34}$$

It follows from (2.5) and (2.9) that

$$\begin{aligned}&M(c)+o(1)=\Phi(u_n)-\frac{1}{4}\mathcal{P}(u_n)\\&=\frac{1}{4}\|\nabla u_n\|_2^2+\frac{\gamma(2q-6-\alpha)}{8q}\int\limits_{\mathbb{R}^2}\left(I_\alpha*|u_n|^q\right)|u_n|^q\mathrm{d}x+\frac{\mu}{4}I_0(u_n)+\frac{\mu c^2}{16}\\&\quad+\frac{1}{4}\int\limits_{\mathbb{R}^2}\left[(u_n^2-3)e^{u_n^2}+3+2u_n^2+\frac{u_n^4}{2}\right]\mathrm{d}x\\&\geq\frac{1}{4}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_1(u_n)-\frac{\mu}{4}I_2(u_n)+\frac{\gamma(2q-6-\alpha)}{8q}\int\limits_{\mathbb{R}^2}\left(I_\alpha*|u_n|^q\right)|u_n|^q\mathrm{d}x\\&\quad+\frac{1}{4}\sum_{k=4}^{\infty}\frac{k-3}{k!}\int\limits_{\mathbb{R}^2}u_n^{2k}\mathrm{d}x\\&\geq\frac{1}{4}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_1(u_n)-C_1c^{3/2}\|\nabla u_n\|_2+\frac{\gamma(2q-6-\alpha)}{8q}\int\limits_{\mathbb{R}^2}\left(I_\alpha*|u_n|^q\right)|u_n|^q\mathrm{d}x.\end{aligned}\tag{6.35}$$

We now proceed by considering the following cases to prove that $\{\|\nabla u_n\|_2\}$ and $\{I_1(u_n)\}$ are both bounded when $q\in[1+\frac{\alpha}{2},2+\frac{\alpha}{2}]\cup[3+\frac{\alpha}{2},\infty)$.

Case (i). $q=1+\frac{\alpha}{2}$. Then by $2q-6-\alpha=-4$ and (2.11), we get

$$M(c)+o(1)\geq\frac{1}{4}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_1(u_n)-C_1c^{3/2}\|\nabla u_n\|_2-\frac{\gamma}{2q}\mathcal{K}_\alpha C(\alpha,q)c^{\frac{\alpha+2}{2}}.$$

Case (ii). $q\in(1+\frac{\alpha}{2},2+\frac{\alpha}{2})$. From $2q-\alpha-2\in(0,2)$ and (2.12), we have

$$\begin{aligned}M(c)+o(1)\geq&\frac{1}{4}\|\nabla u_n\|_2^2+\frac{\mu}{4}I_1(u_n)-C_1c^{3/2}\|\nabla u_n\|_2\\&-\frac{\gamma(6+\alpha-2q)}{8q}C_0(\alpha,q)c^{\frac{\alpha+2}{2}}\|\nabla u\|_2^{2q-\alpha-2}.\end{aligned}$$

Case (iii). $q = 2 + \frac{\alpha}{2}$. It follows from $\frac{2q-6-\alpha}{8q} = -\frac{1}{8+2\alpha}$ that

$$\begin{aligned} M(c) + o(1) &\geq \frac{1}{4}\|\nabla u_n\|_2^2 + \frac{\mu}{4} I_1(u_n) - C_1 c^{3/2}\|\nabla u_n\|_2 - \frac{\gamma}{8+2\alpha} C_0(\alpha, q) c^{\frac{\alpha+2}{2}} \|\nabla u\|_2^2 \\ &\geq \left(\frac{1}{4} - \frac{\gamma}{8+2\alpha} C_0(\alpha, q) c^{\frac{\alpha+2}{2}}\right) \|\nabla u_n\|_2^2 + \frac{\mu}{4} I_1(u_n) - C_1 c^{3/2}\|\nabla u_n\|_2. \end{aligned}$$

Case (iv). $q \in [3 + \frac{\alpha}{2}, \infty)$. In this case, $2q - 6 - \alpha \geq 0$. Then by (6.35), one has

$$M(c) + o(1) \geq \frac{1}{4}\|\nabla u_n\|_2^2 + \frac{\mu}{4} I_1(u_n) - C_1 c^{3/2}\|\nabla u_n\|_2.$$

Therefore, for any $q \in [1 + \frac{\alpha}{2}, 2 + \frac{\alpha}{2}] \cup [3 + \frac{\alpha}{2}, \infty)$, it follows from Cases (i)–(iv) that $\{\|\nabla u_n\|_2\}$ and $\{I_1(u_n)\}$ are bounded.

Furthermore, if $q \in (2 + \frac{\alpha}{2}, 3 + \frac{\alpha}{2})$, then $2q - \alpha - 2 \in (2, 4)$. Therefore, we have

$$\begin{aligned} M(c) + o(1) &= \Phi(u_n) - \frac{1}{2q-2-\alpha}\mathcal{P}(u_n) \\ &= \left(\frac{1}{2} - \frac{1}{2q-2-\alpha}\right)\|\nabla u_n\|_2^2 + \frac{\mu}{4} I_1(u_n) - \frac{\mu}{4} I_2(u_n) + \frac{\mu c^2}{4(2q-2-\alpha)} \\ &\quad + \frac{1}{2q-2-\alpha}\sum_{k=3}^{\infty}\frac{k-1}{k!}\int_{\mathbb{R}^2} u_n^{2k}\mathrm{d}x - \frac{1}{2}\sum_{k=3}^{\infty}\frac{1}{k!}\int_{\mathbb{R}^2} u_n^{2k}\mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{2q-2-\alpha}\right)\|\nabla u_n\|_2^2 + \frac{\mu}{4} I_1(u_n) - \frac{\mu}{4} I_2(u_n) \\ &\geq \left(\frac{1}{2} - \frac{1}{2q-2-\alpha}\right)\|\nabla u_n\|_2^2 + \frac{\mu}{4} I_1(u_n) - C_1 c^{3/2}\|\nabla u_n\|_2, \end{aligned}$$

which implies that $\{\|\nabla u_n\|_2\}$ and $\{I_1(u_n)\}$ are bounded.

From Lemma 2.7, we get that $\{\|u_n\|_*\}$ is bounded, hence $\{u_n\}$ is bounded in E_{as}. Then, there exists $\bar{u} \in E_{as}$ such that, up to a subsequence, $u_n \rightharpoonup \bar{u} \neq 0$ in E_{as} as $n \to \infty$. This finishes the proof of the lemma. □

Lemma 6.3. *Assume $\mu > 0$, $1 + \frac{\alpha}{2} \leq q < +\infty$ and $0 < \gamma < \gamma(q)$. Let $\{u_n\} \subset \hat{\mathcal{S}}_c$ be a sequence which satisfies* (5.17)*. If $u_n \rightharpoonup \bar{u}$ in E_{as}, $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s \geq 2$, $\int_{\mathbb{R}^2}\left(e^{u_n^2} - 1 - u_n^2\right)u_n^2\mathrm{d}x \leq C$ and $\mathcal{P}(\bar{u}) \geq 0$, then*

$$\int_{\mathbb{R}^2} (e^{u_n^2} - 1 - u_n^2)u_n(u_n - \bar{u})\mathrm{d}x = o(1). \tag{6.36}$$

Proof. Using (2.7), (2.8) and the fact that $\mathcal{P}(\bar{u}) \geq 0$, we have $g'_{\bar{u}}(1) \geq 0$, where $g_{\bar{u}}(t) = \Phi(t\bar{u}_t)$. In view of Lemma 5.1, there exist unique $0 < s_{\bar{u}}^+ \leq 1$ and $1 \leq s_{\bar{u}}^- < +\infty$ such that

$$g_{\bar{u}}(s_{\bar{u}}^+) < g_{\bar{u}}(t) < g_{\bar{u}}(s_{\bar{u}}^-) < +\infty, \quad \forall t \in (s_{\bar{u}}^+, s_{\bar{u}}^-).$$

This means for any $t \in (s_{\bar{u}}^{+}, s_{\bar{u}}^{-})$,

$$\Phi(s_{\bar{u}}^{+}\bar{u}_{s_{\bar{u}}^{+}}) < \Phi(t\bar{u}_t) < \Phi(s_{\bar{u}}^{-}\bar{u}_{s_{\bar{u}}^{-}}) < +\infty. \tag{6.37}$$

Let $\bar{v} := \tau\bar{u}_\tau$ with $\tau = \frac{\sqrt{\pi}}{\sqrt{3}\|\nabla\bar{u}\|_2}$. Then $\bar{v} \in \hat{\mathcal{S}}_c \cap \partial A_{\pi/3}$. Using Lemma 3.1(ii), there exists $t_{\bar{v}} \in (0,1)$ satisfying $\mathcal{P}(t_{\bar{v}}\bar{v}_{t_{\bar{v}}}) = 0$. Observe that $t_{\bar{v}}\bar{v}_{t_{\bar{v}}} = (t_{\bar{v}}\tau)\bar{u}_{t_{\bar{v}}\tau}$. Hence, $s_{\bar{u}}^{+} = t_{\bar{v}}\tau$. Using (6.37), we obtain that

$$m(c) \le \Phi(t_{\bar{v}}\bar{v}_{t_{\bar{v}}}) = \Phi(s_{\bar{u}}^{+}\bar{u}_{s_{\bar{u}}^{+}}) \le \Phi(\bar{u}).$$

Since $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s \ge 2$, an application of [15, Lemma 4.5] gives

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(e^{u_n^2} - 1 - u_n^2 - \frac{u_n^4}{2}\right)\mathrm{d}x = \int_{\mathbb{R}^2}\left(e^{\bar{u}^2} - 1 - \bar{u}^2 - \frac{\bar{u}^4}{2}\right)\mathrm{d}x. \tag{6.38}$$

Let $v_n := u_n - \bar{u}$. Using (2.4), (2.5), (4.2), (4.3), (6.33), (6.38), together with [29, Lemma 2.4], one has

$$\begin{aligned}
M(c) + o(1) &= \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{\mu}{4}I_0(u_n) - \frac{\gamma}{2q}\int_{\mathbb{R}^2}(I_\alpha * |u_n|^q)\,|u_n|^q\mathrm{d}x \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}\left(e^{u_n^2} - 1 - u_n^2 - \frac{u_n^4}{2}\right)\mathrm{d}x \\
&= \frac{1}{2}(\|\nabla\bar{u}\|_2^2 + \|\nabla v_n\|_2^2) + \frac{\mu}{4}\left[I_0(\bar{u}) + I_0(v_n) + 2A_0(\bar{u}^2, v_n^2)\right] \\
&\quad - \frac{\gamma}{2q}\left(\int_{\mathbb{R}^2}(I_\alpha * |\bar{u}|^q)\,|\bar{u}|^q\mathrm{d}x + \int_{\mathbb{R}^2}(I_\alpha * |v_n|^q)\,|v_n|^q\mathrm{d}x\right) \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2}\left(e^{\bar{u}^2} - 1 - \bar{u}^2 - \frac{\bar{u}^4}{2}\right)\mathrm{d}x + o(1) \\
&= \Phi(\bar{u}) + \frac{1}{2}\|\nabla v_n\|_2^2 + \frac{\mu}{4}I_0(v_n) + \frac{\mu}{2}A_0(\bar{u}^2, v_n^2) + o(1) \\
&\ge \frac{1}{2}\|\nabla(u_n - \bar{u})\|_2^2 + m(c) + \frac{\mu}{2}A_1(\bar{u}^2, v_n^2) + o(1) \\
&\ge \frac{1}{2}\|\nabla(u_n - \bar{u})\|_2^2 + m(c) + o(1).
\end{aligned} \tag{6.39}$$

Note that for any $c \in (0, c_0)$, we have $M(c) < m(c) + 2\pi$. Thus, from (6.39), we deduce that there exists $\tilde{\varepsilon} > 0$ satisfying

$$\|\nabla(u_n - \bar{u})\|_2^2 \le 4\pi(1 - 3\tilde{\varepsilon}), \text{ for large } n \in \mathbb{N}. \tag{6.40}$$

Let $p \in (1,2)$ such that $p^2(1-3\tilde{\varepsilon}) \le (1-\tilde{\varepsilon})$. From (6.40), Young's inequality and Lemma 2.4, one has

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)^p \mathrm{d}x \le \int_{\mathbb{R}^2}\left(e^{pu_n^2}-1\right)\mathrm{d}x \le \int_{\mathbb{R}^2}\left[e^{(1+\tilde{\varepsilon}^{-1})p\bar{u}^2}e^{(1+\tilde{\varepsilon})p(u_n-\bar{u})^2}-1\right]\mathrm{d}x\\ &\le \frac{p-1}{p}\int_{\mathbb{R}^2}\left[e^{(1+\tilde{\varepsilon}^{-1})p^2(p-1)^{-1}\bar{u}^2}-1\right]\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^2}\left[e^{(1+\tilde{\varepsilon})p^2(u_n-\bar{u})^2}-1\right]\mathrm{d}x \le C_1.\end{aligned} \tag{6.41}$$

Noting that $p/(p-1) > 1$, using the Hölder inequality and (6.41), we get that

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n(u_n-\bar{u})\mathrm{d}x\\ &\le \left[\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)^p\mathrm{d}x\right]^{1/p}\|u_n\|_{2p/(p-1)}\|u_n-\bar{u}\|_{2p/(p-1)} = o(1).\end{aligned}$$

This implies that (6.36) holds. □

Lemma 6.4. *Assume $\mu > 0$, $1+\frac{\alpha}{2} \le q < +\infty$ and $0 < \gamma < \gamma(q)$. Let $\{u_n\} \subset \hat{\mathcal{S}}_c$ be a sequence which satisfies* (5.17)*. Then there exists $\bar{u} \in E_{as}\setminus\{0\}$ and $\bar{\lambda}_c \in \mathbb{R}$ such that $u_n \to \bar{u}$ in E_{as} and $\lambda_n \to \bar{\lambda}_c$ in $\mathbb{R}$.*

Proof. By Lemma 6.2, we may thus assume, passing to a subsequence again if necessary, that

$$u_n, \bar{u} \in \hat{\mathcal{S}}_c, \quad u_n \rightharpoonup \bar{u} \text{ in } E_{as}, \quad u_n \to \bar{u} \text{ in } L^s(\mathbb{R}^2) \text{ for } s \ge 2, \quad u_n \to \bar{u} \text{ a.e. on } \mathbb{R}^2. \tag{6.42}$$

From (2.5), (6.33), (6.34) and the boundedness of $\{u_n\}$ in E_{as}, one can get that

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n^2\mathrm{d}x \le 3\int_{\mathbb{R}^2}\left[\left(e^{u_n^2}-1-u_n^2\right)u_n^2-2\left(e^{u_n^2}-1-u_n^2-\frac{u_n^4}{2}\right)\right]\mathrm{d}x\\ &= 6M(c)-\frac{3\mu}{2}I_0(u_n)-\frac{3\mu c^2}{4}-\frac{3\gamma(2q-4-\alpha)}{2q}\int_{\mathbb{R}^2}\left(I_\alpha * |u_n|^q\right)|u_n|^q\mathrm{d}x+o(1).\end{aligned} \tag{6.43}$$

If $q \in [2+\frac{\alpha}{2}, \infty)$, we immediately obtain that

$$I_1(u_n) \le C_1, \quad \int_{\mathbb{R}^2}\left(e^{u_n^2}-1-u_n^2\right)u_n^2\mathrm{d}x \le C_2. \tag{6.44}$$

If $q \in [1+\frac{\alpha}{2}, 2+\frac{\alpha}{2})$, using (2.11), it is easy to get that $\int_{\mathbb{R}^2}(I_\alpha * |u_n|^q)|u_n|^q\mathrm{d}x$ is bounded. Thus, it follows from (6.43) that (6.44) holds. Using [4, Lemma 3], we have

$$\Phi'(u_n)+\lambda_n u_n \to 0, \tag{6.45}$$

where

$$\begin{aligned}-\lambda_n &= \frac{1}{\|u_n\|_2^2}\langle \Phi'(u_n), u_n\rangle \\ &= \frac{1}{c}\left[\|\nabla u_n\|_2^2 + \mu I_0(u_n) - \gamma\int_{\mathbb{R}^2}(I_\alpha * |u_n|^q)\,|u_n|^q \mathrm{d}x - \int_{\mathbb{R}^2}(e^{u_n^2}-1-u_n^2)u_n^2\mathrm{d}x\right].\end{aligned} \tag{6.46}$$

Since $\{\|u_n\|_{E_{as}}\}$ is bounded, from (6.43), (6.44) and (6.46), we get that $\{|\lambda_n|\}$ is also bounded. Thus, we may thus assume, passing to a subsequence if necessary, that $\lambda_n \to \bar{\lambda}_c$.

Motivated by [11, Assertion 3], we make the following claim

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left[\left(e^{u_n^2}-1-u_n^2\right)u_n\bar{u} - \left(e^{\bar{u}^2}-1-\bar{u}^2\right)\bar{u}^2\right]\mathrm{d}x = 0. \tag{6.47}$$

Because $\bar{u}$ belongs to E_{as}, for any given $\varepsilon > 0$, we can choose $\phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^2) \subset E_{as}$ satisfying $\|\phi_\varepsilon - \bar{u}\|_{E_{as}} < \varepsilon$. It follows from (2.6) and the boundedness of $\{\|u_n\|_{E_{as}}^2\} = \{\|\nabla u_n\|_2^2 + \|u_n\|_*^2\}$, one can deduce that

$$\begin{aligned}&|A_1(u_n^2, u_n(\phi_\varepsilon - \bar{u}))| \\ &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[\log(2+|x|)+\log(2+|y|)]u_n^2(x)|u_n(y)||\phi_\varepsilon(y)-\bar{u}(y)|\mathrm{d}x\mathrm{d}y \\ &\le \|u_n\|_*^2\|u_n\|_2\|\phi_\varepsilon-\bar{u}\|_2 + \|u_n\|_2^2\|u_n\|_*\|\phi_\varepsilon-\bar{u}\|_* < C_3\varepsilon.\end{aligned} \tag{6.48}$$

By (2.4), we also have

$$|A_2(u_n^2, u_n(\phi_\varepsilon - \bar{u}))| < C_4\varepsilon. \tag{6.49}$$

Combining (1.7) with (6.45), we obtain

$$\begin{aligned}o(1) &= \langle \Phi'(u_n) + \lambda_n u_n, \phi_\varepsilon - \bar{u}\rangle \\ &= \int_{\mathbb{R}^2}[\nabla u_n\cdot\nabla(\phi_\varepsilon-\bar{u}) + \lambda_n u_n(\phi_\varepsilon-\bar{u})]\mathrm{d}x + \mu A_1(u_n^2, u_n(\phi_\varepsilon-\bar{u})) \\ &\quad - \mu A_2(u_n^2, u_n(\phi_\varepsilon-\bar{u})) - \gamma\int_{\mathbb{R}^2}(I_\alpha * |u_n|^q)\,|u_n|^{q-2}u_n(\phi_\varepsilon-\bar{u})\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^2}(\mathrm{e}^{u_n^2}-1-u_n^2)u_n(\phi_\varepsilon-\bar{u})\mathrm{d}x.\end{aligned} \tag{6.50}$$

From (6.48), (6.49), (6.50) and Lemma 2.2, one has

$$\begin{aligned}
&\left|\int\limits_{\mathbb{R}^2}(\mathrm{e}^{u_n^2}-1-u_n^2)u_n(\phi_\varepsilon-\bar{u})\mathrm{d}x\right| \\
&\leq \left|\int\limits_{\mathbb{R}^2}[\nabla u_n\cdot\nabla(\phi_\varepsilon-\bar{u})+\lambda_n u_n(\phi_\varepsilon-\bar{u})]\mathrm{d}x\right|+\mu|A_1(u_n^2,u_n(\phi_\varepsilon-\bar{u}))| \\
&\quad+\mu|A_2(u_n^2,u_n(\phi_\varepsilon-\bar{u}))|+\gamma\int\limits_{\mathbb{R}^2}(I_\alpha*|u_n|^q)\,|u_n|^{q-2}|u_n(\phi_\varepsilon-\bar{u})|\mathrm{d}x+o(1) \\
&\leq \|u_n\|_{E_{as}}\|\phi_\varepsilon-\bar{u}\|_{E_{as}}+C_5\varepsilon+o(1)\leq C_6\varepsilon+o(1).
\end{aligned} \tag{6.51}$$

Moreover, in view of Lemma 2.4(i), we obtain

$$\left|\int\limits_{\mathbb{R}^2}(\mathrm{e}^{\bar{u}^2}-1-\bar{u}^2)\bar{u}(\phi_\varepsilon-\bar{u})\mathrm{d}x\right|\leq\left[\int\limits_{\mathbb{R}^2}(\mathrm{e}^{2\bar{u}^2}-1)\mathrm{d}x\right]^{1/2}\|\bar{u}\|_4\|\phi_\varepsilon-\bar{u}\|_4\leq C_7\varepsilon. \tag{6.52}$$

Since $\phi_\varepsilon\in\mathcal{C}_0^\infty(\mathbb{R}^2)$, then by [15, Lemma 4.5], we have

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}(\mathrm{e}^{u_n^2}-1-u_n^2)u_n\phi_\varepsilon\mathrm{d}x=\int\limits_{\mathbb{R}^2}(\mathrm{e}^{\bar{u}^2}-1-\bar{u}^2)\bar{u}\phi_\varepsilon\mathrm{d}x. \tag{6.53}$$

From (6.51), (6.52) and (6.53), one has

$$\left|\int\limits_{\mathbb{R}^2}\left[(e^{u_n^2}-1-u_n^2)u_n-(\mathrm{e}^{\bar{u}^2}-1-\bar{u}^2)\bar{u}\right]\bar{u}\mathrm{d}x\right|\leq(C_6+C_7)\varepsilon+o(1). \tag{6.54}$$

According to the arbitrariness of $\varepsilon>0$, we deduce (6.47) from (6.54). By (1.7), (2.5), (6.45), (6.47) and Lemma 2.8 and [22, Lemma 2.7], we obtain

$$\begin{aligned}
0&=\lim_{n\to\infty}\langle\Phi'(u_n)+\lambda_n u_n,\bar{u}\rangle \\
&=\|\nabla\bar{u}\|_2^2+\bar{\lambda}_c\|\bar{u}\|_2^2+\mu\lim_{n\to\infty}A_0(u_n^2,u_n\bar{u}) \\
&\quad-\gamma\lim_{n\to\infty}\int\limits_{\mathbb{R}^2}(I_\alpha*|u_n|^q)\,|u_n|^{q-2}u_n\bar{u}\mathrm{d}x-\int\limits_{\mathbb{R}^2}(e^{\bar{u}^2}-1-\bar{u}^2)\bar{u}^2\mathrm{d}x \\
&=\|\nabla\bar{u}\|_2^2+\bar{\lambda}_c\|\bar{u}\|_2^2+\mu\lim_{n\to\infty}A_0(u_n^2,\bar{u}^2) \\
&\quad-\gamma\int\limits_{\mathbb{R}^2}(I_\alpha*|\bar{u}|^q)\,|\bar{u}|^q\mathrm{d}x-\int\limits_{\mathbb{R}^2}(e^{\bar{u}^2}-1-\bar{u}^2)\bar{u}^2\mathrm{d}x.
\end{aligned} \tag{6.55}$$

Using (2.1) and Lemma 2.8, one can deduce

$$\begin{aligned}
\lim_{n\to\infty}A_1(u_n^2,u_n^2)&=\lim_{n\to\infty}\left[A_1(u_n^2,(u_n-\bar{u})^2)+2A_1(u_n^2,u_n\bar{u})-A_1(u_n^2,\bar{u}^2)\right] \\
&\geq\lim_{n\to\infty}A_1(u_n^2,\bar{u}^2).
\end{aligned}$$

It follows from (2.5), (6.34), (6.38), (6.46) and (6.55) that

$$
\begin{aligned}
0 &= \lim_{n\to\infty}\Bigg\{\|\nabla u_n\|_2^2-\frac{\mu c^2}{4}-\gamma\frac{2q-(2+\alpha)}{2q}\int_{\mathbb{R}^2}\left(I_\alpha*|u_n|^q\right)|u_n|^q\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^2}\left[(u_n^2-1)e^{u_n^2}+1-\frac{u_n^4}{2}\right]\mathrm{d}x\Bigg\}\\
&= \lim_{n\to\infty}\left[\|\nabla u_n\|_2^2-\int_{\mathbb{R}^2}(e^{u_n^2}-1-u_n^2)u_n^2\mathrm{d}x\right]\\
&\quad-\gamma\frac{2q-(2+\alpha)}{2q}\int_{\mathbb{R}^2}\left(I_\alpha*|\bar{u}|^q\right)|\bar{u}|^q\mathrm{d}x\\
&\quad-\frac{\mu c^2}{4}+\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2-\frac{\bar{u}^4}{2}\right)\mathrm{d}x\\
&= \lim_{n\to\infty}\left[-\lambda_n\|u_n\|_2^2-\mu I_0(u_n)+\gamma\int_{\mathbb{R}^2}\left(I_\alpha*|u_n|^q\right)|u_n|^q\mathrm{d}x\right]-\frac{\mu c^2}{4}\\
&\quad-\gamma\frac{2q-(2+\alpha)}{2q}\int_{\mathbb{R}^2}\left(I_\alpha*|\bar{u}|^q\right)|\bar{u}|^q\mathrm{d}x+\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2-\frac{\bar{u}^4}{2}\right)\mathrm{d}x\\
&= -\bar{\lambda}_c\|\bar{u}\|_2^2-\mu\lim_{n\to\infty}I_0(u_n)-\frac{\mu c^2}{4}+\frac{(2+\alpha)\gamma}{2q}\int_{\mathbb{R}^2}\left(I_\alpha*|\bar{u}|^q\right)|\bar{u}|^q\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2-\frac{\bar{u}^4}{2}\right)\mathrm{d}x\\
&\le -\bar{\lambda}_c\|\bar{u}\|_2^2-\mu\lim_{n\to\infty}A_0(u_n^2,\bar{u}^2)-\frac{\mu c^2}{4}+\frac{(2+\alpha)\gamma}{2q}\int_{\mathbb{R}^2}\left(I_\alpha*|\bar{u}|^q\right)|\bar{u}|^q\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^2}\left(e^{\bar{u}^2}-1-\bar{u}^2-\frac{\bar{u}^4}{2}\right)\mathrm{d}x\\
&= \|\nabla\bar{u}\|_2^2-\frac{\mu c^2}{4}-\gamma\frac{2q-(2+\alpha)}{2q}\int_{\mathbb{R}^2}\left(I_\alpha*|\bar{u}|^q\right)|\bar{u}|^q\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^2}\left[(\bar{u}^2-1)e^{\bar{u}^2}+1-\frac{\bar{u}^4}{2}\right]\mathrm{d}x\\
&= \mathcal{P}(\bar{u}).
\end{aligned}
$$

This means $\mathcal{P}(\bar{u}) \geq 0$. It follows from (1.7), (2.1)–(2.3), (2.5), (6.36), (6.42), (6.45), Lemmas 2.8 and 6.3 that

$$\begin{aligned} o(1) &= \langle \Phi'(u_n) + \lambda_n u_n, u_n - \bar{u} \rangle \\ &= \|\nabla u_n\|_2^2 - \|\nabla \bar{u}\|_2^2 \\ &\quad + \mu A_1(u_n^2, (u_n - \bar{u})^2) + \mu A_1(u_n^2, (u_n - \bar{u})\bar{u}) - \mu A_2(u_n^2, u_n(u_n - \bar{u})) \\ &\quad - \gamma \int\limits_{\mathbb{R}^2} (I_\alpha * |u_n|^q) |u_n|^{q-2} u_n (u_n - \bar{u}) \mathrm{d}x \\ &\quad - \int\limits_{\mathbb{R}^2} \left(e^{u_n^2} - 1 - u_n^2 \right) u_n (u_n - \bar{u}) \mathrm{d}x + o(1) \\ &= \|\nabla (u_n - \bar{u})\|_2^2 + \mu A_1 \left(u_n^2, (u_n - \bar{u})^2 \right) + o(1). \end{aligned} \tag{6.56}$$

Using (6.56) and Lemma 2.7, we can deduce that $u_n \to \bar{u}$ in E_{as}. □

Proof of Theorem 1.2. In view of the conditions of Theorem 1.2, from Lemmas 5.3 and 6.1, we can obtain that there exists a sequence $\{u_n\} \subset \hat{\mathcal{S}}_c$ satisfying

$$\Phi(u_n) \to M(c) < 2\pi + m(c), \quad \Phi\big|'_{\hat{\mathcal{S}}_c}(u_n) \to 0 \quad \text{and} \quad \mathcal{P}(u_n) \to 0. \tag{6.57}$$

Using Lemma 6.4, there exist $\bar{u} \in \hat{\mathcal{S}}_c$ and $\bar{\lambda}_c \in \mathbb{R}$ such that $u_n \to \bar{u}$ in E_{as} and $\lambda_n \to \bar{\lambda}_c$ in $\mathbb{R}$. Therefore, $\Phi(\bar{u}) = M(c)$ and $\Phi'(\bar{u}) + \bar{\lambda}_c \bar{u} = 0$. This completes the proof. □

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