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# EXISTENCE OF SOLUTIONS FOR A DOUBLY CRITICAL SCHRÖDINGER–POISSON SYSTEM ON THE FIRST HEISENBERG GROUP

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**Abstract.** This work is devoted to the study of a class of Schrödinger–Poisson system with doubly critical growth on the first Heisenberg group. Utilizing the concentration-compactness principle associated with classical Sobolev space on the Heisenberg group and mountain pass theorem, we prove that the system admits multiple nontrivial solutions.

**Keywords:** Heisenberg group, Schrödinger–Poisson system, concentration-compactness principle, mountain pass theorem, nontrivial solutions.

**Mathematics Subject Classification:** 35A15, 35B33, 47G20.

## 1. INTRODUCTION

In the present paper, we focus on the following doubly critical Schrödinger–Poisson system on the first Heisenberg group:

$$\begin{cases} -\Delta_H u + V(\xi)u - \kappa\phi u^2 = \lambda f(\xi, u) + |u|^2 u & \text{in } \mathbb{H}^1, \\ -\Delta_H \phi = |u|^3 & \text{in } \mathbb{H}^1, \end{cases} \tag{1.1}$$

where $\Delta_H$ is the Kohn-Laplacian on the first Heisenberg group $\mathbb{H}^1$, $\kappa, \lambda > 0$ are real parameters, $Q = 4$ is the homogeneous dimension of $\mathbb{H}^1$, $Q^* := \frac{2Q}{Q-2} = 4$ is the critical Sobolev exponent, $V$ and $f$ are continuous functions and satisfy certain conditions.

Over the past decades, research on the geometrical analysis of the Heisenberg group has revealed its exceptional value and extensive prospects for application across domains like partial differential equations, quantum mechanics and number theory. Many scholars have attempted to establish partial differential equations on the Heisenberg group in order to explore the existence and multiplicity of solutions. For more application scenarios, please refer to [9, 11].

Furthermore, the Schrödinger–Poisson system represents a fundamental theoretical framework that bridges quantum mechanics and classical mechanics. This coupled system was initially formulated to accurately describe the dynamics of interacting particles within a potential field. Specifically, the Schrödinger equation governs the quantum mechanical behavior of particles, whereas the Poisson equation characterizes the electrostatic potential generated by the particle density distribution. Please see [15, 28]. The Schrödinger–Poisson system has demonstrated significant applicability across diverse scientific domains, including quantum mechanics, astrophysics, plasma physics, and nonlinear optics. Given its broad utility in various physical contexts, this system has emerged as a pivotal research focus, thereby capturing considerable attention in the scientific community.

In the pioneering work [6], Benci and Fortunato addressed the following eigenvalue problem:

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = \lambda f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)|u|^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

With the help of the variational methods, they proved the multiplicity results for this problem (1.2). Subsequently, many scholars began to study the existence and multiplicity of solutions to the Schrödinger–Poisson system under different hypothetical conditions (regarding $f$, $K$ and $V$), and established a series of effective methods for addressing equations or systems involving nonlocal terms.

Liu [25] investigated the following Schrödinger–Poisson system with the nonlocal critical term:

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3 u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = K(x)|u|^5 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

The author used the mountain pass theorem together with the concentration-compactness principle to prove that positive solutions of the equation exist. For other interesting results, the readers may consult references [3, 4, 10, 12, 20, 22, 24, 26, 32, 33].

There are also many very interesting results for partial differential equations on the Heisenberg group. For example, An and Liu [2] investigated the following class of Schrödinger–Poisson type system on the Heisenberg group:

$$\begin{cases} -\Delta_H u + \mu\phi u = \lambda|u|^{q-2}u + |u|^2 u & \text{in } \Omega, \\ -\Delta_H \phi = |u|^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain. By applying the critical point theory and the concentration-compactness principle, they proved that system (1.4) has at least two positive solutions and one positive ground state solution.

For the nonlocal term with critical exponent in system (1.4) also constitutes a very interesting problem. Recently, Guo and Shi [13] studied the following Schrödinger–Poisson system in the Heisenberg group of the form:

$$\begin{cases} -\Delta_H u - \phi|u|u = \mu|u|^{q-2}u & \text{in } \Omega, \\ -\Delta_H \phi = u^3 & \text{in } \Omega, \\ \phi = u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Using the same method as [2], they obtained the existence of the ground state solution for this system.

Liang *et al.* [21] considered the following Schrödinger–Poisson systems with doubly critical growth on the first Heisenberg group:

$$\begin{cases} -\Delta_H u - \phi|u|u = \kappa|u|^{q-2}u + |u|^2 u & \text{in } \Omega, \\ -\Delta_H \phi = |u|^3 & \text{in } \Omega, \\ \phi = u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

By variational methods, together with the concentration-compactness principle on the Heisenberg group, the existence of multiple solutions for this problem is proved. Other interesting results on this topic can also be found in [5, 18, 19, 23, 29, 36].

However, there are very few existence results for solutions of doubly critical Schrödinger–Poisson system on the whole space. Motivated by the aforementioned findings, the present work is dedicated to establishing the existence of nontrivial solutions for this kind of problem.

Next, to elucidate the principal results of this study, we give that the potential function $V$ and nonlinearity $f$ satisfy the following conditions:

($V$) $V\colon \mathbb{H}^1 \to \mathbb{R}^+$ is continuous, and there exists a positive constant $V_0$ satisfying $V \geq V_0$ on $\mathbb{H}^1$.

($f_1$) The Carathéodory function $f : \mathbb{H}^1 \times \mathbb{R} \to \mathbb{R}$ is odd in its second variable.

($f_2$) For $q \in (2, 4)$, we have

$$|f(\xi, t)| \leq a(\xi)|t|^{q-1} \quad \text{for a.e. } \xi \in \mathbb{H}^1 \text{ and } \ t \in \mathbb{R},$$

with $0 \leq a(\xi) \in L^l(\mathbb{H}^1) \cap L^\infty(\mathbb{H}^1)$ and $l = 4/(4 - q)$.

($f_3$) There exists a constant $\theta \in (2, 4)$ that satisfies $0 < \theta F(\xi, t) \leq f(\xi, t)t$ for any $t \in \mathbb{R}^+$, where $F(\xi, t) = \int_0^t f(\xi, s)ds$.

Now, we shall present the key findings of this paper.

**Theorem 1.1.** *Let* ($V$) *hold. If $f$ verifies* ($f_1$)–($f_3$)*, then there exists $\lambda_1 > 0$ such that for any $\lambda > \lambda_1$, problem* (1.1) *has a nontrivial solution.*

**Theorem 1.2.** *Let* ($V$) *hold. If $f$ verifies* ($f_1$)–($f_3$)*, then there exists $\kappa_1 > 0$ such that for any $\kappa < \kappa_1$, problem* (1.1) *has at least $n$ pairs of nontrivial weak solutions.*

**Remark 1.3.** Compared to prior accomplishments, this study demonstrates several distinctive characteristics:

(1) Given that the first equation in system (1.1) involves the nonlocal term $\kappa\phi u^2$, proving the existence of solutions for this system presents significantly greater complexity and difficulty than the corresponding analysis for a single equation with critical nonlinearity.

(2) While the Kohn-Laplacian $\Delta_H$ shares certain similarities with the classical Laplacian operator $\Delta$, these similarities can be misleading, which has introduced substantial challenges in the proof process.

(3) The introduction of potential functions has significantly increased the complexity of problem (1.1), consequently exacerbating the challenges in obtaining their solutions.

The organizational framework of this paper is structured as follows. Section 2 provides a review of fundamental concepts pertaining to the Heisenberg group. Section 3 demonstrates the verification of the Palais-Smale condition at specific energy levels through the application of concentration-compactness principle within the classical Sobolev space on the Heisenberg group. Section 4 presents the proof of Theorem 1.1 utilizing the mountain pass theorem. Finally, Section 5 establishes the existence of multiple solutions to system (1.1) by employing the Krasnoselskii genus theory.

## 2. PRELIMINARIES

In this section, we commence by reviewing fundamental concepts pertaining to the first Heisenberg group. Let $\mathbb{H}^1 = (\mathbb{R}^3, \circ)$ be the first Heisenberg group. Set $\xi = (x, y, t) \in \mathbb{H}^1$ and $\xi' = (x', y', t') \in \mathbb{H}^1$, then we can define the group law using the following formula:

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x'y - y'x))$$

and the inverse is given by $\xi^{-1} = -\xi$, so that $(\xi \circ \xi^{'})^{-1} = (\xi^{'})^{-1} \circ \xi^{-1}$.

The natural dilation group on $\mathbb{H}^1$ is given by

$$\delta_s(\xi) = (sx, sy, s^2 t), \quad \forall\ \xi \in \mathbb{H}^1$$

with $s > 0$, which immediately yields $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$. Further, take any $\xi = (x, y, t) \in \mathbb{H}^1$ evaluating the Jacobian determinant, it is easy to see that the determinant of the dilation $\delta_s : \mathbb{H}^1 \to \mathbb{H}^1$ is uniformly $s^Q$, with $Q = 4$ being the homogeneous dimension of $\mathbb{H}^1$. Then for all $\xi \in \mathbb{H}^1$, the following gauge norm holds:

$$|\xi|_H = [(x^2 + y^2)^2 + t^2]^{\frac{1}{4}}.$$

Obviously, it is also a Korányi norm. Hence, for the dilating operation, the Korányi norm possesses a homogeneous of degree 1 and $\delta_s : (x, y, t) \mapsto (sx, sy, s^2 t)$.

Derived from the real Lie algebra linked to the horizontal left-invariant vector field, $\mathbb{H}^1$ has the set $\{X, Y\}$ as its fundamental basis for the Heisenberg orthogonal transformation. To be more precise, it can be mathematically expressed as:

$$T = \frac{\partial}{\partial t}, \quad X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}.$$

A vector field in the span of $[X, Y]$ is called horizontal. For a $C^1$ function $u : \mathbb{H}^1 \to \mathbb{R}$, its horizontal gradient is defined as

$$\nabla_H u = (X, Y).$$

Let $div_H(v_{1,v_2}) = Xv_1 + Yv_2$ for any vector valued function $(v_1, v_2)$, then the Kohn-Laplacian $\Delta_H$ is given by the definition

$$\Delta_H u = div_H(\nabla_H u), \quad \forall\ u \in C^2(\mathbb{H}^1).$$

We formulate the definition of the left-invariant distance $d_H$ over $\mathbb{H}^1$ as

$$d_H(\xi_0, \xi) = |\xi^{-1} \circ \xi_0|_H.$$

According to the celebrated theorem of Hörmamder [14, Theorem 1.1], it follows that $\Delta_H$ is hypoelliptic, and the maximum principle of Bony [7] holds. Define the Heisenberg open ball centered at $\xi_0$ and with radius $R$:

$$B_H(\xi_0, R) = \{\xi \in \mathbb{H}^N : d_H(\xi_0, \xi) < R\}.$$

To streamline notation, let $B_R$ represent the open ball centered at 0 with radius $R$, where $O = (0, 0)$ is the natural origin of $\mathbb{H}^1$. Therefore, there is

$$|B_H(\xi_0, R)| = |B_1|R^Q.$$

A complete discussion on the content framework of the Heisenberg group can be found in [17, 27].

Next, we will review some of the content about the classical Sobolev spaces. Let symbol $\|\cdot\|_s$ represent the usual $L^s$-norm, which is specifically expressed as

$$\|u\|_s^s = \int_{\mathbb{H}^1} |u|^s d\xi \quad \text{for all } s \in [1, +\infty).$$

Furthermore, we denote by $B_\rho$ the open ball in $\mathbb{H}^1$ with center 0 and radius $\rho$. At the same time, we denote the complement of $\mathbb{H}^1 \backslash B_\rho$ as $B_\rho^c$.

Now, the classical horizontal Sobolev space is denoted by $S^{1,2}(\mathbb{H}^1)$, which is constructed as the closure of $C_0^\infty(\mathbb{H}^1)$ under the corresponding norm

$$\|u\|^2_{S^{1,2}(\mathbb{H}^1)} = \int_{\mathbb{H}^1} |\nabla_H u|^2 d\xi.$$

Define a function space for equation (1.1) is

$$S_V^1(\mathbb{H}^1) := \left\{ u \in S^{1,2}(\mathbb{H}^1) : \int_{\mathbb{H}^1} V(\xi)|u|^2 d\xi < \infty \right\}$$

equipped with norm

$$\|u\| = \|u\|_{S_V^1(\mathbb{H}^1)} := (\|\nabla_H u\|_2^2 + \|u\|_{2,V}^2)^{\frac{1}{2}}, \quad \|u\|_{2,V}^2 = \int_{\mathbb{H}^1} V(\xi)|u|^2 d\xi.$$

Combined with the assumption $V \geq V_0 > 0$, $S_V^1(\mathbb{H}^1)$ constitutes a separable reflexive Banach space for the proof of Lemma 10 in [30] from the Euclidean context, we know that there exists a continuous embedding $S_V^1(\mathbb{H}^1) \hookrightarrow L^s(\mathbb{H}^1)$ for all $2 \leq s \leq 4$.

Moreover, as elucidated by Jerison and Lee [16], the best Sobolev constant

$$S = \inf_{u \in S_V^1(\mathbb{H}^1), u \neq 0} \frac{\int_{\mathbb{H}^1} |\nabla_H u|^2 d\xi}{(\int_{\mathbb{H}^1} |u|^4 d\xi)^{\frac{1}{2}}} \tag{2.1}$$

is attained by the $C^\infty$ function $U(x, y, t) = \frac{c_0}{\sqrt{(1+x^2+y^2)^2+t^2}}$. In other words, $U$ satisfies the equation $-\Delta u = |u|^3$ as a positive solution, with $u \in S_V^1(\mathbb{H}^1)$. Furthermore, we define

$$S_{HG} = \inf_{u \in S_V^1(\mathbb{H}^1) \setminus \{0\}} \frac{\|\nabla_H u\|_2^2}{\|u\|_{FL}^2}, \quad \text{where } \|u\|_{FL} = \Big( \iint\limits_{\mathbb{H}^1 \times \mathbb{H}^1} \frac{|u(\eta)|^3 |u(\xi)|^3}{|\eta^{-1}\xi|^2} d\eta d\xi \Big)^{\frac{1}{6}}. \tag{2.2}$$

According to the Lax–Milgram theorem, the Poisson equation $-\Delta_H \phi = |u|^3$ has a unique solution $\phi = \phi_u$. Thus, substituting $\phi_u$ into the first equation in system (1.1) yields the following single equation:

$$-\Delta_H u + V(\xi)u - \kappa \phi_u u^2 = \lambda f(\xi, u) + |u|^2 u \quad \text{in } \mathbb{H}^1. \tag{2.3}$$

Next, we say that $(u, \phi) \in S_V^1(\mathbb{H}^1) \times S_V^1(\mathbb{H}^1)$ is a solution to system (1.1) if and only if

$$\int\limits_{\mathbb{H}^1} \nabla_H u \nabla_H v d\xi + \int\limits_{\mathbb{H}^1} V(\xi) uv d\xi - \kappa \int\limits_{\mathbb{H}^1} \phi u^2 v d\xi - \lambda \int\limits_{\mathbb{H}^1} f(\xi, u) v d\xi - \int\limits_{\mathbb{H}^1} |u|^2 uv d\xi = 0$$

and

$$\int\limits_{\mathbb{H}^1} \nabla_H \phi \nabla_H w d\xi - \int\limits_{\mathbb{H}^1} |u|^3 w d\xi = 0$$

for all $v, w \in S_V^1(\mathbb{H}^1) \times S_V^1(\mathbb{H}^1)$. Thus, the functional $J(u, \phi) : S_V^1(\mathbb{H}^1) \times S_V^1(\mathbb{H}^1) \to \mathbb{R}$ related to system (1.1) is as follows:

$$\begin{aligned} J(u, \phi) = \frac{1}{2} \int\limits_{\mathbb{H}^1} (|\nabla_H u|^2 + V(\xi)|u|^2) d\xi + \frac{\kappa}{6} \int\limits_{\mathbb{H}^1} |\nabla_H \phi|^2 d\xi \\ - \frac{\kappa}{3} \int\limits_{\mathbb{H}^1} \phi |u|^3 d\xi - \lambda \int\limits_{\mathbb{H}^1} F(\xi, u) d\xi - \frac{1}{4} \int\limits_{\mathbb{H}^1} |u|^4 d\xi. \end{aligned}$$

It can be shown by standard proof methods that the functional $J$ is of class $C^1$ on $S_V^1(\mathbb{H}^1) \times S_V^1(\mathbb{H}^1)$, with its critical points corresponding to the solutions of system (1.1). Furthermore, let $J'_u(u, \phi)$ and $J'_\phi(u, \phi)$ stand for the partial derivatives of $J$ at $(u, \phi)$, which admit the following expressions:

$$\begin{aligned} \langle J'_u(u, \phi), v \rangle = \int\limits_{\mathbb{H}^1} (\nabla_H u \nabla_H v + V(\xi) uv) d\xi - \kappa \int\limits_{\mathbb{H}^1} \phi u^2 v d\xi \\ - \lambda \int\limits_{\mathbb{H}^1} f(\xi, u) v d\xi - \int\limits_{\mathbb{H}^1} |u|^2 uv d\xi \end{aligned}$$

and

$$\langle J'_\phi(u,\phi), w\rangle = \frac{\kappa}{3}\int\limits_{\mathbb{H}^1} \nabla_H \phi \nabla_H w d\xi - \frac{\kappa}{3}\int\limits_{\mathbb{H}^1} |u|^3 w d\xi$$

for any $(v,w) \in S^1_V(\mathbb{H}^1) \times S^1_V(\mathbb{H}^1)$. Moreover, we have the continuous embedding $S^1_V(\mathbb{H}^1)$ into $L^4(\mathbb{H}^1)$, it can be readily demonstrated that the functional $J'_u$ (or $J'_\phi$) maps $S^1_V(\mathbb{H}^1) \times S^1_V(\mathbb{H}^1)$ to the dual space $S^{-1}_V(\mathbb{H}^1)$ of $S^1_V(\mathbb{H}^1)$. Consequently, we establish that $J$ is $C^1$ on $S^1_V(\mathbb{H}^1) \times S^1_V(\mathbb{H}^1)$ and $J'_u(u,\phi) = J'_\phi(u,\phi) = 0$ if and only if $(u,\phi)$ represents the solution to system (1.1).

**Lemma 2.1.** *For all $u \in S^1_V(\mathbb{H}^1)$, there exists a unique nonnegative function $\phi_u \in S^1_V(\mathbb{H}^1)$ such that*

$$-\Delta_H \phi = |u|^3 \quad \text{in } \mathbb{H}^1. \tag{2.4}$$

*In addition, $\phi_u > 0$ if $u \neq 0$ and:*

(a) $\phi_{tu} = t^3 \phi_u$ *for any $t > 0$ and*

$$\int\limits_{\mathbb{H}^1} \phi_u |u|^3 d\xi = \int\limits_{\mathbb{H}^1} |\nabla_H \phi_u|^2 d\xi \leq S^{-1}|u|_4^6; \tag{2.5}$$

(b) $\|\phi_u\| \leq \hat{C}\|u\|^3$, *where $\hat{C} > 0$;*

(c) *let $u_n \rightharpoonup u$ in $S^1_V(\mathbb{H}^1)$. Then, $\phi_{u_n} \rightharpoonup \phi_u$ in $S^1_V(\mathbb{H}^1)$, and*

$$\int\limits_{\mathbb{H}^1} \phi_{u_n} u_n^2 v d\xi \to \int\limits_{\mathbb{H}^1} \phi_u u^2 v d\xi \quad \text{for every } v \in S^1_V(\mathbb{H}^1). \tag{2.6}$$

*Proof.* For all $u \in S^1_V(\mathbb{H}^1)$, define $T : S^1_V(\mathbb{H}^1) \to \mathbb{R}$ as follows:

$$T(v) = \int\limits_{\mathbb{H}^1} v|u|^3 d\xi \quad \text{for every } v \in S^1_V(\mathbb{H}^1).$$

Let $v_n \to v \in S^1_V(\mathbb{H}^1)$ as $n \to \infty$. According to the Hölder inequality, there is

$$\begin{aligned} |T(v_n) - T(v)| &= \int\limits_{\mathbb{H}^1} (v_n - v)|u|^3 d\xi \\ &\leq \Big(\int\limits_{\mathbb{H}^1} |v_n - v|^4 d\xi\Big)^{\frac{1}{4}} \Big(\int\limits_{\mathbb{H}^1} |u|^4 d\xi\Big)^{\frac{3}{4}} \\ &\leq S^{-\frac{1}{2}}|u|_4^3 \|v_n - v\| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This means that $T$ constitutes a continuous linear functional. By applying the Lax–Milgram theorem, we can establish the existence of a unique $\phi_u \in S^1_V(\mathbb{H}^1)$ such that

$$\int\limits_{\mathbb{H}^1} \nabla_H \phi_u \nabla_H v d\xi = \int\limits_{\mathbb{H}^1} v|u|^3 d\xi \quad \text{for every } v \in S^1_V(\mathbb{H}^1). \tag{2.7}$$

Hence, $\phi_u \in S_V^1(\mathbb{H}^1)$ is the unique solution of problem (2.4). In addition, according to the maximum principle, we have $\phi_u \geq 0$ and $\phi_u > 0$ if $u \neq 0$. In reality, for any positive constant $t$, we obtain

$$-\Delta_H \phi_{tu} = t^3|u|^3 = t^3(-\Delta_H \phi_u) = -\Delta_H(t^3\phi_u).$$

Therefore, from the uniqueness of $\phi_u$, it follows that $\phi_{tu} = t^3\phi_u$. Moreover, since $\phi_u \in S_V^1(\mathbb{H}^1)$, we can regard it as the test function for problem (2.4). By using the Hölder inequality and (2.1), we have

$$\int\limits_{\mathbb{H}^1} |\nabla_H \phi_u|^2 d\xi \leq \Big(\int\limits_{\mathbb{H}^1} |\phi_u|^4 d\xi\Big)^{\frac{1}{4}} \Big(\int\limits_{\mathbb{H}^1} |u|^4 d\xi\Big)^{\frac{3}{4}} \leq S^{-\frac{1}{2}} \Big(\int\limits_{\mathbb{H}^1} |\nabla_H \phi_u|^2 d\xi\Big)^{\frac{1}{2}} |u|_4^3.$$

Thus,

$$\int\limits_{\mathbb{H}^1} \phi_u |u|^3 d\xi = \int\limits_{\mathbb{H}^1} |\nabla_H \phi_u|^2 d\xi \leq S^{-1}|u|_4^6.$$

Meanwhile, from (2.7), the Hölder inequality and the Sobolev inequality, we derive that

$$\|\phi_u\|^2 = \int\limits_{\mathbb{H}^1} \phi_u |u|^3 d\xi \leq \|\phi_u\|_4 \|u\|_4^3 \leq C\|\phi_u\|\|u\|^3.$$

Hence, we obtain $\|\phi_u\| \leq \hat{C}\|u\|^3$.

Since $u_n \rightharpoonup u$ in $S_V^1(\mathbb{H}^1)$, it follows that $u_n \to u$ a.e. in $\mathbb{H}^1$ and $\{|u_n|^3\}$ is bounded in $L^4(\mathbb{H}^1)$. Besides, $|u_n|^3 \rightharpoonup |u|^3$ in $L^4(\mathbb{H}^1)$. Then, for any $v \in S_V^1(\mathbb{H}^1)$, we have

$$\int\limits_{\mathbb{H}^1} v|u_n|^3 d\xi \to \int\limits_{\mathbb{H}^1} v|u|^3 d\xi \quad as\ n \to \infty.$$

Thus, we arrive at the result that $\phi_{u_n} \rightharpoonup \phi_u$ in $S_V^1(\mathbb{H}^1)$. According to the Hölder inequality, Sobolev inequality and (b), it follows that

$$\begin{aligned}\int\limits_{\mathbb{H}^1} |\phi_{u_n} u_n^2|^{\frac{4}{3}} d\xi &\leq \|\phi_{u_n}\|_4^{\frac{4}{3}} \Big(\int\limits_{\mathbb{H}^1} |u_n|^4 d\xi\Big)^{\frac{2}{3}} \\ &\leq C_1 \|\phi_{u_n}\|_{S_V^1(\mathbb{H}^1)}^{\frac{4}{3}} \Big(\int\limits_{\mathbb{H}^1} |u_n|^4 d\xi\Big)^{\frac{2}{3}} \\ &\leq C_2 \|\phi_{u_n}\|^{\frac{4}{3}} \Big(\int\limits_{\mathbb{H}^1} |u_n|^4 d\xi\Big)^{\frac{2}{3}}.\end{aligned}$$

Therefore, we conclude that $\{\phi_{u_n} u_n^2\}$ is bounded in $L^{\frac{4}{3}}(\mathbb{H}^1)$ and

$$\phi_{u_n} u_n^2 \to \phi_u u^2 \quad \text{a.e. in } \mathbb{H}^1.$$

Thus,

$$\int_{\mathbb{H}^1} \phi_{u_n} u_n^2 v d\xi \to \int_{\mathbb{H}^1} \phi_u u^2 v d\xi \quad \text{for every } v \in S_V^1(\mathbb{H}^1).$$

The proof of Lemma 2.1 is completed. □

Based on the aforementioned arguments, we define the functional $\mathcal{I} : S_V^1(\mathbb{H}^1) \to \mathbb{R}$ corresponding to system (1.1), whose explicit formulation is presented as follows:

$$\mathcal{I}(u) = \frac{1}{2}\int_{\mathbb{H}^1} (|\nabla_H u|^2 + V(\xi)|u|^2) d\xi - \frac{\kappa}{6}\int_{\mathbb{H}^1} \phi_u |u|^3 d\xi - \lambda \int_{\mathbb{H}^1} F(\xi, u) d\xi - \frac{1}{4}\int_{\mathbb{H}^1} |u|^4 d\xi. \tag{2.8}$$

Thus, from [2], it is known that the solution $(u, \phi_u)$ to problem (1.1) corresponds to the critical point $u$ of the functional $\mathcal{I}$ and

$$\langle \mathcal{I}'(u), v\rangle = \int_{\mathbb{H}^1} (\nabla_H u \nabla_H v + V(\xi) uv) d\xi - \kappa \int_{\mathbb{H}^1} \phi_u u^2 v d\xi - \lambda \int_{\mathbb{H}^1} f(\xi, u) v d\xi - \int_{\mathbb{H}^1} |u|^2 uv d\xi.$$

## 3. $(PS)_c$ CONDITION

In this section, our primary objective is to demonstrate that the energy functional associated with system (1.1) fulfills the compactness condition, specifically the $(PS)_c$ condition. To accomplish this objective, we take the concentration-compactness principle within the framework of the Heisenberg group, thereby deducing the subsequent compactness outcome:

**Lemma 3.1.** *Suppose that $(V)$ and $(f_1)$–$(f_3)$ hold, then the functional $\mathcal{I}$ satisfies the $(PS)_c$ condition for all $c \in (0, c^*)$, where*

$$c^* = \Big(\frac{1}{2} - \frac{1}{\theta}\Big) \frac{S_{HG}^3 (\sqrt{S^{-4} + 4\kappa S_{HG}^{-3}} - S^{-2})}{2\kappa}. \tag{3.1}$$

*Proof.* Let $\{u_n\} \subset S_V^1(\mathbb{H}^1)$ be a $(PS)_c$ sequence related to the functional $\mathcal{I}$ and satisfies

$$\mathcal{I}(u_n) \to c \ \text{ and } \ \mathcal{I}'(u_n) \to 0 \ \text{ as } n \to \infty. \tag{3.2}$$

Next, we will first prove that the sequence $\{u_n\}$ is bounded in $S_V^1(\mathbb{H}^1)$. In fact, we have

$$\begin{aligned}
c+1+o(1)\|u_n\| &= \mathcal{I}(u_n) - \frac{1}{\theta}\langle \mathcal{I}'(u_n), u_n\rangle \\
&= \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_n\|^2 + \kappa\Big(\frac{1}{\theta}-\frac{1}{6}\Big)\int_{\mathbb{H}^1} \phi_{u_n}|u_n|^3 d\xi \\
&\quad + \lambda \int_{\mathbb{H}^1} \Big(\frac{1}{\theta} f(\xi,u_n)u_n - F(\xi,u_n)\Big)d\xi + \Big(\frac{1}{\theta}-\frac{1}{4}\Big)\int_{\mathbb{H}^1} |u_n|^4 d\xi \\
&\geq \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|u_n\|^2.
\end{aligned}$$

This fact, combined with $2<\theta<4$ and $(f_3)$, implies that $\{u_n\}$ is bounded in $S_V^1(\mathbb{H}^1)$. Therefore, up to subsequence, there exists $u \in S_V^1(\mathbb{H}^1)$ such that

$$\begin{aligned}
&u_n \rightharpoonup u \ \text{ in } S_V^1(\mathbb{H}^1), \\
&u_n \rightharpoonup u \ \text{ in } L^s(\mathbb{H}^1) \text{ for all } s\in[1,4), \\
&u_n \to u \ \text{ a.e. in } \mathbb{H}^1.
\end{aligned} \tag{3.3}$$

Additionally, according to the concentration-compactness principle [34], we can similarly obtain

$$\begin{aligned}
&\nu = \left(\int_{\mathbb{H}^1} \frac{|u(\eta)|^3}{|\eta^{-1}\xi|^2} d\eta\right)|u(\xi)|^3 d\xi + \sum_{j\in J}\nu_j\delta_{\xi_j}, \\
&\omega \geq |\nabla_H u|^2 d\xi + \sum_{j\in J}\omega_j\delta_{\xi_j}, \\
&\zeta = |u|^4 d\xi + \sum_{j\in J}\zeta_j\delta_{\xi_j}.
\end{aligned} \tag{3.4}$$

and

$$S_{HG}\nu_j^{\frac{1}{3}} \leq \omega_j, \quad \nu_j^{\frac{2}{3}} \leq C^{\frac{2}{3}}\zeta_j, \quad S\zeta_j^{\frac{1}{2}} \leq \omega_j, \tag{3.5}$$

where $J$ is a countable index set at most, $\xi_j \in \mathbb{H}^1$, $\delta_{\xi_j}$ is the Dirac mass at $\xi_j$, $\nu,\omega$ and $\zeta$ in $\mathbb{H}^1$ are positive Radon measures and $\{\nu_j\}_{j\in J}, \{\omega_j\}_{j\in J}, \{\zeta_j\}_{j\in J}$ are nonnegative numbers.

Besides, by applying the same argumentation method as in [34], we can arrive at the concentration-compactness principle at infinity:

$$\begin{aligned}
&\limsup_{n\to\infty} \iint_{\mathbb{H}^1\times\mathbb{H}^1} \frac{|u_n(\eta)|^3|u_n(\xi)|^3}{|\eta^{-1}\xi|^2} d\eta d\xi = \nu(\mathbb{H}^1) + \nu_\infty, \\
&\limsup_{n\to\infty} \int_{\mathbb{H}^1} |\nabla_H u_n|^2 d\xi = \omega(\mathbb{H}^1) + \omega_\infty, \\
&\limsup_{n\to\infty} \int_{\mathbb{H}^1} |u_n|^4 d\xi = \zeta(\mathbb{H}^1) + \zeta_\infty.
\end{aligned} \tag{3.6}$$

and

$$S_{HG}\nu_\infty^{\frac{1}{3}} \le \omega_\infty, \quad \nu_\infty^{\frac{2}{3}} \le C^{\frac{2}{3}}\zeta_\infty, \quad S\zeta_\infty^{\frac{1}{2}} \le \omega_\infty. \tag{3.7}$$

From now on, we will prove that $J = \emptyset$. By using the method of contradiction, suppose $J \ne \emptyset$. Set an element $j \in J$ and consider a smooth cut-off function $\psi \in C_0^\infty(\mathbb{H}^1)$ as Lemma 3.2 in [8] with $0 \le \psi \le 1$, $\psi(O) = 1$ and $supp(\psi) = \bar{B}_1$. Take $\varepsilon > 0$ sufficiently small and put $\psi_\varepsilon(\xi) = \psi(\delta_{1/\varepsilon}(\xi)),\ \xi \in \mathbb{H}^1$. Distinctly, the sequence $\{u_n\psi_\varepsilon\}$ is bounded in $S_V^1(\mathbb{H}^1)$. Hence, invoking (3.2), we have $\langle \mathcal{I}'(u_n), u_n\psi_\varepsilon\rangle \to 0$ as $n \to \infty$ which gives

$$\begin{aligned}&\int_{\mathbb{H}^1} |\nabla_H u_n|^2\psi_\varepsilon d\xi + \int_{\mathbb{H}^1} \nabla_H u_n \nabla_H \psi_\varepsilon u_n d\xi + \int_{\mathbb{H}^1} V(\xi)|u_n|^2\psi_\varepsilon d\xi\\ &- \kappa\int_{\mathbb{H}^1} \phi_{u_n}|u_n|^3\psi_\varepsilon d\xi = \lambda\int_{\mathbb{H}^1} f(\xi,u_n)u_n\psi_\varepsilon d\xi + \int_{\mathbb{H}^1} |u_n|^4\psi_\varepsilon d\xi + o(1).\end{aligned} \tag{3.8}$$

It follows from the Hölder inequality that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\int_{\mathbb{H}^1} u_n\nabla_H u_n \nabla_H\psi_\varepsilon d\xi\Big|\\ &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big(\int_{\mathbb{H}^1}|\nabla_H u_n|^2 d\xi\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{H}^1}|u_n|^2|\nabla_H\psi_\varepsilon|^2 d\xi\Big)^{\frac{1}{2}}\\ &\le C\lim_{\varepsilon\to 0}\Big(\int_{\mathbb{H}^1}|u|^4 d\xi\Big)^{\frac{1}{4}}\Big(\int_{\mathbb{H}^1}|\nabla_H\psi_\varepsilon|^4 d\xi\Big)^{\frac{1}{4}} = 0.\end{aligned} \tag{3.9}$$

In addition, we can conclude that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^1} V(\xi)|u_n|^2\psi_\varepsilon d\xi = 0. \tag{3.10}$$

From $(f_2)$ and the Hölder inequality, we can obtain

$$\Big|\int_{\mathbb{H}^1} f(\xi,u_n)u_n\psi_\varepsilon d\xi\Big| \le \int_{\mathbb{H}^1} a(\xi)|u_n|^q\psi_\varepsilon d\xi \le 2\|a\|_l\|u_n\|_4^q \le C\|a\|_l.$$

Thus,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{H}^1} f(\xi,u_n)u_n\psi_\varepsilon d\xi = 0. \tag{3.11}$$

For any arbitrary constant $\delta > 0$, it follows from (3.4) that

$$\begin{aligned} \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int\limits_{\mathbb{H}^1}|\nabla_H u_n|^2\psi_\varepsilon d\xi &\geq \lim_{\varepsilon\to 0}\Big(\omega_j+\int\limits_{\mathbb{H}^1}|\nabla_H u|^2\psi_\varepsilon d\xi\Big)=\omega_j,\\ \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int\limits_{\mathbb{H}^1}\phi_{u_n}|u_n|^3\psi_\varepsilon d\xi &= \lim_{\varepsilon\to 0}\Big(\nu_j+\int\limits_{\mathbb{H}^1}\phi_u|u|^3\psi_\varepsilon d\xi\Big)=\nu_j,\\ \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int\limits_{\mathbb{H}^1}|u_n|^4\psi_\varepsilon d\xi &= \lim_{\varepsilon\to 0}\Big(\zeta_j+\int\limits_{\mathbb{H}^1}|u|^4\psi_\varepsilon d\xi\Big)=\zeta_j. \end{aligned} \tag{3.12}$$

Substituting (3.9)–(3.12) into (3.8), we conclude that $\omega_j \leq \kappa\nu_j + \zeta_j$. Furthermore, as stated in (3.5),

$$\omega_j = 0 \quad\text{or}\quad \omega_j \geq \frac{S_{HG}^3(\sqrt{S^{-4}+4\kappa S_{HG}^{-3}}-S^{-2})}{2\kappa}.$$

Therefore by (3.2) and (3.4), we have

$$\begin{aligned} c &= \lim_{n\to\infty}(\mathcal{I}(u_n)-\frac{1}{\theta}\langle\mathcal{I}'(u_n),u_n\rangle)\\ &\geq \lim_{n\to\infty}\Big\{\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\int\limits_{\mathbb{H}^1}|\nabla_H u_n|^2 d\xi+\Big(\frac{1}{\theta}-\frac{1}{6}\Big)\int\limits_{\mathbb{H}^1}\phi_{u_n}|u_n|^3 d\xi\\ &\quad+\lambda\int\limits_{\mathbb{H}^1}\Big(\frac{1}{\theta}f(\xi,u_n)u_n-F(\xi,u_n)\Big)d\xi+\Big(\frac{1}{\theta}-\frac{1}{4}\Big)\int\limits_{\mathbb{H}^1}|u_n|^4 d\xi\Big\}\\ &\geq \lim_{n\to\infty}\Big(\frac{1}{2}-\frac{1}{\theta}\Big)\int\limits_{\mathbb{H}^1}|\nabla_H u_n|^2 d\xi \geq \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\omega_j\\ &\geq \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\frac{S_{HG}^3(\sqrt{S^{-4}+4\kappa S_{HG}^{-3}}-S^{-2})}{2\kappa}=c^* \end{aligned} \tag{3.13}$$

which contradicts (3.1). Hence, $J = \emptyset$.

Then, we will prove that $\omega_\infty = 0$. By using the method of contradiction, we suppose $\omega_\infty > 0$. For the purpose of verifying the potential mass concentration at infinity, we analogously construct a cut-off function $\varphi \in C_0^\infty(\mathbb{H}^1)$ with $0 \leq \varphi \leq 1$ and

$$\begin{cases} \varphi = 1 & \text{in } B_2^c,\\ \varphi = 0 & \text{in } B_1. \end{cases}$$

Taking $R > 0$ and put $\varphi_R(\xi) = \varphi(\delta_{1/R}(\xi))$, $\xi \in \mathbb{H}^1$. Obviously, the sequence $\{u_n\varphi_R\}$ is bounded in $S_V^1(\mathbb{H}^1)$. Hence, we have $\langle\mathcal{I}'(u_n), u_n\varphi_R\rangle \to 0$ as $n \to \infty$ which gives

$$\int_{\mathbb{H}^1} |\nabla_H u_n|^2 \varphi_R d\xi + \int_{\mathbb{H}^1} \nabla_H u_n \nabla_H \varphi_R u_n d\xi + \int_{\mathbb{H}^1} V(\xi)|u_n|^2 \varphi_R d\xi$$
$$- \kappa \int_{\mathbb{H}^1} \phi_{u_n} |u_n|^3 \varphi_R d\xi = \lambda \int_{\mathbb{H}^1} f(\xi, u_n) u_n \varphi_R d\xi + \int_{\mathbb{H}^1} |u_n|^4 \varphi_R d\xi + o(1). \tag{3.14}$$

Obviously, we can also conclude that

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{H}^1} V(\xi)|u_n|^2 \varphi_R d\xi = 0 \tag{3.15}$$

and

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{H}^1} f(\xi, u_n) u_n \varphi_R d\xi = 0, \tag{3.16}$$

From the above facts we conclude that $\omega_\infty \leq \kappa\nu_\infty + \zeta_\infty$. Furthermore, as stated in (3.7),

$$\omega_\infty = 0 \quad \text{or} \quad \omega_\infty \geq \frac{S_{HG}^3(\sqrt{S^{-4} + 4\kappa S_{HG}^{-3}} - S^{-2})}{2\kappa}.$$

If the latter case holds, therefore by (3.2) and (3.6), we have

$$\begin{aligned} c &= \lim_{n\to\infty} (\mathcal{I}(u_n) - \frac{1}{\theta}\langle \mathcal{I}'(u_n), u_n\rangle) \\ &\geq \lim_{n\to\infty} \Big\{ \Big(\frac{1}{2} - \frac{1}{\theta}\Big) \int_{\mathbb{H}^1} |\nabla_H u_n|^2 d\xi + \Big(\frac{1}{\theta} - \frac{1}{6}\Big) \int_{\mathbb{H}^1} \phi_{u_n} |u_n|^3 d\xi \\ &\quad + \lambda \int_{\mathbb{H}^1} \Big(\frac{1}{\theta} f(\xi, u_n) u_n - F(\xi, u_n)\Big) d\xi + \Big(\frac{1}{\theta} - \frac{1}{4}\Big) \int_{\mathbb{H}^1} |u_n|^4 d\xi \Big\} \\ &\geq \lim_{n\to\infty} \Big(\frac{1}{2} - \frac{1}{\theta}\Big) \int_{\mathbb{H}^1} |\nabla_H u_n|^2 d\xi \geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\omega_\infty \\ &\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big) \frac{S_{HG}^3(\sqrt{S^{-4} + 4\kappa S_{HG}^{-3}} - S^{-2})}{2\kappa} = c^* \end{aligned} \tag{3.17}$$

which contradicts (3.1). Hence, $\omega_\infty = 0$. Thus, by combining $J = \emptyset$ and $\omega_\infty = 0$, we can conclude that

$$\lim_{n\to\infty} \int_{\mathbb{H}^1} |u_n|^4 d\xi = \int_{\mathbb{H}^1} |u|^4 d\xi. \tag{3.18}$$

According to Vitali's convergence theorem, we infer that

$$\lim_{n\to\infty} \int_{\mathbb{H}^1} V(\xi)|u_n|^2 d\xi = \int_{\mathbb{H}^1} V(\xi)|u|^2 d\xi. \tag{3.19}$$

Thus, in combination with (3.2), we can deduce that

$$\begin{aligned}&\int_{\mathbb{H}^1} \nabla_H u_n \nabla_H v d\xi + \int_{\mathbb{H}^1} V(\xi) u_n v d\xi \\ &- \kappa \int_{\mathbb{H}^1} \phi_{u_n} |u_n|^2 v d\xi - \lambda \int_{\mathbb{H}^1} f(\xi, u_n) v d\xi - \int_{\mathbb{H}^1} |u_n|^2 u v d\xi = o(1).\end{aligned} \tag{3.20}$$

Let $v = u$ in (3.20), then

$$\|u\|^2 - \kappa \int_{\mathbb{H}^1} \phi_u |u|^3 d\xi - \lambda \int_{\mathbb{H}^1} f(\xi, u) u d\xi - \int_{\mathbb{H}^1} |u|^4 d\xi = 0. \tag{3.21}$$

From (3.2), (3.4), (3.6), (3.19) and Lemma 2.1, one has

$$\lim_{n\to\infty} \|u_n\|^2 - \kappa \int_{\mathbb{H}^1} \phi_u |u|^3 d\xi - \lambda \int_{\mathbb{H}^1} f(\xi, u) u d\xi - \int_{\mathbb{H}^1} |u|^4 d\xi = 0. \tag{3.22}$$

So, by (3.21) and (3.22), we derive the convergence $\lim_{n\to\infty} \|u_n\|^2 = \|u\|^2$. This demonstrates that, owing to the uniform convexity of $S_V^1(\mathbb{H}^1)$, we can ascertain that $u_n \to u$ in $S_V^1(\mathbb{H}^1)$. Therefore, we complete the proof of Lemma 3.1. □

## 4. PROOF OF THEOREM 1.1

The objective of this section is to show the proof of Theorem 1.1, for which we need to invoke the general mountain pass theorem (see [1]) to assist in the proof.

**Lemma 4.1.** *Let $E$ be a real Banach space and functional $\mathcal{I} \in C^1(E)$, with $\mathcal{I}(0) = 0$. Then $\mathcal{I}$ satisfies the mountain pass geometry, that is,*

(a) *there exists positive constants $\rho, \alpha$ satisfying $\mathcal{I}(u) \geq \alpha$ for all $u \in E$, which $\|u\|_E = \rho$;*
(b) *there exists $e \in E$ satisfying $\mathcal{I}(e) < 0$, which $\|u\|_E > \rho$.*

*Proof.* Firstly, by applying the condition $(V)$ and Sobolev imbedding inequality, one has

$$\begin{aligned}\mathcal{I}(u) &= \frac{1}{2}\|u\|^2 - \frac{\kappa}{6}\int_{\mathbb{H}^1} \phi_u |u|^3 d\xi - \lambda \int_{\mathbb{H}^1} F(\xi, u) d\xi - \frac{1}{4}\int_{\mathbb{H}^1} |u|^4 d\xi \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\kappa \hat{C}}{6}\|u\|^6 - \frac{2}{q}\lambda \|a\|_\iota \|u\|_4^q - \frac{1}{4}\|u\|_4^4 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\kappa \hat{C}}{6}\|u\|^6 - \frac{2}{q}\lambda c \|u\|^q - \frac{\|u\|^4}{4S^2},\end{aligned}$$

where $c$ is a positive constant. Given that $2 < q < 4$, it is feasible to choose $\rho, \alpha > 0$, where $\|u\| = \rho$ implies $\mathcal{I}(u) \geq \alpha = \frac{1}{4}\rho^2$. So assertion (a) in Lemma 4.1 is valid.

In what follows, we proceed to verify the assertion (b) in Lemma 4.1 holds. According to $(f_1)$–$(f_3)$ and $t > 1$, we can deduce that

$$\begin{aligned}\mathcal{I}(tu) &= \frac{t^2}{2}\|u\|^2 - \frac{\kappa t^6}{6}\int_{\mathbb{H}^1} \phi_u |u|^3 d\xi - \lambda \int_{\mathbb{H}^1} F(\xi, tu) d\xi - \frac{t^4}{4}\int_{\mathbb{H}^1} |u|^4 d\xi \\ &\le \frac{t^2}{2}\|u\|^2 - \frac{t^4}{4}\int_{\mathbb{H}^1} |u|^4 d\xi \to -\infty \quad \text{as } t \to \infty.\end{aligned}$$

Hence, we conclude that $\mathcal{I}(t_0 u) < 0$ and $t_0\|u\| > \rho$ when $t_0$ is sufficiently large. Setting $e = t_0 u$, we can confirm that $e$ is the required function, which thus completes the verification of Lemma 4.1 (b). □

*Proof of Theorem 1.1.* We assert that

$$0 < c_\lambda = \inf_{h\in\Gamma} \max_{0\le t\le 1} \mathcal{I}(h(t)) < \Big(\frac{1}{2} - \frac{1}{\theta}\Big) \frac{S_{HG}^3\big(\sqrt{S^{-4} + 4\kappa S_{HG}^{-3}} - S^{-2}\big)}{2\kappa}, \tag{4.1}$$

where $\Gamma = \{h \in C([0,1], E) : h(0) = 0,\ h(1) = e\}$. To establish the validity of the above inequality, we choose $v_0 \in S_V^1(\mathbb{H}^1)$ satisfying $\|v_0\| = 1$ and $\lim_{t\to\infty} \mathcal{I}(tv_0) = -\infty$. Then

$$\sup_{t\ge 0} \mathcal{I}(tv_0) = \mathcal{I}(t_\lambda v_0) \quad \text{for some } t_\lambda > 0.$$

Therefore, $t_\lambda$ satisfies

$$t_\lambda^2\|v_0\|^2 - t_\lambda^6 \kappa \int_{\mathbb{H}^1} \phi_{v_0}|v_0|^3 d\xi = \lambda \int_{\mathbb{H}^1} f(\xi, t_\lambda v_0)|t_\lambda v_0|^2 d\xi + t_\lambda^4 \int_{\mathbb{H}^1} |v_0|^4 d\xi. \tag{4.2}$$

Next, we will prove that $\{t_\lambda\}$ is bounded. Let $t_\lambda \ge 1$ be satisfied for every $\lambda > 0$. Then, using (4.2), we can obtain

$$t_\lambda^2 = t_\lambda^2\|v_0\|^2 = t_\lambda^6\kappa \int_{\mathbb{H}^1} \phi_{v_0}|v_0|^3 d\xi + \lambda \int_{\mathbb{H}^1} f(\xi, t_\lambda v_0)|t_\lambda v_0|^2 d\xi + t_\lambda^4 \int_{\mathbb{H}^1} |v_0|^4 d\xi \ge t_\lambda^4 \int_{\mathbb{H}^1} |v_0|^4 d\xi. \tag{4.3}$$

Thus, from (4.3), it can be determined that $\{t_\lambda\}$ is bounded.

Then, we will confirm that $t_\lambda \to 0$ as $\lambda \to \infty$. If not, there exists $t_\lambda > 0$ and a sequence $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$, satisfying $t_{\lambda_n} \to t_\lambda$ as $n \to \infty$. According to Lebesgue dominated convergence theorem, we arrive at

$$\int_{\mathbb{H}^1} f(\xi, t_{\lambda_n} v_0)|t_{\lambda_n} v_0|^2 d\xi \to \int_{\mathbb{H}^1} f(\xi, t_\lambda v_0)|t_\lambda v_0|^2 d\xi \quad \text{as } n \to \infty.$$

From this, we can draw the conclusion that

$$\lambda_n \int_{\mathbb{H}^1} f(\xi, t_\lambda v_0)|t_\lambda v_0|^2 d\xi \to \infty \quad \text{as } n \to \infty.$$

Hence, invoking (4.2), this situation is ruled out. Thus, $t_\lambda \to 0$ as $\lambda \to \infty$.
Moreover, (4.2) also allows us to obtain that

$$\lim_{\lambda\to\infty} \lambda \int\limits_{\mathbb{H}^1} f(\xi, t_\lambda v_0)|t_\lambda v_0|^2 d\xi = 0$$

and

$$\lim_{\lambda\to\infty} \int\limits_{\mathbb{H}^1} |t_\lambda v_0|^4 d\xi = 0.$$

Using the definition of $\mathcal{I}$ and the conclusion $t_\lambda \to 0$ as $\lambda \to \infty$, it follows that

$$\lim_{\lambda\to\infty} \Big( \sup_{t\geq 0} \mathcal{I}(tv_0) \Big) = \lim_{\lambda\to\infty} \mathcal{I}(t_\lambda v_0) = 0.$$

Therefore, there exists $\lambda_1 > 0$ such that for all $\lambda > \lambda_1$,

$$\sup_{t\geq 0} \mathcal{I}(tv_0) < \Big(\frac{1}{2} - \frac{1}{\theta}\Big) \frac{S_{HG}^3(\sqrt{S^{-4} + 4\kappa S_{HG}^{-3}} - S^{-2})}{2\kappa}.$$

Let $e = t_1 v_0$, where $t_1$ is chosen sufficiently large to ensure that $\mathcal{I}(e) < 0$. Thus, we have

$$0 < c_\lambda \leq \max_{0\leq t\leq 1} \mathcal{I}(h(t)),$$

where $h(t) = tt_1 v_0$. Thus, when $\lambda$ large enough, there is

$$0 < c_\lambda \leq \sup_{t\geq 0} \mathcal{I}(tv_0) < \Big(\frac{1}{2} - \frac{1}{\theta}\Big) \frac{S_{HG}^3(\sqrt{S^{-4} + 4\kappa S_{HG}^{-3}} - S^{-2})}{2\kappa}.$$

This completes the proof of Theorem 1.1. □

## 5. PROOF OF THEOREM 1.2

To prove Theorem 1.2, we resort to the Krasnoselskii genus theory in this part. For this purpose, we take $E$ as a Banach space and define $\Lambda$ as the collection of all closed subsets $A \subset E \setminus \{0\}$ symmetric with respect to the origin, which means the inclusion $u \in A$ is equivalent to $-u \in A$. Furthermore, consider a $k$-dimensional subspace $X = \text{span}\{z_1, \ldots, z_k\}$. Inductively, we choose $z_{n+1} \notin Z_n = \text{span}\{z_1, \ldots, z_n\}$, for every $n \geq k$. Set $R_n = R(Z_n)$ and $\Upsilon_n = B_{R_n} \cap Z_n$, we define

$$L_n = \{\varphi \in C(\Upsilon_n, E) : \varphi|\partial B_{R_n \cap X_n} = id \text{ and } \varphi \text{ is old}\}$$

and

$$\Gamma_j = \Big\{ \varphi(\overline{\Upsilon_n \setminus V}) : \varphi \in L_n,\ n \geq j,\ V \in \Lambda,\ \Lambda \text{ is closed},\ \gamma(V) \leq n - j \Big\}$$

with $\gamma(V)$ being the Krasnoselskii genus of $V$.

**Theorem 5.1** ([31]). *Let $E$ be an infinite-dimensional Banach space and $F \in C^1(E)$ be even with $F(0) = 0$. Assume that $X$ be an infinite-dimensional space for which $E = X \bigoplus Y$ holds, and that $F$ fulfills the following conditions:*

(a) *There exists $\zeta > 0$ such that $F$ satisfies $(PS)_c$ condition, for each $c \in (0, \zeta)$.*
(b) *There exist $\rho, \alpha > 0$ satisfying $F(u) \geq \alpha$, for each $u \in \partial B_\rho \cap Y$.*
(c) *For each finite-dimensional subspace $\widetilde{E} \subset E$, there exists $R = R(\widetilde{E}) > \rho$ ensuring that $F(u) \leq 0$ on $\widetilde{E} \setminus B_R$.*

*For each $j \in N$, we define*

$$c_j = \inf_{Z \in \Gamma_j} \max_{u \in Z} F(u).$$

*Hence, $0 \leq c_j \leq c_{j+1}$ and $c_j < \zeta$, for every $j > k$, with each $c_j$ being a critical value of $F$. Besides, if $c_j = c_{j+1} = \ldots = c_{j+p} = c < \zeta$ for $j > k$, then $\gamma(G_c) \geq p + 1$, where*

$$G_c = \{u \in E : F(u) = c \text{ and } F'(u) = 0\}.$$

*Proof of Theorem 1.2.* We shall employ Theorem 5.1 to analyze $\mathcal{I}$. It is established that $S^1_V(\mathbb{H}^1)$ constitutes a reflexive Banach space and $\mathcal{I} \in C^1(S^1_V(\mathbb{H}^1))$. As demonstrated in (2.8), the functional $\mathcal{I}$ satisfies $\mathcal{I}(0) = 0$.

Next, similar to the proof of Lemma 4.1, we can obtain that the energy functional $\mathcal{I}$ meets criteria (b) and (c) stated in Theorem 5.1.

Now, we show that there exists a nondecreasing sequence $\{t_n\}$ of positive real numbers for which

$$c^\lambda_n = \inf_{Z \in \Gamma_n} \max_{u \in Z} \mathcal{I}(u) < t_n.$$

To achieve this goal, by using the argument presented in [35] and combining with the definition of $\Gamma_n$, we arrive at

$$c^\lambda_n = \inf_{Z \in \Gamma_n} \max_{u \in Z} \mathcal{I}(u) \leq \inf_{Z \in \Gamma_n} \max_{u \in Z} \left\{ \frac{1}{2}\|u\|^2 - \frac{1}{4} \int\limits_{\mathbb{H}^1} |u|^4 d\xi \right\} := t_n.$$

Then $t_n < \infty$ and $t_n \leq t_{n+1}$.

Finally, we claim that problem (1.1) admits at least $n$ pairs of weak solutions. Paralleling the deduction in (4.1), we may choose $\kappa_1 > 0$ small enough such that for any $\kappa < \kappa_1$, one has

$$c^\lambda_n \leq t_n < \left(\frac{1}{2} - \frac{1}{\theta}\right) \frac{S^3_{HG}(\sqrt{S^{-4} + 4\kappa S^{-3}_{HG}} - S^{-2})}{2\kappa}.$$

Hence, we have

$$0 < c^\lambda_1 \leq c^\lambda_2 \leq \ldots \leq c^\lambda_n < t_n < \left(\frac{1}{2} - \frac{1}{\theta}\right) \frac{S^3_{HG}(\sqrt{S^{-4} + 4\kappa S^{-3}_{HG}} - S^{-2})}{2\kappa}.$$

By Lemma 3.1, we know that $\mathcal{I}$ satisfies $(PS)_{c^\lambda_j}$ condition for $j = 1, 2, \ldots, n$. An application of [31] ensures that the levels $c^\lambda_1 \leq c^\lambda_2 \leq \ldots \leq c^\lambda_n$ constitute critical values of $\mathcal{I}$.

In the event that $c_j^\lambda = c_{j+1}^\lambda$ for $j = 1, 2, \dots, n-1$, according to [1, Theorem 4.2 and Remark 2.12], the set $G_{c_j^\lambda}$ contains an infinite number of distinct points, thereby indicating that problem (1.1) possesses infinitely many weak solutions. Consequently, problem (1.1) admits at least $n$ pairs of solutions, bringing this proof to an end. $\square$

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