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MINIMUM k-CRITICAL-BIPARTITE GRAPHS: THE IRREGULAR CASE

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Abstract. We study the problem of finding a minimum k-critical-bipartite graph of order (n,m): a bipartite graph $G=(U,V;E)$, with $|U|=n$, $|V|=m$, and $n>m>1$, which is k-critical-bipartite, and the tuple $(|E|,\Delta_U,\Delta_V)$, where Δ_U and Δ_V denote the maximum degree in U and V, respectively, is lexicographically minimum over all such graphs. G is k-critical-bipartite if deleting any set of at most $k=n-m$ vertices from U yields G' that has a complete matching, i.e., a matching of size m. Cichacz and Suchan solved the problem for biregular bipartite graphs. Here, we extend their results to bipartite graphs that are not biregular. We prove tight lower bounds on the connectivity of k-critical-bipartite graphs, and we show that k-critical-bipartite graphs are expander graphs.

Keywords: fault-tolerance, interconnection network, bipartite graph, complete matching, algorithm, k-critical-bipartite graph.

Mathematics Subject Classification: 05C35, 05C70, 05C85, 68M10, 68M15.

1. INTRODUCTION

An important body of knowledge has been developed on networks prone to faults. When representing the network as a simple undirected graph $G=(V,E)$[1], the faults can be modeled as vertex or edge deletions, depending if the faults occur to the nodes or links of the network, respectively. In this work, we focus on node faults.

Many applications in diverse fields consider the robustness of assignments. For example, consider a network of m sensing nodes and n relay nodes with $n \geq m$. The sensing nodes need to transmit their readings through the relay nodes, based on a one-to-one assignment (due to some technological considerations), using a pre-established infrastructure of links. It is natural to ask whether we can design a network such that all the sensing nodes can do their work as long as no more than $k=n-m$ relay nodes are faulty. This kind of network is called a k-critical-bipartite graph and is part of the wider field of research related to fault-tolerant networks.

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[1] For standard terms and notations in graph theory, the reader is referred to the textbook by Diestel [7]

Besides applications in the design of fault-tolerant networks, k-critical-bipartite graphs could find applications in the design of supercomputer architectures [9], flexible processes design [17], personnel rostering [1], and other areas of operations research [16]. Section 2 of [4] presents a brief non-exhaustive overview of connections to other areas of research.

1.1. FAULT-TOLERANT GRAPHS

Given a graph H and a positive integer k, a graph G is called *k-fault-tolerant with respect to H*, denoted by k-FT(H), if $G - S$ contains a subgraph isomorphic to H for every $S \subset V(G)$ with $|S| \leq k$. Clearly, under this definition, it is enough to check that the property holds for any $S \subset V(G)$ with $|S| = k$.

Fault-tolerance was introduced by Hayes [10] in 1976 as a graph theoretic model of computer or communication networks working correctly in the presence of faults. Therein, the main motivation for the problem of constructing k-fault-tolerant graphs lies in finding fault-tolerant network architectures. A graph H represents the desired interconnection network and a k-FT(H) graph G allows one to emulate the graph H even in the presence of k vertex (processor) faults.

The problem has been systematically studied in different settings (see, for example, [4]) and one of them is related to the concept of k-factor-critical graphs.

Let G be a graph of order n with a perfect matching M, and let k, $n/2 > k \geq 0$, be an integer. A graph G is called *k-factor-critical* (also called simply *k-critical*) if, after deleting any k vertices, the remaining subgraph has a perfect matching. This concept was first introduced and studied for $k = 2$ by Lovász [14] under the name of a *bicritical graph*. For $k > 2$ it was introduced by Yu in 1993 [19] and independently by Favaron in 1996 [8].

It is straightforward that a bipartite graph cannot be k-critical. Li and Nie amended the definition of a k-critical graph with respect to bipartite graphs [12]. It requires that the k vertices to be deleted lie in the color class with more vertices. Formally, a bipartite graph $G = (U, V; E)$ such that $k = |U| - |V| \geq 0$ is a *k-critical-bipartite graph* if, after deleting any k vertices from the set U, the remaining subgraph has a perfect matching – and this is the definition that we are using.

The problem of designing k-critical graphs (for the class of general graphs) with the minimum number of edges was studied by Zhang *et al.* in [20]. Using the notation k-FT(pK_c), with positive integers k, p, and c, for a graph in which the removal of k vertices leaves a subgraph that contains p disjoint copies of K_c, the authors gave a construction for k-FT(pK_2) graphs of minimum size for any generally feasible values of p and k. This result was extended to higher values of c by Cichacz *et al.* [5], who characterized minimum k-FT(pK_c) graphs for $k = 1$, any positive integer p, and $c > 3$.

1.2. K-CRITICAL-BIPARTITE GRAPHS

It is well known that the bipartite graphs are exactly the graphs that are 2-colorable. Throughout the paper, we will use the notation $G = (U, V; E)$ for a bipartite graph G with color classes U and V. Let $|U| = n$ and $|V| = m$. We say that G is of order (n, m).

We say that G is *biregular* if the degrees of the vertices in both color classes are constant, and *irregular* otherwise. Let $\delta_U(G)$, $\Delta_U(G)$, $\delta_V(G)$, $\Delta_V(G)$, denote the minimum and maximum degree in G of a vertex in U and V, respectively. Where it does not lead to confusion, we do not mention the graph explicitly, for example, stating just δ_U instead of $\delta_U(G)$. If $\delta_U = \Delta_U = a$ and $\delta_V = \Delta_V = b$, then we say that G is (a, b)-regular. A complete graph of order n is denoted K_n and a complete bipartite graph of order (n, m) is denoted $K_{n,m}$.

A k-critical-bipartite graph $G = (U, V; E)$, with $|U| = n$ and $|V| = m$, such that $k = n - m \geq 0$ can be seen as a k-FT(H) graph where H is a matching of size $|V|$ and the k faults can occur only in U. Cichacz and Suchan [4] introduced the problem of finding a minimum k-critical-bipartite graph according to the following definition.

Definition 1.1 ([4])**.** A bipartite graph $G = (U, V; E)$, with $|U| = n$, $|V| = m$, and $n > m > 1$, is a Minimum k-Critical-Bipartite Graph of order (n, m), MkCBG-(n, m), if it is k-critical-bipartite, and the tuple $(|E|, \Delta_U, \Delta_V)$ is lexicographically minimum over all such graphs.

Note that, given integer n and m with $n > m > 1$, the graph $G^*(U, V; E)$ with $|U| = n$ and $|V| = m$, obtained by taking a matching of size m and adding to U another $k = n - m$ vertices adjacent to every vertex in V gives a graph that is minimal k-critical-bipartite, i.e., removing any edge from G^* yields a graph that is not k-critical-bipartite, but is not minimum according to the definition given above. Indeed, $\Delta_U(G^*) = m$, whereas we show in this paper that there exist k-critical-bipartite graphs $G = (U, V; E)$ that also have $|E(G)| = m(n - m + 1)$, but with $\Delta_U(G) = \lceil \frac{m(n-m+1)}{n} \rceil$. So a minimum k-critical-bipartite graph of order (n, m) cannot be obtained by simply taking any k-critical-bipartite graph and removing edges, one by one, as long as the property is preserved.

Cichacz and Suchan [4] solved the problem of finding MkCBG-(n, m) in the case of biregular graphs, leaving open the case of irregular bipartite graphs. We solve it in this paper.

1.3. EXPANDER GRAPHS

An expander is a sparse graph that has strong connectivity properties [13]. In general graphs, roughly speaking, we say that $G = (V, E)$ is an (α, β)-*expander* for positive parameters α and β if, for any $S \subset V$ such that $|S| \leq \alpha|V|$, there is $|N(S)| \geq \beta|S|$. In bipartite graphs, denoted by $G = (U, V; E)$, the *expansion property* has been introduced: that every (or almost every) "small" subset of V is directly connected to a "large" enough subset of U. A family of expander graphs was defined in the 1970's by Bassalygo and Pinsker [3] and has been used in many areas of mathematics and computer science. One of the main advantages of expanders is that they enable fast and effective dissemination of information from a small group of vertices to the outside world [2]. Among others, they are used in the design and analysis of communication networks, theory of error correcting codes or pseudorandom theory.

Several precise definitions of an expander have been proposed, with natural connections between them. We focus on one created for bipartite graphs.

Definition 1.2 ([15])**.** A bipartite graph $G = (U, V; E)$, with $|U| = n$, $|V| = m$ is an (α, ϵ)-*extended-expander* ((α, ϵ)-EE) for $\alpha \in [1/m, 1]$ and $\epsilon \in (0, 1)$ if, for all subsets $S \subset V$ such that $|S| \leq \alpha m$, we have

$$\frac{|N(S)|}{n} \geq \frac{1-\epsilon}{\alpha} \cdot \frac{|S|}{m}.$$

We show that any k-critical-bipartite graph is an (α, ϵ)-EE for some α and ϵ.

1.4. STRUCTURE OF THE PAPER

The structure of this paper is as follows. In Section 2, we present what was previously known about k-critical-bipartite graphs, show that it implies that they are expander graphs, and detail the problem of designing a minimum k-critical-bipartite graph. In Section 3, we give a construction that yields a minimum k-critical-bipartite graph of order (n, m) for any values of n and m such that $n > m > 1$, with $k = n - m$. We show that a k-critical-bipartite graph $G = (U, V; E)$ of order (n, m) is minimum if $|E| = m(n - m + 1)$, $\Delta_U = \lceil \frac{m(n-m+1)}{n} \rceil$, and $\Delta_V = n - m + 1$. In Section 4, we give a construction that yields graphs $G = (U, V; E)$ of order (n, m) that also have $|E| = m(n-m+1)$, $\Delta_U = \lceil \frac{m(n-m+1)}{n} \rceil$, and $\Delta_V = n-m+1$, but are not k-critical-bipartite - so these properties are not sufficient for a graph to be k-critical-bipartite. In Section 5 we present tight lower bounds for the connectivity of k-critical-bipartite graphs. And we conclude with some final remarks in Section 6.

2. MAIN PROBLEM

Let $G = (U, V; E)$ be a bipartite graph, with $|U| = n$ and $|V| = m$, such that $k = n - m > 0$. Li and Nie [12] described the connectivity of k-critical-bipartite graphs in the following theorem.

Theorem 2.1 ([12])**.** *Let $G = (U, V; E)$ be a bipartite graph such that $k = |U| - |V| > 0$. If G is k-critical-bipartite, then G is connected.*

On the other hand, Laroche *et al.* [11] gave a Hall-style characterization of k-critical-bipartite graphs as follows:

Theorem 2.2 ([11])**.** *Let $G = (U, V; E)$ be a bipartite graph such that $k = |U| - |V| > 0$. The graph G is k-critical-bipartite if and only if $|N(V')| \geq |V'| + k$ for all $\emptyset \neq V' \subseteq V$.*

Note that a k-critical-bipartite graph needs to have at least $(k + 1)m$ edges. Indeed, suppose that the total number of edges is smaller. Then at least one vertex v in V is connected to at most k distinct vertices in U, and there is a fault scenario where precisely the vertices in the neighborhood of v are removed, in which case v cannot be matched. A contradiction.

Hence, we have the following result:

Corollary 2.3. *Let n, m be positive integers such that $1 < m < n$. Let $k = n - m$. Any k-critical-bipartite graph $G = (U, V; E)$ with $|U| = n$ and $|V| = m$ is an (α, ϵ)-extended-expander for any $\alpha \in (\frac{1}{m}, 1)$ and any $\epsilon \in (1 - \alpha, 1)$.*

Proof. Let $G = (U, V; E)$ be a k-critical-bipartite graph of order (n, m), thus for any nonempty set $S \subset V$ there is $|N(S)| \geq |S| + k$ by Theorem 2.2. Moreover, $m(|S| + k) \geq n|S|$ since $|S| \leq m$. Therefore, $\frac{|N(S)|}{n} \geq \frac{|S|}{m}$ and thus, for any $\alpha \in (\frac{1}{m}, 1)$ and $\epsilon \in (1 - \alpha, 1)$

$$\frac{|N(S)|}{n} \geq \frac{1-\epsilon}{\alpha} \cdot \frac{|S|}{m}. \qquad \square$$

As in [4] for k-critical-bipartite biregular graphs, we want to study topologies where not only the total number of links is low, but also the maximum number of links per node is small (in both color classes). Thus, for given positive integer values n, m such that $n > m > 1$ and $k = n - m$, we want to find a bipartite graph $G = (U, V; E)$ of order (n, m) that is a k-critical-bipartite graph and is lexicographically minimum with respect to $(|E|, \Delta_U, \Delta_V)$ (see Definition 1.1).

The construction below is a generalization of the construction from [4]. Indeed, the construction was used only for integers m, n such that $n > m > 1$ and $a = \frac{(k+1)m}{n}$ is an integer. In the construction and throughout the paper we use the following notation: $[o] = \{0, 1, \ldots, o - 1\}$ for any positive integer o.

Construction 2.4. Let n, m, a be positive integers such that $n > m > 1$. Let $\widehat{G}^a_{n,m} = (U, V; E)$ be a bipartite graph with

$$U = \{u_i \mid i \in [n]\}, \quad V = \{v_j \mid j \in [m]\},$$

and

$$E = \left\{ (u_i, v_{(j+\alpha) \bmod m}) \mid i \in [n], \alpha \in [a], j = \left\lceil \frac{im}{n} \right\rceil \right\}.$$

Cichacz and Suchan [4] proved that, if n, m and a are positive integers such that $n > m > 1$, $k = n - m$, and $an = (k + 1)m$, then the graph $\widehat{G}^a_{n,m} = (U, V; E)$ is an $(a, k + 1)$-regular k-critical-bipartite graph of size $(k + 1)m$ that is MkCBG-(n, m). Moreover, they stated the following conjecture for irregular k-critical-bipartite graphs.

Conjecture 2.5 ([4])**.** Let n, m be positive integers such that $n > m > 1$, Let $k = n - m$ and $a = \frac{m(k+1)}{n}$ is not an integer. Then $\widehat{G}^{\lceil a \rceil}_{n,m}$ obtained by Construction 2.4 is k-critical-bipartite.

In this paper, we prove that the conjecture is true. Moreover, for any pair n, m of positive integers such that $n > m > 1$, $k = n - m$, we construct a bipartite graph $G = (U, V; E)$ of order (n, m) that is k-critical-bipartite and is lexicographically minimum among all such graphs with respect to $(|E|, \Delta_U, \Delta_V)$. In other words, we solve the problem of finding a Minimum k-Critical Bipartite Graph of order (n, m) (MkCBG-(n, m)) completely.

3. POSITIVE CONSTRUCTION

In this section, we give a construction that yields Minimum k-Critical Bipartite Graphs of order (n, m).

Let us start by recalling the following lemma that was proved in [4].

Lemma 3.1 ([4, Lemma 3.4])**.** *Let x, y, c be positive integers such that $x > y$. Let $n = cx$, $m = cy$, and $j \in [m]$. Then the number of integer solutions to $\lceil \frac{im}{n} \rceil \equiv j \pmod{m}$ with respect to i, with $i \in [n]$, is equal to $\lfloor j\frac{x}{y} \rfloor - \lfloor (j-1)\frac{x}{y} \rfloor$. Moreover:*

(i) $\lfloor j\frac{x}{y} \rfloor - \lfloor (j-1)\frac{x}{y} \rfloor = \lfloor r\frac{x}{y} \rfloor - \lfloor (r-1)\frac{x}{y} \rfloor$, *where* $r = j \bmod y$.

(ii) *For any interval of consecutive y values of j, for $(x \bmod y)$ of them, there are $\lceil x/y \rceil$ solutions and, for the remaining $(y - x \bmod y)$, there are $\lfloor x/y \rfloor$ solutions.*

(iii) *In general, the number of solutions is $\lceil \frac{x}{y} \rceil$ for $(n \bmod m)$, and $\lfloor \frac{x}{y} \rfloor$ for $(m - n \bmod m)$ values of $j \in [m]$.*

With Lemma 3.1, we can prove the following lemma that allows us to further analyze the neighborhoods of vertices in graphs like the ones from Construction 2.4.

Lemma 3.2. *Let n, m be positive integers such that $n > m > 1$. Let $j \in [m]$. Then*

$$\max\left\{ i \mid i \in [n], \left\lceil \frac{im}{n} \right\rceil \equiv j \pmod{m}, \left\lceil \frac{(i+1)m}{n} \right\rceil \not\equiv j \pmod{m} \right\} = \left\lfloor \frac{jn}{m} \right\rfloor.$$

Proof. Let us write

$$i_j = \max\left\{ i \mid i \in [n], \left\lceil \frac{im}{n} \right\rceil \equiv j \pmod{m}, \left\lceil \frac{(i+1)m}{n} \right\rceil \not\equiv j \pmod{m} \right\}.$$

We have $i_0 = 0 = \lfloor \frac{0 \cdot n}{m} \rfloor$. For $j > 0$, it is easy to check that i_j is equal to i_{j-1} plus the number of integer solutions to $\lceil \frac{im}{n} \rceil \equiv j \pmod{m}$. By Lemma 3.1, the number of integer solutions to $\lceil \frac{im}{n} \rceil \equiv j \pmod{m}$, with respect to i, with $i \in [n]$, is equal to $\lfloor j\frac{n}{m} \rfloor - \lfloor (j-1)\frac{n}{m} \rfloor$. Therefore, we have $i_j = i_{j-1} + \lfloor j\frac{n}{m} \rfloor - \lfloor (j-1)\frac{n}{m} \rfloor$. By the telescopic property, we have $i_j = \sum_{i=1}^{j} \left(\lfloor i\frac{n}{m} \rfloor - \lfloor (i-1)\frac{n}{m} \rfloor \right) = \lfloor j\frac{n}{m} \rfloor$ for $j > 0$. ☐

It is easy to check that the edge set of the graph from Construction 2.4, in the case where $an = (k+1)m$, can also be written as:

$$E = \Bigg\{ (u_{(i-\beta) \bmod n}, v_j) \mid j \in [m], \beta \in [k+1], \\ i = \max\left\{ i \mid i \in [n], \left\lceil \frac{im}{n} \right\rceil \equiv j \pmod{m}, \left\lceil \frac{(i+1)m}{n} \right\rceil \not\equiv j \pmod{m} \right\} \Bigg\}.$$

So, by Lemma 3.2, when $a = \frac{m(k+1)}{n}$ is an integer, the edge set of the graph from Construction 2.4 can be described in the following way:

$$E = \left\{ (u_{(i-\beta) \bmod n}, v_j) \mid j \in [m], \beta \in [k+1], i = \left\lfloor \frac{jn}{m} \right\rfloor \right\}.$$

Let us generalize this construction to the case where $a = \frac{m(k+1)}{n}$ is not an integer.

Construction 3.3. Let n, m be positive integers such that $n > m > 1$. Let $k = n - m$. Let $\overline{G}_{n,m} = (U, V; E)$ be a bipartite graph with

$$U = \{u_i \mid i \in [n]\}, \quad V = \{v_j \mid j \in [m]\},$$

and

$$E = \left\{ (u_{(i-\beta) \bmod n}, v_j) \mid j \in [m], \beta \in [k+1], i = \left\lfloor \frac{jn}{m} \right\rfloor \right\}.$$

So the following observation holds.

Observation 3.4. *If* $a = \frac{m(k+1)}{n}$ *is an integer then* $\overline{G}_{n,m} = \widehat{G}^{a}_{n,m}$.

See Figure 1 for a small example of graphs $\widehat{G}^{\lceil a \rceil}_{n,m}$ and $\overline{G}_{n,m}$ in which $a = \frac{m(k+1)}{n}$ is not an integer.

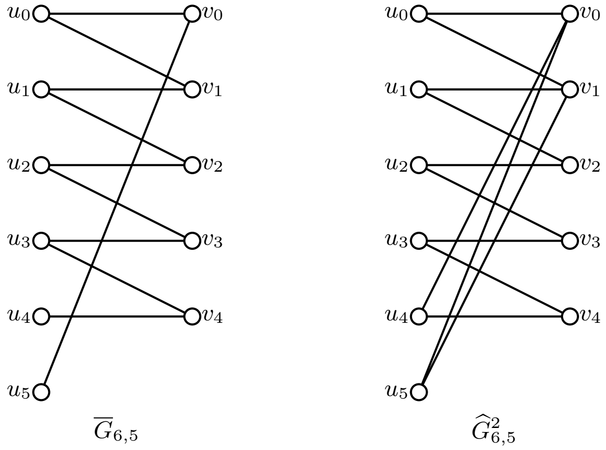

Fig. 1. The graphs $\overline{G}_{6,5}$ and $\widehat{G}^{2}_{6,5}$

Let us present a lemma that relates the graphs obtained by Constructions 2.4 and 3.3 in general.

Lemma 3.5. *Let* n, m *be positive integers such that* $n > m > 1$. *Let* $k = n - m$ *and* $a = \frac{m(k+1)}{n}$. *Then the graph* $\overline{G}_{n,m} = (U, V; \overline{E})$ *obtained by Construction 3.3 is a subgraph of the graph* $\widehat{G}^{\lceil a \rceil}_{n,m} = (U, V; \widehat{E})$ *given in Construction 2.4.*

Proof. Let us consider the graph $\overline{G}_{n,m} = (U, V; \overline{E})$ obtained by Construction 3.3. Let $e_0 \in \overline{E}$. Then there is $j_0 \in [m]$ and $\beta_0 \in [k+1]$ such that $e_0 = (u_{(i_0 - \beta_0) \bmod n}, v_{j_0})$ for $i_0 = \lfloor \frac{j_0 n}{m} \rfloor$. Let us show that e_0 also belongs to $\widehat{E}$.

By Construction 2.4, there is

$$\{(u_{(i_0-\beta_0) \bmod n}, v_{(j_{(i_0-\beta_0) \bmod n} + \alpha) \bmod m}), \alpha \in [\lceil a \rceil]\} \subset \widehat{E},$$

where $j_{(i_0 - \beta_0)} = \lceil \frac{(i_0 - \beta_0)m}{n} \rceil$.

Let us choose $\gamma \in [m]$ such that

$$\frac{(i_0-\beta_0)m}{n} = \frac{(\lfloor \frac{j_0 n}{m} \rfloor - \beta_0)m}{n} = \frac{(\frac{j_0 n-\gamma}{m} - \beta_0)m}{n} = j_0 - \frac{\gamma+\beta_0 m}{n}.$$

So, there is

$$j_{(i_0-\beta_0)} = \left\lceil j_0 - \frac{\gamma+\beta_0 m}{n} \right\rceil = j_0 - \left\lfloor \frac{\gamma+\beta_0 m}{n} \right\rfloor.$$

Note that

$$\left\lfloor \frac{\gamma+\beta_0 m}{n} \right\rfloor \leq \left\lfloor \frac{m-1+km}{n} \right\rfloor = \left\lfloor a - \frac{1}{n} \right\rfloor = \begin{cases} a-1 & \text{if } a \in \mathbb{N}, \\ b \in \{\lceil a \rceil - 2, \lceil a \rceil - 1\} & \text{otherwise.} \end{cases}$$

So there exists $\alpha_0 \in [\lceil a \rceil]$ such that $j_0 = j_{(i_0-\beta_0)} + \alpha_0$, and hence

$$e_0 = (u_{(i_0-\beta_0) \bmod n}, v_{j_0}) \in \{(u_{(i_0-\beta_0) \bmod n}, v_{(j_{(i_0-\beta_0) \bmod n}+\alpha) \bmod m}), \alpha \in [\lceil a \rceil]\} \subset \widehat{E}.$$

□

Theorem 3.6. *Let n, m be positive integers such that $n > m > 1$. Let $k = n - m$. Then $\overline{G}_{n,m} = (U, V; E)$ obtained by Construction 3.3 is a minimum k-critical-bipartite graph of order (n, m).*

Proof. Let us show that, for any $S \neq \emptyset$, $S \subset V$, there is $|N_{\overline{G}_{n,m}}(S)| \geq |S| + k$, which, by Theorem 2.2, implies that $\overline{G}_{n,m}$ is k-critical. The proof is by induction on $|S|$.

Let $|S| = 1$. Let v_j, $j \in [m]$ be the vertex in S. By definition,

$$N_{\overline{G}_{n,m}}(v_j) = \left\{ (u_{(i-\beta) \bmod n}, v_j) \mid i = \left\lfloor \frac{jn}{m} \right\rfloor, \beta \in [k+1] \right\},$$

so the conclusion is true.

Given an integer p, $1 \leq p \leq m-1$, suppose that $|N_{\overline{G}_{n,m}}(S')| \geq p + k$ holds for any S', $S' \subset V$, such that $|S'| = p$.

Take any S, $S \subset V$, with $|S| = p+1$. Suppose first that there exists $v \in S$ such that $N_{\overline{G}_{n,m}}(v) \setminus N_{\overline{G}_{n,m}}(S \setminus \{v\}) \neq \emptyset$ and hence

$$|N_{\overline{G}_{n,m}}(v) \cap N_{\overline{G}_{n,m}}(S \setminus \{v\})| < \deg_{\overline{G}_{n,m}}(v).$$

Let $S' = S \setminus \{v\}$. Then $|S'| = p$ and, by the induction hypothesis, $|N_{\overline{G}_{n,m}}(S')| \geq p + k$. Hence,

$$|N_{\overline{G}_{n,m}}(S)| \geq p + k + |N_{\overline{G}_{n,m}}(v) \setminus N_{\overline{G}_{n,m}}(S')| \geq p + k + 1 = |S| + k.$$

Assume now that $N_{\overline{G}_{n,m}}(v) \subset N_{\overline{G}_{n,m}}(S \setminus \{v\})$ for every $v \in S$. Let us show that it implies that $N_{\overline{G}_{n,m}}(S) = U$, and so $|N_{\overline{G}_{n,m}}(S)| = |V| + k \geq |S| + k$.

Suppose, to the contrary, that

$$I = \{i \in [n] \colon u_i \notin N_{\overline{G}_{n,m}}(S)\} \neq \emptyset.$$

Let

$$i'_0 = \max\{i\colon i \in I, (i+1) \bmod n \notin I\}.$$

Then there exists $v_r \in S$ such that $u_{(i'_0+1) \bmod n} \in N_{\overline{G}_{n,m}}(v_r)$. Since $N_{\overline{G}_{n,m}}(v_r) \subset N_{\overline{G}_{n,m}}(S \setminus \{v_r\})$, there exists $v_l \in S$, $r \neq l$, such that $u_{(i'_0+1) \bmod n} \in N_{\overline{G}_{n,m}}(v_l)$. By Construction 3.3,

$$N_{\overline{G}_{n,m}}(v_j) = \{u_{(i_j-k) \bmod n}, u_{(i_j-(k-1)) \bmod n}, \ldots, u_{i_j}\}$$

for every $j \in [m]$. It is easy to check that, for any $j_1, j_2 \in [m]$, $j_1 \neq j_2$ implies $i_{j_1} \neq i_{j_2}$. So there is $\left|N_{\overline{G}_{n,m}}(v_{j_1}) \setminus N_{\overline{G}_{n,m}}(v_{j_2})\right| \geq 1$.

On the other hand, since

$$u_{(i'_0+1) \bmod n} \in N_{\overline{G}_{n,m}}(v_l) \cap N_{\overline{G}_{n,m}}(v_r), u_{i'_0 \bmod n} \notin N_{\overline{G}_{n,m}}(v_l) \cup N_{\overline{G}_{n,m}}(v_r),$$

there is

$$N_{\overline{G}_{n,m}}(v_l) = N_{\overline{G}_{n,m}}(v_r) = \{u_{(i'_0+1) \bmod n}, u_{(i'_0+2) \bmod n}, \ldots, u_{(i'_0+k+1) \bmod n}\},$$

a contradiction.

It is easy to check, by the pigeonhole principle, that there is $\delta_V \geq k+1$ for any k-critical-bipartite graph $G = (U, V; E)$ of order (n, m). By definition, $\overline{G}_{n,m}$ has $(k+1)m$ edges. So the construction is minimum. $\square$

Combining Theorem 3.6 and Lemma 3.5, we obtain that the Conjecture 2.5 is true.

Corollary 3.7. *Let n, m be positive integers such that $n > m > 1$. Let $k = n - m$ and $a = \frac{m(k+1)}{n}$ not be integers. Then $\widehat{G}^{\lceil a \rceil}_{n,m}$ obtained by Construction 2.4 is k-critical-bipartite.*

4. NEGATIVE CONSTRUCTION

In this section we give a construction that yields graphs $G = (U, V; E)$ of order (n, m) that also have $|E| = m(k+1)$, $\Delta_U = \lceil \frac{m(k+1)}{n} \rceil$, and $\Delta_V = k+1$, where $k = n - m$, but are not k-critical-bipartite. So these properties are not sufficient for a graph to be k-critical-bipartite.

Note that Theorem 3.6, together with the simple observation that there is $\delta_V \geq k+1$ in any k-critical-bipartite $G = (U, V; E)$ with $n > m > 1$ and $k = n - m$, implies that there is $\delta_V = \Delta_V = k+1$, $\Delta_U = \lceil \frac{m(k+1)}{n} \rceil$, and $\delta_U \leq k$ if $G = (U, V; E)$ is minimum k-critical-bipartite. Both in Construction 3.3 and Construction 4.5 that follows, there is $\delta_U = \lfloor \frac{m(n-m+1)}{n} \rfloor$. The graphs obtained by the two constructions have the same degree sequences, so even fixing the vertex degrees does not make a graph k-critical-bipartite.

Cichacz and Suchan in [4] gave the following construction of a class of biregular graphs.

Construction 4.1 ([4]). Let n, m be positive integers such that $n > m > 1$, and $a = \frac{(n-m+1)m}{n}$ is an integer. Let $\gcd(n,m) = c$, $n = cx$, and $m = cy$. Let $\check{G}^a_{n,m} = (U, V; E)$ be the bipartite graph with

$$U = \{u_i \mid i \in [n]\}, \quad V = \{v_j \mid j \in [m]\},$$

$$E = \left\{ (u_i, v_{(j+\alpha) \bmod m}) \mid i \in [n], \alpha \in [a], j = \left\lfloor \frac{i}{x} y \right\rfloor \right\}.$$

It is easy to check that the graph $\check{G}^a_{n,m} = (U, V; E)$ can also be constructed in the following way. Let $b = n - m + 1$ and $d = \gcd(a, b)$. Note that there is $a = dy$ and $b = dx$ (see [4]). Let $\check{G}$ be a d-regular bipartite graph having color classes $U' = \{u_i \mid i \in [c]\}$ and $V' = \{v_j \mid j \in [c]\}$, where $c = \gcd(n,m)$, such that $E(\check{G}) = \{u_i v_{(i+\delta) \bmod c},\ i \in [c], \delta \in [d]\}$. We construct the graph $\check{G}^a_{n,m} = (U, V; E)$ by "blowing up" each vertex u_i into $x = n/c$ vertices $u_{i,\alpha}$, $\alpha \in [x]$, and each vertex v_j into $y = m/c > 1$ vertices $v_{j,\beta}$, $\beta \in [y]$. Each edge from $\check{G}$ is substituted by the corresponding complete bipartite graph $K_{x,y}$. Note that $\check{G}^a_{n,m}$ is (a, b)-regular.

The authors showed that, despite having the same degrees as minimum biregular k-critical-bipartite graphs, the graphs obtained by Construction 4.1 tend not to be k-critical-bipartite.

Observation 4.2 ([4]). *Let n, m be positive integers such that $n > m > 1$, $k = n - m$, and $a = \frac{m(k+1)}{n}$ is an integer. The graph $\check{G}^a_{n,m}$ given in Construction 4.1 is biregular k-critical-bipartite if and only if $\gcd(n,m) = m$.*

Cichacz and Suchan [4] complemented Construction 4.1 with another construction for the case where $\gcd(n,m) = m$ and $a = \frac{m(n-m+1)}{n}$ is an integer to get the following result.

Observation 4.3 ([4]). *Let $n = |U|$, $m = |V| \in$ be such positive integers that $1 < m < n$, $k = n - m$ and $a = \frac{m(k+1)}{n}$ is an integer. There exists an $(a, k+1)$-regular bipartite graph $G = (U, V; E)$ that is not k-critical if and only if $a < m - 1$.*

So we know constructions of (a, b)-regular bipartite graphs of order (n, m) that are not k-critical-bipartite, where $b = n - m + 1$ and $a = \frac{m(k+1)}{n}$, whenever such graphs exist. In what follows, we focus on the cases where $a = \frac{m(k+1)}{n}$ is not an integer.

Let us recall the results of Havel–Hakimi on constructing bipartite graphs based on degree sequences that are useful for constructing graphs that are not k-critical-bipartite. Let $P : p_0 \geq p_1 \geq \ldots \geq p_{n-1}$ and $Q : q_0 \geq q_1 \geq \ldots \geq q_{m-1}$ be sequences of non-negative integers. The pair (P, Q) is *bigraphic* if there exists a bipartite graph $G = (U, V; E)$ with $|U| = n$ and $|V| = m$ in which P and Q describe the degrees of the vertices in U and V, respectively. The following theorem is a version of Havel–Hakimi's theorem for bigraphic sequences.

Theorem 4.4 ([18])**.** *The pair (P, Q) is bigraphic if and only if the pair (P', Q') is bigraphic, where (P', Q') is obtained from (P, Q) by deleting the largest element p_0 of P and subtracting one from each of the p_0 largest elements of Q.*

Let x, y, b be positive integers such that $x > y > 1$ and $b \leq x$. Let $l = by - x\lfloor \frac{yb}{x} \rfloor$. Let $p_i = \lceil \frac{yb}{x} \rceil$ for $i \in [l]$ and $p_i = \lfloor \frac{yb}{x} \rfloor$ for $i \in \{l, \ldots, x-1\}$, let $q_j = b$ for $j \in [y]$. Observe that $P = (p_0, p_1, \ldots, p_{x-1})$ and $Q = (q_0, q_1, \ldots, q_{y-1})$ is the degree sequence of the graph obtained by Construction 3.3, therefore it is bigraphic. Moreover, note that (P, Q) is bigraphic if and only if (Q, P) is.

Let us present two constructions based on P and Q defined above: one for (P, Q) and the other for (Q, P). The graphs thus obtained have different properties.

Construction 4.5. Let x, y, b be positive integers such that $x > y > 1$ and $b \leq x$. Let $l = by - x\lfloor \frac{yb}{x} \rfloor$. Let $p_i = \lceil \frac{yb}{x} \rceil$ for $i \in [l]$ and $p_i = \lfloor \frac{yb}{x} \rfloor$ for $i \in \{l, \ldots, x-1\}$. Let $D_0 = 0$ and $D_i = (\sum_{j=0}^{i-1} p_i) \bmod y$ for $i \in \{1, \ldots, x-1\}$. Let $\dot{G}^b_{x,y} = (U, V; E)$ be the bipartite graph with $U = \{u_i \mid i \in [x]\}$, $V = \{v_j \mid j \in [y]\}$ such that for every $i \in [x]$:

$$N_{\dot{G}^b_{x,y}}(u_i) = \{v_{(D_i + \pi) \bmod y} \mid \pi \in [p_i]\}.$$

Lemma 4.6. *Let x, y, b be positive integers such that $x > y > 1$ and $b \leq x$. Then the graph $\dot{G}^b_{x,y} = (U, V; E)$ obtained by Construction 4.5 has size $|E| = by$, $\deg(u) \in \left\{\lfloor \frac{yb}{x} \rfloor, \lceil \frac{yb}{x} \rceil\right\}$ for every $u \in U$, and $\deg(v) = b$ for every $v \in V$.*

Proof. Note that the graph $\dot{G}^b_{x,y}$ is a graph constructed as in Theorem 4.4 for (P, Q) for P and Q defined above, where P and Q describe the degrees of vertices in U and V, respectively. Thus, $\deg(u) \in \left\{\lfloor \frac{yb}{x} \rfloor, \lceil \frac{yb}{x} \rceil\right\}$ for every $u \in U$ and $\deg(v) = b$ for every $v \in V$. ☐

Construction 4.7. Let x, y, b be positive integers such that $b \leq x$. Let $\ddot{G}^b_{x,y} = (U, V; E)$ be the bipartite graph with

$$U = \{u_i \mid i \in [x]\}, \quad V = \{v_j \mid j \in [y]\},$$

such that for each $j \in [y]$:

$$N_{\ddot{G}^b_{x,y}}(v_j) = \{u_{(jb + \beta) \bmod x} \mid \beta \in [b]\}.$$

Lemma 4.8. *Let x, y, b be positive integers such that $b \leq x$. Then $\ddot{G}^b_{x,y} = (U, V; E)$ has size $|E| = by$, $\deg(u) \in \left\{\lfloor \frac{yb}{x} \rfloor, \lceil \frac{yb}{x} \rceil\right\}$ for every $u \in U$ and $\deg(v) = b$ for every $v \in V$.*

Proof. Note that the graph $\ddot{G}^b_{x,y}$ is a graph constructed as in Theorem 4.4 for (Q, P) for Q and P defined above, where Q and P describe the degrees of vertices in V and U, respectively. Thus, $\deg(u) \in \left\{\lfloor \frac{yb}{x} \rfloor, \lceil \frac{yb}{x} \rceil\right\}$ for every $u \in U$ and $\deg(v) = b$ for every $v \in V$. ☐

Note that, in general, $\dot{G}^b_{x,y} \ncong \ddot{G}^b_{x,y}$. For example, $\dot{G}^2_{6,5}$ is connected, whereas $\ddot{G}^2_{6,5}$ is not (see Figure 2). Based on the above constructions we can show the following result.

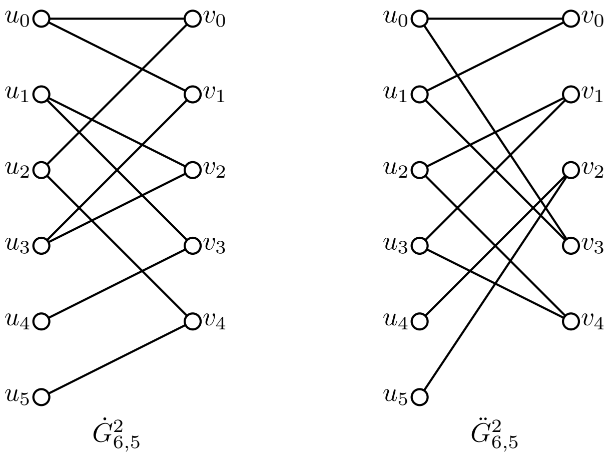

Fig. 2. The graphs $\dot{G}^2_{6,5}$ and $\ddot{G}^2_{6,5}$

Observation 4.9. *Let n, m be positive integers such that $n > m > 1$. Let $k = n - m$. Let c be a positive integer such that $n = cx$ and $m = cy$. If $c > 1$ and $k + 1 \leq x$, or $d = \frac{n}{k+1} > 1$ and d is an integer, then there exists a graph $G = (U, V; E)$ such that $\deg(u) \in \left\{\lfloor\frac{m(k+1)}{n}\rfloor, \lceil\frac{m(k+1)}{n}\rceil\right\}$ for any $u \in U$ and $\deg(v) = k + 1$ for any $v \in V$ which is not k-critical-bipartite.*

Proof. If $c > 1$ and $k+1 \leq x$, define a graph $\dddot{G}^{k+1}_{n,m} = (U, V; E)$ as the disjoint union of c copies of $\dot{G}^{k+1}_{x,y} = (U', V'; E')$. Since $c > 1$, the graph $\dddot{G}^{k+1}_{n,m}$ is disconnected, therefore is not k-critical-bipartite by Theorem 2.1.

Suppose now that $d = \frac{n}{k+1} > 1$ and d is an integer. Then the graph $\ddot{G}^{k+1}_{n,m}$ is disconnected. Indeed, since

$$N_{\ddot{G}^{k+1}_{n,m}}(v_j) = \{u_{(j(k+1)+\beta) \bmod n} \mid \beta \in [k+1]\},$$

every vertex in V has one of $d = \frac{n}{k+1} > 1$ disjoint neighborhoods in U. So $\ddot{G}^{k+1}_{n,m}$ has d connected components. By Theorem 2.1, it means that it is not k-critical-bipartite. □

Note that Observation 4.9 does not cover all cases of n and m. The cases that are left open have that $\frac{n}{n-m+1}$ is not an integer, and $\gcd(n, m) = 1$ or $n - m + 1 > \frac{n}{c}$ for any c non-trivial common divisor of n and m.

5. CONNECTIVITY

A graph $G = (V, E)$ is said to be k-*connected* if it has more than k vertices and remains connected whenever strictly fewer than k vertices are removed. The *connectivity* of G, denoted $\kappa(G)$, is the maximum k such that G is k-connected.

Given a graph G and two vertices u and v that belong to the same component of G, a *vertex cut* in G separating u and v is a set S of vertices of G whose removal leaves u and v in different components of $G - S$. The *local connectivity* $\kappa_{u,v}(G)$ of u and v in G is the size of a smallest vertex cut separating u and v. Given a graph G, $\kappa(G)$ equals the minimum $\kappa_{u,v}(G)$ over all nonadjacent pairs of vertices u, v (except for complete graphs).

For a set $S \subset V(G)$, the *set connectivity* of S, denoted by $\kappa_S(G)$, is the size of a smallest vertex cut separating any $u, v \in S$.

Favaron [8] showed that every k-critical graph G of order $n > k$ is k-connected and this result is sharp. On the other hand, Li and Nie [12] only showed that every k-critical-bipartite graph G is 1-connected. We improve this result to show that for any k-critical-bipartite graph $G = (U, V; E)$ with $|U| = n$, $|V| = m$, and $k = n-m > 0$, there is:

1. $\kappa_V(G) \geq k$,
2. $\kappa_U(G) \geq \min\{\delta_U(G), k\}$,
3. $\kappa(G) \geq \min\{\delta(G), k\}$.

Theorem 5.1. *Let* n, m *be positive integers such that* $1 < m < n$. *Let* $k = n-m$. *Then* $\kappa_V(G) \geq k$ *for any* k*-critical-bipartite graph* $G = (U, V; E)$ *with* $|U| = n$ *and* $|V| = m$.

Proof. Let $G = (U, V; E)$ be a k-critical-bipartite graph of order (n, m). Towards a contradiction, suppose that $\kappa_V(G) < k$. So there exists a set $Z \subset U \cup V$ with $|Z| < k$ that separates two vertices v_1, v_2 in V. Let $Z_1 = Z \cap U$ and $Z_2 = Z \cap V$. Note that there is no path between v_1 and v_2 in $G' = (U', V'; E') = G[(U \setminus Z_1) \cup (V \setminus Z_2)]$. So we can choose a partition of G' into two subgraphs $G'_1 = (U'_1, V'_1; E'_1)$ and $G'_2 = (U'_2, V'_2; E'_2)$ that are unions of components of G' with $v_1 \in V'_1$ and $v_2 \in V'_2$.

Let $|U'_i| = |V'_i| + \varepsilon_i$ for $i = 1, 2$. So there is $|V'_1| + |V'_2| + |Z_2| + k = |U'_1| + |U'_2| + |Z_1| = |V'_1| + |V'_2| + |Z_1| + \varepsilon_1 + \varepsilon_2$. Thus, $|Z_2| + k = |Z_1| + \varepsilon_1 + \varepsilon_2$.

By Theorem 2.2, there is $|N_G(V'_1)| \geq |V'_1| + k$. On the other hand, since $U'_1 \cup Z_1 \supseteq N_G(V'_1)$, there is $|U'_1 \cup Z_1| \geq |N_G(V'_1)|$. So, by simplifying $|V'_1| + \varepsilon_1 + |Z_1| = |U'_1| + |Z_1| \geq |V'_1| + k$, we get $\varepsilon_1 + |Z_1| \geq k$. Similarly, we get that $\varepsilon_2 + |Z_1| \geq k$.

On one hand, since $|Z_2| + k = |Z_1| + \varepsilon_1 + \varepsilon_2 \geq k + \varepsilon_2$, we obtain that $|Z_2| \geq \varepsilon_2$. On the other hand, there is $|Z_1| + |Z_2| < k \leq |Z_1| + \varepsilon_2$, so $|Z_2| < \varepsilon_2$, a contradiction. □

Theorem 5.2. *Let* n, m *be positive integers such that* $1 < m < n$. *Let* $k = n - m$. *Then* $\kappa_U(G) \geq \min\{\delta_U(G), k\}$ *for any* k*-critical-bipartite graph* $G = (U, V; E)$ *with* $|U| = n$ *and* $|V| = m$. *Moreover, for every separator* Z *in* G *with* $|Z| < k$, *there exists a vertex* $u \in U$ *with* $N(u) \subseteq Z$.

Proof. Let $G = (U, V; E)$ be a k-critical-bipartite graph of order (n, m). Towards a contradiction, suppose that $\kappa_U(G) < \min\{\delta_U(G), k\}$. So there exists a set $Z \subset U \cup V$ with $|Z| < \min\{\delta_U(G), k\}$ that separates two vertices u_1, u_2 in U. Let $Z_1 = Z \cap U$ and $Z_2 = Z \cap V$. Note that there is no path between u_1 and u_2 in $G' = (U', V'; E') = G[(U \setminus Z_1) \cup (V \setminus Z_2)]$. So we can choose a partition of G' into two subgraphs $G'_1 = (U'_1, V'_1; E'_1)$ and $G'_2 = (U'_2, V'_2; E'_2)$ that are unions of components of G' with $u_1 \in U'_1$ and $u_2 \in U'_2$.

Suppose first that the graph G' contains an isolated vertex $u \in U'$, then $|Z_2| \geq \delta_U$ since $N(u) \subset Z_2$, a contradiction.

Assume now that the graph G' does not contain an isolated vertex $u \in U'$, hence $|V'_i| > 0$ for $i = 1, 2$. Suppose that $|Z| = |Z_1| + |Z_2| < k$. We will proceed now like in the proof of Theorem 5.1.

Let $|U'_i| = |V'_i| + \varepsilon_i$ for $i = 1, 2$. So there is $|V'_1| + |V'_2| + |Z_2| + k = |U'_1| + |U'_2| + |Z_1| = |V'_1| + |V'_2| + |Z_1| + \varepsilon_1 + \varepsilon_2$. Thus, $|Z_2| + k = |Z_1| + \varepsilon_1 + \varepsilon_2$.

Since $V'_1 \neq \emptyset$ by Theorem 2.2, there is $|N_G(V'_1)| \geq |V'_1| + k$. On the other hand, since $U'_1 \cup Z_1 \supseteq N_G(V'_1)$, there is $|U'_1 \cup Z_1| \geq |N_G(V'_1)|$. So, by simplifying $|V'_1| + \varepsilon_1 + |Z_1| = |U'_1| + |Z_1| \geq |V'_1| + k$, we get $\varepsilon_1 + |Z_1| \geq k$. Similarly, we get that $\varepsilon_2 + |Z_1| \geq k$.

On one hand, since $|Z_2| + k = |Z_1| + \varepsilon_1 + \varepsilon_2 \geq k + \varepsilon_2$, we obtain that $|Z_2| \geq \varepsilon_2$. On the other hand, there is $|Z_1| + |Z_2| < k \leq |Z_1| + \varepsilon_2$, so $|Z_2| < \varepsilon_2$, a contradiction.

Finally, the last part of the thesis of the theorem follows from the previous analyses. ☐

Theorem 5.3. *Let n, m be positive integers such that $1 < m < n$. Let $k = n - m$. Then, for any k-critical-bipartite graph $G = (U, V; E)$ with $|U| = n$ and $|V| = m$, there is $\kappa(G) \geq \min\{\delta(G), k\}$.*

Proof. Let $G = (U, V; E)$ be a k-critical-bipartite graph of order (n, m). Towards a contradiction, suppose that Z is a vertex cut for two vertices x and y in G with $|Z| < \min\{\delta_U(G), k\}$. Let $G' = G - Z = (U', V'; E')$.

If $x, y \in V$, then $|Z| \geq k$ by Theorem 5.1, a contradiction. For $x, y \in U$, we have $|Z| \geq \min\{\delta_U(G), k\}$ by Theorem 5.2, a contradiction. Finally, consider the case where $x \in U$ and $y \in V$. Choose $x' \in U' \setminus \{x\} \cap N_{G'}(y)$. Such a vertex exists since G is k-critical and $n \geq k + 2 > |Z| + 2$. So Z is a vertex cut for x and x', and the thesis holds by Theorem 5.2. ☐

By applying Theorems 5.1, 5.2, and 5.3 to Construction 3.3, we get the following corollary that shows that the given lower bounds are tight.

Corollary 5.4. *Let n, m be positive integers such that $1 < m < n$. Let $k = n - m$ and $\overline{G}_{n,m} = (U, V; E)$ be a graph given by Construction 3.3. Then the following properties hold:*

1. $\kappa_V(\overline{G}_{n,m}) \in \{k, k + 1\}$,
2. $\kappa_U(\overline{G}_{n,m}) = \delta_U(\overline{G}_{n,m})$,
3. $\kappa(\overline{G}_{n,m}) = \delta(\overline{G}_{n,m})$.

Moreover, $\kappa_V(\overline{G}_{n,m}) = k$ if $k = 1$.

Proof. Since $\delta_V(\overline{G}_{n,m}) = k+1$, by Theorem 5.1, there is $\kappa_V(\overline{G}_{n,m}) \in \{k, k+1\}$. Since $\delta_U(\overline{G}_{n,m}) = \lfloor \frac{m(k+1)}{n} \rfloor$ and $\frac{m(k+1)}{n} < k+1$ for $n > m > 1$, by Theorem 5.2, there is $\kappa_U(\overline{G}_{n,m}) = \delta_U(\overline{G}_{n,m})$. By Theorem 5.3, there is $\kappa(\overline{G}_{n,m}) = \delta(\overline{G}_{n,m})$. Finally, the case where $k = 1$ is easy to check (for example, consider removing u_0 in the graph $\overline{G}_{6,5}$ in Figure 1). □

Note that given positive integer values n, m such that $n > m > 1$ and $k = n - m$, for any $\kappa \in \{1, \ldots, m\}$, there exists a k-critical-bipartite graph $G = (U, V; E)$ of order (n, m) with connectivity $\kappa(G) = \kappa$. Indeed, if $\kappa = m$, then $G = K_{n,m}$, otherwise (i.e. $\kappa < m$) let $G' = (V, U; E)$ be a complete bipartite graph $K_{n,m}$. If we pick any vertex $u \in U$ and delete $m - \kappa$ edges incident with u, the obtained graph $G = (U, V; E)$ is k-critical and κ-connected.

6. FINAL REMARKS

Let $G = (U, V; E)$ with $|U| = n$, $|V| = m$ $n > m > 1$, $k = n - m$ be a minimum k-critical-bipartite graph. Then $\delta_V = k + 1$, therefore $\kappa_V \in \{k, k+1\}$ by Theorem 5.1. For example, note that $\kappa_V(\overline{G}_{6,5}) = 1$ (see Figure 1). Therefore, we pose the following open problem.

Problem 6.1. Characterize all minimum k-critical-bipartite graphs for which $\kappa_V = k$.

Recall that for any minimum k-critical-bipartite graph $G = (U, V; E)$ of order (n, m), with $k = n - m + 1$, there is $|E| = (k+1)m$ and $\Delta_U = \lceil \frac{(k+1)m}{n} \rceil$. And, to have $(k+1)m$ edges, the number of vertices in U of degree Δ_U has to be at least $(n - m + 1)m - n\lfloor \frac{(k+1)m}{n} \rfloor$. But there is some flexibility for the degree of other vertices in U: there may be $\delta_U < \lfloor \frac{(k+1)m}{n} \rfloor$ (see Figure 3 for an example). So we pose the following open problem.

Problem 6.2. Determine if there exist minimum k-critical-bipartite graphs $G = (U, V; E)$ of order (n, m) with $\delta_U = \delta$ for any $\delta \in \{1, 2, \ldots, \lfloor \frac{m(k+1)}{n} \rfloor - 1\}$.

The property of being k-critical-bipartite, and fault-tolerance in general, has strong relations with connectivity (besides the results presented in this paper, see, for example, the work of Cichacz *et al.* [5]). Given a bipartite graph $G = (U, V; E)$, besides testing if G is k-critical-bipartite, it is valuable for applications to find a minimum supergraph (adding edges, augmentation) or minimum subgraph (removing edges, sparsification) of G that is k-critical-bipartite. However, unlike testing connectivity, the corresponding edge modification problems tend to be harder and not well understood (see the work of Crespelle *et al.* [6] for a recent review). We believe that the relations between k-critical-bipartiteness and connectivity will permit us to adapt methods developed for edge modification problems related to connectivity to work with k-critical-bipartiteness. Moreover, characterizing minimum k-critical-bipartite graphs is a valuable step in this direction. We terminate with the following open problem.

Problem 6.3. Given a bipartite graph $G = (U, V; E)$ of order (n, m), what is the complexity of finding a minimum supergraph (subgraph) of G that is k-critical-bipartite.

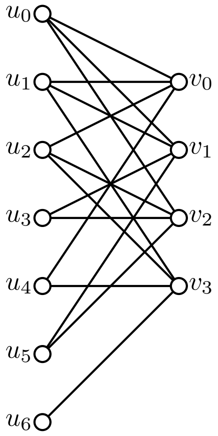

Fig. 3. Minimum k-critical-bipartite graph with $\delta_U < \lfloor \frac{(n-m+1)m}{n} \rfloor$

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