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SOLUTIONS WITH PRESCRIBED MASS FOR A CRITICAL CHOQUARD EQUATION DRIVEN BY A LOCAL-NONLOCAL OPERATOR

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Abstract. In this paper, we study the normalized solutions of the following critical growth Choquard equation with mixed local and nonlocal operators:

$$\begin{aligned} -\Delta u + (-\Delta)^s u &= \lambda u + \mu |u|^{p-2}u + (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha - 2}u \quad \text{in} \quad \mathbb{R}^N, \\ \|u\|_2 &= \tau, \end{aligned}$$

where $N \geq 3$, $\tau > 0$, I_α is the Riesz potential of order $\alpha \in (0, N)$, $2^*_\alpha = \frac{N+\alpha}{N-2}$ is the critical exponent corresponding to the Hardy–Littlewood–Sobolev inequality, $(-\Delta)^s$ is the nonlocal fractional Laplacian operator with $s \in (0,1)$, $\mu > 0$ is a parameter and λ appears as Lagrange multiplier. We show the existence of at least two distinct solutions in the presence of the mass-subcritical perturbation $\mu|u|^{p-2}u$ with $2 < p < 2 + \frac{4s}{N}$ under some assumptions on τ.

Keywords: normalized solution, Choquard equation, critical exponent, mixed local and nonlocal operator, L^2-subcritical perturbation, nonlinear Schrödinger equation driven by local-nonlocal operator.

Mathematics Subject Classification: 35Q55, 35M10, 35J61, 35A01.

1. INTRODUCTION

This article concerns the existence of multiple normalized solutions to the following critical growth Choquard equation involving mixed diffusion-type operator:

$$\begin{aligned} -\Delta u + (-\Delta)^s u &= \lambda u + \mu |u|^{p-2}u + (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha - 2}u \quad \text{in} \quad \mathbb{R}^N \\ \|u\|_2 &= \tau, \end{aligned} \tag{1.1}$$

where $N \geq 3$, $\tau > 0$, $2 < p < 2 + \frac{4s}{N}$, $\mu > 0$ is a parameter and λ appears as Lagrange multiplier. The fractional Laplace operator $(-\Delta)^s$ is defined as follows:

$$(-\Delta)^s u = \frac{C(N,s)}{2} \text{P.V} \int\limits_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

with P.V being the abbreviation for principal value, and $C(N,s)$ is a normalizing constant, refer [17] for a clearer understanding. For the sake of convenience, we will take $C(N,s)=2$. Here, I_α is the Riesz potential of order $\alpha \in (0,N)$ given by

$$I_\alpha(x) = \frac{A_{N,\alpha}}{|x|^{N-\alpha}} \text{ with } A_{N,\alpha} = \frac{\Gamma(\frac{N-2}{2})}{\pi^{\frac{N}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})} \text{ for every } x \in \mathbb{R}^N \setminus \{0\}, \tag{1.2}$$

and $2^*_\alpha = \frac{N+\alpha}{N-2}$, is the critical exponent with respect to the following well known Hardy–Littlewood–Sobolev (HLS) inequality [33, Theorem 4.3]:

Proposition 1.1. *Let $t,r>1$ and $0<\alpha<N$ with $1/t+1/r=1+\alpha/N$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t,r,\alpha,N)$ independent of f and h, such that*

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\alpha}}\, dxdy \le C(t,r,\alpha,N)\|f\|_t\|h\|_r. \tag{1.3}$$

If $t=r=2N/(N+\alpha)$, then

$$C(t,r,\alpha,N) = C(N,\alpha) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right\}^{-\frac{\alpha}{N}}. \tag{1.4}$$

Equality holds in (1.3) *if and only if $\frac{f}{h} \equiv$ constant and*

$$h(x) = A(\gamma^2 + |x-a|^2)^{-\frac{N+\alpha}{2}}$$

for some $A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

From this inequality, it follows that

$$\mathcal{A}_q(u) := \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)|^q|u(y)|^q}{|x-y|^{N-\alpha}}\, dxdy$$

is well defined if $\frac{N+\alpha}{N} \le q \le \frac{N+\alpha}{N-2} = 2^*_\alpha$. The exponent $q=2^*_\alpha$ is known as the Hardy–Littlewood–Sobolev critical exponent. Similar to the usual critical exponent, $H^1_0(\Omega) \ni u \mapsto \mathcal{A}_{2^*_\alpha}(u)$ is continuous with respect to the norm topology but not with respect to the weak topology (see [36]). Thus, the presence of this HLS critical exponent (2^*_α) makes our problem challenging and intriguing to study.

For the equation involving the usual Sobolev critical exponent and classical Laplacian operator, consider the problem of the form:

$$\begin{aligned} -\Delta u + \lambda u &= \mu |u|^{q-2}u + |u|^{2^*-2}u \quad \text{in} \quad \mathbb{R}^N, \\ \|u\|_2 &= c. \end{aligned} \tag{1.5}$$

Then we summarize in Table 1 the results related to (1.5), where S denotes the best constant in the Sobolev inequality.

Table 1

Range of q	Type of solution	Energy level	Reference
$2 < q < 2 + \frac{4}{N}$	a local minimizer	$= m_c < 0$	[15, 27]
	second solution	$< m_c + \frac{1}{N} S^{\frac{N}{2}}$	[15, 26, 49]
$2 + \frac{4}{N} \leq q < 2^*$	a mountain pass type solution	$< \frac{1}{N} S^{\frac{N}{2}}$	[15, 30, 49]

A similar analysis with the Choquard term treated as a perturbation in the case $N = 3$ was carried out by Jin *et al.* in [28]. Also, for a similar problem involving the fractional Laplacian operator

$$\begin{aligned} &(-\Delta)^s u + \lambda u = \mu |u|^{p-2} u + |u|^{2_s^*-2} u \quad \text{in} \quad \mathbb{R}^N, \\ &\|u\|_2 = a \end{aligned}$$

the information summarized in Table 2 is well known, where $\mathcal{S}_s$ is the best constant of the embedding $D^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$.

Table 2

Range of p	Type of solution	Energy level	Reference
$2 < p < 2 + \frac{4s}{N}$	a local minimizer	$= m_a < 0$	[53]
	second solution	$< m_a + \frac{s}{N} \mathcal{S}_s^{\frac{N}{2s}}$	[47]
$2 + \frac{4s}{N} \leq p < 2^*$	a mountain pass type solution	$< \frac{s}{N} \mathcal{S}_s^{\frac{N}{2s}}$	[45, 53, 54]

We would like to extend this theory for the case of critical Choquard equation involving mixed operator, hence we study (1.1). For the case of $q \in (2 + \frac{4}{N}, 2^*)$ one can refer to [22], and see the existence of a mountain pass type solution with energy level strictly less than $\left(\frac{2_\alpha^*-1}{22_\alpha^*}\right) S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}$. This work aims to show the existence of a local minimizer with negative energy level, that is, $m_\tau^+ < 0$, and a second solution with energy level strictly lesser than $m_\tau^+ + \left(\frac{2_\alpha^*-1}{22_\alpha^*}\right) S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}$, when $q \in (2, 2 + \frac{4s}{N})$. To the best of our knowledge, this has not been addressed in the previous literature.

Equations involving nonlinearities of the form $(I_\alpha * |u|^q)|u|^{q-2}u$ are called *Choquard equations*. In 1976, Choquard, at the Symposium on Coulomb Systems, employed the energy functional associated with the equation

$$\begin{cases} -\Delta u + u = (I_2 * |u|^2) u & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3) \end{cases}$$

to construct a viable approximation to the Hartree–Fock theory for a one-component plasma (see [32]). This equation has various applications in quantum physics. For example, it is used to describe an electron confined within its own vacancy (see [42] and related sources). Since then, numerous works have investigated the existence, multiplicity, and qualitative properties of solutions to the problem

$$-\Delta u + \lambda u = \mu(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

as detailed in [18, 34, 35]. Moreover, normalized solutions to (1.6) have been studied in [16, 52]. We are interested in discussing the multiplicity of normalized solutions to a critical growth Choquard equation involving mixed local (Δ) and nonlocal operator $(-\Delta)^s$. Multiplicity results for Kirchhoff problems involving mixed local and nonlocal operators can be found in [48]. Moreover, one may refer to [31] for the existence of multiple normalized solutions to a Choquard equation involving the biharmonic operator.

The mixed operator $\mathcal{L} = -\Delta + (-\Delta)^s$naturally arises in situations where a physical phenomenon is influenced by both local and nonlocal effects. Some of its applications appear in bimodal power-law distribution processes (see [40]). A variety of contributions have addressed issues related to the existence of solutions, their regularity and symmetry properties, Neumann problems, Green's function estimates, and eigenvalues (see, for example, [1, 3, 10, 11, 23]).

The study of (1.1) has physical relevance, as it provides us the standing wave solution for the nonlinear Schrödinger (NLS) equation driven by mixed local and nonlocal operators given as follows:

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + (-\Delta)^s\psi - \mu|\psi|^{p-2}\psi - (I_\alpha * |\psi|^{2^*_\alpha})|\psi|^{2^*_\alpha-2}\psi. \tag{1.7}$$

A standing wave solution is of the form $\psi(x,t) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^N)$ solves:

$$-\Delta u + (-\Delta)^s u = \lambda u + \mu|u|^{p-2}u + (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha-2}u \quad \text{in} \quad \mathbb{R}^N. \tag{1.8}$$

The additional L^2-norm constraint in (1.1) gives us a standing wave with prescribed mass. While addressing solutions to (1.8), there are two main approaches. The first consists of fixing $\lambda \in \mathbb{R}$ and then seeking critical points of the associated energy functional. The second approach, which we adopt here, is to fix the L^2-norm, that is, to search for the critical points of

$$E(u) := \frac{\|\nabla u\|_2^2}{2} + \frac{[u]^2}{2} - \mu\frac{\|u\|_p^p}{p} - \frac{A(u)}{22^*_\alpha},$$

where

$$[u]^2 = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy,$$

restricted to the manifold $S(\tau) := \{u \in H^1(\mathbb{R}^N) : \|u\|_2 = \tau\}$. Here, $A(u) = \mathcal{A}_{2^*_\alpha}(u)$. The previous method has already been extensively employed. However, the latter one is new and appears more captivating. In this case, λ, which plays the role of a Lagrange multiplier, is also an unknown. The solution obtained in this way is called normalized solution. Recently, the study of normalized solutions has attracted significant attention from researchers. Formally, a solution to the following constrained problem is called a normalized solution:

$$\begin{cases} -\Delta u = \lambda u + g(u) \quad \text{in } \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} |u|^2 dx = c. \end{cases} \tag{1.9}$$

Jeanjean [26] demonstrated the existence of a radial solution for equation (1.9) under certain assumptions on the function g. Furthermore, Bartsch and De Valeriola [8] established the existence of infinitely many solutions to (1.9) with $c = 1$ under the same assumptions on g. In [37], Noris *et al.* explored normalized solutions in the context of bounded domains with Dirichlet boundary conditions. They showed that normalized solutions exist for p values in the intervals $(1, 1 + \frac{4}{N})$, $(1 + \frac{4}{N}, 2^* - 1)$, and $p = 1 + \frac{4}{N}$, under certain conditions on c, with the domain being the unit ball and $g(t) = |t|^{p-1}t$. Furthermore, the authors in [43] addressed the problem in general bounded domains. Readers seeking additional details may consult [6, 7, 9, 24, 38, 39] as well as recent publications [13, 28, 29, 51]. The study of quadratic ergodic mean field game systems also involves normalized solutions, as discussed in [41].

Let us formally initiate our study by discussing the variational framework of (1.1).

Definition 1.2. A function $u \in S(\tau)$ is said to be a solution to (1.1) if it satisfies the following:

$$\int\limits_{\mathbb{R}^N} \nabla u \nabla v + \ll u, v \gg = \lambda \int\limits_{\mathbb{R}^N} uv + \mu \int\limits_{\mathbb{R}^N} |u|^{p-2}uv + \int\limits_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha - 2}uv, \tag{1.10}$$

for all $v \in H^1(\mathbb{R}^N)$. Here

$$\ll u, v \gg := \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy,$$

and the space $H^1(\mathbb{R}^N)$ is equipped with the norm:

$$\|u\| = \left(T(u)^2 + \|u\|_2^2\right)^{\frac{1}{2}}, \text{ where } T(u)^2 = \|\nabla u\|_2^2 + [u]^2.$$

Using the Pohozaev identity, it is seen that a solution to (1.1) lies on the Pohozaev manifold

$$\mathcal{M}_\tau := \{u \in S(\tau) : M(u) = 0\}, \tag{1.11}$$

where

$$M(u) = \|\nabla u\|_2^2 + s[u]^2 - \mu\gamma_p\|u\|_p^p - A(u)$$

with $\gamma_p := \frac{N(p-2)}{2p}$. Further, using the fiber maps technique described in Section 2, we subdivided $\mathcal{M}_\tau$ into disjoint subsets $\mathcal{M}_\tau^+$ and $\mathcal{M}_\tau^-$. The idea is to look for distinct solutions in these disjoint subsets.

Let S be the best constant corresponding to the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. By [44], we know that

$$S_\alpha = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{A(u)^{\frac{1}{2^*_\alpha}}} = \frac{S}{(A_\alpha C_\alpha)^{\frac{1}{2^*_\alpha}}}, \tag{1.12}$$

and S_α is achieved by the family of functions of the form

$$U_{\epsilon,x_0}(x) = \frac{(N(N-2)\epsilon^2)^{\frac{N-2}{4}}}{(\epsilon^2 + |x - x_0|^2)^{\frac{N-2}{2}}}, \text{ for } x_0 \in \mathbb{R}^N \text{ and } \epsilon > 0. \tag{1.13}$$

Here, $A_\alpha = A_{N,\alpha}$ and $C_\alpha = C(N,\alpha)$ are given in (1.2) and (1.4), respectively. Thanks to the symmetric decreasing rearrangement, the Gagliardo–Nirenberg inequality (see [19, Theorem 1.1]), namely

$$\|u\|_\beta \le C_{N,\beta} \|\nabla u\|_2^\theta \|u\|_2^{1-\theta} \text{ where } \theta = \frac{N(\beta-2)}{2\beta}, \tag{1.14}$$

for all $\beta \in [2, 2^*]$, and compact embedding $H_r(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*)$ [4, Lemma 3.1.4], the Ekeland variational principle allows us to deduce the existence of a first solution. Taking

$$\tau_0 = \left(\frac{p(2^*_\alpha - 1)}{\mu C_{N,p}(22^*_\alpha - p\gamma_p)} \left(\frac{(2 - p\gamma_p)2^*_\alpha S_\alpha^{2^*_\alpha}}{22^*_\alpha - p\gamma_p} \right)^{\frac{2-p\gamma_p}{2(2^*_\alpha - 1)}} \right)^{\frac{1}{p(1-\gamma_p)}}$$

and

$$\tau_1 = \left(\frac{2(2^*_\alpha - 1)}{\gamma_p^{\frac{p\gamma_p}{2}} \mu C_{N,p}(22^*_\alpha - p\gamma_p)} \left(\frac{p S_\alpha^{\frac{2^*_\alpha}{2^*_\alpha - 1}}}{2 - p\gamma_p} \right)^{\frac{2-p\gamma_p}{2}} \right)^{\frac{1}{p(1-\gamma_p)}},$$

we have the following theorem:

Theorem 1.3. *For $N \ge 3$, $s \in (0,1)$, $2 < p < 2 + \frac{4s}{N}$ and $0 < \tau < \min\{\tau_0, \tau_1\}$, there exists a radially symmetric function $u_\tau^+ \in H^1(\mathbb{R}^N)$ that attains $m_\tau^+ := \inf_{u \in \mathcal{M}_\tau^+} E(u)$, that is, $E(u_\tau^+) = m_\tau^+ < 0$. Moreover, u_τ^+ solves (1.1) corresponding to some $\lambda_\tau^+ < 0$, for sufficiently large $\mu > 0$.*

Since our problem involves a mass subcritical perturbation, $2 < q < 2 + \frac{4s}{N}$, the work [27] motivates us to expect a second solution. Denoting $m_\tau^- = \inf_{u \in \mathcal{M}_\tau^-} E(u)$, in Section 4 we deduced a relation between m_τ^+ and m_τ^-, that helped us to prove the existence of the second solution to (1.1). Precisely, we have the following result:

Theorem 1.4. *Let $N \ge 3$, $2 < p < 2 + \frac{4s}{N}$, $0 < \tau < \min\{\tau_0, \tau_1\}$ and $\mu > 0$ be sufficiently large, then m_τ^- is achieved by a radially symmetric function $u_\tau^- \in H^1(\mathbb{R}^N)$. Furthermore, u_τ^- solves (1.1) corresponding to some $\lambda_\tau^- < 0$.*

2. PRELIMINARIES

In this section, we will establish the necessary groundwork required to deduce the final existence results.

Lemma 2.1. *If* $u \in S(\tau)$ *is a solution of* (1.1), *corresponding to some* $\lambda \in \mathbb{R}$, *then* $u \in \mathcal{M}_\tau$.

Proof. Since $u \in S(\tau)$ solves (1.1) for some $\lambda \in \mathbb{R}$, we have

$$\lambda \|u\|_2^2 = \|u\|_2^2 + [u]^2 - \mu \|u\|_p^p - A(u). \tag{2.1}$$

Also, u satisfies the following Pohozaev identity:

$$\left(\frac{N-2}{2}\right) \|\nabla u\|_2^2 + \left(\frac{N-2s}{2}\right) [u]^2 = \frac{N\lambda}{2} \|u\|_2^2 + \frac{N}{p} \mu \|u\|_p^p + \left(\frac{N+\alpha}{22_\alpha^*}\right) A(u), \tag{2.2}$$

see [21, Theorem A1] and [2, Theorem 2.5]. Using (2.1) in (2.2), we get

$$M(u) = \|\nabla u\|_2^2 + s[u]^2 - \mu \gamma_p \|u\|_p^p - A(u) = 0,$$

where $\gamma_p = \frac{N(p-2)}{2p}$. ☐

This Pohozaev manifold $\mathcal{M}_\tau$ will play a crucial role in the study of existence and multiplicity results. We will further subdivide it into the following three disjoint subsets:

$$\begin{aligned}
\mathcal{M}_\tau^0 &:= \{u \in \mathcal{M}_\tau : 2\,\|\nabla u\|_2^2 + 2s^2[u]^2 = p\gamma_p^2 \mu \|u\|_p^p + 2\cdot 2_\alpha^* A(u)\},\\
\mathcal{M}_\tau^+ &:= \{u \in \mathcal{M}_\tau : 2\,\|\nabla u\|_2^2 + 2s^2[u]^2 > p\gamma_p^2 \mu \|u\|_p^p + 2\cdot 2_\alpha^* A(u)\},\\
\mathcal{M}_\tau^- &:= \{u \in \mathcal{M}_\tau : 2\,\|\nabla u\|_2^2 + 2s^2[u]^2 < p\gamma_p^2 \mu \|u\|_p^p + 2\cdot 2_\alpha^* A(u)\},
\end{aligned}$$

and deduce the existence of a solution in $\mathcal{M}_\tau^+$ and another one in $\mathcal{M}_\tau^-$. As we move forward, it will become clearer why $\mathcal{M}_\tau^+$, $\mathcal{M}_\tau^-$, and $\mathcal{M}_\tau^0$ were chosen in this way. Now, for any $u \in S(\tau)$, by (1.12) and the Gagliardo–Nirenberg inequality (1.14), we have

$$\begin{aligned}
E(u) &= \frac{T(u)^2}{2} - \mu \frac{\|u\|_p^p}{p} - \frac{A(u)}{22_\alpha^*}\\
&\geq \frac{T(u)^2}{2} - \frac{\mu C_{N,p}}{p} T(u)^{p\gamma_p} \tau^{p-p\gamma_p} - \frac{T(u)^{22_\alpha^*}}{22_\alpha^* S_\alpha^{2_\alpha^*}}.
\end{aligned} \tag{2.3}$$

Defining

$$h(t) := \frac{t^2}{2} - \frac{\mu C_{N,p} t^{p\gamma_p} \tau^{p-p\gamma_p}}{p} - \frac{t^{22_\alpha^*}}{22_\alpha^* S_\alpha^{2_\alpha^*}} \text{ for all } t > 0,$$

we get $E(u) \geq h(T(u))$. Let us discuss some properties of the function h that will be helpful for us.

Lemma 2.2. *There exists $\tau_0 > 0$ such that, for $\tau < \tau_0$, h has a strict local minimum at a negative level, a global maximum at a positive level, and we can find two positive constants $R_1 > R_0$ such that $h(R_0) = 0 = h(R_1)$ with $h(t) > 0$ if and only if $t \in (R_0, R_1)$.*

Proof. Define

$$\bar{h}(t) := \frac{t^{2-p\gamma_p}}{2} - \frac{\mu C_{N,p}}{p}\tau^{p(1-\gamma_p)} - \frac{t^{22^*_\alpha - p\gamma_p}}{22^*_\alpha S_\alpha^{2^*_\alpha}} \text{ for } t > 0,$$

then $h(t) = t^{p\gamma_p}\bar{h}(t)$, and hence $h(t) > 0$ if and only if $\bar{h}(t) > 0$. Clearly, since $\bar{h}$ has a unique critical point

$$t_0 = \left(\frac{(2 - p\gamma_p)2^*_\alpha S_\alpha^{2^*_\alpha}}{22^*_\alpha - p\gamma_p}\right)^{\frac{1}{2(2^*_\alpha - 1)}},$$

$\bar{h}$ is increasing in $(0, t_0)$, decreasing in (t_0, ∞),

$$\bar{h}(0) = -\frac{\mu C_{N,p}}{p}\tau^{p(1-\gamma_p)},$$

and $\bar{h}(t_0) > 0$ for all $\tau < \tau_0$. Its curvature can be visualized in Figure 1.

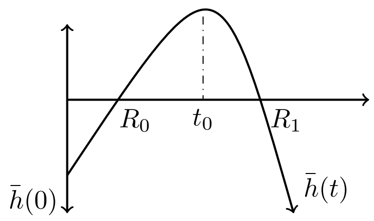

Fig. 1. Qualitative behavior of $\bar{h}(t)$

Thus, there exist $0 < R_0 < R_1$ such that $h(R_0) = 0 = h(R_1)$ and $h(t) > 0$ if and only if $t \in (R_0, R_1)$. Next, we claim that h has exactly two non-zero critical points. Now, since

$$h'(t) = t^{p\gamma_p - 1}\left(t^{2-p\gamma_p} - \mu\gamma_p C_{N,p}\tau^{p(1-\gamma_p)} - \frac{t^{22^*_\alpha - p\gamma_p}}{S_\alpha^{2^*_\alpha}}\right),$$

if h had more than two non-zero critical points, then the function g, defined as

$$g(t) := t^{2-p\gamma_p} - \frac{t^{22^*_\alpha - p\gamma_p}}{S_\alpha^{2^*_\alpha}},$$

would attain $C_\tau = \mu\gamma_p C_{N,p}\tau^{p(1-\gamma_p)}$ at least three times and therefore would have at least two critical points. But, since

$$\bar{t} = \left(\frac{(2-p\gamma_p)S_\alpha^{2_\alpha^*}}{22_\alpha^* - p\gamma_p} \right)^{\frac{1}{2(2_\alpha^*-1)}}$$

is the unique critical point of g, we obtain a contradiction. Thus, h has at most two non-zero critical points. Also, since $h(t) \to 0^-$ as $t \to 0^+$ and $h(t) \to -\infty$ as $t \to \infty$, h can exhibit the following geometry, as shown in Figure 2.

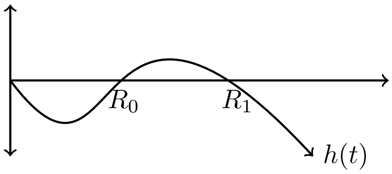

Fig. 2. Qualitative behavior of $h(t)$

The proof is complete. □

For any $u \in H^1(\mathbb{R}^N)$, let us define the fiber maps $\star$ and $\circledast$ as follows:

$$(t \star u)(x) := e^{\frac{Nt}{2}} u(e^t x) \text{ for } t \in \mathbb{R} \quad \text{ and } \quad (t \circledast u)(x) := t^{\frac{N}{2}} u(tx) \text{ for } t \geq 0.$$

Clearly, $e^t \circledast u = t \star u$. Now, defining $\psi_u(t) := E(t \star u)$, one can notice that $M(t \star u) = \psi_u'(t)$. Also, we have the following results about ψ_u.

Lemma 2.3. *Let $u \in S(\tau)$ and $\tau < \tau_0$, then ψ_u has exactly two zeroes and two critical points, that is, we can find unique $a_u < b_u < c_u < d_u$ such that $\psi_u'(a_u) = 0 = \psi_u'(c_u)$ and $\psi_u(b_u) = 0 = \psi_u(d_u)$. Moreover, we have the following:*

1. *$a_u \star u \in \mathcal{M}_\tau^+$ and $c_u \star u \in \mathcal{M}_\tau^-$. If $t \star u \in \mathcal{M}_\tau$, then either $t = a_u$ or $t = c_u$ and hence $\mathcal{M}_\tau^0$ is empty,*
2. *$E(c_u \star u) = \max\{E(t \star u) : t \in \mathbb{R}\} > 0$ and ψ_u is strictly decreasing in (c_u, ∞),*
3. *$T(t \star u) \leq R_0$ for every $t < b_u$ and*

$$E(a_u \star u) = \min\{E(t \star u) : t \in \mathbb{R} \text{ and } T(t \star u) \leq R_0\} < 0,$$

4. *the maps $\Phi_1 : \mathcal{M}_\tau \to \mathbb{R}$ and $\Phi_2 : \mathcal{M}_\tau \to \mathbb{R}$ defined as $\Phi_1(u) := a_u$ and $\Phi_2(u) := c_u$ are of class C^1.*

Proof. Since

$$\psi_u(t) = E(t \star u) = \frac{e^{2t}}{2} \|\nabla u\|_2^2 + \frac{e^{2st}}{2}[u]^2 - \frac{\mu e^{p\gamma_p t}}{p}\|u\|_p^p - \frac{e^{22^*_\alpha t}}{22^*_\alpha}A(u),$$

we get

$$\psi_u'(t) = e^{22^*_\alpha t}\big(e^{(2-22^*_\alpha)t} \|\nabla u\|_2^2 + se^{(2s-22^*_\alpha)t}[u]^2 - A(u) \\ - \gamma_p \mu e^{(p\gamma_p - 22^*_\alpha)t}\|u\|_p^p\big).$$

If ψ_u has more than two critical points, then the function g defined as

$$g(t) := e^{(2-22^*_\alpha)t} \|\nabla u\|_2^2 + se^{(2s-22^*_\alpha)t}[u]^2 - \gamma_p \mu e^{(p\gamma_p - 22^*_\alpha)t}\|u\|_p^p$$

attains $A(u)$ at least three times and hence has at least two critical points. Now, since

$$g'(t) = e^{(p\gamma_p - 22^*_\alpha)t}(\bar{g}(t) - C_p),$$

where

$$\bar{g}(t) = (2 - 22^*_\alpha)e^{(2-p\gamma_p)t} \|\nabla u\|_2^2 + s(2s - 22^*_\alpha)e^{(2s-p\gamma_p)t}[u]^2$$

and

$$C_p = \mu\gamma_p(p\gamma_p - 22^*_\alpha)\|u\|_p^p,$$

$\bar{g}$ must attain C_p at least twice and hence have at least one critical point. But,

$$\bar{g}'(t) = (2 - 2.2^*_\alpha)(2 - p\gamma_p)e^{(2-p\gamma_p)t} \|\nabla u\|_2^2 \\ + s(2s - 22^*_\alpha)(2s - p\gamma_p)e^{(2s-p\gamma_p)t}[u]^2 > 0,$$

for all $t \in \mathbb{R}$, thus we get contradiction. Hence, ψ_u has atmost two critical points. Further, since $t \mapsto T(t \star u)$ is continuous and increasing map from $\mathbb{R}$ onto $(0, +\infty)$, we can find $t_1, t_2 \in \mathbb{R}$ such that $R_0 = T(t_1 \star u) < T(t \star u) < T(t_2 \star u) = R_1$ for all $t \in (t_1, t_2)$, by (2.3) and Lemma 2.2

$$\psi_u(t) = E(t \star u) \geq h(T(t \star u)) > 0 \text{ for all } t \in (t_1, t_2).$$

Also, one can see that $\psi_u(t) \to -\infty$ as $t \to +\infty$ and $\psi_u(t) \to 0^-$ as $t \to -\infty$, because $p\gamma_p < 2s < 2 < 22^*_\alpha$. Thus, ψ_u can have the curvature shown in Figure 3.

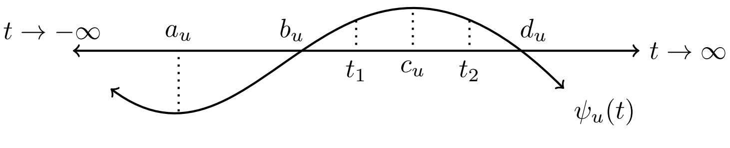

Fig. 3. Qualitative behavior of ψ_u

Therefore, ψ_u has exactly two critical points, corresponding to a local minima (a_u) at negative level and global maxima (c_u) at positive level, and exactly two roots (b_u and d_u).

1. Since a_u is a strict local minimum of ψ_u, $M(a_u \star u) = \psi_u'(a_u) = 0$, and

$$\begin{aligned} 0 < \psi_u''(a_u) &= 2e^{2a_u}\left\|\nabla u\right\|_2^2 + 2s^2 e^{2sa_u}[u]^2 - \mu p\gamma_p^2 e^{p\gamma_p a_u}\left\|u\right\|_p^p - 22^*_\alpha e^{22^*_\alpha} A(u) \\ &= 2\left\|\nabla a_u \star u\right\|_2^2 + 2s^2[a_u \star u]^2 - \mu p\gamma_p^2\left\|a_u \star u\right\|_p^p - 22^*_\alpha A(a_u \star u), \end{aligned}$$

thus $a_u \star u \in \mathcal{M}_\tau^+$. Similarly, since c_u is the global maximum of ψ_u, we will get $c_u \star u \in \mathcal{M}_\tau^-$. Now, if $t \star u \in \mathcal{M}_\tau$, then clearly t is a critical point of ψ_u, hence either $t = a_u$ or $t = c_u$. Moreover, since ψ_u has exactly two critical points, both corresponding to its extremas, $\mathcal{M}_\tau^0$ must be an empty set.

2. It is evident by the curvature of ψ_u.

3. By the monotonicity of the surjective map $t \mapsto T(t \star u)$ onto $(0, \infty)$, it is clear that $T(t \star u) \le T(t_1 \star u) = R_0$ for all $t < b_u \le t_1$. Moreover, since ψ_u is decreasing in $(-\infty, a_u)$ and increasing in $(a_u, t_1]$,

$$\begin{aligned} 0 > E(a_u \star u) = \psi_u(a_u) &= \min\{\psi_u(t) : t \le t_1\} \\ &= \min\{E(t \star u) : T(t \star u) \le T(t_1 \star u) = R_0\}. \end{aligned}$$

4. By the implicit function theorem, as in the proof of Lemma 3.3 in [25], it follows that Φ_1 and Φ_2 are of class C^1. ☐

Lemma 2.4. *If $u \in \mathcal{M}_\tau$ is a critical point of $E|_{\mathcal{M}_\tau}$, then u is a critical point of $E|_{S(\tau)}$.*

Proof. For a critical point u of $E|_{\mathcal{M}_\tau}$, by the Lagrange multiplier's rule, there exist λ_1 and $\lambda_2 \in \mathbb{R}$ such that

$$E'(u)(v) - \lambda_1 \int\limits_{\mathbb{R}^N} uv - \lambda_2 M'(u)(v) = 0 \text{ for all } v \in H^1(\mathbb{R}^N),$$

that is,

$$\begin{aligned} &(1 - 2\lambda_2)\int\limits_{\mathbb{R}^N} \nabla u \nabla v + (1 - 2\lambda_2 s) \ll u, v \gg \\ &= \mu(1 - \lambda_2 p\gamma_p)\int\limits_{\mathbb{R}^N} |u|^{p-2}uv + \lambda_1 \int\limits_{\mathbb{R}^N} uv + (1 - \lambda_2 22^*_\alpha)\int\limits_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha - 2}uv, \end{aligned}$$

for all $v \in H^1(\mathbb{R}^N)$, and hence u solves

$$\begin{aligned} &-(1 - 2\lambda_2)\Delta u + (1 - 2\lambda_2 s)(-\Delta)^s u \\ &= \lambda_1 u + \mu(1 - \lambda_2 p\gamma_p)|u|^{p-2}u + (1 - \lambda_2 22^*_\alpha)(I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha - 2}u \end{aligned} \tag{2.4}$$

in $\mathbb{R}^N$.

We will show that $\lambda_2 = 0$. As in the proof of Lemma 2.1, by (2.4) we have

$$(1-2\lambda_2)\left\|\nabla u\right\|_2^2 + (1-2s\lambda_2)[u]^2 = \lambda_1\|u\|_2^2 + \mu(1-\lambda_2 p\gamma_p)\|u\|_p^p + (1-\lambda_2 22{*}_\alpha)A(u),$$

and

$$\begin{aligned}\lambda_1\|u\|_2^2 = \frac{2}{N}\Bigg((1-2\lambda_2)\left(\frac{N-2}{2}\right)\left\|\nabla u\right\|_2^2 + (1-2s\lambda_2)\left(\frac{N-2s}{2}\right)[u]^2 \\ - \mu(1-\lambda_2 p\gamma_p)\frac{N}{p}\|u\|_p^p - (1-\lambda_2 22^*_\alpha)\left(\frac{N+\alpha}{22^*_\alpha}\right)A(u)\Bigg),\end{aligned}$$

thus

$$\lambda_2\Big(2\left\|\nabla u\right\|_2^2 + 2s^2[u]^2 - \mu p\gamma_p^2\|u\|_p^p - 22^*_\alpha A(u)\Big) = 0.$$

Since $\mathcal{M}_\tau^0$ is an empty set, we have $\lambda_2 = 0$. Therefore, u is a critical point of $E|_{S(\tau)}$. ☐

For any $k > 0$, denoting $A_k = \{u \in S(\tau) : T(u) < k\}$, we define

$$m_\tau := \inf_{u \in A_{R_0}} E(u),$$

where R_0 is as deduced in Lemma 2.2. Then we have the following results for m_τ, m_τ^- and m_τ^+.

Lemma 2.5. $m_\tau^- > 0$.

Proof. For any $u \in \mathcal{M}_\tau^-$, we have $0 \star u = u \in \mathcal{M}_\tau^-$. Then, by Lemma 2.3, 0 is the global maximum of ψ_u at a positive level and

$$E(u) = \psi_u(0) = \max\{E(t \star u) : t \in \mathbb{R}\} > 0,$$

hence $m_\tau^- \geq 0$. Moreover, for every $u \in \mathcal{M}_\tau^-$, we can find some $t_u \in \mathbb{R}$ such that $T(t_u \star u) = t_0$, where t_0 is the global maximum of h deduced in Lemma 2.2. Thus,

$$E(u) = \max\{E(t \star u) : t \in \mathbb{R}\} \geq E(t_u \star u) \geq h(T(t_u \star u)) = h(t_0) > 0$$

for all $u \in \mathcal{M}_\tau^-$, hence $m_\tau^- \geq h(t_0) > 0$. ☐

Lemma 2.6. $\sup\limits_{u \in \mathcal{M}_\tau^+} E(u) \leq 0 < m_\tau^-$ *and* $\mathcal{M}_\tau^+ \subset A_{R_0}$.

Proof. Clearly, for any $u \in \mathcal{M}_\tau^+$, $a_u = 0$. Thus, by Lemma 2.3, $E(u) < 0$, and hence, by Lemma 2.5, $\sup_{u \in \mathcal{M}_\tau^+} E(u) \leq 0 < m_\tau^-$. Further, $T(u) = T(a_u \star u) < T(t_1 \star u) = R_0$, for all $u \in \mathcal{M}_\tau^+$, since $0 = a_u < t_1$. Hence, $\mathcal{M}_\tau^+ \subset A_{R_0}$. ☐

Lemma 2.7. $-\infty < m_\tau = \inf\limits_{u \in \mathcal{M}_\tau} E(u) = m_\tau^+ < 0$, *and for* $\delta > 0$ *small enough*

$$m_\tau < \inf_{\bar{A}_{R_0} \setminus A_{R_0 - \delta}} E(u). \tag{2.5}$$

Proof. For any $u \in A_{R_0}$, we have

$$E(u) \geq h(T(u)) \geq \min_{t \in [0,R_0]} h(t) > -\infty,$$

and hence $m_\tau > -\infty$. Also, since $a_u \star u \in \mathcal{M}_\tau^+ \subset A_{R_0}$,

$$-\infty < m_\tau = \inf_{u \in A_{R_0}} E(u) \leq E(a_u \star u) = \psi_u(a_u) < 0.$$

Further, if $u \in A_{R_0}$, then, by Lemma 2.3, we have

$$E(u) = E(0 \star u) \geq E(a_u \star u) \geq m_\tau^+,$$

hence $m_\tau \geq m_\tau^+$. Also, since $\mathcal{M}_\tau^+ \subset A_{R_0}$, it follows that $m_\tau = m_\tau^+$. Now, since $\mathcal{M}_\tau = \mathcal{M}_\tau^+ \cup \mathcal{M}_\tau^+ \cup \mathcal{M}_\tau^0$, we conclude that $\mathcal{M}_\tau^0$ is an empty set, and

$$m_\tau^+ = \inf_{u \in \mathcal{M}_\tau^+} E(u) \leq \sup_{u \in \mathcal{M}_\tau^+} E(u) \leq 0 < \inf_{u \in \mathcal{M}_\tau^-} E(u),$$

by Lemma 2.6, then clearly

$$\inf_{u \in \mathcal{M}_\tau} E(u) = \inf_{u \in \mathcal{M}_\tau^+} E(u) = m_\tau^+.$$

Therefore,

$$-\infty < m_\tau = \inf_{u \in \mathcal{M}_\tau} E(u) = m_\tau^+ < 0.$$

Now, since h is continuous, $h(R_0) = 0$, $h(t) < 0$ for all $t \in (0, R_0)$ and $m_\tau < 0$, we can find $\delta > 0$ small enough so that $h(t) \geq \frac{m_\tau}{2}$ for all $t \in [R_0 - \delta, R_0]$. Hence, for all $u \in \bar{A}_{R_0} \setminus A_{R_0 - \delta}$,

$$R_0 - \delta < T(u) \leq R_0 \Rightarrow E(u) \geq h(T(u)) \geq \frac{m_\tau}{2} > m_\tau.$$

Thus, we get (2.5). $\square$

3. FIRST SOLUTION

In this section, using the above prerequisite results, the symmetric decreasing rearrangement, and the Ekeland variational principle, we will show the existence of a radially symmetric function $u_\tau^+ \in \mathcal{M}_\tau^+$ and $\lambda_\tau^+ < 0$, such that $(u_\tau^+, \lambda_\tau^+)$ solves (1.1). The subsequent rearrangement inequalities will be useful for this purpose.

Remark 3.1. For any $u \in H^1(\mathbb{R}^N)$, let u^* denote its symmetric decreasing rearrangement. Then we have:

1. $\|u\|_q = \|u^*\|_q$ for all $q \in [2, 2^*]$,
2. $A(u) \leq A(u^*)$,
3. $\|\nabla u^*\|_2 \leq \|\nabla u\|_2$ and $[u^*]^2 \leq [u]^2$.

Interested readers may refer to [5, 12, 33] and [22, Remark 2.1] for the proof.

Proof of Theorem 1.3. Let $\{w_n\} \subset A_{R_0}$ be a minimizing sequence for E on A_{R_0}, and let w_n^* denote the symmetric decreasing rearrangement of w_n. By the rearrangement inequalities and Remark 3.1, we have $\{w_n^*\} \subset A_{R_0}$ and $E(w_n^*) \leq E(w_n)$ for each $n \in \mathbb{N}$. Thus, $\{w_n^*\}$ is also a minimizing sequence. Now, for each $n \in \mathbb{N}$, by Lemma 2.3, there exists $a_n \in \mathbb{R}$ such that $a_n \star w_n^* \in \mathcal{M}_\tau^+$ and $E(w_n^*) = E(0 \star w_n^*) \geq E(a_n \star w_n^*)$. Taking $v_n = a_n \star w_n^*$ as the minimizing sequence for E on $\mathcal{M}_\tau^+$, and hence also for E on A_{R_0}, we see that v_n is radially symmetric and $T(v_n) < R_0 - \delta$ for all $n \in \mathbb{N}$.

Applying the Ekeland variational principle (see Theorem 1.1 and its corollaries in [20]), we can find a sequence of radially symmetric functions $\{u_n\}$ such that

$$\begin{cases} E(u_n) \to m_\tau & \text{as } n \to \infty, \\ E(u_n) \leq E(v_n) & \text{for all } n \in \mathbb{N}, \\ M(u_n) \to 0 & \text{as } n \to \infty, \\ E'_{S(\tau)}(u_n) \to 0 & \text{as } n \to \infty. \end{cases} \tag{3.1}$$

Here, $E'_{S(\tau)}(u_n) \to 0$ means that the sequence $y_n = \sup\left\{\frac{E'(u_n)(w)}{\|w\|} : w \in S(\tau)\right\}$ converges to 0. Now, by (3.1) and the method of Lagrange multipliers, we can find a sequence $\{\lambda_n\}$ such that

$$E'(u_n) - \lambda_n \Phi'(u_n) \to 0, \text{ where } \Phi(u) = \frac{1}{2}\|u\|_2^2. \tag{3.2}$$

Clearly, since $\{u_n\} \subset A_{R_0}$, it is bounded in $H^1(\mathbb{R}^N)$ and hence, up to a subsequence, weakly convergent in $H^1(\mathbb{R}^N)$. Denoting the subsequence by $\{u_n\}$, let $u_0 \in H^1(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u_0$. Clearly, $u_0 \in H_r(\mathbb{R}^N)$.

Claim 1. $\lambda_n \to \lambda < 0$, up to a subsequence.

Clearly,

$$o_n(1) = \|\nabla u_n\|_2^2 + [u_n]^2 - \mu \|u_n\|_p^p - A(u_n) - \lambda_n \tau^2, \tag{3.3}$$

by weak convergence of $\{u_n\}$ and (3.2).

Then, by Fatou's lemma and the compact embedding of $H_r(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ (see [4, Lemma 3.1.4]), we have

$$\lambda_n \leq \frac{T(u_n)^2}{\tau^2} - \frac{\mu \|u_0\|_p^p}{\tau^2} - \frac{A(u_0)}{\tau^2} + o(1).$$

Hence, by the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^N)$,

$$|\tau^2 \lambda_n| \leq |T(u_n)^2| + \mu \|u_0\|_p^p + |A(u_0)| + o(1) < +\infty.$$

Thus, $\{\lambda_n\}$ is is bounded and hence, up to a subsequence, convergent. Denoting the subsequence $\{\lambda_n\}$, let $\lambda_0 \in \mathbb{R}$ be such that $\lambda_n \to \lambda_0$. Now, since $u_n \in \mathcal{M}_\tau$, by (3.3) and the fact that $\gamma_p < 1$, we get

$$\begin{aligned} \lambda_0\tau^2 &= \lim_{n\to\infty}\left(\|\nabla u_n\|_2^2 + [u_n]^2 - \mu\,\|u_n\|_p^p - A(u_n)\right) \\ &= \lim_{n\to\infty}\left((1-s)[u_n]^2 + \mu\,(\gamma_p - 1)\,\|u_n\|_p^p\right) < 0, \end{aligned}$$

for sufficiently large $\mu > 0$.

Claim 2. $u_0 \neq 0$.

Suppose $u_0 = 0$, then by the compact embedding $H_r(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*)$ and (3.1), we get $\lim\limits_{n\to\infty} A(u_n) = \lim\limits_{n\to\infty} T_s(u_n)^2$, where $T_s(u) := (\|\nabla u\|_2^2 + s[u]^2)^{\frac{1}{2}}$. Suppose $T_s(u_n)^2 \to l$, then by (1.12)

$$l \leq \frac{l^{2^*_\alpha}}{S_\alpha^{2^*_\alpha}} \Rightarrow l(S_\alpha^{2^*_\alpha} - l^{2^*_\alpha - 1}) \leq 0.$$

Since $m_\tau < 0$, $l = 0$ will lead us to a contradiction, because if $l = 0$, then

$$m_\tau = \lim_{n\to\infty} E(u_n) \geq \lim_{n\to\infty}\left(\frac{T_s(u_n)^2}{2} - \frac{\mu\,\|u_n\|_p^p}{p} - \frac{A(u_n)}{22^*_\alpha}\right) = 0.$$

Hence, we must have $l \geq S_\alpha^{\frac{N+\alpha}{\alpha+2}}$. Now,

$$\begin{aligned} m_\tau &= \lim_{n\to\infty} E(u_n) = \lim_{n\to\infty}\left(E(u_n) - \frac{M(u_n)}{22^*_\alpha}\right) \\ &= \lim_{n\to\infty}\left(\left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right)\|\nabla u_n\|_2^2 + \left(\frac{2^*_\alpha - s}{22^*_\alpha}\right)[u_n]^2 - \mu\left(\frac{1}{p} - \frac{\gamma_p}{22^*_\alpha}\right)\|u_n\|_p^p\right) \\ &\geq \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right)\lim_{n\to\infty} T_s(u_n)^2 = \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right) l \geq \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right) S_\alpha^{\frac{N+\alpha}{\alpha+2}} \geq 0, \end{aligned}$$

thus, we are again lead to a contradiction. Therefore, $u_0 \neq 0$.

Claim 3. (u_0, λ_0) solves (1.1).

Since $\lambda_0 < 0$, we can define the following equivalent norm on $H^1(\mathbb{R}^N)$:

$$\|u\|_{\lambda_0} := (\|\nabla u\|_2^2 + [u]^2 - \lambda_0\|u\|_2^2)^{\frac{1}{2}}.$$

Then, for any $v \in H^1(\mathbb{R}^N)$, by (3.2), we have

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(E'(u_n)(v) - \lambda_n \Phi'(u_n)(v)\right) \\ &= \int\limits_{\mathbb{R}^N} \nabla u_0 \nabla v + \ll u_0, v \gg -\lambda_0 \int\limits_{\mathbb{R}^N} u_0 v - A'(u_0)(v) - \mu \int\limits_{\mathbb{R}^N} |u_0|^{p-2} u_0 v, \end{aligned} \tag{3.4}$$

since the mappings $u \mapsto \frac{\|u\|_p^p}{p}$ and A defined on $H^1(\mathbb{R}^N)$ are of class C^1. Thus, u_0 solves

$$-\Delta u_0 + (-\Delta)^s u_0 = \lambda_0 u_0 + \mu |u_0|^{p-2} u_0 + (I_\alpha * |u_0|^{2^*_\alpha}) |u|^{2^*_\alpha - 2} u_0 \quad \text{in} \quad \mathbb{R}^N.$$

Next, we will show that $\|u_0\|_2 = \tau$. Following the proof of Lemma 2.1, we have $M(u_0) = 0$. Now, define $\bar{u}_n := u_n - u_0$. Since $\bar{u}_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, and hence in $H_r(\mathbb{R}^N)$, then, by the Brezis–Lieb lemma, Lemma 2.4 of [35], and the compact embedding of $H_r(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$, we get

$$\begin{aligned} \|\nabla \bar{u}_n\|_2^2 &= \|\nabla u_n\|_2^2 - \|u_0\|_2^2 + o_n(1), \\ [\bar{u}_n]^2 &= [u_n]^2 - [u_0]^2 + o_n(1), \\ A(\bar{u}_n) &= A(u_n) - A(u_0) + o_n(1), \\ \|\bar{u}_n\|_p^p &= o_n(1). \end{aligned} \tag{3.5}$$

Now, by (3.5), it follows that

$$\begin{aligned} &\lim_{n\to\infty} M(\bar{u}_n) \\ &= \lim_{n\to\infty} \Big(\|\nabla \bar{u}_n\|_2^2 + s[\bar{u}_n]^2 - \mu\gamma_p \|\bar{u}_n\|_p^p - A(\bar{u}_n) \Big) \\ &= \lim_{n\to\infty} \Big(\|\nabla u_n\|_2^2 + s[u_n]^2 - A(u_n) \Big) - (\|\nabla u_0\|_2^2 + s[u_0]^2 - A(u_0)) \\ &= \lim_{n\to\infty} \Big(M(u_n) - \mu\gamma_p \|u_n\|_p^p - M(u_0) + \mu\gamma_p \|u_0\|_p^p \Big) = 0. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \Big(\|\nabla \bar{u}_n\|_2^2 + s[\bar{u}_n]^2 \Big) = \lim_{n\to\infty} \Big(\mu\gamma_p \|\bar{u}_n\|_p^p + A(\bar{u}_n) \Big) = \lim_{n\to\infty} A(\bar{u}_n).$$

Since $\bar{u}_n$ is bounded in $H^1(\mathbb{R}^N)$, up to a subsequence, $|\nabla \bar{u}_n|_2^2 + s[\bar{u}_n]^2$ is convergent. Denoting this convergent subsequence again by $|\nabla \bar{u}_n|_2^2 + s[\bar{u}_n]^2$, let $l \geq 0$ be such that

$$l = \lim_{n\to\infty} \Big(\|\nabla \bar{u}_n\|_2^2 + s[\bar{u}_n]^2 \Big) = \lim_{n\to\infty} A(\bar{u}_n). \tag{3.6}$$

Then, by (1.12), we have either $l = 0$ or $l \geq S_\alpha^{\frac{2^*_\alpha}{2^*_\alpha - 1}}$.

Subclaim. $l = 0$.

Let, if possible, $l \geq S_\alpha^{\frac{2^*_\alpha}{2^*_\alpha - 1}}$. Then, by (3.5), Fatou's lemma, and the Gagliardo–Nirenberg inequality (1.14),

$$\begin{aligned}
m_\tau &= \lim_{n\to\infty} E(u_n)\\
&= \lim_{n\to\infty}\left(\frac{\|\nabla \bar{u}_n\|_2^2+\|\nabla u_0\|_2^2}{2}+\frac{[\bar{u}_n]^2+[u_0]^2}{2}-\mu\frac{\|u_n\|_p^p}{p}-\frac{A(\bar{u}_n)+A(u_0)}{22_\alpha^*}\right)\\
&\geq \lim_{n\to\infty}\left(\frac{\|\nabla \bar{u}_n\|_2^2+s[\bar{u}_n]^2}{2}-\frac{A(\bar{u}_n)}{22_\alpha^*}\right)+E(u_0)\\
&= \left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)l+E(u_0)\geq \left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}+E(u_0)\\
&= \left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}+E(u_0)-\frac{M(u_0)}{22_\alpha^*}\\
&\geq \left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)T(u_0)^2+\mu\left(\frac{p\gamma_p-22_\alpha^*}{22_\alpha^* p}\right)\|u_0\|_p^p+\left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}\\
&\geq \left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)T(u_0)^2+\mu\left(\frac{p\gamma_p-22_\alpha^*}{22_\alpha^* p}\right)C_{N,p}T(u_0)^{p\gamma_p}\tau^{p(1-\gamma_p)}\\
&\quad +\left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}\\
&= f(T(u_0))+\left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}},
\end{aligned}$$

where

$$f(t)=\left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)t^2+\mu\left(\frac{p\gamma_p-22_\alpha^*}{22_\alpha^* p}\right)C_{N,p}t^{p\gamma_p}\tau^{p-p\gamma_p}.$$

Now, since $t_0=\left(\frac{(22_\alpha^*-p\gamma_p)\gamma_p\mu C_{N,p}\tau^{p-p\gamma_p}}{2(2_\alpha^*-1)}\right)^{\frac{1}{2-p\gamma_p}}$ is the point of global minimum of f. Thus,

$$\begin{aligned}
m_\tau &\geq f(t_0)+\left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}\\
&= -\left(\frac{\gamma_p}{2_\alpha^*-1}\right)^{\frac{p\gamma_p}{2-p\gamma_p}}\left(\frac{2-p\gamma_p}{22_\alpha^* p}\right)\left(\frac{(22_\alpha^*-p\gamma_p)\mu C_{N,p}\tau^{p(1-\gamma_p)}}{2}\right)^{\frac{2}{2-p\gamma_p}}\\
&\quad +\left(\frac{2_\alpha^*-1}{22_\alpha^*}\right)S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}\\
&> 0,\ \text{for } \tau<\tau_1.
\end{aligned}$$

But this contradicts Lemma 2.7. Therefore, $l=0$. Now, by (3.5) and (3.6), we have $\lim\limits_{n\to\infty} A(u_n)=A(u_0)$ and $\lim\limits_{n\to\infty} T(u_n)=T(u_0)$. Then, taking u_0 as a test function in (3.4) and using (3.1), we get

$$\begin{aligned}\lambda_0 \|u_0\|_2^2 &= E'(u_0)(u_0) - \lim_{n\to\infty} (E'(u_n)(u_n) - \lambda_n \Phi'(u_n)(u_n)) \\ &= \lambda_0 \lim_{n\to\infty} \|u_n\|_2^2 = \lambda_0 \tau^2.\end{aligned}$$

Hence, u_0 is a solution of (1.1) and $u_n \to u_0$ strongly in $H^1(\mathbb{R}^N)$. Taking $u_\tau^+ = u_0$ and $\lambda_\tau^+ = \lambda_0$, we are done. □

4. SECOND SOLUTION

Until now, we have seen that the infimum of E on $\mathcal{M}\tau^+$ is achieved and is a solution of (1.1). In this section, we will show that the infimum over $\mathcal{M}\tau^-$, that is, m_τ^-, is also achieved. Since the spaces $\mathcal{M}\tau^+$ and $\mathcal{M}\tau^-$ are disjoint, this corresponds to the second normalized solution. The following result will play a crucial role in proving the convergence of the Palais–Smale sequence, by providing an upper bound for m_τ^-.

Lemma 4.1. *For all* $\tau < \min\{\tau_0, \tau_1\}$,

$$m_\tau^- = \inf_{u\in\mathcal{M}_\tau^-} E(u) < m_\tau + \left(\frac{2_\alpha^* - 1}{22_\alpha^*}\right) S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}. \tag{4.1}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that

$$\phi(x) = \begin{cases} 0 \le \phi(x) \le 1 & \text{for all } x \in \mathbb{R}^N, \\ \phi(x) = 1 & \text{for } x \in B_1(0), \\ \phi(x) = 0 & \text{for } x \in \mathbb{R}^N \setminus B_2(0). \end{cases} \tag{4.2}$$

Then, taking $u_\epsilon = \phi U_{\epsilon,0}$, where $U_{\epsilon,0}$ is as defined in (1.13), by [50, Lemma 1.46], [14, Lemma 3.3, eq. 3.7], and [46, Lemma 5.3], we have:

$$\|\nabla u_\epsilon\|_2^2 = S^{\frac{N}{2}} + O(\epsilon^{N-2}), \tag{4.3}$$

$$\|u_\epsilon\|_2^2 = \begin{cases} K_1\epsilon^2 + O(\epsilon^{N-2}) & \text{for } N \ge 5, \\ K_1\epsilon^2|\ln(\epsilon)| + O(\epsilon^2) & \text{for } N = 4, \\ K_1\epsilon + O(\epsilon^2) & \text{for } N = 3. \end{cases} \tag{4.4}$$

$$A(u_\epsilon) \ge (A_\alpha C_\alpha)^{\frac{N}{2}} S_\alpha^{\frac{N+\alpha}{2}} - O(\epsilon^{\frac{N+\alpha}{2}}), \tag{4.5}$$

$$[u_\epsilon]^2 = O(\epsilon^{m_{N,s}}), \quad \text{where } m_{N,s} = \begin{cases} 2(1-s) & \text{for } N \ge 4, \\ 2(1-s) & \text{for } N = 3 \text{ with } s > \frac{1}{2}, \\ 1 & \text{for } N = 3 \text{ with } s \le \frac{1}{2}. \end{cases} \tag{4.6}$$

and

$$\|u_\epsilon\|_p^p = \begin{cases} K_2\epsilon^{N-\frac{p(N-2)}{2}} + O(\epsilon^{\frac{p(N-2)}{2}}) & \text{for } N > \frac{2p}{p-1}, \\ K_2\epsilon^{\frac{N}{2}} \ln(1/\epsilon) + O(\epsilon^{\frac{N}{2}}) & \text{for } N = \frac{2p}{p-1}, \\ O(\epsilon^{\frac{p(N-2)}{2}}) & \text{for } N < \frac{2p}{p-1}. \end{cases} \tag{4.7}$$

For $\zeta, t \geq 0$, define

$$\hat{u}_{\epsilon,t}(x) := u_\tau^+(x) + tu_\epsilon(x) \quad \text{and} \quad \bar{u}_{\epsilon,t}(x) := \zeta^{\frac{N-2}{2}}\hat{u}(\zeta x),$$

with u_τ^+ being the radial solution deduced in Theorem 1.3. We will see that

$$m_\tau^- \leq \sup_{t\geq 0} E(\bar{u}_{\epsilon,t}) \quad \text{and} \quad E(\bar{u}_{\epsilon,t}) < m_\tau + \left(\frac{2_\alpha^* - 1}{22_\alpha^*}\right) S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}$$

for all $t > 0$ and small enough $\epsilon > 0$. Clearly,

$$\begin{cases} \|\nabla\bar{u}_{\epsilon,t}\|_2^2 = \|\nabla\hat{u}_{\epsilon,t}\|_2^2,\ [\bar{u}_{\epsilon,t}]^2 = \zeta^{2(s-1)}[\hat{u}_{\epsilon,t}]^2,\ \|\bar{u}_{\epsilon,t}\|_2^2 = \zeta^{-2}\|\hat{u}_{\epsilon,t}\|_2^2, \\ \|\bar{u}_{\epsilon,t}\|_p^p = \zeta^{p\gamma_p - p}\|\hat{u}_{\epsilon,t}\|_p^p, \quad A(\bar{u}_{\epsilon,t}) = A(\hat{u}_{\epsilon,t}). \end{cases} \tag{4.8}$$

Then, taking $\zeta = \zeta_{\epsilon,t} = \frac{\|\hat{u}_{\epsilon,t}\|_2}{\tau}$, we get $\bar{u}_{\epsilon,t} \in S(\tau)$. Thus, by Lemma 2.3, we can find $\bar{q}\epsilon, t \in \mathbb{R}$ such that $\bar{q}\epsilon, t \star \bar{u}\epsilon, t \in \mathcal{M}\tau^-$, or $q_{\epsilon,t} \circledast \bar{u}\epsilon, t \in \mathcal{M}\tau^-$, where $q_{\epsilon,t} = e^{\bar{q}_{\epsilon,t}} > 0$. Then

$$0 = M(q_{\epsilon,t} \circledast \bar{u}_{\epsilon,t}) = q_{\epsilon,t}^2 \|\nabla\bar{u}_{\epsilon,t}\|_2^2 + sq_{\epsilon,t}^{2s}[\bar{u}_{\epsilon,t}]^2 - \mu\gamma_p q_{\epsilon,t}^{p\gamma_p} \|\bar{u}_{\epsilon,t}\|_p^p - q_{\epsilon,t}^{22_\alpha^*} A(\bar{u}_{\epsilon,t}),$$

and hence

$$q_{\epsilon,t}^{2-p\gamma_p} \|\nabla\bar{u}_{\epsilon,t}\|_2^2 + sq_{\epsilon,t}^{2s-p\gamma_p}[\bar{u}_{\epsilon,t}]^2 = \mu\gamma_p \|\bar{u}_{\epsilon,t}\|_p^p + q_{\epsilon,t}^{22_\alpha^*-p\gamma_p} A(\bar{u}_{\epsilon,t}). \tag{4.9}$$

Now, since $0 \star \hat{u}_{\epsilon,0} = u_\tau^+ \in \mathcal{M}_\tau^+$, by Lemma 2.3, $\bar{q}_{\epsilon,0} > 0$, that is, $q_{\epsilon,0} > 1$. Also, by (4.9),

$$q_{\epsilon,t}^{22_\alpha^*} \leq \frac{q_{\epsilon,t}^2 \|\nabla\bar{u}_{\epsilon,t}\|_2^2 + sq_{\epsilon,t}^{2s}[\bar{u}_{\epsilon,t}]^2}{A(\bar{u}_{\epsilon,t})}.$$

Defining $B_{\epsilon,t} := \frac{\|\nabla\bar{u}_{\epsilon,t}\|_2^2 + s[\bar{u}_{\epsilon,t}]^2}{A(\bar{u}_{\epsilon,t})}$, we get $0 < q_{\epsilon,t} \leq \max\{B_{\epsilon,t}^{\frac{1}{2(2_\alpha^*-1)}}, B_{\epsilon,t}^{\frac{1}{2(2_\alpha^*-s)}}\}$. By (4.8), we have

$$\begin{aligned} B_{\epsilon,t} &= \frac{\|\nabla\bar{u}_{\epsilon,t}\|_2^2 + s[\bar{u}_{\epsilon,t}]^2}{A(\bar{u}_{\epsilon,t})} = \frac{\|\nabla\hat{u}_{\epsilon,t}\|_2^2 + s\zeta_{\epsilon,t}^{2(s-1)}[\hat{u}_{\epsilon,t}]^2}{A(\hat{u}_{\epsilon,t})} \\ &= \frac{1}{A(\hat{u}_{\epsilon,t})}\left(\|\nabla\hat{u}_{\epsilon,t}\|_2^2 + s\Big(\frac{\tau}{\|\hat{u}_{\epsilon,t}\|_2^2}\Big)^{2(1-s)}[\hat{u}_{\epsilon,t}]^2\right) \leq \frac{\|\nabla\hat{u}_{\epsilon,t}\|_2^2 + s[\hat{u}_{\epsilon,t}]^2}{A(\hat{u}_{\epsilon,t})} \\ &\leq C\frac{\|\nabla u_\tau^+\|_2^2 + t^2\|\nabla u_\epsilon\|_2^2 + s[u_\tau^+]^2 + st^2[u_\epsilon]^2}{t^{22_\alpha^*}A(u_\epsilon)} \longrightarrow 0 \quad \text{as } t \to \infty, \end{aligned}$$

and hence $q_{\epsilon,t} \to 0$ as $t \to \infty$. Since $q_{\epsilon,0} > 1$, there exists some $t_\epsilon > 0$ such that $q_{\epsilon,t_\epsilon} = 1$, which implies that

$$m_\tau^- = \inf_{u \in \mathcal{M}_\tau^-} E(u) \leq E(q_{\epsilon,t_\epsilon} \circledast \bar{u}_{\epsilon,t_\epsilon}) = E(\bar{u}_{\epsilon,t_\epsilon}) \leq \sup_{t \geq 0} E(\bar{u}_{\epsilon,t}). \tag{4.10}$$

Now, since $\hat{u}\epsilon, t \geq u\tau^+$ by (4.8) and the definition of $\bar{u}_{\epsilon,t}$, we have

$$\begin{aligned} E(\bar{u}_{\epsilon,t}) &= \frac{\|\nabla u_\tau^+ + t\nabla u_\epsilon\|_2^2}{2} + \frac{\zeta_{\epsilon,t}^{2(s-1)}}{2}[u_\tau^+ + tu_\epsilon]^2 - \frac{\zeta_{\epsilon,t}^{p(\gamma_p-1)}\mu}{p}\left\|u_\tau^+ + tu_\epsilon\right\|_p^p \\ &\quad - \frac{A(u_\tau^+ + tu_\epsilon)}{22_\alpha^*} \\ &\leq \frac{\|\nabla u_\tau^+\|_2^2}{2} + \frac{t^2\left\|\nabla u_\epsilon\right\|_2^2}{2} + t\int\limits_{\mathbb{R}^N} \nabla u_\tau^+ \nabla u_\epsilon + \frac{[u_\tau^+]^2}{2} + \frac{t^2[u_\epsilon]^2}{2} \\ &\quad + t \ll u_\tau^+, u_\epsilon \gg -\mu\frac{\|u_\tau^+\|_p^p}{p} - \frac{A(u_\tau^+)}{22_\alpha^*} \\ &= E(u_\tau^+) + \frac{t^2\left\|\nabla u_\epsilon\right\|_2^2}{2} + t\int\limits_{\mathbb{R}^N} \nabla u_\tau^+ \nabla u_\epsilon + \frac{t^2[u_\epsilon]^2}{2} + t \ll u_\tau^+, u_\epsilon \gg \\ &\to E(u_\tau^+) = m_\tau < 0 \text{ as } t \to 0^+. \end{aligned} \tag{4.11}$$

Also,

$$\begin{aligned} E(\bar{u}_{\epsilon,t}) \leq{}& \frac{\|\nabla u_\tau^+\|_2^2}{2} + \frac{[u_\tau^+]^2}{2} - \mu\frac{\|u_\tau^+\|_p^p}{p} + t^2\frac{\|\nabla u_\epsilon\|_2^2}{2} + t^2\frac{[u_\epsilon]^2}{2} \\ &+ t\int\limits_{\mathbb{R}^N} \nabla u_\tau^+ \nabla u_\epsilon + t \ll u_\tau^+, u_\epsilon \gg -\frac{t^{22_\alpha^*}}{22_\alpha^*}A(u_\epsilon) \\ \to{}& -\infty \text{ as } t \to +\infty, \end{aligned} \tag{4.12}$$

and, by Lemma 2.3, $E(\bar{u}_{\epsilon,t_\epsilon}) = E(0 \star \bar{u}_{\epsilon,t_\epsilon}) = E(\bar{q}_{\epsilon,t_\epsilon} \star \bar{u}_{\epsilon,t_\epsilon}) > 0$. Thus, there exists some $t_0 > 0$ large enough such that $E(\bar{u}\epsilon, t) < 0$ for $t \in \left(0, \frac{1}{t_0}\right) \cup (t_0, \infty)$. Therefore, we need to estimate $E(\bar{u}\epsilon, t)$ for $t \in \left[\frac{1}{t_0}, t_0\right]$. The above analysis can be summarized by the following plot in Figure 4.

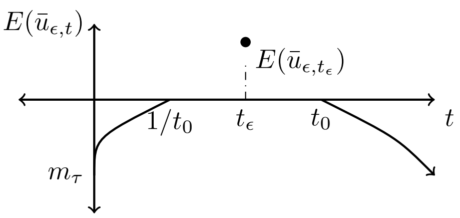

Fig. 4. Qualitative behavior of $E(\bar{u}_{\epsilon,t})$

Now, let us study $E(\bar{u}_{\epsilon,t})$ for $t \in [1/t_0, t_0]$. Since

$$\zeta_{\epsilon,t}^2 = \frac{\|\hat{u}_{\epsilon,t}\|_2^2}{\tau^2} = 1 + \frac{t^2}{\tau^2}\int\limits_{\mathbb{R}^N} |u_\epsilon|^2 + \frac{2t}{\tau^2}\int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon,$$

and hence

$$\begin{aligned}\zeta_{\epsilon,t}^{p\gamma_p - p} &= \left(1 + \left(\frac{t^2}{\tau^2}\int\limits_{\mathbb{R}^N} |u_\epsilon|^2 + \frac{2t}{\tau^2}\int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon\right)\right)^{\frac{p(\gamma_p-1)}{2}} \\ &\geq 1 + \frac{p(\gamma_p - 1)}{2}\left(\frac{t^2}{\tau^2}\int\limits_{\mathbb{R}^N} |u_\epsilon|^2 + \frac{2t}{\tau^2}\int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon\right),\end{aligned}$$

by (4.8) and the fact that $\hat{u}_{\epsilon,t} \geq u_\tau^+$, we get

$$\begin{aligned}E(\bar{u}_{\epsilon,t}) \leq{}& \frac{\|\nabla \hat{u}_{\epsilon,t}\|_2^2}{2} + \frac{[\hat{u}_{\epsilon,t}]^2}{2} - \frac{A(\hat{u}_{\epsilon,t})}{22_\alpha^*} \\ &- \left(1 + \frac{p(\gamma_p - 1)}{2}\left(\frac{t^2}{\tau^2}\int\limits_{\mathbb{R}^N} |u_\epsilon|^2 + \frac{2t}{\tau^2}\int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon\right)\right)\frac{\mu\,\|\hat{u}_{\epsilon,t}\|_p^p}{p}.\end{aligned}$$

Further, we have

$$\begin{aligned}A(\hat{u}_{\epsilon,t}) &= A(u_\tau^+ + tu_\epsilon) \\ &\geq A(u_\tau^+) + A(tu_\epsilon) + 22_\alpha^*\int\limits_{\mathbb{R}^N} (I_\alpha * |u_\tau^+|^{2_\alpha^*})|u_\tau^+|^{2_\alpha^*-2}u_\tau^+(tu_\epsilon),\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\|\hat{u}_{\epsilon,t}\|_p^p &\geq \left\|u_\tau^+\right\|_p^p + \|tu_\epsilon\|_p^p = \left\|u_\tau^+ + tu_\epsilon\right\|_p^p \\ &\geq \left\|u_\tau^+\right\|_p^p + \|tu_\epsilon\|_p^p + pt\int\limits_{\mathbb{R}^N} |u_\tau^+|^{p-2}u_\tau^+ u_\epsilon.\end{aligned}$$

Thus, using (4.13) and (4.14) in (4.13), we have

$$\begin{aligned}E(\bar{u}_{\epsilon,t}) \leq{}& E(u_\tau^+) + E(tu_\epsilon) + \Bigg(t\int\limits_{\mathbb{R}^N} \nabla u_\tau^+ \nabla u_\epsilon + t \ll u_\tau^+, u_\epsilon \gg \\ &-t\mu\int\limits_{\mathbb{R}^N} |u_\tau^+|^{p-2}u_\tau^+ u_\epsilon - \int\limits_{\mathbb{R}^N} (I_\alpha * |u_\tau^+|^{2_\alpha^*})|u_\tau^+|^{2_\alpha^*-2}u_\tau^+(tu_\epsilon)\Bigg) \\ &+ \frac{(1-\gamma_p)t^2}{2\tau^2}\mu\,\|u_\epsilon\|_2^2\,\|\hat{u}_{\epsilon,t}\|_p^p + \mu\frac{t(1-\gamma_p)}{\tau^2}\,\|\hat{u}_{\epsilon,t}\|_p^p\int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon.\end{aligned}$$

Moreover, since u_τ^+ solves (1.1), we get

$$\begin{aligned}E(\bar{u}_{\epsilon,t}) &\leq E(u_\tau^+) + E(tu_\epsilon) + \lambda_\tau^+ \int\limits_{\mathbb{R}^N} u_\tau^+(tu_\epsilon) + \frac{\mu(1-\gamma_p)t^2}{2\tau^2}\|u_\epsilon\|_2^2\,\|\hat{u}_{\epsilon,t}\|_p^p \\ &\quad + \frac{\mu t(1-\gamma_p)}{\tau^2}\|\hat{u}_{\epsilon,t}\|_p^p \int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon \\ &= m_\tau + E(tu_\epsilon) + \frac{\mu t(1-\gamma_p)}{\tau^2}\left(\|\hat{u}_{\epsilon,t}\|_p^p - \left\|u_\tau^+\right\|_p^p\right)\int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon \\ &\quad + \frac{\mu t^2(1-\gamma_p)}{2\tau^2}\|u_\epsilon\|_2^2\,\|\hat{u}_{\epsilon,t}\|_p^p + \frac{t(1-s)}{\tau^2}[u_\tau^+]^2 \int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon.\end{aligned}$$

Since u_τ^+ is a radially symmetric solution of (1.1), as done in [27, Lemma 5.5], one can deduce that

$$\int\limits_{\mathbb{R}^N} u_\tau^+ u_\epsilon = O(\epsilon^{\frac{N-2}{2}}) \quad \text{and} \quad \int\limits_{\mathbb{R}^N} |u_\tau^+|^{p-1} u_\epsilon = O(\epsilon^{\frac{N-2}{2}}).$$

Then (4.14) becomes

$$\begin{aligned}E(\bar{u}_{\epsilon,t}) &\leq m_\tau + E(tu_\epsilon) + \frac{\mu t(1-\gamma_p)}{\tau^2}\left(O(\epsilon^{\frac{N-2}{2}}) + \|tu_\epsilon\|_p^p\right)O(\epsilon^{\frac{N-2}{2}}) \\ &\quad + \frac{\mu t^2(1-\gamma_p)}{2\tau^2}\|u_\epsilon\|_2^2\,\left\|u_\tau^+ + tu_\epsilon\right\|_p^p + \frac{t(1-s)}{\tau^2}[u_\tau^+]O(\epsilon^{\frac{N-2}{2}}) \\ &= m_\tau + E(tu_\epsilon) + O(\epsilon^{N-2}) + O(\|u_\epsilon\|_p^p)O(\epsilon^{\frac{N-2}{2}}) + O(\|u_\epsilon\|_2^2) \\ &\quad + O(\|u_\epsilon\|_2^2)O(\|u_\epsilon\|_p^p) + O(\epsilon^{\frac{N-2}{2}}) \\ &\leq m_\tau + E(tu_\epsilon) \\ &< m_\tau + f_{u_\epsilon}(t) \text{ for small } \epsilon > 0,\end{aligned}$$

where

$$f_u(t) := \frac{t^2 T(u)^2}{2} - \frac{t^{22_\alpha^*}A(u)}{22_\alpha^*}.$$

Also, since f_u has a global maximum at $t_u = \left(\frac{T(u)^2}{A(u)}\right)^{\frac{1}{2(2^*_\alpha-1)}}$, by (4.3), (4.6), and (4.5), we get

$$
\begin{aligned}
E(\bar{u}_{\epsilon,t}) < m_\tau + f_{u_\epsilon}(t_{u_\epsilon}) &= m_\tau + \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right)\left(\frac{T(u_\epsilon)^2}{A(u_\epsilon)^{\frac{1}{2^*_\alpha}}}\right)^{\frac{2^*_\alpha}{2^*_\alpha-1}} \\
&\leq m_\tau + \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right)\left(\frac{S^{\frac{N}{2}} + O(\epsilon^{N-2}) + O(\epsilon^{m_{N,s}})}{\left((A_\alpha C_\alpha)^{\frac{N}{2}} S_\alpha^{\frac{N+\alpha}{2}} - O(\epsilon^{\frac{N+\alpha}{2}})\right)^{\frac{1}{2^*_\alpha}}}\right)^{\frac{2^*_\alpha}{2^*_\alpha-1}} \\
&\leq m_\tau + \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right) S_\alpha^{\frac{2^*_\alpha}{2^*_\alpha-1}} \text{ as } \epsilon \text{ goes to zero, for all } t \in [1/t_0, t_0].
\end{aligned}
$$

Therefore, by (4.10), we are done. ☐

For $0 < \tau < \min \tau_0, \tau_1$, let $u \in \mathcal{M}\tau^{\pm}$. Then, $v\beta := \frac{\beta}{\tau}u \in S(\beta)$ for all $\beta > 0$. Now, for $0 < \beta < \min \tau_0, \tau_1$, by Lemma 2.3, there exists $t_\pm(\beta) > 0$ such that $t_\pm(\beta) \circledast v_\beta \in \mathcal{M}\beta^{\pm}$. Clearly, since $v\tau = u \in \mathcal{M}\tau^{\pm}$, we have $t\pm(\tau) = 1$. Further, we have the following results for $t_\pm(\beta)$.

Lemma 4.2. *For $N \geq 3$, $2 < p < 2 + \frac{4s}{N}$ and $0 < \tau < \min\{\tau_0, \tau_1\}$, $t_\pm$ is differentiable at τ, with*

$$
t'_\pm(\tau) = \frac{p\gamma_p\mu\|u\|_p^p + 22^*_\alpha A(u) - 2s[u]^2 - 2\left\|\nabla u\right\|_2^2}{\tau\left(2s^2[u]^2 + 2\left\|\nabla u\right\|_2^2 - \mu p\gamma_p^2\|u\|_p^p - 22^*_\alpha A(u)\right)},
$$

Moreover, for sufficiently large $\mu > 0$, we have $E(t_\pm(\beta) \circledast v_\beta) < E(u)$ whenever $\tau < \beta < \min \tau_0, \tau_1$.

Proof. Since $M(t_\pm(\beta) \circledast v_\beta) = 0$ and $v_\beta = \frac{\beta}{\tau}u$, for all $0 < \beta < \min\{\tau_0, \tau_1\}$,

$$
\begin{aligned}
0 = &\left(\frac{\beta t_\pm(\beta)}{\tau}\right)^2 \left\|\nabla u\right\|_2^2 + s\left(\frac{\beta t_\pm^s(\beta)}{\tau}\right)^2 [u]^2 - \mu\gamma_p\left(\frac{\beta t_\pm^{\gamma_p}(\beta)}{\tau}\right)^p \|u\|_p^p \\
&- \left(\frac{\beta t_\pm(\beta)}{\tau}\right)^{22^*_\alpha} A(u).
\end{aligned}
$$

Defining $\Phi : (0, \min\{\tau_0, \tau_1\}) \times (0, \infty) \to \mathbb{R}$ as

$$
\Phi(\beta, t) := \left(\frac{\beta t}{\tau}\right)^2 \left\|\nabla u\right\|_2^2 + s\left(\frac{\beta t^s}{\tau}\right)^2 [u]^2 - \mu\gamma_p\left(\frac{\beta t^{\gamma_p}}{\tau}\right)^p \|u\|_p^p - \left(\frac{\beta t}{\tau}\right)^{22^*_\alpha} A(u),
$$

we get $\Phi(\beta, t_\pm(\beta)) = 0$, for all $0 < \beta < \min\{\tau_0, \tau_1\}$. Since $u \in \mathcal{M}_\tau^{\pm}$, we have

$$
\frac{\partial}{\partial t}\Phi(\tau, 1) = 2s^2[u]^2 + 2\left\|\nabla u\right\|_2^2 - \mu p\gamma_p^2\|u\|_p^p - 22^*_\alpha A(u) \neq 0.
$$

Thus, by the implicit function theorem, the map $\beta \mapsto t_\pm(\beta)$ is differentiable at τ, and

$$t'_\pm(\tau) = -\frac{\frac{\partial}{\partial \beta}\Phi(\tau,1)}{\frac{\partial}{\partial t}\Phi(\tau,1)} = \frac{\mu p\gamma_p\|u\|_p^p + 22^*_\alpha A(u) - 2s[u]^2 - 2\left\|\nabla u\right\|_2^2}{\tau\left(2s^2[u]^2 + 2\left\|\nabla u\right\|_2^2 - \mu p\gamma_p^2\|u\|_p^p - 22^*_\alpha A(u)\right)}.$$

It follows that

$$1 + \tau t'_\pm(\tau) = \frac{2s(s-1)[u]^2 + \mu p\gamma_p(1-\gamma_p)\|u\|_p^p}{2s^2[u]^2 + 2\left\|\nabla u\right\|_2^2 - \mu p\gamma_p^2\|u\|_p^p - 22^*_\alpha A(u)}. \tag{4.14}$$

Now,

$$\begin{aligned} E(t_\pm(\beta) \circledast v_\beta) = {} & \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right)\left\|\nabla t_\pm(\beta) \circledast v_\beta\right\|_2^2 + \left(\frac{1}{2} - \frac{s}{p\gamma_p}\right)[t_\pm(\beta) \circledast v_\beta]^2 \\ & + \left(\frac{1}{p\gamma_p} - \frac{1}{22^*_\alpha}\right) A(t_\pm(\beta) \circledast v_\beta). \end{aligned}$$

Then, we have

$$\begin{aligned} & E(t_\pm(\beta) \circledast v_\beta) \\ & = \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right)\left(\frac{t_\pm(\beta)\beta}{\tau}\right)^2 \|\nabla u\|_2^2 + \left(\frac{1}{2} - \frac{s}{p\gamma_p}\right)\left(\frac{t^s_\pm(\beta)\beta}{\tau}\right)^2 [u]^2 \\ & \quad + \left(\frac{1}{p\gamma_p} - \frac{1}{22^*_\alpha}\right)\left(\frac{t_\pm(\beta)\beta}{\tau}\right)^{22^*_\alpha} A(u) \\ & = \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right)\left(1 + (\beta-\tau)\left(\frac{t_\pm(\beta)\beta - \tau t_\pm(\tau)}{\tau(\beta-\tau)}\right)\right)^2 \|\nabla u\|_2^2 \\ & \quad + \left(\frac{1}{2} - \frac{s}{p\gamma_p}\right)\left(1 + (\beta-\tau)\left(\frac{t^s_\pm(\beta)\beta - \tau t^s_\pm(\tau)}{\tau(\beta-\tau)}\right)\right)^2 [u]^2 \\ & \quad + \left(\frac{1}{p\gamma_p} - \frac{1}{22^*_\alpha}\right)\left(1 + (\beta-\tau)\left(\frac{t_\pm(\beta)\beta - \tau t_\pm(\tau)}{\tau(\beta-\tau)}\right)\right)^{22^*_\alpha} A(u). \end{aligned}$$

Thus,

$$\begin{aligned} E(t_\pm(\beta) \circledast v_\beta) = {} & \left(\frac{1}{2} - \frac{1}{p\gamma_p}\right)\|\nabla u\|_2^2 + \left(\frac{1}{2} - \frac{s}{p\gamma_p}\right)[u]^2 + \left(\frac{1}{p\gamma_p} - \frac{1}{22^*_\alpha}\right) A(u) \\ & + \left(2\frac{(1+\tau t'_\pm(\tau))}{\tau}\left(\frac{1}{2} - \frac{1}{p\gamma_p}\right)\|\nabla u\|_2^2\right. \\ & + 2\frac{(1+s\tau t'_\pm(\tau))}{\tau}\left(\frac{1}{2} - \frac{s}{p\gamma_p}\right)[u]^2 \\ & \left. + 22^*_\alpha\frac{(1+\tau t'_\pm(\tau))}{\tau}\left(\frac{1}{p\gamma_p} - \frac{1}{22^*_\alpha}\right) A(u)\right)(\beta-\tau) + o(\beta-\tau)^2. \end{aligned}$$

Further, since $M(u) = 0$, one can deduce that

$$\begin{aligned}E(t_\pm(\beta) \circledast v_\beta) = \frac{2(\beta-\tau)}{\tau}&\left(\gamma_p(1+\tau t'_\pm(\tau))\left(\frac{1}{2}-\frac{1}{p\gamma_p}\right)\|u\|_p^p\right.\\&+\frac{(2^*_\alpha-1)(1+\tau t'_\pm(\tau))}{p\gamma_p}A(u)\\&\left.+\frac{(1-s)s\tau t'_\pm(\tau)}{p\gamma_p}[u]^2\right)+E(u)+o(\beta-\tau)^2,\end{aligned}$$

and hence, by (4.14),

$$E(t_\pm(\beta) \circledast v_\beta) = E(u) - \mu\frac{(1-\gamma_p)(\beta-\tau)}{\tau}\|u\|_p^p + \frac{(\beta-\tau)(1-s)}{\tau}[u]^2 + o(\beta-\tau)^2.$$

For sufficiently large $\mu > 0$, we have

$$\frac{\partial}{\partial\beta}E(t_\pm(\beta) \circledast v_\beta)_{|\beta=\tau} = -\frac{\mu(1-\gamma_p)\|u\|_p^p}{\tau} + \frac{(1-s)}{\tau}[u]^2 < 0,$$

thus, for $\tau < \beta < \min \tau_0, \tau_1$, $E(t_\pm(\beta) \circledast v_\beta) < E(u)$. ☐

Denoting $\mathcal{M}^-_{r,\tau} := \mathcal{M}^-_\tau \cap H_r(\mathbb{R}^N)$, we get

$$m^-_{r,\tau} := \inf_{u\in\mathcal{M}^-_{r,\tau}} E(u) = \inf_{u\in\mathcal{M}^-_{r,\tau}} E(u) = m^-_\tau,$$

by symmetrization and the fact that $\mathcal{M}^-_{r,\tau} \subset \mathcal{M}^-_\tau$. Now, let us prove our final result.

Proof of Theorem 1.4. Let $\{\bar{u}_n\}$ be a minimizing sequence for E on $\mathcal{M}^-_{r,\tau}$, then by the Ekeland variational principle, [20, Theorem 1.1], we can find a sequence $\{u_n\} \in \mathcal{M}^-_{r,\tau}$ such that

$$\begin{cases}\|\bar{u}_n - u_n\|_{H^1(\mathbb{R}^N)} \to 0 & \text{as } n \to \infty,\\ E(u_n) \to m^-_{r,\tau} & \text{as } n \to \infty,\\ M(u_n) \to 0 & \text{as } n \to \infty,\\ E'|_{\mathcal{M}^-_{r,\tau}}(u_n) \to 0 & \text{as } n \to \infty.\end{cases} \tag{4.15}$$

Now, by (4.15), we have

$$\begin{aligned}m^-_{r,\tau} &= \lim_{n\to\infty} E(u_n) = \lim_{n\to\infty}\left(E(u_n) - \frac{M(u_n)}{2}\right)\\&= \lim_{n\to\infty}\left(\frac{1}{p}\left(\frac{p\gamma_p}{2}-1\right)\|u_n\|_p^p + \frac{(1-s)}{2}[u_n]^2 + \left(\frac{2^*_\alpha-1}{22^*_\alpha}\right)A(u_n)\right),\end{aligned}$$

and, since $E(u_n) \leq m^-_{r,\tau} + 1$, for large $n \in \mathbb{N}$, by the Gagliardo–Nirenberg inequality (1.14), we obtain

$$\begin{aligned}
\frac{(2^*_\alpha - 1)}{22^*_\alpha} T(u_n)^2 &\leq \frac{(2^*_\alpha - 1)}{22^*_\alpha} \|\nabla u_n\|_2^2 + \frac{(2^*_\alpha - s)}{22^*_\alpha} [u_n]^2 \\
&= E(u_n) - \frac{1}{22^*_\alpha} M(u_n) + \frac{1}{p}\left(1 - \frac{p\gamma_p}{22^*_\alpha}\right) \|u_n\|_p^p \\
&\leq m^-_{r,\tau} + 1 + \frac{C_{N,p}(22^*_\alpha - p\gamma_p)}{p22^*_\alpha} \tau^{p(1-\gamma_p)} T(u_n)^{p\gamma_p},
\end{aligned}$$

Thus, $\{u_n\}$ is bounded and hence, up to a subsequence, weakly convergent in $H^1(\mathbb{R}^N)$. Denoting the weakly convergent subsequence again by $\{u_n\}$, let $u_0 \in H_r(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u_0$ weakly. Thanks to the compact embedding $H_r(\mathbb{R}) \hookrightarrow L^q(\mathbb{R}^N)$, for all $q \in (2, 2^*)$, we get $u_n \to u_0$ in $L^p(\mathbb{R}^N)$.

Next, we claim that $u_0 \neq 0$. Suppose $u_0 = 0$. Then

$$0 = \lim_{n\to\infty} M(u_n) = \lim_{n\to\infty} \left(\|\nabla u_n\|_2^2 + s[u_n]^2 - A(u_n) \right),$$

and hence $\lim\limits_{n\to\infty} \left(\|\nabla u_n\|_2^2 + s[u_n]^2 \right) = \lim\limits_{n\to\infty} A(u_n)$. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, the sequence $\{\|\nabla u_n\|_2^2 + s[u_n]^2\}$ is, up to a subsequence, convergent in $\mathbb{R}$. Now, let

$$l = \lim_{n\to\infty} \left(\|\nabla u_n\|_2^2 + s[u_n]^2 \right) = \lim_{n\to\infty} A(u_n).$$

Then, by (1.12), we get $l(S_\alpha^{2^*_\alpha} - l^{2^*_\alpha - 1}) \leq 0$, thus, either $l = 0$ or $l \geq S_\alpha^{\frac{2^*_\alpha}{2^*_\alpha - 1}}$. For $l \geq S_\alpha^{\frac{2^*_\alpha}{2^*_\alpha - 1}}$, by (4.16), we get

$$\begin{aligned}
m^-_\tau = m^-_{r,\tau} &= \lim_{n\to\infty} \left(\frac{1}{p}\left(\frac{p\gamma_p}{2} - 1\right) \|u_n\|_p^p + \frac{(1-s)}{2}[u_n]^2 + \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right) A(u_n) \right) \\
&\geq \lim_{n\to\infty} \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right) A(u_n) \geq \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right) S_\alpha^{\frac{2^*_\alpha}{2^*_\alpha - 1}} \\
&> m_\tau + \left(\frac{2^*_\alpha - 1}{22^*_\alpha}\right) S_\alpha^{\frac{2^*_\alpha}{2^*_\alpha - 1}},
\end{aligned}$$

but this contradicts Lemma 4.1. Also, if $l = 0$, we will end up with $m^-_{r,\tau} = 0$, but since $0 < m^-_\tau = m^-_{r,\tau}$, we get a contradiction. Therefore, $u_0 \neq 0$. Now, define $v_n := u_n - u_0$, clearly $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$.

Case 1. $\|v_n\|_{H^1(\mathbb{R}^N)} \to 0$.

In this case, we get strong convergence of $\{u_n\}$ in $H^1(\mathbb{R}^N)$, and hence $u_0 \in \mathcal{M}^-_{r,\tau}$ with $E(u_0) = m^-_\tau$ and hence $E'_{\mathcal{M}_\tau}(u_0) = 0$. Thus, by Lemma 2.4, u_0 solves (1.1) for some $\lambda_0 \in \mathbb{R}$, and since $M(u) = 0$, we have

$$\lambda_0 \tau^2 = \|\nabla u_0\|_2^2 + [u_0]^2 - \mu \|u_0\|_p^p - A(u_0) = (1-s)[u_0]^2 + \mu(\gamma_p - 1)\|u_0\|_p^p < 0,$$

for sufficiently large $\mu > 0$. Hence, taking $u^-_\tau = u_0$ and $\lambda^-_\tau = \lambda_0$, we are done.

Case 2. $\lim\limits_{n\to\infty} \|v_n\|_{H^1(\mathbb{R}^N)} \neq 0$, that is, $\|v_n\|_{H^1(\mathbb{R}^N)} \geq \tilde{C} > 0$ for large $n \in \mathbb{N}$.

Let $\|u_0\|_2 = r_0$. Then, by Fatou's lemma, we have $0 < r_0 \leq \tau$. Now, either $A(v_n) \to 0$ or there exists a constant $\bar{C} > 0$ such that $A(v_n) \geq \bar{C}$ for large $n \in \mathbb{N}$. Let us analyse the two subcases separately:

Subcase 1. $A(v_n) \to 0$ as $n \to \infty$.

Since $u_0 \in S(r_0)$, by Lemma 2.3, there exists $c_0 > 0$ such that $c_0 \circledast u_0 \in \mathcal{M}^-_{r,r_0}$. Thus, by [35, Lemma 2.4], the compact embedding of $H_r(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, Fatou's lemma, and Lemma 2.3, we get

$$\begin{aligned} m^-_\tau &= \lim_{n\to\infty} E(u_n) \geq \lim_{n\to\infty} E(c_0 \circledast u_n) \\ &= \lim_{n\to\infty} \left(\frac{c_0^2 \|\nabla u_n\|_2^2}{2} + \frac{c_0^{2s}[u_n]^2}{2} - \frac{\mu c_0^{p\gamma_p} \|u_n\|_p^p}{p} - \frac{c_0^{22^*_\alpha} A(u_n)}{22^*_\alpha} \right) \\ &\geq \frac{c_0^2 \|\nabla u_0\|_2^2}{2} + \frac{c_0^{2s}[u_0]^2}{2} - \frac{\mu c_0^{p\gamma_p} \|u_0\|_p^p}{p} - \frac{c_0^{22^*_\alpha} A(u_0)}{22^*_\alpha} \\ &= E(c_0 \circledast u_0) \geq m^-_{r_0}. \end{aligned} \tag{4.16}$$

Also, since $0 < r_0 \leq \tau$, for any $u \in \mathcal{M}^-_{r_0}$, by Lemma 4.2, we can find $v \in \mathcal{M}^-_\tau$ such that $E(u) > E(v) \geq \inf_{u \in \mathcal{M}^-_\tau} E(u)$ and hence $m^-_{r_0} \geq m^-_\tau$. Therefore, $m^-_\tau = m^-_{r_0}$. Now, we claim that $r_0 = \tau$, and hence $u^-_\tau = c_0 \circledast u_0$ is the required solution to (1.1) corresponding to some λ^-_τ with $\lambda^-_\tau < 0$ for sufficiently large $\mu > 0$ as done in Case 1. Suppose that $0 < r_0 < \tau < \min \tau_0, \tau_1$. Then, by Lemma 4.2, there exists $\bar{v} \in \mathcal{M}^-_\tau$ such that $E(c_0 \circledast u_0) > E(\bar{v})$. Then, by (4.16), we have

$$m^-_{r_0} = E(c_0 \circledast u_0) > E(\bar{v}) \geq m^-_\tau,$$

but since $m^-_{r_0} = m^-_\tau$, we get contradiction, thus $r_0 = \tau$.

Subcase 2. $A(v_n) \geq \bar{C} > 0$ for large $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, define

$$s_n := \left(\frac{\|\nabla v_n\|_2^2}{A(v_n)} \right)^{\frac{1}{2(2^*_\alpha - 1)}},$$

clearly, by boundedness of $\{\frac{1}{A(v_n)}\}$ and $\{u_n\}$ in $H^1(\mathbb{R}^N)$, $\{s_n\}$ is a bounded sequence in $\mathbb{R}$. Now, since $u_0 \in S(r_0)$, by Lemma 2.3, there exists $c_0 > 0$ such that $c_0 \circledast u_0 \in \mathcal{M}^-_{r_0}$. We claim that $s_n \geq c_0$, up to a subsequence.

Suppose $s_n < c_0$ for all $n \in \mathbb{N}$. by Lemma 2.3, the Brezis–Lieb lemma, and [35, Lemma 2.4], we get

$$\begin{aligned} m^-_\tau &= \lim_{n\to\infty} E(u_n) \geq \lim_{n\to\infty} E(s_n \circledast u_n) \\ &= \lim_{n\to\infty} \left(E(s_n \circledast u_0) + E(s_n \circledast v_n) \right) \\ &\geq \lim_{n\to\infty} \left(E(s_n \circledast u_0) + E_0(s_n \circledast v_n) \right) \\ &\geq m^+_{r_0} + \lim_{n\to\infty} E_0(s_n \circledast v_n). \end{aligned} \tag{4.17}$$

Now, by (1.12),

$$E_0(s_n \circledast v_n) = \left(\frac{2_\alpha^* - 1}{22_\alpha^*}\right)\left(\frac{\|\nabla v_n\|_2^2}{A(v_n)^{\frac{1}{2_\alpha^*}}}\right)^{\frac{2_\alpha^*}{2_\alpha^*-1}} \geq \left(\frac{2_\alpha^* - 1}{22_\alpha^*}\right) S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}},$$

thus, by Lemma 4.2,

$$m_\tau^- \geq m_{r_0}^+ + \left(\frac{2_\alpha^* - 1}{22_\alpha^*}\right) S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}} \geq m_\tau^+ + \left(\frac{2_\alpha^* - 1}{22_\alpha^*}\right) S_\alpha^{\frac{2_\alpha^*}{2_\alpha^*-1}}.$$

But this contradicts Lemma 4.1. Thus, there exists a subsequence (denoted as $\{s_n\}$), such that $s_n \geq c_0$ for all $n \in \mathbb{N}$. Now, proceeding again as in (4.17),

$$\begin{aligned} m_\tau^- &= \lim_{n\to\infty} E(u_n) \geq \lim_{n\to\infty} E(c_0 \circledast u_n) \\ &\geq \lim_{n\to\infty} \left(E(c_0 \circledast u_0) + E_0(c_0 \circledast v_n)\right) \geq E(c_0 \circledast u_0), \end{aligned}$$

because $c_0 \leq s_n$, which implies that

$$\frac{c_0^{22_\alpha^*} A(v_n)}{\|\nabla v_n\|_2^2} \leq c_0^2,$$

and hence

$$E_0(c_0 \circledast v_n) \geq \left(\frac{2_\alpha^* - 1}{22_\alpha^*}\right) c_0^{22_\alpha^*} A(v_n) \geq 0.$$

Therefore, $E(c_0 \circledast u_0) \leq m_\tau^-$. Also, since $c_0 \circledast u_0 \in \mathcal{M}_{r_0}^-$, by Lemma 4.2, we have

$$m_\tau^- \geq E(c_0 \circledast u_0) \geq m_{r_0}^- \geq m_\tau^-.$$

Hence, $E(c_0 \circledast u_0) = m_\tau^-$. Thus, taking $u_\tau^- = c_0 \circledast u_0$, we get the required result. ☐

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