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# ON THE EXISTENCE OF INDEPENDENT $(1,k)$-DOMINATING SETS FOR $k \in \{1,2\}$ IN TWO PRODUCTS OF GRAPHS

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**Abstract.** A subset $J$ of vertices is said to be a $(1,k)$-dominating set if every vertex $v$ not belonging to the set $J$ has a neighbour in $J$ and there exists also another vertex in $J$ within the distance at most $k$ from $v$. In this paper, we study the problem of the existence of independent $(1,k)$-dominating sets for $k \in \{1,2\}$ in the tensor product and in the strong product of two graphs. We give complete characterisations of these graph products, which have independent $(1,1)$-dominating sets or independent $(1,2)$-dominating sets, with respect to the properties of their factors.

**Keywords:** dominating set, independent set, multiple domination, secondary domination, tensor product, strong product.

**Mathematics Subject Classification:** 05C69, 05C76.

## 1. INTRODUCTION

In general, we use the standard terminology and notation of graph theory; see [14].

Let $G$ be an undirected, connected, simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* of the graph $G$ is the number of vertices in $G$. We say that a vertex $x \in V(G)$ is a *neighbour* of a vertex $y \in V(G)$ in a graph $G$ if there is an edge $xy \in E(G)$. The set of all neighbours of $x \in V(G)$ is called a *neighbourhood* of the vertex $x$ and is denoted by $N_G(x)$. We also denote $N_G[x] = N_G(x) \cup \{x\}$. By $d_G(x)$ we mean the *degree* of the vertex $x$ in $G$. A vertex of degree one is called a *leaf*. The set of all leaves of a graph $G$ is denoted by $L(G)$. By $P_n$, $n \geq 2$, $C_n$, $n \geq 3$ and $K_n$, $n \geq 2$, we mean a *path*, a *cycle* and a *complete graph* of order $n$, respectively. The distance $d_G(x,y)$ between two vertices $x$ and $y$ of $G$ is the length of the shortest path joining them. A set of vertices that are within the distance at most $k$ from the vertex $x \in V(G)$ is denoted by $N_G^k(x)$.

 

The union of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is a graph $G \cup H$ such that $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. In our paper, whenever we consider the union of graphs, we assume that $V(G)$ and $V(H)$ are disjoint.

A subset $D \subseteq V(G)$ is a *dominating set* of $G$ if every vertex is either in $D$ or has a neighbour in $D$. A subset $S \subseteq V(G)$ is called an *independent set* of $G$ if no two vertices of $S$ are adjacent in $G$. An empty set and a set containing only one vertex are also independent. An independent set which is not a proper subset of any other independent set is called a *maximal* independent set. It is well known that every maximal independent set is a dominating set.

A subset $J$ that is both independent and dominating is a *kernel* of $G$. The classical concept of kernels was introduced for digraphs by J. von Neumann and O. Morgenstern in the context of game theory; see [29]. In the next decades this topic was studied for various reasons, for example, in list colourings, perfectness or location problems. C. Berge was a mathematician who intensively studied kernels in digraphs and also used them to solve different mathematical problems, some fundamental results can be found in [9] and [10]. In the literature there are many variants and generalisations of kernels, as examples we can indicate [1, 24] and [34].

In the literature, we can find many kinds of dominating sets, obtained either by adding some restrictions or generalising the concept of classical domination, see [19]. In particular, one of the most studied types of dominating sets are multiple dominating sets introduced by J.M. Fink and M.S. Jacobson in [15]. Let $p \geq 1$ be an integer. A subset $S$ is said to be *$p$-dominating* if every vertex outside $S$ has at least $p$ neighbours in $S$. If $p = 1$, then we obtain a dominating set in the classical sense. If $p = 2$, we get 2-dominating sets, which were considered in [11] and [13].

Using the definition of the 2-dominating set, A. Włoch introduced and studied the concept of a 2-dominating kernel; see [31]. A set which is 2-dominating and independent is named a *2-dominating kernel*. The concept of 2-dominating kernels was studied in [3–6] and [7].

Another widely researched type of dominating sets are their generalisations in the distance sense, which was introduced by A. Meir and J.W. Moon in [25]. Based on this definition, Borowiecki and Kuzak in [12] introduced the concept of the $(k, l)$-kernel, which was studied in [16, 17, 23, 32] and [33].

In [20] Hedetniemi and others combined multiple domination and distance domination by defining $(1, k)$-dominating sets. Let $k \geq 1$ be an integer. A subset $S$ is a *$(1, k)$-dominating set* of $G$ if for every vertex $v \in V(G)$ there are distinct vertices $u, w \in S$ such that $uv \in E(G)$ and $d_G(v, w) \leq k$. Some problems concerning $(1, 2)$-dominating sets were considered in [21, 22, 28] and [30].

Hedetniemi and others studied also $(1, k)$-dominating sets which are independent simultaneously, see [20]. If $k = 1$, then we obtain independent $(1, 1)$-dominating sets (2-dominating kernels). If $k = 2$ we obtain independent $(1, 2)$-dominating sets. Clearly, every independent $(1, 1)$-dominating set is an independent $(1, 2)$-dominating set.

A graph does not always have an independent $(1, 1)$-dominating sets or an independent $(1, 2)$-dominating set. In general, the problem of the existence of these sets is $\mathcal{NP}$-complete, see [4, 20]. Of course, any complete graph does not have such a set.

Moreover, in [31] A. Włoch provided, among others, the following result.

**Theorem 1.1** ([31]). *Let $n \geq 3$ be an integer. Then:*

1. *the path $P_n$ has an independent $(1,1)$-dominating set if and only if $n$ is odd,*
2. *the cycle $C_n$ has an independent $(1,1)$-dominating set if and only if $n$ is even.*

A simple example of a graph which does not have an independent $(1,2)$-dominating set in presented in the Figure 1. It consists of two triangles joined by an edge. Since we will use this graph in our next considerations, we denote it by $H^*$.

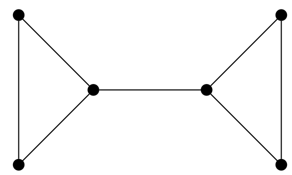

**Fig. 1.** A graph $H^*$ which does not have an independent $(1,2)$-dominating set

A sufficient condition for a graph having an independent $(1,2)$-dominating set was given in [20].

**Theorem 1.2** ([20]). *Every connected graph $G$ with at least two nonadjacent vertices and without triangles has an independent $(1,2)$-dominating set.*

The natural consequence of this theorem is to study graphs with a triangle. In particular, different graph products belong to the class of graphs with triangles. It was the motivation to study the problem of the existence of independent $(1,2)$-dominating sets in certain graph products. Hedetniemi and others in [20] gave a necessary and sufficient condition for a Cartesian product with an independent $(1,2)$-dominating set. Then Michalski and I. Włoch studied these sets in the $G$-join of graphs, which is also known as the lexicographic product of graphs, see [27]. Next, Michalski and Bednarz gave a complete characterisation of independent $(1,2)$-dominating sets and $(1,1)$-dominating sets in the generalised corona of graphs, see [26]. These results were a motivation to study the existence of independent $(1,1)$-dominating sets and $(1,2)$-dominating sets in well-known products such as the tensor product and the strong product.

Let $G$ and $H$ be two disjoint graphs. *The tensor product* of the graphs $G$ and $H$ is the graph $G \times H$ such that $V(G \times H) = V(G) \times V(H)$ and

$$E(G \times H) = \{(x_i, y_p)(x_j, y_q) \colon x_i x_j \in E(G) \text{ and } y_p y_q \in E(H)\}.$$

The tensor product is also called a direct product or a categorical product; see [18] for details.

For a given subset $S \subseteq V(G \times H)$ we denote $\pi_G(S) = \{x \in V(G) \colon (x,y) \in S\}$. Analogously, let $\pi_H(S) = \{y \in V(H) \colon (x,y) \in S\}$.

*The strong product* of graphs $G$ and $H$ is the graph $G \boxtimes H$ such that $V(G \boxtimes H) = V(G) \times V(H)$ and

$$E(G \boxtimes H) = \{(x_i, y_p)(x_j, y_q) \colon (x_i = x_j \text{ and } y_p y_q \in E(H)) \\ \text{or } (y_p = y_q \text{ and } x_i x_j \in E(G)) \text{ or } (x_i x_j \in E(G) \text{ and } y_p y_q \in E(H))\}.$$

For a given subset $S \subseteq V(G \boxtimes H)$ we denote $\pi_G(S) = \{x \in V(G) \colon (x, y) \in S\}$. Analogously, let $\pi_H(S) = \{y \in V(H) \colon (x, y) \in S\}$.

This paper is a sequel to the results obtained in [26] and [27] and provides complete characterisations of the tensor product and the strong product of graphs which have independent $(1,1)$-dominating sets and independent $(1,2)$-dominating sets with respect to properties of factors of these products.

## 2. MAIN RESULTS

### 2.1. EXISTENCE OF INDEPENDENT (1,2)-DOMINATING SETS

Our first result gives the complete characterization of tensor products which have an independent $(1,2)$-dominating set.

**Theorem 2.1.** *Let $G$ and $H$ be connected graphs. The graph $G \times H$ has an independent $(1,2)$-dominating set if and only if $G \ncong P_2$ or $H \ncong P_2$.*

*Proof.* Let $G$ and $H$ be connected graphs and let $V(G) = \{x_1, x_2, \ldots, x_n\}$, $V(H) = \{y_1, y_2, \ldots, y_m\}$, where $n \geq 1, m \geq 1$ are fixed integers. If $n = 1$ or $m = 1$, then the graph $G \times H$ has only isolated vertices, so the set $V(G \times H)$ is an independent $(1,2)$-dominating set. Let $n, m \geq 2$. First, we prove the sufficient condition. Let us consider the following cases:

*Case 1.* $L(G) = \emptyset$ and $L(H) = \emptyset$.

Then the degree of each vertex from $G$ and $H$ is greater or equal than two. Without loss of generality, suppose that $M$ is a maximal independent set of the graph $G$. We will show that a set $J = M \times V(H)$ is an independent $(1,2)$-dominating set of a graph $G \times H$. From the independence of the set $M$ and from the definition of the tensor product, it follows that $J$ is an independent set. Moreover, because $M$ is a maximal independent set of $G$, it is also a dominating set of $G$. That is why, for every vertex $x_i \notin M$, there exists a vertex $x_k \in M$ such that $x_i x_k \in E(G)$. Thus, for every vertex $v \in V(G \times H)$ that does not belong to the set $J$ we have $|N_{G \times H}(v) \cap J| \geq 2$. Therefore, the vertex $v \in V(G \times H)$, $v \notin J$, is $(1,1)$-dominated by $J$. This means that $J$ is the independent $(1,2)$-dominating set of the graph $G \times H$.

*Case 2.* Either $L(G) \neq \emptyset$ or $L(H) \neq \emptyset$.

Without loss of generality, suppose that the graph $G$ has leaves. If $G \cong P_2$ and $V(G) = \{x_1, x_2\}$, then it is easy to prove that a set $J = \{x_1\} \times V(H)$ is an independent $(1,2)$-dominating set of the graph $G \times H$. Suppose now that $G \ncong P_2$. Let $M$ be a maximal independent set of $G$ containing all leaves. The existence of such a set is obvious. Then, proving analogously to Case 1, we can show that $J = M \times V(H)$ is the independent $(1,2)$-dominating set of $G$.

*Case 3.* $L(G) \neq \emptyset$ and $L(H) \neq \emptyset$.

At least one of the graphs $G, H$ is not isomorphic to $P_2$. Without loss of generality, suppose that $H \ncong P_2$. Let $M$ be a maximal independent set of $H$ containing all leaves. The set $M$ is also a dominating set in $H$. We will show that $J = V(G) \times M$ is an independent $(1,2)$-dominating set of $G \times H$. By the definition of the tensor product and by the independence of $M$ it follows that $J$ is independent in $G \times H$. Let $v = (x_i, y_j) \in V(G \times H) \setminus J$. Since $M$ is a dominating set, the vertex $y_j \in V(H) \setminus M$ has a neighbour $y_k \in M$. The graph $G$ is connected, so $x_i$ has at least one neighbour in $G$, let us denote this neighbour by $x_p$. Hence, $v = (x_i, y_j)$ is adjacent to $(x_p, y_k) \in J$. Moreover, we know that $d_H(y_j) \geq 2$. Let $y_l, l \neq k$ be a neighbour of $y_j$. If $y_l \in M$, then $v = (x_i, y_j)$ is adjacent to $(x_p, y_l) \in J$ and therefore $v$ is $(1,1)$-dominated by $J$. If $y_l \notin M$, then there exists $y_r \in M$ such that $y_l y_r \in E(H)$. In this case we obtain the path $v - (x_p, y_l) - (x_i, y_r)$, where $(x_i, y_r) \in J$. This means that $v$ is $(1,2)$-dominated by $J$, which completes the proof.

To prove the necessary condition, let us suppose that $G \times H$ has an independent $(1,2)$-dominating set. For the contradiction let $G \cong P_2$ and $H \cong P_2$. Then $G \times H \cong P_2 \cup P_2$ but $P_2$ does not have an independent $(1,2)$-dominating set, a contradiction, which completes the proof. □

Figure 2 provides an example of an independent $(1,2)$-dominating set in the tensor product $H^* \times P_3$, this set consists of white vertices.

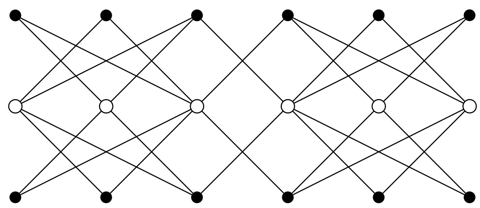

**Fig. 2.** The graph $H^* \times P_3$ with an independent $(1,2)$-dominating set

The proven theorem tells us that, in fact, the only tensor product of connected graphs that does not have an independent $(1,2)$-dominating set is the product $P_2 \times P_2$. However, if we consider the same problem in the strong product of two graphs, its solution is more complicated.

**Theorem 2.2.** *Let $G$ and $H$ be connected graphs. The graph $G \boxtimes H$ has an independent $(1,2)$-dominating set if and only if at least one of the following conditions hold:*

1. *$G$ or $H$ has an independent $(1,1)$-dominating set,*
2. *$G$ and $H$ has an independent $(1,2)$-dominating set.*

*Proof.* Let us begin with the sufficient condition. Assume that at least one of the conditions holds. We will show that the graph $G \boxtimes H$ has an independent $(1,2)$-dominating set. Let us consider two cases.

*Case 1.* $G$ and $H$ have an independent $(1,2)$-dominating set.

By $J_G$ we denote the independent $(1,2)$-dominating set of the graph $G$ and by $J_H$ the independent $(1,2)$-dominating set of the graph $H$. We will show that the set $J = J_G \times J_H$ is an independent $(1,2)$-dominating set of the graph $G \boxtimes H$. The independence of the set $J$ follows directly from the independence of the sets $J_G$, $J_H$ and the definition of the strong product. Hence, it suffices to show that $J$ is a $(1,2)$-dominating set in the graph $G \boxtimes H$. We can see that $V(G \boxtimes H) = J \cup V_1 \cup V_2 \cup V_3$, where $V_1 = (V(G) \setminus J_G) \times J_H$, $V_2 = J_G \times (V(H) \setminus J_H)$ and $V_3 = (V(G) \setminus J_G) \times (V(H) \setminus J_H)$. Since $J_G$ is an independent $(1,2)$-dominating set, the vertex $(x_i, y_l)$, $x_i \in V(G) \setminus J_G$, $y_l \in J_H$ is $(1,2)$-dominated by $J$. Therefore, all vertices of the set $V_1$ are $(1,2)$-dominated by $J$. By analogous reasoning, we know that all vertices of $V_2$ are $(1,2)$-dominated by $J$. Let now $(x_i, y_j) \in V_3$. By definition of an independent $(1,2)$-dominating set we obtain that for every vertex $x_i$ there is a vertex $x_k \in J_G$ such that $x_i x_k \in E(G)$ and another vertex $x_u \in J_G$, $x_u \neq x_k$ such that $d_G(x_i, x_u) \leq 2$. Similarly, for every vertex $y_j$ we can find a vertex $y_l$ such that $y_j y_l \in E(H)$ and another vertex $y_v \in J_H$, $y_v \neq y_l$ such that $d_H(y_j, y_v) \leq 2$. Hence, by definition of the strong product we conclude that for every vertex $(x_i, y_j) \in V_3$ there exist two vertices $(x_k, y_l), (x_u, y_v) \in J$ such that $(x_i, y_j)(x_k, y_l) \in E(G \boxtimes H)$ and $d_{G \boxtimes H}((x_i, y_j), (x_u, y_v)) \leq 2$. This means that $J$ is a $(1,2)$-dominating set of the graph $G \boxtimes H$ which is also independent, so $J$ is an independent $(1,2)$-dominating set.

*Case 2.* $G$ or $H$ has an independent $(1,1)$-dominating set.

Without loss of generality, let us assume that $G$ has an independent $(1,1)$-dominating set $J_G$. Let $M$ be a maximal independent set of the graph $H$. We will show that the set $J = J_G \times M$ is an independent $(1,2)$-dominating set of the graph $G \boxtimes H$. The fact that $J$ is independent follows directly from the independence of sets $J_G$, $M$ and from the definition of the strong product. Hence, we need to show that $J$ is a $(1,2)$-dominating set in $G \boxtimes H$. We can see that $V(G \boxtimes H) = J \cup V_1 \cup V_2 \cup V_3$, where $V_1 = (V(G) \setminus J_G) \times M$, $V_2 = (V(G) \setminus J_G) \times (V(H) \setminus M)$, $V_3 = J_G \times (V(H) \setminus M)$. Let $x_i \in V(G) \setminus J_G$ and $y_j \in V(H) \setminus M$. Since $J_G$ is an independent $(1,1)$-dominating set of $G$ we know that for any $y_l \in M$ the vertex $(x_i, y_l)$ is $(1,1)$-dominated by $J$ for all $x_i \in V(G) \setminus J_G$. Therefore, all vertices from the set $V_1$ are $(1,2)$-dominated by $J$. Moreover, for every $x_i \in V(G) \setminus J_G$ we can find vertices $x_k, x_n \in J_G$, $x_k \neq x_n$ such that $x_i x_k, x_i x_n \in E(G)$. Furthermore, since $M$ is a maximal independent set, for every $y_j \in V(H) \setminus M$ there is a vertex $y_l \in M$ such that $y_j y_l \in E(H)$. This means, by the definition of the strong product, that every vertex of $(x_i, y_j) \in V_2$ has two neighbours $(x_k, y_l), (x_n, y_l) \in J$. So, all vertices from $V_2$ are $(1,2)$-dominated by $J$. Let $x_k \in J_G$ and $y_l \in M$. Finally, since $M$ is a maximal independent set in $H$, we know that for any $x_k \in J_G$, $y_j \in V(H) \setminus M$ the vertex $(x_k, y_j)$ is dominated by $J$. Moreover, because $J_G$ is a $(1,1)$-dominating set, then for every $x_k \in J_G$ there is $x_t \in J_G$, $x_t \neq x_k$ such that $d_G(x_k, x_t) = 2$. Hence, for every vertex $(x_k, y_j)$ we can find a vertex $(x_t, y_l)$ such that $d_{G \boxtimes H}((x_k, y_j), (x_t, y_l)) = 2$. This means that all vertices in the set $V_3$ are $(1, 2)$-dominated by the set $J$. We have shown that $J$ is an independent $(1, 2)$-dominating set in the graph $G \boxtimes H$. The case where the graph $H$ instead of $G$ has an independent $(1, 1)$-dominating set is analogous.

Conversely, let us assume that the graph $G \boxtimes H$ has an independent $(1, 2)$-dominating set $J$. Let $J_G = \pi_G(J)$, $J'_G = V(G) \setminus J_G$, $J_H = \pi_H(J)$, $J'_H = V(H) \setminus J_H$. The set of vertices of the graph $G \boxtimes H$ we can partition into four subsets as follows: $V(G \boxtimes H) = J \cup (J_G \times J'_H) \cup (J'_G \times J_H) \cup (J'_G \times J'_H)$. Since $J$ is independent, by definition of $G \boxtimes H$, the sets $J_G$ and $J_H$ must also be independent. We will show that they are maximal independent sets. Since the proofs for $J_G$ and $J_H$ are analogous, we will prove it only for $J_G$. Suppose, for the contradiction, that $J_G$ is not maximal. It means that there exists a vertex $x' \in J'_G$ such that $N_G[x'] \cap J_G = \emptyset$. Then the vertex $(x', y) \in V(G \boxtimes H)$, $y \in V(H)$ is not dominated by $J$, a contradiction. Hence, $J_G$ is a maximal independent set.

Consider the following cases.

(a) Every vertex $(x', y) \in J'_G \times V(H)$ is $(1, 1)$-dominated by $J$. Then there are two vertices $(x_1, y_1), (x_2, y_2) \in J$ such that $(x_1, y_1)(x', y)$, $(x_2, y_2)(x', y) \in E(G \boxtimes H)$. By definition of $G \boxtimes H$ it follows that $x_1, x_2 \in J_G$ and $x_1x', x_2x' \in E(G)$. Hence, $J_G$ is an independent $(1, 1)$-dominating set.
(b) Every vertex $(x, y') \in V(G) \times J'_H$ is $(1, 1)$-dominated by $J$. Then, analogously to (a), we show that $J_H$ is an independent $(1, 1)$-dominating set in $H$.
(c) There is a vertex $(x', y) \in J'_G \times V(H)$, which is not $(1, 1)$-dominated by $J$. Then $N_{G \boxtimes H}(x', y) \cap J = \{(u, v)\}$ and $N_G(x') \cap J_G = \{u\}$. To ensure that the vertex $(u, y') \in J_G \cap J'_H$ is $(1, 2)$-dominated by $J$, it must be true that $|N^2_H(y') \cap J_H| \geq 2$. Since $J_H$ is a maximal independent set, it is a dominating set and this means that $J_H$ is an independent $(1, 2)$-dominating set.
(d) There is a vertex $(x, y') \in V(G) \times J'_H$, which is not $(1, 1)$-dominated by $J$. Then, analogously to (c), we prove that $J_G$ is an independent $(1, 2)$-dominating set in $G$.

We can see that cases (a) and (c) are mutually exclusive, the same holds for (b) and (d). Therefore, we have four possibilities. If both (a) and (b) are true, then there is an independent $(1, 1)$-dominating set both in $G$ and in $H$. If (a) and (d) hold, then there is an independent $(1, 1)$-dominating set in $G$. If (b) and (c) hold, then there is an independent $(1, 1)$-dominating set in $H$. If both (c) and (d) are true, then there is an independent $(1, 2)$-dominating set both in $G$ and $H$. In summary, we then obtain that $G$ or $H$ have an independent $(1, 1)$-dominating set or both $G$ and $H$ have independent $(1, 2)$-dominating sets, which completes the proof. $\square$

White vertices in Figure 3 indicate the example of an $(1, 2)$-independent set in the strong product $H^* \boxtimes P_3$. The product $H^* \boxtimes P_2$ presented in Figure 4 does not have an $(1, 2)$-independent set since none of the factors have neither an independent $(1, 1)$-dominating set nor an independent $(1, 2)$-dominating set.

Theorem 1.1 and Theorem 2.2 provide us a corollary regarding the strong products of paths and cycles.

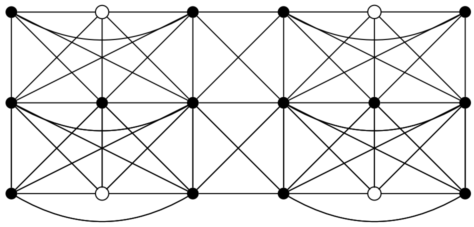

**Fig. 3.** The graph $H^* \boxtimes P_3$ with an independent $(1,2)$-dominating set

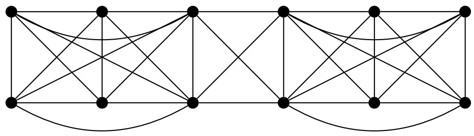

**Fig. 4.** The graph $H^* \boxtimes P_2$ which does not have an independent $(1,2)$-dominating set

**Corollary 2.3.** *Let $n, m$ be integers. Then the graph*

1. $P_n \boxtimes P_m$, *for* $n, m \geq 3$,
2. $P_n \boxtimes C_m$, *for* $n \geq 3, m \geq 4$,
3. $P_n \boxtimes K_m$, *for* $m \geq 3$ *and odd* $n$,
4. $C_n \boxtimes C_m$, *for* $n, m \geq 4$,
5. $C_n \boxtimes K_m$, *for* $m \geq 3$ *and even* $n$

*has an independent* $(1,2)$*-dominating set.*

Using Theorem 1.2 we can also conclude that every strong product of two connected, non-complete graphs of order at least three without triangles has an independent $(1,2)$-dominating set. In particular, the strong product of two trees of order at least three has such a set.

## 2.2. EXISTENCE OF INDEPENDENT (1,1)-DOMINATING SETS

Our next result is the complete characterization of tensor products with an independent $(1,1)$-dominating set.

**Theorem 2.4.** *Let $G$ and $H$ be connected graphs. The graph $G \times H$ has an independent* $(1,1)$*-dominating set if and only if one of the following conditions holds:*

1. *$G$ or $H$ do not have leaves,*
2. *both $G$ and $H$ have leaves and at least one of graphs $G$ or $H$ has an independent* $(1,1)$*-dominating set.*

*Proof.* At the beginning, let us assume that one of the conditions holds. We shall show that $G \times H$ has an independent $(1,1)$-dominating set. We consider the following cases.

*Case 1.* At least one of the graphs $G$, $H$ does not have leaves.

Without loss of generality we can assume that $H$ does not have leaves, that is, $\delta(H) \geq 2$. This means that $\delta(G \times H) \geq 2$. Let $M$ be a maximal independent set of the graph $G$. We will show that the set $J = M \times V(H)$ is an independent $(1, 1)$-dominating set in the graph $G \times H$. By the independence of $M$ and by definition of the tensor product, we immediately find that $J$ is independent. Moreover, because $M$ is a maximal independent set of $G$, it is also a dominating set of $G$. That is why, for every vertex $x_i \notin M$, there exists a vertex $x_k \in M$ such that $x_i x_k \in E(G)$. Therefore, for every vertex of the form $(x_i, y) \in V(G \times H)$, where $x_i \notin M$, $y \in V(H)$ we can always find two vertices $(x_k, y_1), (x_k, y_2)$ where $y_1 y, y_2 y \in E(H)$ and $x_k \in M$. Therefore, $(x_i, y)$ has at least two neighbours $(x_k, y_1), (x_k, y_2)$ in $J$. So, every vertex of the graph $G \times H$ not belonging to $J$ is $(1, 1)$-dominated by $J$. This means that $J$ is the independent $(1, 1)$-dominating set of the graph $G \times H$.

*Case 2.* Both graphs $G$ and $H$ have leaves, and at least one of them has an independent $(1, 1)$-dominating set.

Without loss of generality, let $G$ have an independent $(1, 1)$-dominating set $J_G$. Let $J = J_G \times V(H)$. By definition of a $(1, 1)$-dominating set, all leaves of the graph $G$ must belong to the set $J_G$. Moreover, $d_{G\times H}((x, y)) = d_G(x) \cdot d_H(y)$, therefore all leaves of $G \times H$ must belong to $J$. By reasoning analogous to the previous case it can be proved that $J$ is an independent $(1, 1)$-dominating set of the graph $G \times H$.

Conversely, let us assume now that the graph $G \times H$ has an independent $(1, 1)$-dominating set $J$. Obviously, $J$ must contain all leaves. Let us denote $J_G = \pi_G(J)$ and $J_H = \pi_H(J)$. First, we show $J_G = V(G)$ or $J_H = V(H)$. For the contradiction, let us assume that $J_G \neq V(G)$ and $J_H \neq V(H)$. This means that we can partition the set $V(G \times H)$ into four pairwise disjoint sets $V_1, V_2, V_3, J$, where $V_1 = (V(G) \setminus J_G) \times J_H$, $V_2 = J_G \times (V(H) \setminus J_H)$ and $V_3 = (V(G) \setminus J_G) \times (V(H) \setminus J_H)$. By definition of the tensor product we see that there can be edges only between $V_1$ and $J$ or between $V_2$ and $V_3$. Therefore, $J$ is not an independent $(1, 1)$-dominating set, a contradiction. Therefore, $J_G = V(G)$ or $J_H = V(H)$. Now for the contradiction we suppose that any of the conditions from the statement of the proven theorem does not hold. This means that $G$ and $H$ have leaves and neither of them has an independent $(1, 1)$-dominating set. The existence of leaves in both graphs $G$, $H$ implies that there are leaves in $G \times H$ and that every leaf must belong to $J$. Therefore, we can assume that $J_G = V(G)$ and $J_H$ is a maximal independent set containing all leaves (the reasoning in the converse case is identical). Hence, there is a vertex $y \in V(H) \setminus J_H$, which is not $(1, 1)$-dominated in $H$, so it has only one neighbour $y'$ in $J_H$. Let $x \in V(G) = J_G$ be a leaf of the graph $G$ with its unique vertex $x' \in V(G)$. Then the vertex $(x, y)$ has only one neighbour $(x', y')$ in the set $J$. This means that $J$ is not an independent $(1, 1)$-dominating set, a contradiction, which completes the proof. □

We know that paths of even order do not have an independent $(1, 1)$-dominating set, while paths of odd order do. Therefore, according to Theorem 2.4, a simple example of a tensor product without an independent $(1, 1)$-dominating set is the graph $P_4 \times P_4$, presented in Figure 5.

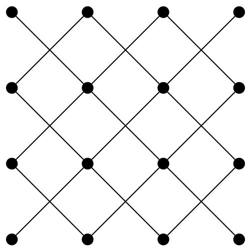

**Fig. 5.** The graph $P_4 \times P_4$ which does not have an independent $(1,1)$-dominating set

On the other hand, the graph $P_5 \times P_4$ has such sets. One of them is indicated by the white colour in Figure 6.

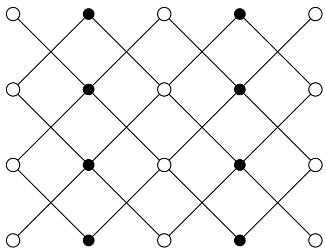

**Fig. 6.** The graph $P_5 \times P_4$ with an independent $(1,1)$-dominating set

From Theorems 1.1 and 2.4 we obtain a corollary concerning the existence of independent $(1,1)$-dominating sets in tensor products of path, cycles and complete graphs.

**Corollary 2.5.** *Let $n, m$ be integers. Then the graph*

1. $K_n \times K_m$, *for* $n, m \geq 3$,
2. $K_n \times P_m$, *for* $n \geq 3, m \geq 2$,
3. $K_n \times C_m$, *for* $n, m \geq 3$,
4. $C_n \times C_m$, *for* $n, m \geq 3$,
5. $C_n \times P_m$, *for* $n \geq 3, m \geq 2$,
6. $P_n \times P_m$, *for* $n, m \geq 2$ *and at least one of the numbers* $n, m$ *is odd*

*has an independent* $(1,1)$-*dominating set.*

Finally, we prove the necessary and sufficient conditions for strong products with an independent $(1,1)$-dominating set.

**Theorem 2.6.** *Let $G$ and $H$ be connected graphs. The graph $G \boxtimes H$ has an independent $(1,1)$-dominating set if and only if $G$ and $H$ have independent $(1,1)$-dominating sets.*

*Proof.* First, assume that the graphs $G$ and $H$ have independent $(1,1)$-dominating sets $J_G$ and $J_H$, respectively. We will show that the set $J = J_G \times J_H \subset V(G \boxtimes H)$

is an independent $(1,1)$-dominating set of the graph $G \boxtimes H$. By definition of the strong product, we immediately find that $J$ is independent. Since $J_G$ is an independent $(1,1)$-dominating set of $G$, every vertex $x' \in V(G) \setminus J_G$ is $(1,1)$-dominated by the set $J_G$. Therefore, $(x',y) \in V(G \boxtimes H)$, $y \in V(H)$ is $(1,1)$-dominated by $J$. By analogous reasoning every vertex $y' \in V(H) \setminus J_H$ is $(1,1)$-dominated by $J_H$ and this means that $(x,y') \in V(G \boxtimes H)$, $x \in V(G)$ is $(1,1)$-dominated by $J$. Hence, we have that $J$ is an independent $(1,1)$-dominating set of the graph $G \boxtimes H$.

Now we assume that $G \boxtimes H$ has an independent $(1,1)$-dominating set $J$. Let $J_G = \pi_G(J)$ and $J'_G = V(G) \setminus J_G$, $J_H = \pi_H(J)$ and $J'_H = V(H) \setminus J_H$. The set of vertices of $G \boxtimes H$ can be written as

$$V(G \boxtimes H) = J \cup (J_G \times J'_H) \cup (J'_G \times J_H) \cup (J'_G \times J'_H).$$

Since $J$ is an independent set, by the definition of $G \boxtimes H$ we see that $J_G$ and $J_H$ are independent sets. We will show that they are maximal independent sets. For the contradiction, suppose that $J_G$ is not maximal. Then there exists a vertex $x' \in J'_G$ such that $N_G[x'] \cap J_G = \emptyset$. This would mean that the vertex $(x',y) \in V(G \boxtimes H)$, $y \in V(H)$ was not dominated, a contradiction. Analogously, it is proved that $J_H$ is also maximal. Now we will show that both graphs $G$ and $H$ must have an independent $(1,1)$-dominating set. Every vertex $(u,v) \in V(G \boxtimes H) \setminus J$ is $(1,1)$-dominated, so we have $|N_G(u) \cap J_G| \geq 2$ or $|N_H(v) \cap J_H| \geq 2$. We will prove that $J_G$ and $J_H$ are independent $(1,1)$-dominating sets. For the contradiction, let us assume that one of these sets is not an independent $(1,1)$-dominating set; without loss of generality let us assume that this is the set $J_G$. Then there exists a vertex $u' \in V(G) \setminus J_G$, which is not $(1,1)$-dominated by $J_G$. Let us consider a vertex $(u',y) \in J'_G \times J_H$. The set $J_H$ is independent, so $(u',y)$ must be $(1,1)$-dominated $J_G$, a contradiction. Therefore, $J_G$ and $J_H$ are independent $(1,1)$-dominating sets, which completes the proof. □

From Theorem 2.6 we conclude that the product $P_5 \boxtimes P_4$, presented in Figure 7, does not have an independent $(1,1)$-dominating set, because $P_4$ does not have one.

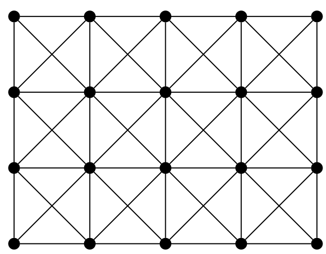

**Fig. 7.** The graph $P_5 \boxtimes P_4$ which does not have an independent $(1,1)$-dominating set

On the other hand, the graph $P_5$ has an independent $(1,1)$-dominating set, so the product $P_5 \boxtimes P_5$ also has one. It is indicated by the white colour in Figure 8.

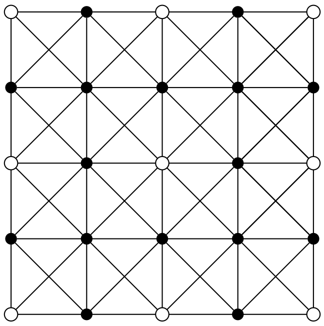

**Fig. 8.** The graph $P_5 \boxtimes P_5$ with an independent $(1,1)$-dominating set

From Theorem 1.1 and Theorem 2.4 we obtain a corollary concerning the existence of independent $(1,1)$-dominating sets in strong products of path, cycles and complete graphs.

**Corollary 2.7.** *Let $n, m$ be integers. Then the graph*

1. $C_n \boxtimes C_m$*, for* $n, m \geq 3$ *and both* $n, m$ *are even,*
2. $C_n \boxtimes P_m$*, for* $n, m \geq 3$ *and* $n$ *is even,* $m$ *is odd,*
3. $P_n \boxtimes P_m$*, for* $n, m \geq 3$ *and both* $n, m$ *are odd*

*has an independent* $(1,1)$*-dominating set.*

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