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A PRIORI ESTIMATES AND EXISTENCE OF POSITIVE SOLUTIONS FOR ELLIPTIC PROBLEMS UNDER INTEGRAL NEUMANN BOUNDARY CONDITIONS

Francisco J.S.A. Corrêa, Giovany M. Figueiredo, and Joelma Morbach

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Abstract. In this paper, we establish a priori estimates and existence of positive solutions for elliptic problems under integral Neumann boundary conditions.

Keywords: a priori estimates, positive solution, integral Neumann boundary condition.

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1. INTRODUCTION

In this paper, we are concerned with questions related to a priori estimates and the existence of positive solutions for some classes of elliptic equations under integral Neumann boundary conditions. More precisely, we will focus our attention on problems of the form

$$\begin{cases} a\left(x, \int\limits_{\Omega} u\,dx\right)(-\Delta u + u) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \int\limits_{\partial\Omega} g(u)\,dS & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

and

$$\begin{cases} a\left(x, \int\limits_{\Omega} u\,dx\right)(-\Delta u + u) = \lambda u + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \int\limits_{\Omega} h(u)\,dx & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N, N \geq 3$, is a bounded smooth domain, $a : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}, f, g, h : \mathbb{R} \to \mathbb{R}$ are functions whose precise properties will be given later, $\lambda \geq 0$ is a real parameter, $p > 0$ is a real number related to the Sobolev embeddings and η is the outward unit normal vector on $\partial\Omega$. These problems are called nonlocal because of the presence of terms as $\int_{\partial\Omega} g(u)dS$ and $\int_\Omega h(u)dx$ which make problems (1.1) and (1.2) no longer pointwise equalities.

Problems like (1.1) and (1.2), under Dirichlet boundary conditions, have been studied by several authors. For instance, Chipot and Lovat [12, 13], Chipot and Rodrigues [14], Corrêa [15] and Figueiredo-Sousa, Morales-Rodrigo and Suárez [20] attacked problems whose prototype is

$$\begin{cases} -a\left(\int\limits_\Omega u^q\,dx\right)\Delta u = f\left(x, u, \int\limits_\Omega u\,dx\right) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

using some techniques as fixed point theory, sub and supersolutions and so on. Note that, in the Dirichlet problem (1.3), the nonlocal term a does not depend on $x \in \Omega$. For nonhomogeneous case, $a = a(x,t)$, we refer to Chipot and Corrêa [11]. See also Figueiredo, Morales-Rodrigo, Santos Junior and Suárez [19] in which the authors study the Kirchhoff Equation when the Kirchhoff nonlocal term depends on $x \in \Omega$. In this work, the authors use a fixed point approach in a very interesting way. Also, using a fixed point technique, Morbach [28] studied a problem like (1.1) and (1.2) which are not much found in the literature, at least for the stationary case.

The reader may consult several papers in which the authors study problems under nonlocal boundary condition. We cite some of them. For instance, Amster and De Nápoli [4] consider a problem like

$$\Delta_p u + g(u) = f(x) \text{ in } \Omega, \quad u = C \text{ on } \partial\Omega, \quad \int\limits_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial \eta}dS = h(C),$$

where C is an unknown constant which appears in the description of a plasma confined in a toroidal cavity. Under appropriate conditions this model can be reduced to the nonhomogeneous boundary value problem

$$\Delta u + h(x,u) = 0 \text{ in } \Omega, \quad u|_{\partial\Omega} = \text{constant}, \quad -\int\limits_{\partial\Omega} \frac{\partial u}{\partial \eta}dS = I.$$

In Slodička and Dehilis [35], the authors focus their attention on the evolutionary counterpart of the elliptic problem

$$-\Delta u = f(u) \text{ in } \Omega, \quad -\nabla u \cdot \eta = \alpha(x)u + \beta(x) + \int\limits_\Omega K(x)u dx \text{ on } \partial\Omega$$

which arises in the theory of thermoelasticity.

In Pao [30], it is considered problems like

$$-\Delta u = f(x,u) \text{ in } \Omega, \quad \alpha_0 \frac{\partial u}{\partial \eta} = \int\limits_{\Omega} K(x,y)u(y)dy \text{ on } \partial\Omega$$

which is motivated by a model problem arising from quasi-static thermoelasticity.

Some other relevant problem in which there are integrals on the boundary (and in the equation) arises in Stellators. See, e.g., Díaz and Rakotoson [17].

Note that in our problems (1.1) and (1.2) we consider two integral terms: one of them in the equation and the other on the boundary condition. It is worthwhile to cite the article by Chabrowski [10] in which the author considers the problem

$$\begin{cases} -\left(\int\limits_{\Omega} |\nabla u|^2\, dx\right)^s \Delta u = Q(x)|u|^{p-2}u + \left(\int\limits_{\Omega} |u|^q\, dx\right)^r |u|^{q-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases}$$

containing two nonlocal terms. As the author point out:

In recent years, nonlocal problems governed by differential equations have attracted considerable interest. In particular, the importance of non-local problems in partial differential equations appears to have been first noted in mathematical literature by Bitsadze and Samarskii [6].

Hence, it is not a novelty to deal with problems with two nonlocal terms.

Another point to remark is that we may consider Dirichlet integral boundary conditions as it is studied in Wang [38].

Other examples may be seen in the references of these papers and others.

Problems like (1.1), (1.2) and (1.3), which contain nonlocal terms, appear in several physical situations. When studying problems related to Biology, we have that u sometimes describes the population of bacteria and so $u > 0$ in Ω. Because of this, at least to our knowledge, positive solutions are more relevant from the physical point of view. See the previous cited papers and the references therein. In problem (1.3), when $q = 2$ we get the well-known Carrier's equation which is an appropriate model to analyze some questions related to nonlinear deflection of beams.

In particular, in the present paper, we are interested, among other things, in seeking a priori estimates for positive solutions of elliptic problems since this is an area of interest of several researchers because, having a priori bounds, one may deduce information on the existence and emplacement of solutions of nonlinear eigenvalue problems.

In what follows, we will make a brief review on a priori bounds for the usual (local) Dirichlet problems.

For the unidimensional case, a priori bounds were obtained by Kuiper and Turner [23] for certain classes of nonlinearities as given in the prototype problem

$$\begin{cases} Ly \equiv -(p(x)y'(x))' + q(x)y = \lambda f(x, y(x), y'(x)), \quad 0 < x < 1, \\ C_0 y \equiv \alpha y(0) + \beta y'(0) = 0, \quad \alpha^2 + \beta^2 > 0, \\ C_1 y \equiv \gamma y(1) + \delta y'(1) = 0, \quad \gamma^2 + \delta^2 > 0, \end{cases}$$

where $p(x) > 0$ is a continuously differentiable function on $[0, 1]$ and $q(x)$ is a continuous function on $[0, 1]$. The authors also assume some technical hypotheses on f concerning its behavior near $y = 0$ and near "$y = +\infty$" to obtain a priori estimates for positive solutions of (1). See [23].

In case $N = 2$, a priori bounds were obtained by Turner [37] for a problem like

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in which $f(x, u)$ enjoys the following property: $Au^p \le f(x, u) \le \max\{BC^p, Bu^p\}$ for $u \ge 0$, $p, A, B > 0$, $C \ge 0$ and $1 < p < 3$. For $1 < p < 2$, the Laplacian Δ may be replaced by a more general elliptic operator.

Brezis and Turner [8] studied problems like

$$\begin{cases} -\Delta u = f(x, u, \nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Thus, the authors permit a more general nonlinear function also depending on the gradient: $f = f(x, u, \nabla u)$. In this case such a function has a smaller growth, namely

$$\frac{f(\cdot, u, \cdot)}{u^{\frac{N+1}{N-1}}} \longrightarrow 0 \quad \text{as} \quad u \longrightarrow +\infty, \quad N \ge 2.$$

Then solutions of problem (1.4) are a priori bounded. Note that, in this case, the growth is below the critical one $\sigma = \frac{N+2}{N-2}, \quad N \ge 3$.

This critical case was considered by De Figueiredo, Lions and Nussbaum [18] in which is used as a main tool a symmetry result due to Gidas, Ni and Nirenberg [22]. See also Gidas and Spruck [21] in which the authors use blow-up techniques in order to obtain a priori estimates. In Souto [36], the author considers questions on a priori estimates for elliptic systems in the subcritical case by using blow-up techniques. Finally, we mention Oswald [29] in which is considered a priori estimates for a biharmonic problem in a ball under the boundary condition

$$u = \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \partial\Omega.$$

We believe that our a priori estimates are new results concerning problems like (1.1) and (1.2). Furthermore, at least to our knowledge, the existence of a solution, given in Section 5, by using truncation in combination with a priori estimates is also a novelty for this kind of problem.

The interested reader on a priori estimates for solutions of elliptic problems may consult Pardo [31, 32] for excellent surveys on this subject.

The plan for this paper is as follows: In Section 2, we present some preliminary results. In Section 3, we establish a priori estimates, a la Brezis and Turner, for a superlinear and subcritical problem like (1.1). In Section 4, we consider a priori estimates for problem (1.2) by means of bootstrapping arguments. In Section 5, we consider the question of existence of positive solution, and we show that under some natural assumptions, problem (1.1) possesses at least one solution.

2. AUXILIARY RESULTS

We are going to state some results concerning the existence and regularity for linear elliptic problems of the form

$$\begin{cases} -\Delta u + u = f(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = h(x) & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

See, for instance, Amann [3], Morales-Rodrigo [27] and Rossi [34].

Theorem 2.1. *Assume $f \in L^p(\Omega)$ and $h \in W^{1-\frac{1}{p},p}(\partial\Omega)$. Then problem (2.1) possesses a unique solution $u \in W^{2,p}(\Omega)$. Moreover, u satisfies*

$$\|u\|_{W^{2,p}(\Omega)} \le C\left(\|f\|_{L^p(\Omega)} + \|h\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right), \tag{2.2}$$

where C is independent of u.

We recall that, for $0 < s < 1$, the fractional Sobolev space $W^{s,p}(\partial\Omega)$ is defined by

$$W^{s,p}(\partial\Omega) = \left\{ u \in L^p(\partial\Omega) \,:\, [u]^p_{s,p} := \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dS(x)dS(y) < +\infty \right\}$$

with the norm

$$\|u\|_{W^{s,p}(A)} = \|u\|_{L^p(A)} + (p(1-s))^{\frac{1}{p}}[u]^p_{s,p}$$

which appears in (2.2) related to the space $W^{1-\frac{1}{p},p}(\partial\Omega)$. In our case, in Theorem 2.1, one has $s = 1 - \frac{1}{p}$.

The quantity $[u]^p_{s,p}$ is known as the Gagliardo seminorm. See Bourgain, Brezis and Minorescu [7]. For more details on Trace Theory, $W^{s,p}(\partial\Omega)$, Besov Spaces and related topics, we cite Lamberti and Provenzano [24], Burenkov [9], Adams and Fournier [1] and Leoni [25], among others.

Theorem 2.2. *Assume $f \in C^{\alpha}(\overline{\Omega})$ and $h \in C^{1+\alpha}(\overline{\Omega})$. Then problem* (2.1) *possesses a unique solution $u \in C^{2+\alpha}(\overline{\Omega})$. Moreover, u satisfies*

$$\|u\|_{C^{2+\alpha}(\overline{\Omega})} \leq C(\|f\|_{C^{\alpha}(\overline{\Omega})} + \|h\|_{C^{1+\alpha}(\partial\Omega)}), \tag{2.3}$$

where C is independent of u.

Theorem 2.3. *Assume $f \in C(\overline{\Omega})$ and $h \in W^{1-\frac{1}{p},p}(\partial\Omega)$. If $u \in C^2(\overline{\Omega})$ is a solution of the problem* (2.1)*, then*

$$\|u\|_{W^{1,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\partial\Omega)}), \tag{2.4}$$

where C is independent of u.

We now recall a result due to Rabinowitz [33] which is a main tool in the proof of the existence result. Let E be a real Banach space and let us consider the equation $u = T(\lambda, u)$ in $\mathbb{R}^+ \times E$, where the operator $T : \mathbb{R}^+ \times E \to E$ is compact and continuous. We have the following result contained in [33].

Theorem 2.4. *Let T be an operator as described above and suppose that $T(0,u) \equiv 0$. Then*

$$\mathcal{S} := \left\{(\lambda, u) \in \mathbb{R}^+ \times E \,:\, u = T(\lambda, u)\right\}$$

contains an unbounded component $\mathcal{S}^+ \subset \mathcal{S}$ such that $(0,0) \in \mathbb{R}^+ \times E$ belongs to $\mathcal{S}^+$.

3. A PRIORI ESTIMATES A LA BREZIS AND TURNER

We now establish a priori estimates for positive solutions for the problem

$$\begin{cases} a\left(x, \int\limits_{\Omega} u dx\right)(-\Delta u + u) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \int\limits_{\partial\Omega} g(u) dS & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $a : \overline{\Omega} \times \mathbb{R}^+ \to (0, +\infty)$, $f, g : \mathbb{R}^+ \to (0, +\infty)$ are functions whose properties will be presented below. We follow some ideas contained in [8]. Throughout the proofs of a priori estimates we consider weak solutions $u \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ of (3.1). More precisely, we say that $u \in W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ is a weak solution of problem (3.1) if

$$\int\limits_{\Omega} \nabla u \cdot \nabla \varphi dx + \int\limits_{\Omega} u \varphi dx = \int\limits_{\Omega} \frac{f(u)}{a\left(x, \int_{\Omega} u dx\right)} \varphi dx + \int\limits_{\partial\Omega} g(u) dS \int\limits_{\partial\Omega} \varphi dS \quad \forall \varphi \in W^{1,2}(\Omega). \tag{3.2}$$

Note that, under appropriate assumptions on f and g we may use Theorem 2.1 (respectively, Theorem 2.2) to show that the weak solution is a strong solution in $W^{2,2}(\Omega)$ (respectively, a classical solution in $C^{2+\alpha}(\overline{\Omega})$).

Here, the notation $u > 0$ in Ω means that $u(x) \geq 0$ for all $x \in \Omega$ and $u \not\equiv 0$ in Ω.

Remark 3.1. We point out that possible solutions of problem (3.1) are nonnegative. Indeed, using identity (3.2) and taking $\varphi = u^- = -\min\{u, 0\}$ as a test-function and using the positiveness of a, f, g, we obtain $u^- = 0$ a.e. in Ω. Thus, $u \geq 0$ a.e. in Ω.

We suppose that $a : \overline{\Omega} \times \mathbb{R}^+ \to (0, +\infty)$, $f, g : \mathbb{R}^+ \to (0, +\infty)$ are continuous functions and

$$(a_1) \qquad 0 < a_0 \leq a(x,t) \leq a_\infty \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}^+,$$

where a_0, a_∞ are real constants,

$$(f_1) \qquad K := \liminf_{t \to +\infty} \frac{f(t)}{t} > a_\infty,$$

$$(f_2) \qquad \lim_{t \to +\infty} \frac{f(t)}{t^\sigma} = 0, \quad \text{where} \quad \begin{cases} \sigma \in [2, +\infty) & \text{if} \quad N = 2, \\ \sigma = \frac{N}{N-2} & \text{if} \quad N \geq 3. \end{cases}$$

We should point out that assumption (f_1) implies that there is $t_0 > 0$ such that $f(t) \geq Kt$ for all $t \geq t_0$ and, in view of the continuity of f, there is a positive constant C_1 such that

$$(f_1') \qquad f(t) \geq Kt - C_1 \quad \forall t \geq 0,$$

and hypothesis (f_2) implies that for each $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$(f_2') \qquad f(t) \leq \epsilon t^\sigma + C_\epsilon \quad \forall t \geq 0.$$

Note that problem (3.1) may be written as

$$\begin{cases} -\Delta u + u = \dfrac{f(u)}{a\left(x, \int_\Omega u dx\right)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \displaystyle\int_{\partial\Omega} g(u) dS & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

We remark that in some points of the proofs below we are assuming $N \geq 3$. However, the results hold true for $N = 2$ with suitable adaptations with respect to the Sobolev embeddings.

Theorem 3.2. *There is a constant $C > 0$ such that*

$$\int_\Omega f(u) dx \leq C \text{ and } \int_{\partial\Omega} g(u) dS \leq C$$

for all solution u of problem (3.1)

Proof. To begin with, we consider a solution u of problem (3.1) and take $\varphi \equiv 1$ as a test function in the identity (3.2) to obtain

$$\int_\Omega u dx = \int_\Omega \frac{f(u)}{a\left(x, \int_\Omega u dx\right)} dx + \int_{\partial\Omega} g(u) dS \int_{\partial\Omega} dS. \tag{3.4}$$

Using assumption (a_1),

$$\int_\Omega u dx \geq \frac{1}{a_\infty} \int_\Omega f(u) dx + |\partial\Omega| \int_{\partial\Omega} g(u) dS,$$

$|\partial\Omega| = \int_{\partial\Omega} dS$ is the $(N-1)$-Lebesgue measure of the boundary $\partial\Omega$. In view of (f_1') and $g(t) \geq 0$, for all $t \geq 0$, we obtain

$$\int_\Omega u dx \geq \frac{K}{a_\infty} \int_\Omega u dx - \frac{C_1|\Omega|}{a_\infty},$$

where $|\Omega|$ is the Lebesgue measure of Ω. Then

$$\int_\Omega u dx \leq \frac{C_1|\Omega|}{K - a_\infty} \equiv C,$$

where $C > 0$ is a constant that does not depend on the solution u. Using equality (3.4), one has

$$\frac{1}{a_\infty} \int_\Omega f(u) dx + |\partial\Omega| \int_{\partial\Omega} g(u) dS \leq C.$$

Thanks to the positivity of both f, g, we conclude that

$$\int_\Omega f(u) dx \leq C \text{ and } \int_{\partial\Omega} g(u) dS \leq C$$

and the proof of this theorem is over. □

Note that, on the contrary of f, we do not have imposed any growth condition on the function g.

In what follows, $\|\cdot\|$ will denote the usual norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right) dx$$

in $W^{1,2}(\Omega)$.

Theorem 3.3. *The following estimates on the solutions u of* (3.1) *holds:*

$$\|u\| \leq C,$$

where $C > 0$ is a constant independent of the solution u.

Proof. Firstly, we note that if u is a solution of (3.3) we have

$$\|u\|^2 = \int_\Omega \frac{f(u)u}{a\left(x, \int_\Omega u dx\right)} dx + \int_{\partial\Omega} g(u) dS \int_{\partial\Omega} u dS.$$

Using assumption (a_1),

$$\|u\|^2 \leq \frac{1}{a_0} \int_\Omega f(u) u dx + \int_{\partial\Omega} g(u) dS \int_{\partial\Omega} u dS. \tag{3.5}$$

We now take $0 < \alpha = \frac{2}{N} < 1, N \geq 3$, such that $\frac{N}{2}$ and $\frac{N}{N-2}$ are conjugate exponent, and note that

$$\int_\Omega f(u) u dx = \int_\Omega [f(u)]^\alpha [f(u)^{1-\alpha} u] dx.$$

Using the Hölder inequality for the above exponents, we get

$$\int_\Omega f(u) u dx \leq \left(\int_\Omega f(u) dx \right)^\alpha \left(\int_\Omega f(u) u^{\frac{1}{1-\alpha}} dx \right)^{1-\alpha}.$$

From the preceding result, $\int_\Omega f(u) dx \leq C$, which leads us to

$$\int_\Omega f(u) u dx \leq C \left(\int_\Omega f(u) u^{\frac{1}{1-\alpha}} dx \right)^{1-\alpha}.$$

Using (f_2'), we get

$$\begin{aligned}
\int_\Omega f(u) u dx &\leq C \left[\int_\Omega (\epsilon u^\sigma + C_\epsilon) u^{\frac{1}{1-\alpha}} dx \right]^{1-\alpha} \\
&\leq C \left[\int_\Omega (\epsilon u^{\sigma + \frac{1}{1-\alpha}} + C_\epsilon u^{\frac{1}{1-\alpha}}) dx \right]^{1-\alpha} \\
&\leq C \epsilon^{1-\alpha} \left(\int_\Omega u^{\sigma + \frac{1}{1-\alpha}} dx \right)^{1-\alpha} + C_\epsilon' \left(\int_\Omega u^{\frac{1}{1-\alpha}} dx \right)^{1-\alpha}.
\end{aligned}$$

In view of the value of α and the Sobolev embedding we obtain

$$\int_\Omega f(u) u dx \leq C \epsilon^{1-\alpha} \|u\|^2 + C_\epsilon \|u\|.$$

Using the trace theorem, we obtain

$$\int_{\partial\Omega} g(u)dS \int_{\partial\Omega} udS \leq C_1\|u\|.$$

From (3.5) we have

$$\|u\|^2 \leq \frac{C\epsilon^{1-\alpha}}{a_0}\|u\|^2 + \frac{C_\epsilon}{a_0}\|u\| + C_1\|u\|.$$

For $\epsilon > 0$ small enough such that

$$K_\epsilon = 1 - \frac{C\epsilon^{1-\alpha}}{a_0} > 0,$$

we conclude that

$$\|u\| \leq C,$$

where C does not depend on u and the proof of the theorem is over. □

Theorem 3.4. *The following estimate on solutions u of problem* (3.1) *holds*

$$\|u\|_{C^0(\overline{\Omega})} \leq C_\infty,$$

where C_∞ does not depend on the particular solution u.

Proof. From Theorems 3.2 and 2.1, we have $u \in W^{2,p}(\Omega)$ for all $p > 1$. Taking $p = N > \frac{N}{2}$ we get $u \in C^1(\overline{\Omega})$. Hence, $f(u) \in C^\alpha(\overline{\Omega})$. Thanks to Theorem 2.2, we obtain, in particular, $u \in C^2(\overline{\Omega})$.

Note that

$$\frac{1}{a\left(., \int_\Omega udx\right)} f(u) \in L^p(\Omega), \quad \int_{\partial\Omega} g(u)dS \in W^{1-\frac{1}{p}}(\partial\Omega)$$

and recall that $\int_\Omega udx \leq C$.

So,

$$\|u\|_{W^{2,p}}(\Omega) \leq C[\|f(u)\|_{L^p(\Omega)} + C] \leq \left[\left(\int_\Omega (\epsilon u^\sigma + C_\epsilon)^p dx\right)^{\frac{1}{p}} + C\right]$$

$$\leq C\epsilon\left(\int_\Omega u^{\sigma p} dx\right)^{\frac{1}{p}} + CC_\epsilon + C.$$

Since $p = N > \frac{N}{2}$, we have $W^{2,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, and so

$$\|u\|_{C^0(\overline{\Omega})} \leq C\epsilon\left(\int_\Omega u^{(\sigma-1)p} u^p dx\right)^{\frac{1}{p}} + C'_\epsilon$$

which leads us to

$$\|u\|_{C^0(\overline{\Omega})} \leq C\epsilon \|u\|_{C^0(\overline{\Omega})} \left(\int\limits_{\Omega} u^{(\sigma-1)p} dx \right)^{\frac{1}{p}} + C'_\epsilon.$$

Consequently, and recalling that $(\sigma - 1)p = \frac{2N}{N-2} = 2^*$ and $W^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$, we get

$$\|u\|_{C^0(\overline{\Omega})} \leq C\epsilon \|u\|_{C^0(\overline{\Omega})} \|u\|^{\sigma-1}_{L^{2^*}(\Omega)} + C'_\epsilon.$$

The a priori estimate $\|u\| \leq C$ and the Sobolev embedding imply

$$\|u\|_{C^0(\overline{\Omega})} \leq C\epsilon \|u\|_{C^0(\overline{\Omega})} + C'_\epsilon.$$

Choosing $C\epsilon < 1$ and noticing that C does not depend on ϵ, we conclude that

$$\|u\|_{C^0(\overline{\Omega})} \leq C_\infty,$$

where the constant C_∞ does not depend on u. This concludes the proof of this theorem. □

Remark 3.5. Note that problem (1.1) does not possess variational structure. Because of this it is not possible to use techniques as minimization, mountain pass theorem and so on. However, there are several nonlocal elliptic problems which are of variational type as, for instance, the Kirchhoff problem. In this way, we may consider the problem

$$\begin{cases} M\left(\|u\|^2\right)(-\Delta u + u) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ M\left(\|u\|^2\right)\dfrac{\partial u}{\partial \eta} = \int\limits_{\partial\Omega} g(u)dS \int\limits_{\partial\Omega} G(u)dS & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function that, in the original Kirchhoff problem, satisfies $M(t) \geq M_0$, for all $t \geq M_0$. In some particular cases, we may consider $0 < M_0 \leq M(t) \leq M_\infty$ for $t \geq 0$ and M_0, M_∞ are constants. See, for instance, Alves *et al.* [2], Andrade and Ma [5] and Ma and Muñoz Rivera [26].

The energy functional $J : W^{1,2}(\Omega) \to \mathbb{R}$ associated to the problem (3.6), is given by

$$J(u) = \frac{1}{2}\hat{M}\left(\|u\|^2\right) - \int\limits_{\Omega} F(u)dx - \frac{1}{2}\left(\int\limits_{\partial\Omega} G(u)dS \right)^2,$$

where

$$F(t) = \int\limits_0^t f(\xi)d\xi \quad \text{and} \quad G(t) = \int\limits_0^t g(\xi)d\xi.$$

We assume on f, g the same assumptions of Theorem 3.4. In this case $J : W^{1,2}(\Omega) \to \mathbb{R}$ is of C^1-class and its Fréchet derivative is

$$J'(u)\varphi = M\left(\|u\|^2\right) \int\limits_{\Omega} (\nabla u \nabla \varphi)\, dx - \int\limits_{\Omega} f(u)\varphi dx$$
$$+ \int\limits_{\partial\Omega} g(u) dS \cdot \int\limits_{\partial\Omega} G(u) dS \cdot \int\limits_{\partial\Omega} \varphi dS \quad \forall \varphi \in W^{1,2}(\Omega).$$

Consequently, weak solutions of the problem (3.6) are critical points of the functional J. Reasoning as in the above a priori estimates results, it is possible to prove that possible solutions of the problem (3.6) are a priori bounded. So, using variational techniques we may prove an existence result for the problem (3.6) in the same way as it is done in [18]. The details concerning these remarks will appear in a forthcoming paper which, among other things, will consider Moser Iteration Method to attack a supercritical problem.

4. A PRIORI ESTIMATES VIA BOOTSTRAPPING ARGUMENTS

We now consider the following problem

$$\begin{cases} -\Delta u + u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \displaystyle\int\limits_{\Omega} h(u) dx & \text{on } \partial\Omega, \end{cases}$$

in which we omit the diffusion coefficient $a = a(x,t)$ in order to facilitate the calculations, where Ω is as before and $f, h : \mathbb{R}^+ \to (0, +\infty)$ satisfies

(f_3) $$0 \le f(t) \le a + bt^p, \ a, b \ge 0, \ p > 1, \ \forall t \ge 0,$$
(h_1) $$h(t) \ge t^\beta \quad \forall t \ge 0,$$

where p and β are related as in the next result.

In the next result, we follow ideas contained in Corrêa and Suárez [16].

Theorem 4.1. *Let suppose that $\beta > 1$ and $\beta > \max\left\{p, \frac{N}{2}(p-1)\right\}$ and $(f_3), (h_1)$ hold true. Then there exists a positive constant $C > 0$ such that $\|u\|_{C^0(\overline{\Omega})} \le C$.*

Proof. To begin with, let e be the only positive solution of

$$-\Delta e + e = 0 \text{ in } \Omega, \quad \frac{\partial e}{\partial \eta} = 1 \text{ on } \partial\Omega.$$

Since $f \ge 0$, one has

$$-\Delta\left(\frac{u}{\int_\Omega u^\beta dx}\right) + \frac{u}{\int_\Omega u^\beta dx} \ge 0 = -\Delta e + e \text{ in } \Omega,$$
$$\frac{\partial}{\partial \eta}\left(\frac{u}{\int_\Omega u^\beta dx}\right) \ge 1 = \frac{\partial e}{\partial \eta} \text{ on } \partial\Omega$$

and so, by the comparison principle,

$$\frac{u}{\int_\Omega u^\beta dx} \geq e \text{ in } \Omega.$$

Because $\beta > 1$, one gets

$$\int_\Omega u^\beta dx \leq \left(\int_\Omega e^\beta dx \right)^{-\frac{1}{\beta-1}}. \tag{4.1}$$

Inequality (4.1) says that u is a priori bounded in $L^\beta(\Omega)$. From (f_3), it follows that $f(u)$ is a priori bounded in $L^{\frac{\beta}{p}}(\Omega)$. Invoking elliptic regularity, we get the boundedness of u in $W^{2,\frac{\beta}{p}}(\Omega)$. If $\frac{\beta}{p} > \frac{N}{2}$, then u is a priori bounded in $L^\infty(\Omega)$. If $\frac{\beta}{p} = \frac{N}{2}$, then u is a priori bounded in $L^q(\Omega)$ for all $1 \leq q < +\infty$ and we may conclude that u is a priori bounded in $L^\infty(\Omega)$.

In case $\frac{\beta}{p} < \frac{N}{2}$, it follows that u is a priori bounded in $L^{\beta_1}(\Omega)$ with

$$\beta_1 = \frac{\beta N}{pN - 2\beta}.$$

Invoking, again, condition (f_3), we obtain the boundedness of $f(u)$ in $L^{\frac{\beta_1}{p}}(\Omega)$, from which u is a priori bounded in $W^{2,\frac{\beta_1}{p}}(\Omega)$. Reasoning as in the previous case, we obtain the boundedness of u in $L^\infty(\Omega)$, if $\frac{\beta_1}{p} < \frac{N}{2}$ or in $L^{\frac{\beta_2}{p}}(\Omega)$ for

$$\beta_2 = \frac{\beta N}{p^2 N - 2\beta(p+1)},$$

and then $f(u)$ is a priori bounded in $L^{\frac{\beta_2}{p}}(\Omega)$. Reasoning as above n times we will get an a priori bound for u in $L^\infty(\Omega)$. □

5. AN EXISTENCE RESULT

We now consider the question of existence. We suppose, along this section, that f is a C^1-function satisfying (f_1)–(f_2) and (f_3') below:

(f_3')
$$K := \liminf_{t\to+\infty} \frac{f(t)}{t} > \hat{\lambda}_1,$$

where $\hat{\lambda}_1$ is the principal eigenvalue of a linear eigenvalue problem

$$\begin{cases} -\Delta u + u = \lambda \frac{K}{A_\infty(x)} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases}$$

with

(a_2)
$$A_\infty(x) := \lim_{t\to+\infty} a(x,t)$$

uniformly on $x \in \overline{\Omega}$. It is worthy to emphasize that the principal eigenvalue $\hat{\lambda}_1$ of the linear eigenvalue (5) is simple and their associated eigenfunctions are the only with defined sign. We denote by $\hat{\varphi}_1 > 0$ an eigenfunction normalized as $\|\hat{\varphi}_1\|_{C^0(\overline{\Omega})} = 1$.

Theorem 5.1. *Under assumptions* (a_1)–(a_2)–(f_1)–(f_2)–(f_3') *and* g *bounded, problem* (1.1) *possesses at least one solution.*

Proof. Let us consider a positive constant $C > C\infty$ and suppose that $C > t_0$. Since $f \in C^1[0,+\infty)$, there is a straight line, with negative slope, through $(C, f(C) - KC)$ such that the points $(t, f(t) - Kt)$, $t \geq C$, are above of this straight line. Set $\tilde{h}$ such a straight line. So, if $t \geq C$, $\tilde{h}(t) \leq f(t) - Kt$. Let $C_1 > C$ such that $\tilde{h}(C_1) = 0$. We now consider another function $h : \mathbb{R}^+ \to \mathbb{R}$ as

$$h(t) = \begin{cases} f(t) - Kt & \text{if } \ 0 \leq t \leq C, \\ \tilde{h}(t) & \text{if } \ C \leq t \leq C_1, \\ 0 & \text{if } \ C_1 \leq t. \end{cases}$$

Clearly, h is a bounded and locally Lipschitz. Let us consider the function $\tilde{f} : \mathbb{R}^+ \to \mathbb{R}$ given by

$$\tilde{f}(t) = Kt + h(t).$$

We now consider the auxiliary problem

$$\begin{cases} -\Delta u + u = \dfrac{\tilde{f}(u)}{a\left(x, \int_\Omega u dx\right)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \displaystyle\int_{\partial\Omega} g(u) d\sigma & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

As it is easy to see, a priori estimates like those established in Theorems 3.2, 3.3 and 3.4 hold for the problem (5.1). Besides of this, these a priori bounds depends essentially on the inequalities (f_1')–(f_2'). Furthermore, in view of the construction of $\tilde{f}$, both f and $\tilde{f}$ satisfy inequalities like (f_1')–(f_2') with the same constants. Consequently, if u is a solution of (1.1) and v is a solution of (5.1), then $\|u\|_{C^0(\overline{\Omega})}, \|v\|_{C^0(\overline{\Omega})} \leq C_\infty$. Because of this, the set of positive solutions of these two problems coincide.

We now use Theorem 2.4. For this, let us consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta u + u = \lambda \dfrac{\tilde{f}(u)}{a\left(x, \int_\Omega u dx\right)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \lambda \displaystyle\int_{\partial\Omega} g(u) d\sigma & \text{on } \partial\Omega. \end{cases}$$

For each $(\lambda, v) \in \mathbb{R}^+ \times C^0(\overline{\Omega})$, let us consider $u := T(\lambda, v) \in C^0(\overline{\Omega})$ the only solution of the problem

$$\begin{cases} -\Delta u + u = \lambda \dfrac{\tilde{f}(v)}{a\left(x, \int_\Omega u dx\right)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \lambda \displaystyle\int_{\partial\Omega} g(v) d\sigma & \text{on } \partial\Omega. \end{cases}$$

Straightforward computations show that the nonlinear operator T enjoys the assumptions of Theorem 2.4, recalling that, in particular, $g(0) > 0$. Consequently, there is a component $\mathcal{S}^+ \subset \mathbb{R}^+ \times C^0(\overline{\Omega})$ of positive solutions of the problem

$$\begin{cases} -\Delta u + u = \dfrac{\lambda}{a\left(x, \int_\Omega u dx\right)} (Ku + h(u)) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta} = \lambda \displaystyle\int_{\partial\Omega} g(u) d\sigma & \text{on } \partial\Omega. \end{cases}$$

Since $\mathcal{S}^+$ is unbounded we have two possibilities not mutually exclusive.

Case 1. $\mathcal{S}^+$ is unbounded with respect to λ. In this case, $\mathcal{S}^+$ crosses the fiber $\{1\} \times C^0(\overline{\Omega})$ and so we obtain a solution of our problem.

Case 2. $\mathcal{S}^+$ is bounded with respect to λ. Since $\mathcal{S}^+$ is unbounded, there is a sequence $(\lambda_j, u_j) \in \mathcal{S}^+$ such that $\lambda_j \to \lambda_0$ and $\|u_j\|_{C^0(\overline{\Omega})} \to +\infty$. Setting $v_j = \frac{u_j}{\|u_j\|_{C^0(\overline{\Omega})}}$, we obtain

$$\begin{cases} -\Delta v_j + v_j = \dfrac{\lambda_j}{a\left(x, \int_\Omega u_j dx\right)} \left(K v_j + \dfrac{h(u_j)}{\|u_j\|_{C^0(\overline{\Omega})}} \right) & \text{in } \Omega, \\ u_j > 0 & \text{in } \Omega, \\ \dfrac{\partial v_j}{\partial \eta} = \lambda_j \dfrac{1}{\|u_j\|_{C^0(\overline{\Omega})}} \displaystyle\int_{\partial\Omega} g(u_j) d\sigma & \text{on } \partial\Omega. \end{cases}$$

Using the elliptic regularity, $\|v_j\|_{W^{2,p}(\Omega)} \le C_p$ for all $p \ge 1$. Taking p large enough, we get $v_j \to v$ in $C^1(\overline{\Omega})$. In this way, the equality bellow holds for all $\varphi \in W^{1,2}(\Omega)$:

$$\begin{aligned} \int_\Omega (\nabla v_j \nabla \varphi + v_j \varphi) dx = \lambda_j K \int_\Omega & \frac{1}{a(x, \int_\Omega u_j dx)} \\ & + \frac{\lambda_j}{\|u_j\|_{C^0(\overline{\Omega})}} \int_\Omega \frac{1}{a(x, \int_\Omega u_j dx)} h(u_j) \varphi dx \\ & + \frac{\lambda_j}{\|u_j\|_{C^0(\overline{\Omega})}} \int_{\partial\Omega} g(u_j) d\sigma \int_{\partial\Omega} \varphi d\sigma. \end{aligned}$$

We now remark that $\|v\|_{C^0(\overline{\Omega})} = 1$, $u_j(x) = \|u_j\|_{C^0(\overline{\Omega})} v_j(x)$ and $\int_\Omega u_j dx = \|u_j\|_{C^0(\overline{\Omega})} \int_\Omega v_j dx$. Consequently, $\int_\Omega u_j dx \to +\infty$. In view of assumption (a_2), we get

$$\begin{cases} -\Delta v + v = \lambda_0 \dfrac{K}{A_\infty(x)} v & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, $\lambda_0 = \hat{\lambda}_1$ and because $\hat{\lambda}_1 > 1$ we obtain, like in the previous case, a solution of our problem. □

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Francisco J.S.A. Corrêa
fjsacorrea@gmail.com

Universidade Federal do Pará, Instituto de Ciências Exatas e Naturais
Programa de Doutorado em Matemática
CEP: 66.075-110, Belém-PA, Brazil

Giovany M. Figueiredo (corresponding author)
giovany@unb.br
https://orcid.org/0000-0003-1697-1592

Universidade de Brasília
Departamento de Matemática
CEP: 70.910-900, Brasília-DF, Brazil

Joelma Morbach
joelmam@ufpa.br

Universidade Federal do Pará
Faculdade de Matemática
66.075-110, Belém-PA, Brazil

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