

ON MIXED LOCAL-NONLOCAL
SOBOLEV-TYPE INEQUALITIES
AND THEIR CONNECTION
WITH SINGULAR EQUATIONS
IN THE HEISENBERG GROUP

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Abstract. In this work, we establish a mixed local-nonlocal Sobolev-type inequality in the Heisenberg group and demonstrate that its extremals coincide with solutions to the corresponding mixed local-nonlocal singular p -Laplace equations. We further show that these inequalities serve as a necessary and sufficient condition for the existence of weak solutions to the associated singular problems. Notably, the same characterization remains valid in both the purely local and purely nonlocal settings. Our results thus provide a unified framework linking the existence theory for singular equations across local, nonlocal, and mixed regimes.

Keywords: Sobolev type inequality, extremal, mixed local-nonlocal singular problem, Heisenberg group.

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1. INTRODUCTION

In the Euclidean setting, Sobolev-type inequalities connected to singular elliptic problems have been recently investigated in the local [1], nonlocal [12], and mixed local-nonlocal frameworks [18]. More precisely, in [1] the authors proved that, for a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and parameters $0 < \delta < 1 < p < \infty$, the quantity

$$\mu(\Omega) := \inf_{v \in W_0^{1,p}(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla v|^p dx : \int_{\Omega} |v|^{1-\delta} dx = 1 \right\}$$

is attained by some $u \in W_0^{1,p}(\Omega)$, which is a positive solution to the singular p -Laplace equation

$$-\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \mu(\Omega) u^{-\delta} \quad \text{in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

The corresponding anisotropic setting has been addressed in [3, 4]. In the nonlocal framework, the authors of [12] established that for any $0 < \delta < 1$, $0 < s < 1 < p < \infty$, and for a given nonnegative function $f \in L^m(\Omega) \setminus \{0\}$ with $m \geq 1$, the quantity

$$\nu(\Omega) := \inf_{v \in W_0^{s,p}(\Omega) \setminus \{0\}} \left\{ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dx dy : \int_{\Omega} |v|^{1-\delta} f dx = 1 \right\} \quad (1.1)$$

is achieved at some $u \in W_0^{s,p}(\Omega)$, which satisfies the singular fractional p -Laplace equation

$$(-\Delta_p)^s u = \nu(\Omega) f(x) u^{-\delta} \quad \text{in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \quad (1.2)$$

Here, $(-\Delta_p)^s$ denotes the fractional p -Laplace operator, defined by

$$(-\Delta_p)^s u(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+ps}} dy,$$

where P.V. stands for the principal value. In a recent development, the authors in [18] proved that the infimum

$$\theta(\Omega) := \inf_{v \in W_0^{1,p}(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla v|^p dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy : \int_{\Omega} |v|^{1-\delta} f dx = 1 \right\}, \quad (1.3)$$

is attained by a function u solving the mixed local-nonlocal p -Laplace equation

$$-\Delta_p u + (-\Delta_p)^s u = \theta(\Omega) f(x) u^{-\delta} \quad \text{in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \quad (1.4)$$

Furthermore, the mixed anisotropic setting has been addressed in [15]. We also note that singular elliptic problems have been extensively studied over the past decades, resulting in a vast body of literature. For the purely local case, we refer to [7, 9, 11], while the purely nonlocal case is treated in [5, 8, 13] and the references therein. More recently, mixed local-nonlocal singular problems have been investigated in [2, 6, 14, 18] among others. We also refer to [21], where a multiplicity result is established for a mixed local-nonlocal Kirchhoff problem involving critical growth.

In non-Euclidean settings, Sobolev-type inequalities of this nature have only recently begun to be understood. For the purely local case in Carnot groups, see [19], and for the purely nonlocal case, we refer to [17]. However, to the best of our knowledge, the mixed local-nonlocal case remains unexplored. The primary objective of this article is to bridge this gap.

More precisely, we establish that for $\alpha > 0$ and a given nonnegative function $f \in L^m(\Omega) \setminus \{0\}$ with $m \geq 1$, the quantity

$$\Theta(\Omega) := \inf_{v \in HW_0^{1,p}(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla_H v|^p dx + \alpha \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} \frac{|v(x) - v(y)|^p}{|y^{-1} \circ x|^{Q+sp}} dx dy \right. \\ \left. : \int_{\Omega} |v|^{1-\delta} f dx = 1 \right\}, \tag{1.5}$$

is associated to a function $u \in HW_0^{1,p}(\Omega)$ solving the mixed local-nonlocal singular problem

$$M_\alpha u := -\Delta_{H,p} u + \alpha (-\Delta_{H,p})^s u = f(x) u^{-\delta} \quad \text{in } \Omega, \\ u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{H}^N \setminus \Omega, \tag{1.6}$$

where $0 < \delta < 1$ with $0 < s < 1 < p < Q$ and $Q = 2N + 2$ denotes the homogeneous dimension of the Heisenberg group \mathbb{H}^N . Here,

$$\Delta_{H,p} u = \operatorname{div}(|\nabla_H u|^{p-2} \nabla_H u)$$

is the p -subLaplace operator and

$$(-\Delta_{H,p})^s u(x) := \text{P.V.} \int_{\mathbb{H}^N} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|y^{-1} \circ x|^{Q+ps}} dy$$

is the fractional p -subLaplace operator.

Moreover, we prove that the mixed local-nonlocal Sobolev-type inequality is both necessary and sufficient for the existence of solutions to equation (1.6). Analogous characterizations hold separately in the purely local and purely nonlocal cases. As an immediate consequence, we deduce that the purely singular p -Laplace equation

$$-\Delta_{H,p} u = f(x) u^{-\delta} \quad \text{in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.7}$$

admits a solution if and only if the mixed local-nonlocal singular p -Laplace equation (1.6) has a solution for every $\alpha > 0$.

To establish our main results, we adopt the methodologies developed in [1, 12, 18]. This necessitates a detailed analysis of the corresponding singular mixed problem. We highlight that mixed local-nonlocal problems in non-Euclidean settings remain largely unexplored, even in the nonsingular case. For recent progress in this direction, we refer to [10, 22, 23], where existence and regularity results for nonsingular mixed equations were obtained. Very recently, singular mixed problems in the Heisenberg group have been investigated in [16].

1.1. FUNCTIONAL SETTING

In this subsection, we briefly recall some fundamental properties of the Heisenberg group \mathbb{H}^N and introduce the function spaces relevant to our analysis.

The Euclidean space \mathbb{R}^{2N+1} , equipped with the group law

$$\xi \circ \eta = \left(x_1 + y_1, x_2 + y_2, \dots, x_{2N} + y_{2N}, \tau + \tau' + \frac{1}{2} \sum_{i=1}^N (x_i y_{N+i} - x_{N+i} y_i) \right),$$

where $\xi = (x_1, \dots, x_{2N}, \tau)$ and $\eta = (y_1, \dots, y_{2N}, \tau') \in \mathbb{R}^{2N+1}$, constitutes the Heisenberg group \mathbb{H}^N .

The left-invariant vector fields on \mathbb{H}^N are given by

$$X_i = \partial_{x_i} - \frac{x_{N+i}}{2} \partial_\tau, \quad X_{N+i} = \partial_{x_{N+i}} + \frac{x_i}{2} \partial_\tau, \quad 1 \leq i \leq N,$$

and their only nontrivial commutator is

$$T = \partial_\tau = [X_i, X_{N+i}] = X_i X_{N+i} - X_{N+i} X_i, \quad 1 \leq i \leq N.$$

We refer to X_1, X_2, \dots, X_{2N} as the *horizontal* vector fields on \mathbb{H}^N , and to T as the *vertical* vector field.

The Haar measure on \mathbb{H}^N coincides with the Lebesgue measure on \mathbb{R}^{2N+1} . For any measurable set $E \subset \mathbb{H}^N$, its Lebesgue measure will be denoted by $|E|$.

For a point $\xi = (x_1, \dots, x_{2N}, \tau)$, we introduce the Korányi-type norm

$$|\xi| = \left(\left(\sum_{i=1}^{2N} x_i^2 \right) + \tau^2 \right)^{1/4}.$$

The Carnot–Carathéodory distance between two points $\xi, \eta \in \mathbb{H}^N$ is defined as the infimum of the lengths of horizontal curves joining them and is denoted by $d(\xi, \eta)$. This distance is equivalent to the Korányi metric:

$$d(\xi, \eta) \sim |\xi^{-1} \circ \eta|.$$

The ball of radius $r > 0$ centered at ξ_0 , with respect to the distance d , is given by

$$B_r(\xi_0) = \{\xi \in \mathbb{H}^N : d(\xi, \xi_0) < r\}.$$

When the center is unimportant or clear from context, we simply write $B_r := B_r(\xi_0)$.

The homogeneous dimension of \mathbb{H}^N is $Q = 2N + 2$. For $1 \leq p < \infty$ and an open set $\Omega \subset \mathbb{H}^N$, the Sobolev space

$$HW^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla_H u \in L^p(\Omega)\},$$

is defined via the horizontal gradient

$$\nabla_H u = (X_1 u, X_2 u, \dots, X_{2N} u),$$

and equipped with the norm

$$\|u\|_{HW^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla_H u\|_{L^p(\Omega)}.$$

With this norm, $HW^{1,p}(\Omega)$ becomes a Banach space. The local Sobolev space $HW_{loc}^{1,p}(\Omega)$ is defined by

$$HW_{loc}^{1,p}(\Omega) = \{u : u \in HW^{1,p}(\Omega') \text{ for every open set } \Omega' \Subset \Omega\}.$$

Let $1 \leq p < \infty$, $s \in (0, 1)$, and let $v : \mathbb{H}^N \rightarrow \mathbb{R}$ be a measurable function. The associated Gagliardo seminorm is given by

$$[v]_{HW^{s,p}(\mathbb{H}^N)} = \left(\int_{\mathbb{H}^N} \int_{\mathbb{H}^N} \frac{|v(x) - v(y)|^p}{|y^{-1} \circ x|^{Q+sp}} dx dy \right)^{1/p}.$$

The fractional Sobolev space on the Heisenberg group is defined as

$$HW^{s,p}(\mathbb{H}^N) = \{v \in L^p(\mathbb{H}^N) : [v]_{HW^{s,p}(\mathbb{H}^N)} < \infty\},$$

and it is endowed with the natural norm

$$\|v\|_{HW^{s,p}(\mathbb{H}^N)} = \left(\|v\|_{L^p(\mathbb{H}^N)}^p + [v]_{HW^{s,p}(\mathbb{H}^N)}^p \right)^{1/p}.$$

For any open set $\Omega \subset \mathbb{H}^N$, the fractional Sobolev space $HW^{s,p}(\Omega)$ and its associated norm $\|v\|_{HW^{s,p}(\Omega)}$ are defined analogously to the whole-space case.

To study the mixed problem (1.6) with $\alpha > 0$, we consider the space

$$HW_0^{1,p}(\Omega) = \{u \in HW^{1,p}(\mathbb{H}^N) : u = 0 \text{ in } \mathbb{H}^N \setminus \Omega\}.$$

The following result is a direct consequence of [10, Theorem 2.4].

Lemma 1.1. *Let Ω be a bounded domain in \mathbb{H}^N , and let $1 < p < \infty$ and $0 < s < 1$. Then there exists a constant $C = C(Q, p, s, \Omega) > 0$ such that*

$$\iint_{\mathbb{H}^N \times \mathbb{H}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{Q+ps}} dx dy \leq C \int_{\Omega} |\nabla_H u(x)|^p dx,$$

for every $u \in HW_0^{1,p}(\Omega)$.

For the following embedding result, we refer to [20, Theorem 8.1].

Lemma 1.2. *Let $0 < s < 1$ and $1 < p < Q$. Then the embedding*

$$HW^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$$

is continuous for every $r \in [1, p^*]$ and compact for every $r \in [1, p^*)$.

Taking into account Lemmas 1.1 and 1.2, for a given $\alpha > 0$, we introduce the following norm $\|\cdot\|$ on $HW_0^{1,p}(\Omega)$, which will be used throughout the rest of the paper:

$$\|u\|_{HW_0^{1,p}(\Omega)} := \left(\int_{\Omega} |\nabla_H u|^p dx + \alpha \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} \frac{|u(x) - u(y)|^p}{|y^{-1} \circ x|^{Q+sp}} dx dy \right)^{1/p}. \quad (1.8)$$

1.2. NOTATIONS AND ASSUMPTIONS

We fix the following notations and assumptions, which will be used throughout the paper unless otherwise specified.

For $l > 1$, the conjugate exponent is denoted by $l' = \frac{l}{l-1}$. We assume $\alpha > 0$, $0 < s < 1 < p < Q$, and $\delta \in (0, 1)$, where $Q = 2N + 2$ is the homogeneous dimension of \mathbb{H}^N . The critical Sobolev exponent is

$$p^* = \frac{Qp}{Q-p}, \quad 1 < p < Q.$$

For $q > 1$, we define

$$J_q(t) = |t|^{q-2}t \quad \text{for all } t \in \mathbb{R}.$$

We use the notation

$$d\mu = \frac{dx dy}{|y^{-1} \circ x|^{Q+sp}}.$$

Ω denotes a bounded smooth domain in \mathbb{H}^N . For $\omega \Subset \Omega$, we mean that ω is compactly contained in Ω , i.e.,

$$\omega \subset \bar{\omega} \subset \Omega.$$

For $k \in \mathbb{R}$, we adopt the standard notation

$$k^+ = \max\{k, 0\}, \quad k^- = \max\{-k, 0\}, \quad k_- = \min\{k, 0\}.$$

For a measurable function F on a set S and constants c, d , the notation $c \leq F \leq d$ in S means

$$c \leq F(x) \leq d \quad \text{for almost every } x \in S.$$

The symbol C denotes a generic positive constant, which may change from line to line. If the constant depends on parameters r_1, r_2, \dots, r_k , we write

$$C = C(r_1, r_2, \dots, r_k).$$

Before stating our main results, we mention some auxiliary results below.

1.3. AUXILIARY RESULTS

The following result is a direct consequence of [16, Lemma 4.4].

Lemma 1.3. *Let $g \in L^\infty(\Omega) \setminus \{0\}$ be a nonnegative function in Ω . Then there exists a unique solution*

$$u \in HW_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

to the problem

$$M_\alpha u = g \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{H}^N \setminus \Omega. \quad (1.9)$$

Moreover, for every $\omega \Subset \Omega$, there exists a constant $C(\omega) > 0$ such that

$$u \geq C(\omega) \quad \text{in } \omega.$$

The next result follows from [16, Lemmas 3.1 and 3.6].

Lemma 1.4. *Let $f \in L^1(\Omega) \setminus \{0\}$ be a nonnegative function, and for $n \in \mathbb{N}$, define*

$$f_n(x) := \min\{f(x), n\}.$$

Consider the approximated problem

$$M_\alpha u = f_n(x) \left(u^+ + \frac{1}{n}\right)^{-\delta} \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{H}^N \setminus \Omega. \tag{1.10}$$

Then, for each $n \in \mathbb{N}$, there exists a unique positive solution

$$u_n \in HW_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Furthermore, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is monotone increasing, i.e.,

$$u_{n+1} \geq u_n \quad \text{in } \Omega \text{ for all } n \in \mathbb{N}.$$

In addition, for every compact set $\omega \Subset \Omega$, there exists a constant $C(\omega) > 0$, independent of n , such that

$$u_n \geq C(\omega) > 0 \quad \text{in } \omega.$$

Moreover, if $f \in L^m(\Omega)$, where

$$m = \left(\frac{p^*}{1-\delta}\right)',$$

then the sequence $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $HW_0^{1,p}(\Omega)$.

Before presenting the next result, we introduce the notion of weak solutions for problem (1.6).

Definition 1.5. Let $f \in L^1(\Omega) \setminus \{0\}$ be a nonnegative function. A function

$$u \in HW_0^{1,p}(\Omega)$$

is called a *weak solution* of problem (1.6) if for every subset $\omega \Subset \Omega$, there exists a constant $C = C(\omega) > 0$ such that $u \geq C$ in ω , and for every test function $\varphi \in C_c^1(\Omega)$, it holds that

$$\begin{aligned} & \int_{\Omega} |\nabla_H u|^{p-2} \nabla_H u \cdot \nabla_H \varphi \, dx + \alpha \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} J_p(u(x) - u(y)) (\varphi(x) - \varphi(y)) \, d\mu \\ & = \int_{\Omega} f(x) u^{-\delta} \varphi(x) \, dx. \end{aligned} \tag{1.11}$$

Remark 1.6. By density arguments (see [18, Lemma 5.1]), the identity (1.11) extends to all $\varphi \in HW_0^{1,p}(\Omega)$. Consequently, following the proof of [18, Corollary 5.2], one concludes that problem (1.6) admits at most one weak solution in $HW_0^{1,p}(\Omega)$ whenever $f \in L^1(\Omega) \setminus \{0\}$.

The next result can be found in [16, Theorems 1.2 and 1.7].

Theorem 1.7. *Let $f \in L^m(\Omega) \setminus \{0\}$ be nonnegative, where*

$$m = \left(\frac{p^*}{1-\delta} \right)'$$

Then problem (1.6) admits a unique weak solution

$$u_\delta \in HW_0^{1,p}(\Omega).$$

We present our main results in the following subsection. The remainder of the paper is organized as follows: In Section 2, we discuss preliminary results, and in Section 3, we prove the main theorems.

1.4. MAIN RESULTS

Theorem 1.8. *Let $f \in L^m(\Omega) \setminus \{0\}$ be nonnegative, where*

$$m = \left(\frac{p^*}{1-\delta} \right)'$$

Let $u_\delta \in HW_0^{1,p}(\Omega)$ denote the weak solution of problem (1.6) given by Theorem 1.7. Then the following properties hold:

(a) *(Extremal)*

$$\Theta(\Omega) = \left(\int_{\Omega} |\nabla_H u_\delta|^p dx + \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} |u_\delta(x) - u_\delta(y)|^p d\mu \right)^{\frac{1-\delta-p}{1-\delta}},$$

where $\Theta(\Omega)$ is defined in (1.5).

(b) *(Sobolev-type inequality) For every $v \in HW_0^{1,p}(\Omega)$, the following mixed Sobolev inequality holds:*

$$C \left(\int_{\Omega} |v|^{1-\delta} f dx \right)^{\frac{p}{1-\delta}} \leq \int_{\Omega} |\nabla_H v|^p dx + \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} |v(x) - v(y)|^p d\mu, \quad (1.12)$$

if and only if

$$C \leq \Theta(\Omega).$$

(c) *(Simplicity) If for some $w \in HW_0^{1,p}(\Omega)$ the equality*

$$\Theta(\Omega) \left(\int_{\Omega} |w|^{1-\delta} f dx \right)^{\frac{p}{1-\delta}} = \int_{\Omega} |\nabla_H w|^p dx + \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} |w(x) - w(y)|^p d\mu \quad (1.13)$$

holds, then $w = ku_\delta$ for some constant k .

Corollary 1.9. *Define*

$$S_\delta := \left\{ v \in HW_0^{1,p}(\Omega) : \int_\Omega |v|^{1-\delta} f \, dx = 1 \right\}, \quad \zeta_\delta := \left(\int_\Omega u_\delta^{1-\delta} f \, dx \right)^{-\frac{1}{1-\delta}}.$$

Then

$$V_\delta := \zeta_\delta u_\delta \in S_\delta,$$

and from Theorem 1.8, we have

$$\Theta(\Omega) = \int_\Omega |\nabla_H V_\delta|^p \, dx + \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} |V_\delta(x) - V_\delta(y)|^p \, d\mu.$$

Moreover, V_δ satisfies the singular problem

$$M_\alpha V_\delta = \Theta(\Omega) f V_\delta^{-\delta} \quad \text{in } \Omega, \quad V_\delta > 0 \text{ in } \Omega.$$

Theorem 1.10. *Let $f \in L^1(\Omega) \setminus \{0\}$ be nonnegative. Then, for every $v \in HW_0^{1,p}(\Omega)$, the mixed Sobolev inequality (1.12) holds if and only if the problem (1.6) admits a weak solution in $HW_0^{1,p}(\Omega)$.*

The following results concern the purely local and purely nonlocal cases.

Theorem 1.11. *Let $f \in L^1(\Omega) \setminus \{0\}$ be nonnegative. Then, for every $v \in HW_0^{1,p}(\Omega)$, the local Sobolev inequality*

$$C \left(\int_\Omega |v|^{1-\delta} f \, dx \right)^{\frac{p}{1-\delta}} \leq \int_\Omega |\nabla_H v|^p \, dx, \tag{1.14}$$

holds if and only if the singular p -Laplace equation (1.7) admits a weak solution in the usual Sobolev space $HW_0^{1,p}(\Omega)$.

Theorem 1.12. *Let $f \in L^1(\Omega) \setminus \{0\}$ be nonnegative. Then, for every $v \in HW_0^{s,p}(\Omega)$, the nonlocal Sobolev inequality*

$$C \left(\int_\Omega |v|^{1-\delta} f \, dx \right)^{\frac{p}{1-\delta}} \leq \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} |v(x) - v(y)|^p \, d\mu, \tag{1.15}$$

holds if and only if the singular fractional p -Laplace equation

$$(-\Delta_{H,p})^s u = f(x) u^{-\delta} \quad \text{in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{H}^N \setminus \Omega \tag{1.16}$$

admits a weak solution in $HW_0^{s,p}(\Omega)$.

Remark 1.13. The weak solutions of problems (1.7) and (1.16) are defined analogously to [16, Definition 3.1] and [17, Definition 2.3], respectively.

As a direct consequence of Theorems 1.10, 1.11 and Lemma 1.1, we obtain the following result connecting the problems (1.6) and (1.7).

Corollary 1.14. *Let $f \in L^1(\Omega) \setminus \{0\}$ be nonnegative. Then problem (1.6) admits a weak solution in $HW_0^{1,p}(\Omega)$ if and only if problem (1.7) admits a weak solution in the usual Sobolev space $HW_0^{1,p}(\Omega)$.*

2. PRELIMINARIES FOR THE SOBOLEV-TYPE INEQUALITY

Throughout this section, we assume $f \in L^m(\Omega) \setminus \{0\}$, where

$$m = \left(\frac{p^*}{1-\delta} \right)',$$

unless stated otherwise. Let $u_n \in HW_0^{1,p}(\Omega)$ denote the solution of the approximated problem (1.10) provided by Lemma 1.4. By the proof of Theorem 1.7, the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges pointwise to a function $u_\delta \in HW_0^{1,p}(\Omega)$, which is the weak solution of problem (1.6), satisfying $u_n \leq u_\delta$ in Ω for all $n \in \mathbb{N}$. In the following, we establish several auxiliary results that will be useful for proving the main results.

Lemma 2.1. *Let $n \in \mathbb{N}$. Then, for every $\varphi \in HW_0^{1,p}(\Omega)$, the following inequality holds:*

$$\|u_n\|^p \leq \|\varphi\|^p + p \int_{\Omega} (u_n - \varphi) \left(u_n + \frac{1}{n}\right)^{-\delta} f_n dx. \quad (2.1)$$

Moreover, the sequence $\{\|u_n\|\}_{n \in \mathbb{N}}$ is nondecreasing, i.e.,

$$\|u_n\| \leq \|u_{n+1}\| \quad \text{for every } n \in \mathbb{N}. \quad (2.2)$$

Proof. Let $h \in HW_0^{1,p}(\Omega)$. By Lemma 1.3, there exists a unique solution

$$v \in HW_0^{1,p}(\Omega)$$

to the problem

$$M_\alpha v = f_n(x) \left(h^+ + \frac{1}{n}\right)^{-\delta}, \quad v > 0 \text{ in } \Omega, \quad v = 0 \text{ in } \mathbb{H}^N \setminus \Omega.$$

Furthermore, v minimizes the functional $J : HW_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J(\varphi) := \frac{1}{p} \|\varphi\|^p - \int_{\Omega} f_n \left(h^+ + \frac{1}{n}\right)^{-\delta} \varphi dx.$$

Hence, for every $\varphi \in HW_0^{1,p}(\Omega)$, we have $J(v) \leq J(\varphi)$, which gives

$$\frac{1}{p} \|v\|^p - \int_{\Omega} f_n \left(h^+ + \frac{1}{n}\right)^{-\delta} v dx \leq \frac{1}{p} \|\varphi\|^p - \int_{\Omega} f_n \left(h^+ + \frac{1}{n}\right)^{-\delta} \varphi dx. \quad (2.3)$$

Inequality (2.1) follows by taking $v = h = u_n$ in (2.3). Next, choosing $\varphi = u_{n+1}$ in (2.1) and using the monotonicity property $u_n \leq u_{n+1}$ from Lemma 1.4, we deduce (2.2). \square

Lemma 2.2. *Up to a subsequence, the sequence $\{u_n\}_{n \in \mathbb{N}}$ converges strongly to u_δ in $HW_0^{1,p}(\Omega)$.*

Proof. Note that $u_n \leq u_\delta$. By taking $\varphi = u_\delta$ in (2.1), we obtain

$$\|u_n\| \leq \|u_\delta\|,$$

which, together with the monotonicity property (2.2) from Lemma 2.1, gives

$$\lim_{n \rightarrow \infty} \|u_n\| \leq \|u_\delta\|. \tag{2.4}$$

By Lemma 1.4, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $HW_0^{1,p}(\Omega)$, so up to a subsequence, $u_n \rightharpoonup u_\delta$ weakly in $HW_0^{1,p}(\Omega)$. Consequently,

$$\|u_\delta\| \leq \liminf_{n \rightarrow \infty} \|u_n\|. \tag{2.5}$$

Combining (2.4) and (2.5) and using the uniform convexity of $HW_0^{1,p}(\Omega)$, the strong convergence follows. \square

Lemma 2.3. *Define the functional $I_\delta : HW_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ by*

$$I_\delta(v) := \frac{1}{p} \|v\|^p - \frac{1}{1-\delta} \int_{\Omega} (v^+)^{1-\delta} f \, dx.$$

Then u_δ is a minimizer of I_δ .

Proof. Consider the auxiliary functional $I_n : HW_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$I_n(v) := \frac{1}{p} \|v\|^p - \int_{\Omega} G_n(v) f_n \, dx,$$

where

$$G_n(t) := \frac{1}{1-\delta} \left(t^+ + \frac{1}{n} \right)^{1-\delta} - \left(\frac{1}{n} \right)^{-\delta} t^-.$$

Note that I_n is coercive, bounded from below, and of class C^1 , so it admits a minimizer $v_n \in HW_0^{1,p}(\Omega)$ such that

$$\langle I_n'(v_n), \varphi \rangle = 0 \quad \text{for all } \varphi \in HW_0^{1,p}(\Omega).$$

Since $I_n(v_n) \leq I_n(v_n^+)$, it follows that $v_n \geq 0$ in Ω . Hence, v_n solves the approximated problem (1.10). By the uniqueness result in Lemma 1.4, we have $u_n = v_n$, showing that u_n is a minimizer of I_n . Therefore, for all $v \in HW_0^{1,p}(\Omega)$,

$$I_n(u_n) \leq I_n(v^+). \tag{2.6}$$

Since $u_n \leq u_\delta$, the Lebesgue dominated convergence theorem gives

$$\lim_{n \rightarrow \infty} \int_{\Omega} G_n(u_n) f_n dx = \frac{1}{1-\delta} \int_{\Omega} u_\delta^{1-\delta} f dx.$$

Moreover, by Lemma 2.2,

$$\lim_{n \rightarrow \infty} \|u_n\| = \|u_\delta\|,$$

so that

$$\lim_{n \rightarrow \infty} I_n(u_n) = I_\delta(u_\delta). \quad (2.7)$$

Additionally, for any $v \in HW_0^{1,p}(\Omega)$,

$$\lim_{n \rightarrow \infty} \int_{\Omega} G_n(v^+) f_n dx = \frac{1}{1-\delta} \int_{\Omega} (v^+)^{1-\delta} f dx. \quad (2.8)$$

Since $\|v^+\| \leq \|v\|$, using (2.7) and (2.8) in (2.6), we conclude that

$$I_\delta(u_\delta) \leq I_\delta(v) \quad \text{for all } v \in HW_0^{1,p}(\Omega).$$

This completes the proof. \square

Lemma 2.4. *Suppose $f \in L^1(\Omega) \setminus \{0\}$ is nonnegative. Then, the sequence $\{u_n\}_{n \in \mathbb{N}}$, is uniformly bounded in $HW_0^{1,p}(\Omega)$, provided that (1.12) holds.*

Proof. Taking u_n as a test function in the weak formulation of (1.10) and using inequality (1.12) along with Lemma 1.1, we obtain

$$\|u_n\|_p^p \leq \int_{\Omega} f_n(x) u_n \left(u_n + \frac{1}{n}\right)^{-\delta} dx \leq \int_{\Omega} u_n^{1-\delta} f dx \leq C \|u_n\|^{1-\delta},$$

for some constant $C > 0$ independent of n . This gives the desired uniform bound. \square

3. PROOF OF THE SOBOLEV-TYPE INEQUALITY

Proof of Theorem 1.8. Recall the norm on $HW_0^{1,p}(\Omega)$ defined by

$$\|v\| = \left(\int_{\Omega} |\nabla_H v|^p dx + \alpha \int_{\mathbb{H}^N} \int_{\mathbb{H}^N} |v(x) - v(y)|^p d\mu \right)^{\frac{1}{p}}.$$

(a) It suffices to show that

$$\Theta(\Omega) := \inf_{v \in S_\delta} \|v\|^p = \|u_\delta\|^{\frac{p(1-\delta-p)}{1-\delta}},$$

where

$$S_\delta = \left\{ v \in HW_0^{1,p}(\Omega) : \int_\Omega |v|^{1-\delta} f \, dx = 1 \right\}.$$

Define

$$V_\delta = \tau_\delta u_\delta, \quad \text{with} \quad \tau_\delta = \left(\int_\Omega u_\delta^{1-\delta} f \, dx \right)^{-\frac{1}{1-\delta}},$$

so that $V_\delta \in S_\delta$.

By Remark 1.6, testing (1.11) with u_δ yields

$$\|u_\delta\|^p = \int_\Omega u_\delta^{1-\delta} f \, dx,$$

and therefore

$$\|V_\delta\|^p = \tau_\delta^p \|u_\delta\|^p = \left(\int_\Omega u_\delta^{1-\delta} f \, dx \right)^{-\frac{p}{1-\delta}} \|u_\delta\|^p = \|u_\delta\|^{\frac{p(1-\delta-p)}{1-\delta}}.$$

For any $v \in S_\delta$, set

$$\lambda = \|v\|^{-\frac{p}{p+\delta-1}}.$$

Since u_δ minimizes I_δ , we have

$$I_\delta(u_\delta) \leq I_\delta(\lambda|v|),$$

which implies

$$\|u_\delta\|^{\frac{p(1-\delta-p)}{1-\delta}} \leq \|v\|^p.$$

Taking the infimum over $v \in S_\delta$ gives

$$\Theta(\Omega) = \|u_\delta\|^{\frac{p(1-\delta-p)}{1-\delta}}.$$

(b) Suppose the mixed Sobolev inequality (1.12) holds. If $C > \Theta(\Omega)$, then by part (a),

$$C \left(\int_\Omega V_\delta^{1-\delta} f \, dx \right)^{\frac{p}{1-\delta}} > \|V_\delta\|^p,$$

contradicting (1.12). Conversely, if $C \leq \Theta(\Omega)$, for any $v \in HW_0^{s,p}(\Omega) \setminus \{0\}$, define

$$w = \left(\int_\Omega |v|^{1-\delta} f \, dx \right)^{-\frac{1}{1-\delta}} v \in S_\delta.$$

Then

$$C \leq \|w\|^p = \left(\int_{\Omega} |v|^{1-\delta} f \, dx \right)^{-\frac{p}{1-\delta}} \|v\|^p,$$

which proves (1.12).

(c) From part (a), $\Theta(\Omega) = \|V_{\delta}\|^p$. Suppose $v \in S_{\delta}$ also attains this minimum, i.e., $\|v\|^p = \Theta(\Omega)$. First, v cannot change sign in Ω , because if it did,

$$\| |v| \|^p < \|v\|^p,$$

contradicting the minimality since $|v| \in S_{\delta}$. Without loss of generality, assume $v \geq 0$ and define

$$g = \left(\int_{\Omega} \left(\frac{v}{2} + \frac{V_{\delta}}{2} \right)^{1-\delta} f \, dx \right)^{\frac{1}{1-\delta}}, \quad h = \frac{v + V_{\delta}}{2g} \in S_{\delta}.$$

By strict convexity,

$$\Theta(\Omega) \leq \|h\|^p \leq \frac{1}{g^p} \left\| \frac{v + V_{\delta}}{2} \right\|^p \leq \frac{\Theta(\Omega)}{g^p} \leq \Theta(\Omega),$$

forcing $g = 1$ and equality in the convexity inequality. Hence,

$$v = V_{\delta}.$$

Thus, the minimizer is unique up to sign.

If (1.13) holds for some nonzero w , then

$$\gamma w \in S_{\delta}, \quad \gamma = \left(\int_{\Omega} |w|^{1-\delta} f \, dx \right)^{-\frac{1}{1-\delta}},$$

which implies

$$w = \pm \gamma^{-1} V_{\delta} = \pm \gamma^{-1} \tau_{\delta} u_{\delta}.$$

□

Proof of Theorem 1.10. Assume that the inequality (1.12) holds. Then, by Lemma 2.4, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $HW_0^{1,p}(\Omega)$. Following the same arguments as in the proof of [16, Theorem 1.2], it follows that the problem (1.6) admits a weak solution $u \in HW_0^{1,p}(\Omega)$.

Conversely, let $u \in HW_0^{1,p}(\Omega)$ be a weak solution of equation (1.6). By Remark 1.6, choosing u as a test function in equation (1.6) gives

$$\|u\|^p = \int_{\Omega} u^{1-\delta} f \, dx. \tag{3.1}$$

Next, again using Remark 1.6, take $|v| \in HW_0^{1,p}(\Omega)$ as a test function in equation (1.6). Applying Hölder's inequality, we obtain

$$\int_{\Omega} |v| u^{-\delta} f \, dx \leq C \|u\|^{p-1} \|v\|, \quad (3.2)$$

for some constant $C > 0$.

Combining (3.1) and (3.2), for any $v \in HW_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \int_{\Omega} |v|^{1-\delta} f \, dx &= \int_{\Omega} (|v| u^{-\delta} f)^{1-\delta} (u^{1-\delta} f)^{\delta} \, dx \\ &\leq \left(\int_{\Omega} |v| u^{-\delta} f \, dx \right)^{1-\delta} \left(\int_{\Omega} u^{1-\delta} f \, dx \right)^{\delta} \\ &\leq C \|u\|^{p+\delta-1} \|v\|^{1-\delta}, \end{aligned}$$

which establishes the inequality (1.12). \square

Proof of Theorem 1.11. Using [19, Lemma 4.2] and proceeding as in the proof of Theorem 1.10, the result follows. \square

Proof of Theorem 1.12. Using [17, Lemma 3.1] and following the same arguments as in the proof of Theorem 1.10, the result follows. \square

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