Opuscula Math. 46, no. 1 (2026), 101–119
https://doi.org/10.7494/OpMath.202511231 Opuscula Mathematica

GALERKIN-TYPE MINIMIZERS TO A COMPETING PROBLEM FOR $(\overrightarrow{p},\overrightarrow{q})$-LAPLACIAN WITH VARIABLE EXPONENTS

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Communicated by Marek Galewski

Abstract. This study focuses on a sequence of approximate minimizers for the functional

$$J(u)=\int\limits_{\Omega}\sum_{i=1}^{N}\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx-\mu\int\limits_{\Omega}\sum_{i=1}^{N}\frac{1}{q_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{q_i(x)}dx-\int\limits_{\Omega}F(u(x))dx,$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain, and $p_i,q_i\in C(\overline{\Omega})$ with $1<p_i,q_i<+\infty$ for all $i\in\{1,\ldots,N\}$. We establish the convergence result to the infimum of $J(u)$ when $F:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz function of controlled growth, following the Galerkin method. As an application, we establish the existence of solutions to a class of Dirichlet inclusions associated to the functional.

Keywords: anisotropic Sobolev space, Clarke generalized gradient, Dirichlet problem, Galerkin basis, $(\overrightarrow{p},\overrightarrow{q})$-Laplacian with variable exponents.

Mathematics Subject Classification: 46E30, 47J22.

1. MAIN THEOREM AND ENVIRONMENT

Let $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ be a bounded domain with a smooth boundary $\partial\Omega$. In this paper we consider the following functional

$$J(u)=\int\limits_{\Omega}\sum_{i=1}^{N}\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx-\mu\int\limits_{\Omega}\sum_{i=1}^{N}\frac{1}{q_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{q_i(x)}dx-\int\limits_{\Omega}F(u(x))dx \tag{1.1}$$

for all $u\in W_0^{1,\overrightarrow{p}(x)}(\Omega)$, where $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is the anisotropic Sobolev space determined by the variable exponent $\overrightarrow{p}(x)=(p_1(x),p_2(x),\ldots,p_N(x))$, $p_i,q_i\in C(\overline{\Omega})$ and

$1 < p_i(x), q_i(x) < +\infty$ for all $x \in \overline{\Omega}$ and all $i \in I := \{1, \dots, N\}$, also $\mu \in \mathbb{R}$ is a real parameter, and $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function of the following growth:

(G) there are a function $\alpha \in C(\overline{\Omega})$ with $1 < \alpha^+ < P_-^-$ and constants $c, \widehat{c} > 0$ satisfying

$$|z| \leq c + \widehat{c}|t|^{\alpha(x)-1} \quad \text{for all } t \in \mathbb{R}\ ,\ x \in \Omega \text{ and } z \in \partial F(t),$$

where $\partial F(t)$ denotes the Clarke subdifferential of $F(t)$, $\alpha^+ = \max_{x\in\overline{\Omega}} \alpha(x)$, $P_-^- = \min_{i\in I} p_i^-$ and $p_i^- = \min_{x\in\overline{\Omega}} p_i(x)$, $i \in I$.

Condition (G) is essential to control the third term (reaction) in (1.1), leading to well-posedness and providing a priori estimates (hence, boundedness) for the functional. A relevant feature of our functional is the presence of a parameter $\mu \in \mathbb{R}$ that acts as switching coefficient between the case of an elliptic functional $(0 \geq \mu)$ and the case of a non-elliptic functional $(0 < \mu)$. As for the elliptic case, the analysis of the functional (1.1) may benefit of suitable monotonicity arguments which also have a key role in the proof of existence results to various classes of differential equations and inclusions, see the books by Motreanu *et al.* [18], and by Rădulescu and Repovš [20] for more background. Differently, the non-elliptic case inhibits the usage of monotonicity arguments, hence leading to some technical difficulties. However, we can overcome such problems by the use of a discretized Galerkin approach together with certain convergence arguments. In the framework of separable Banach spaces, we denote by $\{X_n\}_{n\in\mathbb{N}}$ a Galerkin basis of $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, that is a sequence of vector spaces such that

$$\dim(X_n) < +\infty \quad \text{for all } n \in \mathbb{N} \text{ (finite dimension)}, \tag{1.2}$$

$$X_n \subseteq X_{n+1} \quad \text{for all } n \in \mathbb{N} \text{ (nesting property)}, \tag{1.3}$$

$$\overline{\bigcup_{n=1}^{\infty} X_n} = W_0^{1,\overrightarrow{p}(x)}(\Omega) \quad \text{(covering property)}. \tag{1.4}$$

Our primary goal is to establish a convergence result to the infimum of functional (1.1) over $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, by construction of a suitable sequence of approximate minimizers for (1.1) over X_n (for all $n \in \mathbb{N}$). So, we first give the notion of approximate minimizer in the finite-dimensional vector space X_n.

Definition 1.1. We say that $u_n \in X_n$, $n \in \mathbb{N}$, is an approximate minimizer of the functional (1.1) if $J(u_n) = \inf\{J(v) : v \in X_n\}$.

Remark 1.2. If $u_n \in X_n$ is an approximate minimizer for (1.1), as a consequence we get that for some $z_n \in \partial F(u_n)$ a.e. on Ω one has

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_n}{\partial x_i}\frac{\partial h}{\partial x_i} dx - \mu \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial u_n}{\partial x_i}\frac{\partial h}{\partial x_i} dx - \int_\Omega z_n h dx = 0 \tag{1.5}$$

for all $h \in X_n$, $n \in \mathbb{N}$.

Now, let $(W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ denote the topological dual space of $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, then the concept of Galerkin-type minimizer is given as follows, hence it is understood as the generated sequence of approximate minimizers over the Galerkin basis of $W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

Definition 1.3. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$ be a sequence of approximate minimizers of the functional (1.1) and assume that equation (1.5) holds for $z_n \in (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$, $n \in \mathbb{N}$. We say that $\{u_n\}_{n\in\mathbb{N}}$ is a Galerkin-type minimizer of the functional (1.1) if

$$\lim_{n\to+\infty} J(u_n) = \inf\{J(v) : v \in W_0^{1,\overrightarrow{p}(x)}(\Omega)\}.$$

In Definition 1.3, we precisely mean that for each $n \in \mathbb{N}$, equation (1.5) holds for some $z_n \in (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ and $z_n \in \partial F(u_n)$ a.e. on Ω. Hence, we state our main result.

Theorem 1.4. *Let $p_i, q_i \in C(\overline{\Omega})$ for all $i \in I$ such that $1 < Q_-^- \leq Q_+^+ < P_-^- < N$. If (G) is satisfied and $u_n \in X_n \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$ fulfills equation (1.5), then the functional (1.1) has a Galerkin-type minimizer over $W_0^{1,\overrightarrow{p}(x)}(\Omega)$.*

We note that Theorem 1.4 gives the relation between the variable exponents for the $(\overrightarrow{p}, \overrightarrow{q})$-Laplacian defined by

$$-\Delta_{\overrightarrow{p}(x)}u + \mu\Delta_{\overrightarrow{q}(x)}u = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i} - \mu\left|\frac{\partial u}{\partial x_i}\right|^{q_i(x)-2}\frac{\partial u}{\partial x_i}\right)$$

for all $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

However, as far as we know, such operator has not been systematically evaluated. Our analysis here fills-in this gap of the literature, by involving a precise extension of the main arguments for [14, 17] to the anisotropic setting. In detail, we observe that the following functionals can be deduced as particular cases of (1.1):

– If the functional (1.1) involves the (p, q)-Laplacian with variable exponents, hence it reduces to the functional

$$J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \mu\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx - \int_\Omega F(u(x))dx \qquad (1.6)$$

for all $u \in W_0^{1,p(x)}(\Omega)$, also assuming the standard conditions on the variable exponents $p, q \in C(\overline{\Omega})$:

$$\begin{aligned} 1 < q^- &= \min_{x\in\overline{\Omega}} q(x) \leq q(x) \leq q^+ = \max_{x\in\overline{\Omega}} q(x) \\ < p^- &= \min_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ = \max_{x\in\overline{\Omega}} p(x) < +\infty. \end{aligned}$$

Such model is considered by Ghasemi *et al.* [14], who involve embedding results and useful estimates to show first that (1.6) is locally Lipschitz and coercive (see [14, Proposition 4]), then they use the Galerkin approach to show existence and boundedness of local minimizers to (1.6) in [14, Propositions 5 and 6], finally the result corresponding to our Theorem 1.4 can be deduced by [14, Proposition 8], where in the proof the authors argue by contradiction.

– If the functional (1.1) involves the (p, q)-Laplacian with constant exponents, hence it reduces to the functional

$$J(u) = \frac{1}{p}\int_{\Omega} |\nabla u|^p dx - \frac{\mu}{q}\int_{\Omega} |\nabla u|^q dx - \int_{\Omega} F(u(x))dx \tag{1.7}$$

for all $u \in W_0^{1,p}(\Omega)$, $1 < q < p < +\infty$.

Such model is considered by Motreanu [17], who uses embedding results and Hölder inequality to establish first that (1.7) is locally Lipschitz and coercive ([17, Proposition 1]) then, involving the Galerkin basis of $W_0^{1,p}(\Omega)$, proves existence and boundedness of local minimizers to (1.7) in [17, Corollary 2 and Proposition 3]. Reasoning by contradiction, the author concludes the existence of a sequence of minimizers in [17, Corollary 5].

Clearly the analysis of functionals (1.1), (1.6) and (1.7) cannot leave aside imposing an appropriate growth condition on the locally Lipschitz function F (for more details, compare the growth (G) above with the corresponding conditions H_f and (H) in [14] and [17], respectively).

Anisotropic functionals with $\mu = 0$ were investigated by Bonanno *et al.* [1] (constant exponent), Chems Eddine *et al.* [4], Fan [9] and Tavares [25] (variable exponent), further anisotropic functionals with $\mu = -1$ were studied by Tavares [25], Razani and colleagues [23, 24] (constant exponents). These works provide the readers with a solid understanding of topological and variational methods used to evaluate the effects of reaction terms on the structure of functionals. For more results on non-elliptic functionals, we mention the works by Diblík *et al.* [6], Galewski and Motreanu [12], Liu *et al.* [15], Vetro and Efendiev [26], and for elliptic functionals we refer to the recent investigations by El Yazidi *et al.* [8], Papageorgiou *et al.* [19], Zeng *et al.* [27], and the references cited therein. We finally mention the very recent monograph by Galewski and Motreanu [13] devoted to the study of boundary value problems driven by principal operators which lack monotonicity.

The organization of the manuscript is as follows. In Section 2 we briefly review the variable anisotropic Sobolev spaces. In Section 3 we provide the detailed analysis of functional (1.1). In Section 4 we apply Theorem 1.4 to solve the existence problem for a class of differential inclusions with Dirichlet boundary condition.

2. SPACE $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ and embedding in $L^{s(x)}(\Omega)$

The analysis of functional (1.1) is carried out in the variable anisotropic Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, but it also requires the variable Lebesgue space $L^{s(x)}(\Omega)$, for suitable exponent $s \in C(\overline{\Omega})$ as will be precised later on. In this section, we mainly follow the works by Mihăilescu *et al.* [16] and Fan [9], but for a comprehensive covering of variable Lebesgue and Sobolev spaces, and their involvement in the study of PDEs, we suggest the books by Diening *et al.* [7] and Rădulescu and Repovš [20].

The variable Lebesgue space denoted by $L^{r_i(x)}(\Omega)$ is defined as

$$L^{r_i(x)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u(x)|^{r_i(x)} dx < +\infty \right\},$$

where $M(\Omega)$ is the space of measurable functions $u : \Omega \to \mathbb{R}$, and $r_i \in C(\overline{\Omega})$ with $1 < r_i^- := \min\{r_i(x) : x \in \overline{\Omega}\}$, $i \in I$. So, introducing the so-called modular function $\rho_{r_i(x)} : M(\Omega) \to [0, +\infty]$ defined by

$$\rho_{r_i(x)}(u) := \int_\Omega |u(x)|^{r_i(x)} dx \quad \text{for all } u \in M(\Omega),\ i \in I, \tag{2.1}$$

and referring to the Luxemburg norm given by

$$\|u\|_{r_i(x)} := \inf \left\{ \lambda > 0 : \rho_{r_i(x)}\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

we know that the space $(L^{r_i(x)}(\Omega), \|\cdot\|_{r_i(x)})$ is a separable and reflexive Banach space. Letting $r_i' \in C(\overline{\Omega})$ the Hölder conjugate exponent to r_i, that is $r_i'(x) = r_i(x)/(r_i(x)-1)$, for any $x \in \overline{\Omega}$, $i \in I$, it leads to the following Hölder inequality

$$\int_\Omega |u\, h| dx \leq \left(\frac{1}{r_i^-} + \frac{1}{(r_i')^-}\right) \|u\|_{r_i(x)} \|h\|_{r_i'(x)} \leq 2\|u\|_{r_i(x)} \|h\|_{r_i'(x)} \tag{2.2}$$

for $u \in L^{r_i(x)}(\Omega)$ and $h \in L^{r_i'(x)}(\Omega)$.

If $r_i, s \in C(\overline{\Omega})$ with $r_i(x) \geq s(x)$ for all $x \in \overline{\Omega}$, then $L^{r_i(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is a continuous embedding. Involving $L^{r_i(x)}(\Omega)$, we can introduce the variable Sobolev space

$$W^{1,r_i(x)}(\Omega) := \{u \in L^{r_i(x)}(\Omega) : |\nabla u| \in L^{r_i(x)}(\Omega)\}.$$

We endow this space with the norm

$$\|u\|_{1,r_i(x)} = \|u\|_{r_i(x)} + \|\nabla u\|_{r_i(x)},$$

where as usual we set $\|\nabla u\|_{r_i(x)} := \||\nabla u|\|_{r_i(x)}$, and ∇u is the weak gradient of u. It is on this basis that we can consider the Sobolev space $W_0^{1,r_i(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r_i(x)}}$. We know that both the spaces $W^{1,r_i(x)}(\Omega)$ and $W_0^{1,r_i(x)}(\Omega)$ are separable and uniformly convex (hence, reflexive) Banach spaces. Also, for some constant $c_1 > 0$, we have the following version of Poincaré inequality:

$$\|u\|_{r_i(x)} \leq c_1 \|\nabla u\|_{r_i(x)} \text{ for all } u \in W_0^{1,r_i(x)}(\Omega). \tag{2.3}$$

Consequently, on $W_0^{1,r_i(x)}(\Omega)$, we can use the norm

$$\|u\|_{1,r_i(x)} := \|\nabla u\|_{r_i(x)} \text{ for all } u \in W_0^{1,r_i(x)}(\Omega).$$

The norm $\|\cdot\|_{r_i(x)}$ and the modular function $\rho_{r_i(x)}$ (see (2.1)) are closely related by the following proposition.

Proposition 2.1 ([10, Theorem 1.3])**.** *If $r_i \in C(\overline{\Omega})$ with $1 < r_i^-$, $i \in I$, and $u \in L^{r_i(x)}(\Omega)$, then the following hold:*

(i) $\|u\|_{r_i(x)} < 1$ *(resp.* $= 1, > 1) \Leftrightarrow \rho_{r_i(x)}(u) < 1$ *(resp.* $= 1, > 1)$,

(ii) *if* $\|u\|_{r_i(x)} > 1$*, then* $\|u\|_{r_i(x)}^{r_i^-} \le \rho_{r_i(x)}(u) \le \|u\|_{r_i(x)}^{r_i^+}$,

(iii) *if* $\|u\|_{r_i(x)} < 1$*, then* $\|u\|_{r_i(x)}^{r_i^+} \le \rho_{r_i(x)}(u) \le \|u\|_{r_i(x)}^{r_i^-}$.

According to Proposition 2.1, we deduce that

$$\|u\|_{r_i(x)}^{r_i^+} + 1 \ge \rho_{r_i(x)}(u) \ge \|u\|_{r_i(x)}^{r_i^-} - 1 \text{ for all } u \in L^{r_i(x)}(\Omega). \tag{2.4}$$

We introduce the critical Sobolev exponent r_i^* corresponding to $r_i \in C(\overline{\Omega})$ with $1 < r_i^-$, $i \in I$, as follows:

$$r_i^*(x) = \begin{cases} \frac{N r_i(x)}{N - r_i(x)} & \text{if } r_i(x) < N, \\ +\infty & \text{if } N \le r_i(x), \end{cases} \quad \text{for all } x \in \overline{\Omega}. \tag{2.5}$$

The following embedding result holds true.

Proposition 2.2. *Let $i \in I$. If $r_i, s \in C(\overline{\Omega})$ with $1 < r_i^-, s^-$ and $r_i^*(x) > s(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,r_i(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact.*

According to Proposition 2.2, we can find a constant $c_2 > 0$ (depending on $s \in C(\overline{\Omega})$) satisfying the inequality

$$\|u\|_{s(x)} \le c_2 \|\nabla u\|_{r_i(x)} \text{ for all } u \in W_0^{1,r_i(x)}(\Omega). \tag{2.6}$$

We can now extend the above theory to anisotropic Sobolev spaces, considering the vectorial function $\overrightarrow{r} : \overline{\Omega} \to \mathbb{R}^N$ given as $\overrightarrow{r}(x) = (r_1(x), r_2(x), \dots, r_N(x))$, see [9, 16]. We first complete the notions and notation partially introduced in Section 1, but for readers' convenience we also repeat some of the already given ones. For $r_i, \in C(\overline{\Omega})$ with $1 < r_i^-$, $i \in I$, referring to [9, 16] and in view of the definition of critical Sobolev exponent in (2.5), we introduce the following notation

$$\begin{aligned} &R_+^+ = \max\{r_1^+, \dots, r_N^+\}, \quad R_-^+ = \max\{r_1^-, \dots, r_N^-\}, \\ &R_-^- = \min\{r_1^-, \dots, r_N^-\}, \\ &R_-^* = \frac{N}{(\sum_{i=1}^N 1/r_i^-) - 1}, \quad R_{-,\infty} = \max\{R_-^+, R_-^*\}. \end{aligned}$$

Remark 2.3. The quantity R_-^* is properly defined provided that we assume the condition

$$1 < \sum_{i=1}^{N} \frac{1}{r_i^-},$$

see also Fragalà *et al.* [11, Sec. 2.1]. So, it is crucial to establish the embedding result in Proposition 2.4 below. Furthermore, suppose

$$r_i(x) = \begin{cases} p(x) & \text{if } i \in I \setminus \{N\}, \\ 2p(x) & \text{if } i = N, \end{cases}$$

for all $x \in \overline{\Omega}$, some $p \in C(\overline{\Omega})$ such that $N > 3$ and $N - 1 > 2p^- > 2$, so that

$$\begin{aligned} R_-^* &= \frac{N}{(\sum_{i=1}^{N} 1/r_i^-) - 1} = \frac{N}{(2N-1)/2p^- - 1} \\ &= \frac{2Np^-}{2N - 1 - 2p^-} < 2p^- = \max_{i \in I} r_i^- = R_-^+. \end{aligned}$$

Differently, for every $i \in I$ assume

$$r_i(x) = p(x) \quad \text{for all } x \in \overline{\Omega}, \text{ some } p \in C(\overline{\Omega}) : N > p^- > 1.$$

Hence, we have

$$R_-^* = \frac{N}{(\sum_{i=1}^{N} 1/r_i^-) - 1} = \frac{N}{N/p^- - 1} = \frac{Np^-}{N - p^-} > p^- = \max_{i \in I} r_i^- = R_-^+.$$

This motivates introduction of the quantity $R_{-,\infty} = \max\{R_-^+, R_-^*\}$.

We also denote

$$r_M(x) = \max\{r_1(x), \ldots, r_N(x)\}.$$

The variable exponent anisotropic Sobolev space $W^{1,\overrightarrow{r}(x)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{r}(x)}(\Omega) := \left\{ u \in \ L^{r_M(x)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{r_i(x)}(\Omega), \quad i \in I \right\},$$

equipped with the norm

$$\|u\|_{1,\overrightarrow{r}} = \|u\|_{r_M} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{r_i(x)}. \tag{2.7}$$

Similar to the previous setting, we can introduce the space

$$W_0^{1,\overrightarrow{r}(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\overrightarrow{r}}}.$$

So $W_0^{1,\vec{r}(x)}(\Omega)$ endowed with the following norm

$$\|u\|_{\vec{r},0} = \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{r_i(x)}$$

is a separable, reflexive and uniformly convex Banach space for $r_i, \in C(\overline{\Omega})$ with $1 < r_i^-$, $i \in I$, see Rákosník [21, 22]. We recall the following key embedding result from the literature, it can be seen in the work by Mihăilescu *et al.* [16, Theorem 1].

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ be a bounded domain with smooth boundary $\partial\Omega$. Then, for $s, r_i \in C(\overline{\Omega})$ verifying*

$$1 < s(x) < R_{-,\infty}, \quad \textit{for all } x \in \overline{\Omega}, \ \textit{and } 1 < \sum_{i=1}^{N} \frac{1}{r_i^-},$$

we have

$$W_0^{1,\overrightarrow{r}(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega) \ \textit{compactly.}$$

The proof of Proposition 2.4 substantially observes that for every $i \in I$, one can find a constant $C_i > 0$ satisfying the inequality

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{r_i^-} \leq C_i \left\| \frac{\partial u}{\partial x_i} \right\|_{r_i(x)} \quad \text{for all } u \in W_0^{1,\overrightarrow{r}(x)}(\Omega).$$

Summing from 1 to N both the sides of this inequality and denoting $\overrightarrow{R}_- = \{r_1^-, \ldots, r_N^-\}$, and $C := \max_{i \in I} C_i$, one easily has

$$\|u\|_{\overrightarrow{R}_-} \leq C \|u\|_{\overrightarrow{r}},$$

which establishes the continuous embedding $W_0^{1,\overrightarrow{r}(x)}(\Omega) \hookrightarrow W_0^{1,\overrightarrow{R}_-}(\Omega)$. Assumption $1 < s(x) < R_{-,\infty}$, for all $x \in \bar{\Omega}$, together with several manipulations of the exponents (see also [11]), leads to

$$W_0^{1,\overrightarrow{r}(x)}(\Omega) \hookrightarrow W_0^{1,\overrightarrow{R}_-}(\Omega) \hookrightarrow\hookrightarrow L^{s^+}(\Omega) \hookrightarrow L^{s(x)}(\Omega).$$

3. ANALYSIS OF FUNCTIONAL J

According to the spaces and results introduced in Section 2, we start our detailed analysis of functional $J : W_0^{1,\overrightarrow{p}(x)}(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} J(u) &= J_{\overrightarrow{p}}(u) - \mu J_{\overrightarrow{q}}(u) - J_F(u) \\ &= \int_\Omega \sum_{i=1}^{N} \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx - \mu \int_\Omega \sum_{i=1}^{N} \frac{1}{q_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{q_i(x)} dx - \int_\Omega F(u(x)) dx \end{aligned}$$

for all $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

Since in Theorem 1.4 we look for Galerkin-type minimizers, then we have to provide a proper notion of derivative to deal with. So, if we denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair $((W_0^{1,\overrightarrow{r}(x)}(\Omega))^*, W_0^{1,\overrightarrow{r}(x)}(\Omega))$, $r \in \{p, q\}$, we know that $J_{\overrightarrow{r}} : W_0^{1,\overrightarrow{r}(x)}(\Omega) \to (W_0^{1,\overrightarrow{r}(x)}(\Omega))^*$ is a C^1-functional, and we have

$$\langle J'_{\overrightarrow{r}}(u), h \rangle = \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{r_i(x)-2} \frac{\partial u}{\partial x_i} \frac{\partial h}{\partial x_i} dx$$

for all $u, h \in W_0^{1,\overrightarrow{r}(x)}(\Omega)$. Furthermore, it is bounded, continuous, strictly monotone and if $u_n \xrightarrow{w} u$ in $W_0^{1,\overrightarrow{r}(x)}(\Omega)$ and $\limsup_{n\to+\infty} \langle J'_{\overrightarrow{r}}(u_n), u_n - u \rangle \le 0$, then $u_n \to u$ in $W_0^{1,\overrightarrow{r}(x)}(\Omega)$ (that is, it satisfies the $(S)_+$ property)), see Boureanu [2, Lemma 2].

Next, we know that $J_F : W_0^{1,\overrightarrow{p}(x)}(\Omega) \to \mathbb{R}$ defined by

$$J_F(u) = \int_{\Omega} F(u(x))dx \quad \text{for all } u \in W_0^{1,\overrightarrow{p}(x)}(\Omega) \tag{3.1}$$

is Lipschitz on the bounded subsets of $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ because of the function $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz (by hypothesis) and its growth is controlled by assumption (G). According to the Clarke subdifferential theory (see Clarke [5] and Chang [3]), we recall that a real-valued function ϕ, defined on a Banach space X, is locally Lipschitz if for every $u \in X$, there is open neighborhood Y of u and constant $k > 0$ (depending on Y) with

$$|\phi(z_1) - \phi(z_2)| \le k\|z_1 - z_2\| \quad \text{for all } z_1, z_2 \in Y.$$

If $\phi : X \to \mathbb{R}$ is continuous and convex, then it is locally Lipschitz. The generalized directional derivative of ϕ at $u \in X$ in the direction $v \in X$ is defined as

$$\phi^\circ(u; v) = \limsup_{z\to u, t\downarrow 0} \frac{\phi(z + tv) - \phi(z)}{t}.$$

This is a convex function with respect to its second variable, and so one can appeal to Hahn–Banach theorem to write

$$\partial\phi(u) = \{u^* \in X^* \mid \phi^\circ(u; v) \ge \langle u^*, v \rangle \ \text{ for all } v \in X\},$$

being $\langle \cdot, \cdot \rangle$ the duality brackets for (X^*, X). Now, by $u \to \partial\phi(u)$ we denote the subdifferential of $\phi(\cdot)$ in Clarke's sense. Furthermore, we recall that $v \to \phi^\circ(u; v)$ is finite, positively homogeneous, subadditive and $|\phi^\circ(u; v)| \le k\|v\|$ for all $v \in X$. Following this theory, we can properly consider the Clarke subdifferential of $J_F : W_0^{1,\overrightarrow{p}(x)}(\Omega) \to \mathbb{R}$ denoted by $\partial J_F : W_0^{1,\overrightarrow{p}(x)}(\Omega) \to 2^{(W_0^{1,\overrightarrow{p}(x)}(\Omega))^*}$ that is a nonempty, convex, *weak**-compact subset of $(W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ (see [3]). Furthermore, $J_{\overrightarrow{p}}, J_{\overrightarrow{q}} \in C^1(W_0^{1,\overrightarrow{p}(x)}(\Omega))$ and so

$$\partial J_{\overrightarrow{r}}(u) = \{J'_{\overrightarrow{r}}(u)\} \quad \text{for all } u \in W_0^{1,\overrightarrow{r}(x)}(\Omega),\ r \in \{p, q\}.$$

Since every C^1-functional is locally Lipschitz, clearly $J : W_0^{1,\overrightarrow{p}(x)}(\Omega) \to \mathbb{R}$ is locally Lipschitz too, and we have

$$\partial J(u) = \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} dx - \mu \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial u}{\partial x_i} dx - \partial J_F(u) \tag{3.2}$$

for all $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

It follows that $w \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is a critical point (local minimum or local maximum) of J provided that $0 \in \partial J(w)$. Such necessary condition is essential in establishing our proof (recall Definitions 1.1 and 1.3), and we will use it involving the precise equation

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial w}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial w}{\partial x_i}\frac{\partial h}{\partial x_i} dx - \mu \int_\Omega \sum_{i=1}^N \left|\frac{\partial w}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial w}{\partial x_i}\frac{\partial h}{\partial x_i} dx - \int_\Omega z^* h dx = 0 \tag{3.3}$$

for some $z^* \in (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ with $z^* \in \partial F(w)$ a.e. on Ω, all $h \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

The following results (see again [3, 5]) are also needed in the proof.

Lemma 3.1 (Mean-value theorem)**.** *If $\phi : X \to \mathbb{R}$ is locally Lipschitz on an open neighborhood containing the segment $[a, b]$, then there are $c \in (a, b)$ and $\zeta \in \partial\phi(c)$ such that $\phi(b) - \phi(a) = \langle \zeta, b - a\rangle$.*

Lemma 3.2. *If $\{u_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ are two sequences in X and X^*, respectively, such that $\zeta_n \in \partial\psi(u_n)$ and $u_n \to u$ in X and $\zeta_n \xrightarrow{w^*} \zeta$, then we have $\zeta \in \partial\psi(u)$.*

We are ready to prove our main result.

Proof of Theorem 1.4. We divide the proof in some steps, for the sake of clarity.

Claim 1. $J : W_0^{1,\overrightarrow{p}(x)}(\Omega) \to \mathbb{R}$ is coercive.

We recall that coercivity means

$$J(u) \to +\infty \quad \text{as } \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)} \to +\infty.$$

Now, from the inequalities in (2.4), we easily deduce that

$$\begin{aligned}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{q_i(x)} dx &\le \left\|\frac{\partial u}{\partial x_i}\right\|_{q_i(x)}^{q_i^+} + 1 \\ &\le K_i \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)}^{q_i^+} + 1 \quad \text{(for some } K_i > 0\text{)},\end{aligned} \tag{3.4}$$

where $K_i > 0$ is a suitable constant linked to the embedding of $L^{p_i(x)}(\Omega)$ into $L^{q_i(x)}(\Omega)$. Appealing to Lemma 3.1 and involving condition (G), we get

$$|F(t)| \le |F(0)| + c|t| + \widehat{c}|t|^{\alpha(x)} \quad \text{for all } t \in \mathbb{R}, \text{ all } x \in \Omega. \tag{3.5}$$

Using (1.1) and combining the following inequality

$$\int\limits_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \geq \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)}^{p_i^-} - 1,$$

with (3.4) and (3.5), in view of Propositions 2.2 and 2.4 we get the following achievement

$$\begin{aligned}
J(u) &\geq \frac{1}{P_+^+} \int\limits_{\Omega} \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx - \frac{|\mu|}{Q_-^-} \int\limits_{\Omega} \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^{q_i(x)} dx \\
&\quad - \int\limits_{\Omega} (c\,|u| + \widehat{c}|u|^{\alpha(x)}) dx - |F(0)||\Omega| \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^{N} \left(\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)}^{p_i^-} - 1 \right) - \frac{|\mu|}{Q_-^-} \sum_{i=1}^{N} \left(K_i \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)}^{q_i^+} + 1 \right) - c_3 \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)} \\
&\quad - \widehat{c}(\|u\|_{\alpha(x)}^{\alpha^+} + 1) - |F(0)||\Omega| \\
&\geq \frac{1}{P_+^+ N^{P_-^- - 1}} \left(\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)} \right)^{P_-^-} - \frac{|\mu|}{Q_-^-} c_4 N \left(\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)} \right)^{Q_+^+} \\
&\quad - c_3 \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)} - c_5 \left(\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(x)} \right)^{\alpha^+} - c_6
\end{aligned}$$

for suitable constants $c_3, c_4, c_5, c_6 > 0$, where as usual $|\Omega|$ stays for the Lebesgue measure of Ω.

Since $P_-^- > Q_+^+$ and $P_-^- > \alpha^+$, we deduce that (1.1) is a coercive functional.

Claim 2. The functional J admits an approximate minimizer u_n over X_n.

Referring to the notion of Galerkin basis for $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ (recall conditions (1.2)-(1.4)), we know that $\dim(X_n) < +\infty$ for all $n \in \mathbb{N}$. Furthermore, the locally Lipschitzianity and coercivity of J imply there is $u_n \in X_n$ such that

$$J(u_n) = \inf\{J(v) : v \in X_n\}. \tag{3.6}$$

So, the local minimum condition for $u_n \in X_n$ is shown by

$$0 \in \partial(J|_{X_n})(u_n). \tag{3.7}$$

In the framework of Banach spaces, involving (3.7) one can find $z_n' \in \partial J_F(u_n)$ fulfilling the equation

$$\langle J_p'(u_n) - \mu J_q'(u_n) - z_n', h\rangle = 0 \quad \text{for all } h \in X_n. \tag{3.8}$$

Appealing to [5, Theorem 2.7.5, Remark 2.7.6], we get

$$\partial J_F(u_n) \subset \int\limits_{\Omega} \partial F(u_n) dx,$$

which says us that for each $z_n' \in \partial J_F(u_n)$, there is $z_n \in \partial F(u_n)$ a.e. on Ω such that

$$\langle z_n', h \rangle = \int_\Omega z_n h dx. \tag{3.9}$$

Combining (3.8) and (3.9) we establish the validity of equation (1.5), hence in view of Definition 1.1 the claim is proved.

Claim 3. The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded on $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, that is

$$\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i(x)} \le L \text{ for some } L > 0, \text{ all } n \in \mathbb{N}. \tag{3.10}$$

From equation (1.5) with $h = u_n \in X_n$, using the inequalities in (3.4) and condition (G), we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} dx &= \mu \int_\Omega \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^{q_i(x)} dx + \int_\Omega z_n \, u_n dx \\ &\le c_7 \left(\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i(x)} \right)^{Q_+^+} + c_8 \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i(x)} \\ &\quad + c_9 \left(\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i(x)} \right)^{\alpha^+} + c_{10}\end{aligned}$$

for some $c_7, c_8, c_9, c_{10} > 0$.

So, we have

$$\begin{aligned}\left(\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i(x)} \right)^{P_-^-} &\le c_{11} \left(\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i(x)} \right)^{Q_+^+} + c_{12} \sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i(x)} \\ &\quad + c_{13} \left(\sum_{i=1}^{N} \left\| \frac{\partial u_n}{\partial x_i} \right\|_{p_i(x)} \right)^{\alpha^+} + c_{14}\end{aligned}$$

for all $n \in \mathbb{N}$ and for some $c_{11}, c_{12}, c_{13}, c_{14} > 0$.

Since $P_-^- > Q_+^+$ and $P_-^- > \alpha^+$, we deduce that the sequence of approximate minimizers $\{u_n\}_{n\in\mathbb{N}}$ is bounded on $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, indeed (3.10) holds.

Claim 4. The sequence $\{u_n\}_{n\in\mathbb{N}}$ fulfills the following condition

$$\lim_{n\to+\infty} J(u_n) = \inf\{J(v) : v \in W_0^{1,\overrightarrow{p}(x)}(\Omega)\}. \tag{3.11}$$

By Claim 2 we know that $u_n \in X_n$ is an approximate minimizer for the functional J. This means that

$$J(u_n) = \inf\{J(v) : v \in X_n\}.$$

Using the nesting property (1.3) of $\{X_n\}_{n\in\mathbb{N}}$, we conclude that the sequence $\{J(u_n)\}_{n\in\mathbb{N}}$ is nonincreasing and bounded (recall Claim 3). Hence, there is $\ell \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} J(u_n) = \ell.$$

Arguing by contradiction with (3.11), we suppose $\ell > \inf\{J(v) : v \in W_0^{1,\overrightarrow{p}(x)}(\Omega)\}$, so that we can find $v_0 \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ with $\ell > J(v_0)$. But $J \in C(W_0^{1,\overrightarrow{p}(x)}(\Omega))$ and hence there is an open neighborhood of v_0, say $Y \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$, such that we get

$$J(v) < \ell \text{ for all } v \in Y \subset W_0^{1,\overrightarrow{p}(x)}(\Omega). \tag{3.12}$$

Now, using the covering property (1.4) of $\{X_n\}_{n\in\mathbb{N}}$, we easily deduce that

$$\left(\bigcup_{n=1}^{+\infty} X_n\right) \cap Y \neq \emptyset.$$

It follows that we can find $\overline{v} \in Y \cap X_{\overline{n}}$ for some $\overline{n} \in \mathbb{N}$, fulfilling condition (3.12). Finally, combining (3.6) and (3.12), it leads to

$$\inf\{J(v) : v \in X_{\overline{n}}\} \leq J(\overline{v}) < \ell \leq \inf\{J(v) : v \in X_{\overline{n}}\}.$$

This is absurd, hence it permits us to conclude that (3.11) holds true. In view of Definition 1.3, this means that the functional (1.1) has a Galerkin-type minimizer $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$. □

4. APPLICATION TO DIRICHLET PROBLEM

In this section, we consider (1.1) like as the energy functional associated to the following Dirichlet problem for the $(\overrightarrow{p}, \overrightarrow{q})$-Laplacian

$$-\Delta_{\overrightarrow{p}(x)} u(x) + \mu \Delta_{\overrightarrow{q}(x)} u(x) \in \partial F(u) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{4.1}$$

Our goal here is to establish the existence of a suitable solution to problem (4.1), hence we introduce the following notion.

Definition 4.1. A function $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is a generalized solution to problem (4.1) if we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$ fulfilling the conditions:

(i) $u_n \xrightarrow{w} u$ in $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, as $n \to +\infty$;

(ii) $-\Delta_{\overrightarrow{p}(x)} u_n + \mu \Delta_{\overrightarrow{q}(x)} u_n - z_n \xrightarrow{w} 0$ in $(W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$, as $n \to +\infty$, with $z_n \in (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ and $z_n \in \partial F(u_n)$ a.e. in Ω;

(iii) $\langle -\Delta_{\overrightarrow{p}(x)} u_n + \mu \Delta_{\overrightarrow{q}(x)} u_n, u_n - u\rangle \to 0$, as $n \to +\infty$.

This definition extends the corresponding notions given for instance in [14, 17] to the anisotropic setting, and is motivated by the lack of monotonicity for the non-elliptic functional (i.e., $0 < \mu$) discussed in Section 1. We now state and prove the existence result.

Theorem 4.2. *If* (G) *is satisfied, then for all* $\mu \in \mathbb{R}$*, problem* (4.1) *has a generalized solution* $u \in W_0^{1,p(x)}(\Omega)$*.*

Proof. Following Claims 2 and 3 in the proof of Theorem 1.4, we construct a bounded sequence of approximate minimizers to (1.1), namely $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$. It admits a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, converging weakly to some $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$, and hence the first requirement in Definition 4.1 holds true.

Next, for each $h \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$, we observe that

$$\begin{aligned}
&|\langle -\Delta_{\overrightarrow{p}(x)} u_n + \mu \Delta_{\overrightarrow{q}(x)} u_n - z_n, h\rangle| \\
&= \left| \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u_n}{\partial x_i}\frac{\partial h}{\partial x_i} dx - \mu \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial u_n}{\partial x_i}\frac{\partial h}{\partial x_i} dx - \int_\Omega z_n h dx \right| \\
&\leq \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-1} \left|\frac{\partial h}{\partial x_i}\right| dx + |\mu| \int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i(x)-1} \left|\frac{\partial h}{\partial x_i}\right| dx + \int_\Omega |z_n|\,|h| dx.
\end{aligned} \tag{4.2}$$

Using (2.1), we know that

$$\rho_{p_i'(x)}\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-1}\right) = \rho_{p_i(x)}\left(\left|\frac{\partial u_n}{\partial x_i}\right|\right),$$

and so there is $\beta_{i,n} \in [P_-^-, P_+^+]$ satisfying the inequality

$$\left\|\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-1}\right\|_{p_i'(x)} \leq \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i(x)}^{\beta_{i,n}} \quad \text{for all } n \in \mathbb{N}.$$

Based on it, we can obtain useful estimates for the inequality (4.2). Precisely, we first have

$$\begin{aligned}
\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-1} \left|\frac{\partial h}{\partial x_i}\right| dx &\leq 2\sum_{i=1}^N \left\|\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-1}\right\|_{p'(x)} \left\|\frac{\partial h}{\partial x_i}\right\|_{p_i(x)} \\
&\leq 2\sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i(x)}^{\beta_{i,n}} \left\|\frac{\partial h}{\partial x_i}\right\|_{p_i(x)} \\
&\leq \widehat{C} \sum_{i=1}^N \left\|\frac{\partial h}{\partial x_i}\right\|_{p_i(x)}
\end{aligned} \tag{4.3}$$

for some $\widehat{C} > 0$ (recall that $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is bounded, see (3.10)). Then, we also get

$$\begin{aligned}\int\limits_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{q_i(x)-1}\left|\frac{\partial h}{\partial x_i}\right| dx &\le \int\limits_\Omega \sum_{i=1}^N \left[\left(1+\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)-1}\right)\left|\frac{\partial h}{\partial x_i}\right|\right] dx \\ &\le c_{15}\sum_{i=1}^N \left\|\frac{\partial h}{\partial x_i}\right\|_{p_i(x)} + 2\sum_{i=1}^N \left(\left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i(x)}^{\beta_{i,n}}\left\|\frac{\partial h}{\partial x_i}\right\|_{p_i(x)}\right) \\ &\le \widetilde{C}\sum_{i=1}^N \left\|\frac{\partial h}{\partial x_i}\right\|_{p_i(x)}\end{aligned} \tag{4.4}$$

for some $c_{15}, \widetilde{C} > 0$, see again the estimate (3.10).

On the other hand, the Hölder inequality (2.2) leads to the following achievement

$$\begin{aligned}\int\limits_\Omega |z_n||h|dx &\le \int\limits_\Omega (c+\widehat{c}|u_n|^{\alpha(x)-1})|h|dx \\ &\le c\|h\|_1 + 2\widehat{c}\|u_n\|_{\alpha(x)}^{\gamma_n}\|h\|_{\alpha(x)} \quad \text{(for some } 1<\gamma_n<P_-^-) \\ &\le \overline{C}\sum_{i=1}^N \left\|\frac{\partial h}{\partial x_i}\right\|_{p_i(x)}\end{aligned} \tag{4.5}$$

for some $\overline{C} > 0$.

Using (4.3)–(4.5) in (4.2), we can easily deduce that its left-hand side is bounded in $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, namely

$$|\langle -\Delta_{\overrightarrow{p}(x)}u_n + \mu\Delta_{\overrightarrow{q}(x)}u_n - z_n, h\rangle| \le C^*\sum_{i=1}^N \left\|\frac{\partial h}{\partial x_i}\right\|_{p_i(x)} \tag{4.6}$$

with $C^* := \widehat{C} + \widetilde{C} + \overline{C} > 0$. So, up to a subsequence if necessary, we may assume that

$$-\Delta_{\overrightarrow{p}(x)}u_n + \mu\Delta_{\overrightarrow{q}(x)}u_n - z_n \xrightarrow{w} y \text{ in } (W_0^{1,\overrightarrow{p}(x)}(\Omega))^* \text{ as } n\to+\infty \tag{4.7}$$

for some $y \in (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$.

If we set $h \in \bigcup_{n=1}^{+\infty} X_n$ in (1.5) (recall also that $h \in X_n$ for $n > \overline{n}$ for some $\overline{n} \in \mathbb{N}$, for the nesting property), pass to the limit as $n \to +\infty$ and use the above weak convergence, then we get

$$\begin{aligned}&\langle -\Delta_{\overrightarrow{p}(x)}u_n, h\rangle + \langle \mu\Delta_{\overrightarrow{q}(x)}u_n, h\rangle - \int\limits_\Omega z_n h dx = 0, \\ \Rightarrow\quad &\lim_{n\to+\infty}\langle -\Delta_{\overrightarrow{p}(x)}u_n + \mu\Delta_{\overrightarrow{q}(x)}u_n - z_n, h\rangle = 0, \\ \Rightarrow\quad &\langle y, h\rangle = 0,\end{aligned}$$

and so, by the covering property of $\{X_n\}_{n\in\mathbb{N}}$, we conclude $y = 0$. We directly deduce that

$$-\Delta_{\overrightarrow{p}(x)}u_n + \mu\Delta_{\overrightarrow{q}(x)}u_n - z_n \xrightarrow{w} 0 \quad \text{in } (W_0^{1,\overrightarrow{p}(x)}(\Omega))^* \text{ as } n \to +\infty, \tag{4.8}$$

which is the second requirement of Definition 4.1.

We now set $h = u_n - u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ in (1.5). Since the Hölder inequality gives us

$$\lim_{n\to+\infty} \int_\Omega z_n(u_n - u)dx = 0,$$

then we can pass to the limit as $n \to +\infty$ to deduce that

$$\lim_{n\to+\infty} \left[\langle -\Delta_{\overrightarrow{p}(x)}u_n, u_n - u\rangle + \mu\langle \Delta_{\overrightarrow{q}(x)}u_n, u_n - u\rangle - \int_\Omega z_n(u_n - u)dx \right] = 0,$$

implies

$$\lim_{n\to+\infty} \left[\langle -\Delta_{\overrightarrow{p}(x)}u_n, u_n - u\rangle + \mu\langle \Delta_{\overrightarrow{q}(x)}u_n, u_n - u\rangle \right] = 0,$$

which permits us to conclude the validity of the third requirement of Definition 4.1. The existence of a generalized solution $u \in W_0^{1,\overrightarrow{p}(x)}(\Omega)$ to problem (4.1) for all $\mu \in \mathbb{R}$ is proved. □

In the case $\mu \le 0$, we can also establish the existence of weak solutions.

Definition 4.3. We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (4.1) if there exists $z \in \partial F(u) \subset (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ a.e. on Ω such that

$$-\Delta_{\overrightarrow{p}(x)}u + \mu\Delta_{\overrightarrow{q}(x)}u - z = 0 \quad \text{in } (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*.$$

Consequently, we can state and prove the following result, where we will use the monotonicity arguments.

Theorem 4.4. *If* (G) *is satisfied, then for all* $\mu \le 0$*, problem* (4.1) *has a weak solution* $u \in W_0^{1,p(x)}(\Omega)$.

Proof. By Theorem 4.2 we already know that problem (4.1) admits a generalized solution, say $u \in W_0^{1,p(x)}(\Omega)$. It remains to show that using the monotonicity of the involved operators we can now pass from the weak convergence stated in Definition 4.1 to the strong convergence in $W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

We first know that the negative $\overrightarrow{q}$-Laplacian defined by

$$-\Delta_{\overrightarrow{q}(x)}u = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{q_i(x)-2} \frac{\partial u}{\partial x_i}\right) \quad \text{for all } u \in W_0^{1,\overrightarrow{q}(x)}(\Omega),$$

is a monotone operator. Hence, we have

$$\langle -\Delta_{\overrightarrow{q}(x)}u + \Delta_{\overrightarrow{q}(x)}v, u - v\rangle \ge 0 \quad \text{for all } u, v \in W_0^{1,\overrightarrow{q}(x)}(\Omega).$$

By Definition 4.1(i), (iii) for $\mu \leq 0$, we deduce that

$$\begin{aligned}
&\limsup_{n\to+\infty}\langle -\Delta_{\overrightarrow{p}(x)}u_n, u_n-u\rangle \\
&= \limsup_{n\to+\infty}\big[\langle -\Delta_{\overrightarrow{p}(x)}u_n+\mu\Delta_{\overrightarrow{q}(x)}u_n, u_n-u\rangle + \mu\langle -\Delta_{\overrightarrow{q}(x)}u_n+\Delta_{\overrightarrow{q}(x)}u, u_n-u\rangle \\
&\quad + \mu\langle -\Delta_{\overrightarrow{q}(x)}u, u_n-u\rangle\big] \\
&\leq \limsup_{n\to+\infty}\langle -\Delta_{\overrightarrow{p}(x)}u_n+\mu\Delta_{\overrightarrow{q}(x)}u_n, u_n-u\rangle + \mu\lim_{n\to+\infty}\langle -\Delta_{\overrightarrow{q}(x)}u, u_n-u\rangle = 0.
\end{aligned}$$

Since the negative $\overrightarrow{r}(x)$-Laplacian is continuous and fulfills the $(S)_+$-property, then we know that u_n converging to u (as $n\to+\infty$) in $W_0^{1,\overrightarrow{r}(x)}(\Omega)$ leads to

$$\lim_{n\to+\infty} -\Delta_{\overrightarrow{r}(x)}u_n = -\Delta_{\overrightarrow{r}(x)}u \quad \text{in } (W_0^{1,r(x)}(\Omega))^*,\ r\in\{p,q\}.$$

By Claim 3 of Theorem 1.4 and estimate (4.6), we retrieve the boundedness of the sequence $\{z_n\}_{n\in\mathbb{N}}$ in Definition 4.1(ii), in $W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$. Without loss of generality, we may assume that

$$z_n \xrightarrow{w} z \text{ in } (W_0^{1,\overrightarrow{p}(x)}(\Omega))^* \text{ as } n\to+\infty.$$

Referring to Lemma 3.2, we know that $z\in\partial F(u)$ a.e. on Ω. Hence, using (4.8), $u_n\to u$ in $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ and $z_n\in\partial F(u_n)\subset(W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$, it leads to

$$-\Delta_{\overrightarrow{p}(x)}u+\mu\Delta_{\overrightarrow{q}(x)}u - z = 0 \quad \text{in } (W_0^{1,\overrightarrow{p}(x)}(\Omega))^*,$$

where $z\in\partial F(u)\subset(W_0^{1,\overrightarrow{p}(x)}(\Omega))^*$ a.e. on Ω. We conclude that the requirement in Definition 4.3 is fulfilled, namely problem (4.1) admits a weak solution $u\in W_0^{1,\overrightarrow{p}(x)}(\Omega)$. □

Acknowledgments
This study was conducted during the visit of Z. Zhang to the Department of Mathematics and Computer Science, University of Palermo. This research was funded by the China Scholarship Council program (Project ID: 202406710096).

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Received: September 23, 2025.
Revised: November 6, 2025.
Accepted: November 23, 2025.
Published online: January 27, 2026.