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SOME REMARKS AND RESULTS ON THE STANDARD $(2,2)$-CONJECTURE

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Abstract. In this note, we prove that every graph can be edge-labelled with red labels $1,2$ and blue labels $1,2$ so that vertices with any sum of incident red labels induce a 1-degenerate graph, while vertices with any sum of incident blue labels induce a 0-degenerate graph. This result stands as a closer step towards the so-called Standard $(2,2)$-Conjecture (stating that 0-degeneracy can be achieved in both colours), and provides some insight on the surrounding field, which covers the 1-2-3 Conjecture, the 1-2 Conjecture, and other close problems. Along the way, we also describe how many related problems are interconnected, and raise new problems and questions for further work on the topic.

Keywords: 1-2-3 Conjecture, 1-2 Conjecture, proper labelling, labelling.

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1. INTRODUCTION

In this note, we provide some remarks and results on the so-called 1-2-3 Conjecture and related problems For the sake of what is to come later on, we thus start by introducing and recalling some notions related to the 1-2-3 Conjecture.

1.1. 1-2-3 CONJECTURE

Let G be a (simple) graph. A *k-labelling* (or *labelling* for short) ℓ of G is an assignment $\ell : E(G) \rightarrow \{1,\ldots,k\}$ of *labels* (from $\{1,\ldots,k\}$) to the edges of G. Now, for every vertex u of G, one can compute the *sum* of labels assigned to edges incident to u, which we denote by $\sigma_\ell(u)$ (or simply by $\sigma(u)$ when no ambiguities are possible). We say ℓ is *proper* if σ_ℓ is a proper vertex-colouring of G, or, in other words, if $\sigma(u) \neq \sigma(v)$ for every edge $uv \in E(G)$. In case proper labellings of G exist, we say G is *nice* and we denote by $\chi^{\mathrm{e}}_{\Sigma}(G)$ the smallest $k \geq 1$ such that G admits proper k-labellings. Actually, it is known, see [16], that a graph is nice if and only if it has no K_2 as a connected component.

Let us mention to the reader that there are more concrete interpretations behind all these notions, refer e.g. to [5]. On the more theoretical side of things, most of the related investigations in the field have been on the maximum value that $\chi^{\mathrm{e}}_{\Sigma}(G)$ can attain for a nice graph G. Said investigations have been mostly driven by the so-called 1-2-3 Conjecture, introduced by Karoński, łuczak, and Thomason [16], which reads as follows.

1-2-3 Conjecture. If G is a nice graph, then $\chi^{\mathrm{e}}_{\Sigma}(G) \leq 3$.

Several aspects of the 1-2-3 Conjecture have been investigated since its introduction, see e.g. [21] for some survey. Some of these investigations ultimately led Keusch to recently provide a full solution to this problem.

Theorem 1.1 (Keusch [17])**.** *The* 1-2-3 *Conjecture holds true.*

Although the 1-2-3 Conjecture was fully verified, many of its aspects of interest remain to be fully understood. Such aspects include variants of the 1-2-3 Conjecture that have been introduced over the years. Said variants can be mostly regarded as simpler variants of the 1-2-3 Conjecture; however, some of these variants remain open to date, as, as far as we know, they are not implied by the fact that the 1-2-3 Conjecture holds true.

1.2. TOTAL AND DEGENERATE VARIANTS

A very legitimate concern is to wonder about the tightness of the 1-2-3 Conjecture. It has actually been known from the beginning, see [16], that graphs G with $\chi^{\mathrm{e}}_{\Sigma}(G) = 3$ do exist. Even worse is true, namely that deciding whether $\chi^{\mathrm{e}}_{\Sigma}(G) = 2$ holds for a given graph G is NP-complete [14], which implies, unless $\mathsf{P} = \mathsf{NP}$, that there is no "nice" characterisation of graphs requiring label 3 in their optimal proper labellings.

Still, an appealing question is how much needed label 3 actually is. A very first question, for instance, is whether nice graphs admit proper 3-labellings assigning "a few" 3's only. This was exactly studied in [6], in which the authors, through several constructions, show that there is no fixed $k \geq 1$ such that all nice graphs admit proper 3-labellings assigning label 3 to at most k edges only, and that there actually exist arbitrarily large connected graphs in which label 3 must be assigned to at least a $\frac{1}{10}$th of the edges.

Before pursuing, let us introduce a first variant of the 1-2-3 Conjecture, which, although unrelated to these concerns, actually has some interpretation under those considerations. Let G be a graph. A *k-total-labelling* (or *total-labelling* for short) τ is an assignment $\tau : V(G) \cup E(G) \to \{1, \ldots, k\}$ of labels to both the vertices and the edges of G. For any vertex u of G, we now define the sum $\sigma_\tau(u)$ (or $\sigma(u)$) as the sum of the label assigned to u and the labels assigned to its incident edges. We say that τ is proper if $\sigma_\tau(u) \neq \sigma_\tau(v)$ for all $uv \in E(G)$, and we define $\chi^{t}_{\Sigma}(G)$ as the smallest $k \geq 1$ such that G admits proper k-total-labellings. Similarly as for the 1-2-3 Conjecture, one can wonder how large $\chi^{t}_{\Sigma}(G)$ can be for a graph G, which led Przybyło and Woźniak to raise the following [20]:

1-2 Conjecture. If G is a graph, then $\chi^{t}_{\Sigma}(G) \leq 2$.

Proper k-total-labellings can actually be interpreted in terms of regular (i.e., labelling edges only) proper k-labellings. Indeed, if G is a graph, then note that, because we are assigning strictly positive labels, finding a proper k-total-labelling of G is equivalent to finding a proper k-labelling of the corona product $G \odot K_1$, obtained from G by attaching a leaf at every vertex. Thus, the 1-2 Conjecture actually asserts that, in terms of labels, we can do strictly better than the 1-2-3 Conjecture in a restricted class of graphs, or, put differently, if we get a little help from pendant edges yielding more labelling possibilities.

The 1-2 Conjecture, in parallel to the 1-2-3 Conjecture, also received some attention over the years, see [21]. Very recently, a full solution was proposed by Deng and Qiu.

Theorem 1.2 (Deng, Qiu [13])**.** *The* 1-2 *Conjecture holds true.*

To resume our explanations on the importance of label 3 for the 1-2-3 Conjecture, let us observe that a particular solution to the 1-2-3 Conjecture would imply the 1-2 Conjecture.

Observation 1.3. *If a nice graph* G *admits a proper* 3*-labelling where no two adjacent edges are assigned label* 3*, then* G *admits a proper* 2*-total-labelling.*

Proof. Let ℓ be any proper 3-labelling of G where no two adjacent edges are assigned label 3, and let τ be the 2-total-labelling obtained as follows. For every edge e of G set $\tau(e) = \ell(e)$, and for every vertex u of G set $\tau(u) = 1$. Now, for every edge uv of G with $\ell(uv) = 3$, change $\tau(uv)$ to 2, and change both $\tau(u)$ and $\tau(v)$ to 2. Because ℓ has the property that it does not assign label 3 to any two adjacent edges, it can be noticed that, for every vertex u of G, we get $\sigma_\ell(u) = \sigma_\tau(u)$. Since ℓ is proper, so is τ. □

Let us also mention that the solution to the 1-2 Conjecture of Deng and Qiu does not exploit this; thus, through proving that said proper 3-labellings always exist, then we would obtain an alternative proof of the 1-2 Conjecture. To this date, we are not aware of any nice graph in which all proper 3-labellings require to have adjacent edges assigned label 3.

Let us now discuss a second variant of the 1-2-3 Conjecture, where we go back to labelling edges only. For a graph G and a labelling ℓ of G, for any $x \geq 0$ we denote by $\mathcal{S}_x$ the subgraph of G induced by vertices with sum x. That is,

$$V(\mathcal{S}_x) = \{u \in V(G) : \sigma_\ell(u) = x\} \quad \text{and} \quad E(\mathcal{S}_x) = \{uv \in E(G) : \sigma_\ell(u) = \sigma_\ell(v) = x\}.$$

Put differently, the 1-2-3 Conjecture asserts that every nice graph admits a proper 3-labelling where all $\mathcal{S}_x$'s have no edges. As mentioned earlier, some graphs do require the use of label 3 in order to build proper 3-labellings. In other words, there are some graphs for which, in all 2-labellings, we will necessarily get *conflicts*, i.e., adjacent vertices with the same sum. But now, through the concept of $\mathcal{S}_x$'s, one may wonder about the possible structure of conflicts when assigning labels 1 and 2 only. For instance, Observation 1.3 implies that the 1-2 Conjecture would hold, if we could prove that every graph admits a 2-labelling where all $\mathcal{S}_x$'s are graphs of maximum degree at most 1.

These concerns were first investigated by Gao, Wang, and Wu in [15], and then further in [8]. The main wonder there is whether, through 2-labellings, we can keep conflicts sort of under control, namely so that they remain *degenerate* in the graph theoretical sense. More precisely, this leads to the following:

Degenerate 1-2-3 Conjecture. Every graph admits a 2-labelling where all $\mathcal{S}_x$'s have no cycles, i.e., are 1-degenerate.

Regarding this Degenerate 1-2-3 Conjecture, let us just mention that, in [8, 15], the authors verify it for some classes of graphs, but that it is widely open otherwise. And let us insist on the fact that one can just see this conjecture as some guess on the type of conflicts we can restrict ourselves to, when designing labellings with labels 1 and 2 only.

1.3. WEAK, STANDARD, AND STRONG (2,2)-CONJECTURES

We last introduce a set of three conjectures introduced by Baudon, Bensmail, Davot, Hocquard, Przybyło, Senhaji, Sopena, and Woźniak in [1], which, again, provide some interpretation regarding the significance of label 3 for the 1-2-3 Conjecture. For transparency, let us insist on the fact that the terminology introduced and used below, based on labellings, is different from the original one from [1], based on edge-colourings. Our main reason for performing this deviation is that all notions and problems discussed earlier and the upcoming ones are better unified, in our opinion, through the labelling formalism than they are through any edge-colouring or edge-weighting formalism. Any reader familiar to the investigations in [1] should just, in what follows, regard (k, l)-labellings as (k, l)-edge-colourings.

Let G be a graph, and $k, l \geq 1$ be two integers. A *(k, l)-labelling* ℓ of G is an assignment $\ell : E(G) \to \{1, \ldots, k\} \times \{1, \ldots, l\}$ of *coloured labels* to the edges of G, where each assigned label (x, y) has *value* $x \in \{1, \ldots, k\}$ and *colour* $y \in \{1, \ldots, l\}$. Note that there is a straight correspondence between labellings in the usual sense and some of these coloured labellings, as k-labellings are nothing but $(k, 1)$-labellings. This more general terminology of (k, l)-labellings was actually introduced in [1] to generalise a bunch of other problems related to the 1-2-3 Conjecture which we will not mention further here.

For every vertex u of G and any $i \in \{1, \ldots, l\}$, one can now compute the *i-sum* of u, denoted by $\sigma^i_\ell(u)$ (or just $\sigma^i(u)$), being the sum of the values of the coloured labels with colour i assigned to the edges incident to u. So every vertex of G gets now associated l different coloured sums, and there are thus plenty of ways to consider that any two adjacent vertices of G are distinguished w.r.t. ℓ. Furthermore, as mentioned in [1], it turns out that (k, l)-labellings with $\max\{k, l\} \geq 3$ relate to the 1-2-3 Conjecture and other problems. All these concerns led the authors of [1] to narrow their concern to $(2, 2)$-labellings. In this context, to make the terminology more digest, for convenience we can consider that $(2, 2)$-labellings of G assign labels 1 and 2 that are either *red* or *blue*, and that each vertex u gets associated a *red sum* $r(u)$ and a *blue sum* $b(u)$. And, for any $x \geq 0$, we denote by $\mathcal{R}_x$ the subgraph of G with vertex set $\{u \in V(G) : r(u) = x\}$ and edge set $\{uv \in E(G) : r(u) = r(v) = x\}$; that is, the graph with red edges of G

joining vertices with red sum x. We define $\mathcal{B}_x$, the subgraph of G w.r.t. blue edges and blue sums, similarly.

W.r.t. these notions, in [1] the authors proposed to focus on three main conjectures inspired from the 1-2-3 Conjecture, and of increasing complexity. Note that, in the latter two, a *nicer graph* designates a graph with no connected component being K_2 or K_3.

Weak $(2,2)$-Conjecture. Every nice graph admits a $(2,2)$-labelling where we have $r(u) \neq r(v)$ or $b(u) \neq b(v)$ for every edge uv.

Standard $(2,2)$-Conjecture. Every nicer graph admits a $(2,2)$-labelling where we have $r(u) \neq r(v)$ for every red edge uv, and $b(u) \neq b(v)$ for every blue edge uv.

Strong $(2,2)$-Conjecture. Every nicer graph admits a $(2,2)$-labelling where we have $r(u) \neq r(v)$ and $b(u) \neq b(v)$ for every edge uv, except when $0 \in \{r(u), r(v)\}$ and $0 \in \{b(u), b(v)\}$, respectively.

Their own interest apart, these three conjectures are also interesting in that they connect to some of the concerns mentioned earlier on the 1-2-3 Conjecture, and, in particular, with the deep role of label 3. For instance, Bensmail observed in [3] that, given any proper 3-labelling of a graph G, making all 1's and 2's red and turning all 3's into blue 1's results in a $(2,2)$-labelling where $r(u) \neq r(v)$ or $b(u) \neq b(v)$ for all $uv \in E(G)$. From this and Theorem 1.1, this yields that the Weak $(2,2)$-Conjecture holds true.

This apart, the Standard and Strong $(2,2)$-Conjectures are still widely open to date, although some partial results are provided in [1, 4, 19]. In this work, we focus on the Standard $(2,2)$-Conjecture. Our main intent is to investigate the impact of recent Theorem 1.2: just as how Theorem 1.1 implies the Weak $(2,2)$-Conjecture, one may wonder how close to the Standard $(2,2)$-Conjecture we can get, when assuming that the 1-2 Conjecture holds true. W.r.t. our previous words on the significance of label 3 for the 1-2-3 Conjecture, note that the Standard $(2,2)$-Conjecture sort of asks whether label 3 can be assigned to edges inducing a graph admitting proper 2-labellings.

1.4. OUR RESULT

In these lines, our main result incorporates some of the notions behind the Degenerate 1-2-3 Conjecture. Namely, we get closer to the Standard $(2,2)$-Conjecture by proving the following.

Theorem 1.4. *Every graph G can be labelled with red labels $1,2$ and blue labels $1,2$ so that all $\mathcal{R}_x$'s are 1-degenerate, and all $\mathcal{B}_x$'s are 0-degenerate.*

We actually prove a stronger statement; namely, in our proof of Theorem 1.4, all $\mathcal{R}_x$'s are actually forests of trees of diameter at most 3. An interesting aspect in our proof, in our opinion, is that it combines several tools and elements that are classical in the field: maximum independent sets, the fact that the 1-2 Conjecture holds (Theorem 1.2), the characterisation from [22] of bipartite graphs G with $\chi^{\mathrm{e}}_{\Sigma}(G) = 3$, and, when assigning labels 1 and 2 only, changing labels along paths to reach sums with desired parities.

Our proof of Theorem 1.4 is given in Section 2. After that, we provide questions and directions for further work on the topic, in Section 3.

2. PROOF OF THEOREM 1.4

Throughout, we may assume G is connected. Since the Standard $(2,2)$-Conjecture holds for bipartite graphs [1], we may also assume G is not bipartite.

Let X be a maximum independent set of G, and Y be a maximum independent set of $G-X$. Set $G'=G-X-Y$. Since G is not bipartite, G' is not empty. Also, since X and Y are maximum independent sets of G and $G-X$, respectively, all vertices of $G-X$ have at least one neighbour in X, and all vertices of G' also have at least one neighbour in Y.

To reach the desired labelling ℓ of G, we first assign colours to the edges of G and will later focus on assigning values to form coloured labels. Namely, we assign colour

- red to every edge of G',
- red to every edge of (X,Y),
- blue to every edge uv with $u \in V(G')$ and $v \in X \cup Y$,
- for every vertex $u \in V(G')$, we choose any neighbour $p_u \in X$ of u in X (which exists since X is a maximum independent set of G), and change the colour of up_u to red.

Note that $\mathcal{B}$, the subgraph of G induced by all blue edges, is bipartite, as it contains all edges of $(V(G'),Y)$ and most edges of $(V(G'),X)$. In particular, no blue edge belongs to (X,Y). So $\mathcal{B}$ is bipartite with bipartition $(X \cup Y, V(G)-X-Y)$.

For upcoming arguments to work, we need to make sure that $\mathcal{B}$ does not contain an isolated edge (i.e., an edge uv with $d_{\mathcal{B}}(u)=d_{\mathcal{B}}(v)=1$). Assuming $\mathcal{B}$ contains an isolated edge uv with, say, $u \in V(G')$ and $v \in X \cup Y$, we infer that actually $v \in Y$, since u is necessarily incident to an edge going to Y (by maximality of Y), and that edge was assigned colour blue. For uv to be isolated in $\mathcal{B}$, by how we assigned colours we also deduce that u is incident to exactly one edge (assigned colour red) $up_u \neq uv$ going to X, and that, currently, no edge incident to v going to X is assigned colour blue. We here consider w, any neighbour of v in X (which exists by maximality of X), and change the colour of vw to blue. Note that this preserves that $\mathcal{B}$ is bipartite, since, essentially, this adds or attaches (to w) a pendant path of length 2 to $\mathcal{B}$. By repeating this process of changing to blue the colour of some edges of (X,Y), we can thus get rid of isolated edges in $\mathcal{B}$, all the while preserving the bipartiteness of $\mathcal{B}$ and the property that all vertices u of G' are incident to a single red edge up_u with $p_u \in X \cup Y$, where actually $p_u \in X$.

We pursue now by deciding how to assign labels (in $\{1,2\}$) to blue edges (i.e., edges of $\mathcal{B}$). For that, we consider the connected components of $\mathcal{B}$ independently. Let B be any connected component of $\mathcal{B}$. Recall that B is bipartite and is not just an edge. If B can be 2-labelled in a proper/0-degenerate way (i.e., $\chi^{\mathrm{e}}_{\Sigma}(B) \leq 2$), then it suffices to assign labels 1 and 2 to edges of B following a proper 2-labelling. Otherwise, B is a so-called *odd multi-cactus* according to [22]. Odd multi-cacti are a bit tedious to describe, so for a more thorough insight we refer the interested reader to e.g. [10, 11, 22]. In brief, every connected odd multi-cactus different from K_2 is obtained from an initial cycle of length congruent to 2 modulo 4 by repeatedly adding paths of length 1 modulo 4 in a particular fashion (in particular, preserving 2-edge-connectivity). In particular, odd multi-cacti have minimum degree 2. An important point for us, is that every connected odd multi-cactus different from K_2 contains a path $u_1u_2u_3u_4$ where $d(u_1) = d(u_2) = d(u_3) = d(u_4) = 2$. So, B contains such a path $u_1u_2u_3u_4$ where, by the bipartition of B, we have, say, $u_2 \in X \cup Y$ and $u_3 \in V(G')$. Recall that we cannot have $u_2, u_3 \in X \cup Y$, since the only reason this may occur would require u_2u_3 to have been recoloured blue, which, as mentioned earlier, would imply B is of minimum degree 1, thus not an odd multi-cactus. Recall that u_3 is incident to a single red edge $u_3p_{u_3}$ with $p_{u_3} \in X$; we change to blue the colour assigned to $u_3p_{u_3}$ and to red the colour of u_2u_3. Since we swapped these colours around u_3, note that u_3 remains incident to a single red edge going to $X \cup Y$. Regarding the resulting B, note that it may now form a bigger connected component of $\mathcal{B}$ (since $u_3p_{u_3}$ might have connected the original B and another connected component of $\mathcal{B}$); however, in this new B vertex u_2 is now of degree 1. Thus, this new B is not an odd multi-cactus, and can be labelled in a proper way with labels 1 and 2. Note as well that swapping a red edge and a blue edge as we did cannot create an isolated edge in $\mathcal{B}$.

Applying those arguments for all connected components B of $\mathcal{B}$, eventually we have all blue edges assigned labels 1 and 2, in such a way that $b(u) \neq b(v)$ for every blue edge uv. So, every $\mathcal{B}_x$ is 0-degenerate, as required.

It remains to deal with red edges. Recall that, at this point, all edges of G' are assigned colour red, for every vertex u of G' there is a single red edge up_u for some $p_u \in X \cup Y$, and some edges of (X, Y) (but not necessarily all of them) are also assigned colour red.

Since the 1-2 Conjecture holds (recall Theorem 1.2), there is a proper 2-total-labelling τ of G'. We pursue the design of ℓ by first assigning, to every (red) edge e of G', label $\tau(e) \in \{1, 2\}$ to e. Now, for every $u \in V(G')$, we assign label $\tau(u) \in \{1, 2\}$ to the red edge up_u. As a result, all red edges of G' and all red edges of the form up_u are 2-labelled, and for every vertex u of G' we get $r(u) = \sigma_\tau(u)$. In particular, since τ is proper, we already have $r(u) \neq r(v)$ for every (red) edge uv of G'.

The only (red) edges that remain to be labelled are red edges in (X, Y). For convenience, let $\mathcal{R}$ be the subgraph of G induced by the red edges of (X, Y). Let R be any connected component of $\mathcal{R}$. Every vertex u of R currently has a partial red sum $r_u \geq 0$, due to how are labelled all red edges incident to u going to G'.

Start by assigning label 2 to all (red) edges of R. As a result, we can partition the vertices of R into four (possibly empty) parts: X_e contains vertices of X with current red sum even, X_o contains vertices of X with current red sum odd, Y_e contains vertices of Y with current red sum even, and Y_o contains vertices of Y with current red sum odd. Because we assigned label 2 to all edges of R, these four sets X_e, X_o, Y_e, Y_o are actually defined by the parity of the r_u's. We now employ common arguments of the field, being that, in any 2-labelled graph, changing 1's to 2's (and *vice versa*) as traversing a walk joining any two vertices u and v changes the parity of the sums of u and v only (see e.g. [2, 9, 22] and the references therein for more on this process). Using this, we can change labels of R so that (when taking into account the r_u's) all vertices of X have, say, even red sum, while all vertices in Y have odd red sum, except maybe for only one vertex. Indeed:

(1) If $|X_o|$ is even, then we can repeatedly pick any two new vertices u and v of X_o, a path P joining u and v in R, and change labels of P (as described above). This results in all vertices of X_o to have even red sum. Then:
   (i) If $|Y_e|$ is even, then, in the very same way, we can change labels in R so that all vertices of Y_e become of odd red sum. As a result, all vertices (of R) in X have even red sum, while all vertices in Y have odd red sum.
   (ii) If $|Y_e|$ is odd, then we can essentially achieve the same but for one vertex of Y_e.

(2) If $|X_o|$ is odd, then, by the same arguments, we can modify labels in R so that all its vertices in X but one, call it u, have even red sum. Then:
   (i) If $|Y_e|$ is even, then, again, we can modify labels so that, additionally, all vertices in Y have odd red sum.
   (ii) If $|Y_e|$ is odd, then this can also be achieved, except for one vertex v of Y, which has even red sum. In this case, we can correct the parity of the red sums of u and v by changing labels along a path joining u and v. Thus, eventually, all vertices of R in X have even red sum while all vertices in Y have odd red sum.

We claim that, after performing this for all connected components R of $\mathcal{R}$, the $\mathcal{R}_x$'s are 1-degenerate, as required. First off, recall that $r(u) = \tau(u)$ for all $u \in V(G')$, while τ is a proper 2-total-labelling of G'. Therefore, there are no two adjacent vertices of G' (thus joined by a red edge) with the same red sum. Regarding vertices of X and Y, we have that all vertices of X have even red sum while all vertices of Y have odd red sum, with the exception of at most one vertex in every connected component R of $\mathcal{R}$. Also, every vertex of G' is incident to exactly one red edge going to $X \cup Y$. This implies that, in any $\mathcal{R}_x$, any vertex of $V(G')$ is of degree at most 1. Thus, if some $\mathcal{R}_x$ is not 1-degenerate, i.e., contains some cycle, then it must be through vertices in $X \cup Y$. This is not possible, however, as it would imply that $\mathcal{R}$, the graph induced by the red edges of (X, Y), contains a connected component R with at least two vertices not fulfilling their required parity condition for their red sums, which is not possible due to how we dealt with them earlier. Thus, any $\mathcal{R}_x$ is 1-degenerate as desired.

Thus, all $\mathcal{B}_x$'s are 0-degenerate, while all $\mathcal{R}_x$'s are 1-degenerate, as desired. This concludes the proof of Theorem 1.4. □

3. QUESTIONS AND DIRECTIONS FOR FURTHER WORK

With Theorem 1.4, we get quite close to the Standard $(2, 2)$-Conjecture. Although it is interesting that our proof employs the fact that the 1-2 Conjecture holds true (recall

Theorem 1.2), it might not be possible to go further with this approach. Recall that we use Theorem 1.2 to assign red labels, the 1-degenerate part of the result, and we believe going further would require to go farther with the proof of Theorem 1.2 from [13].

As another way to approach the Standard $(2, 2)$-Conjecture, one could try to get closer by improving our result with a better condition for the red subgraph. For instance, we would be curious to see a proof of the same result, but where the $\mathcal{R}_x$'s are e.g. of maximum degree at most 1 (i.e., matchings). The Standard $(2, 2)$-Conjecture apart, one could also wonder about getting closer to the Strong $(2, 2)$-Conjecture in a similar way. The distinguishing requirement in this latter stronger problem being much more constrained, we note that our proof of Theorem 1.4 would just fall apart here.

W.r.t. the discussion given in the introductory section, we believe the following questions might sound appealing to the reader.

The 1-2-3 Conjecture and related problems have also been investigated w.r.t. products. That is, requiring, through a labelling, that adjacent vertices are distinguished w.r.t. the products of labels assigned to their incident edges. In particular, a product version of the 1-2-3 Conjecture was proved in [7], while a product version of the 1-2 Conjecture was proved in [12]. For the latter result, this is because labels 1 and 2 in the product convention act equivalently as labels 0 and 1 in the sum convention.

We also wondered about proving a result akin Theorem 1.4 but under the product convention. While most of our arguments still apply the same way, a complicated point is that, contrarily to the sum context, there is no known characterisation of bipartite graphs admitting no proper $\{0, 1\}$-labellings. As far as we are aware this problem is still open to date, despite dedicated works such as [18], and we believe it would deserve more attention.

As mentioned in the introductory section, the 1-2 Conjecture would also follow from the fact that nice graphs admit proper 3-labellings with no two adjacent edges assigned label 3 (recall Observation 1.3). To date, we are still not aware of any graph G with $\chi^{\mathrm{e}}_{\Sigma}(G) = 3$ having, necessarily, adjacent edges assigned label 3 by any proper 3-labelling. As a first step towards this question, it might be interesting to investigate the existence of proper 3-labellings where assigned 3's have particular properties, for instance induce degenerate structures.

In this vein, we also wonder if the Standard $(2, 2)$-Conjecture could be strengthened further, by allowing labels 1 and 2 to be assigned in one of the two colours, but restricting the other colour to label 1. In other words, we wonder whether it is possible, in any nicer graph G, to find a set $F \subseteq E(G)$ of edges such that $\chi^{\mathrm{e}}_{\Sigma}(G - F) \leq 2$ and $G[F]$ is locally irregular, i.e., has no two adjacent vertices with the same degree. Note that the same question but under the convention of the Weak $(2, 2)$-Conjecture admits a positive answer, by arguments from [3]. Meanwhile, this is not true under the convention of the Strong $(2, 2)$-Conjecture. To see this, consider any complete graph G: whatever set F of edges of G with $G[F]$ locally irregular we consider, because $G[F]$ is simple it must have two vertices u and v with the same degree, which can be checked leads to a contradiction. We were not able to find a similar construction under the convention of the Standard $(2, 2)$-Conjecture, and, due to its intermediate status, we wonder whether the answer is positive here.

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