CALDERÓN–HARDY TYPE SPACES
AND THE HEISENBERG SUB-LAPLACIAN

Pablo Rocha

Communicated by Palle E.T. Jorgensen

Abstract. For $0 < p \leq 1 < q < \infty$ and $\gamma > 0$, we introduce the Calderón–Hardy spaces $\mathcal{H}_{q,\gamma}^p(\mathbb{H}^n)$ on the Heisenberg group $\mathbb{H}^n$, and show for every $f \in H^p(\mathbb{H}^n)$ that the equation
$$\mathcal{L}F = f$$
has a unique solution F in $\mathcal{H}_{q,2}^p(\mathbb{H}^n)$, where $\mathcal{L}$ is the sub-Laplacian on $\mathbb{H}^n$,

$$1 < q < \frac{n+1}{n} \quad \text{and} \quad (2n+2)\left(2 + \frac{2n+2}{q}\right)^{-1} < p \leq 1.$$

Keywords: Calderón–Hardy type spaces, Hardy type spaces, atomic decomposition, Heisenberg group, sub-Laplacian.

Mathematics Subject Classification: 42B25, 42B30, 42B35, 43A80.

1. INTRODUCTION

The Laplace operator or Laplacian Δ on $\mathbb{R}^n$ is defined by

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

The ubiquity and the importance of this operator in physics and mathematics is well known. Needless to say that the study of problems involving the Laplacian are of interest either because of their applications or in their own right.

Given $m \in \mathbb{N}$, consider the inhomogeneous equation

$$\Delta^m F = f, \tag{1.1}$$

where Δ^m is the iterated Laplacian, f is a given data function and F is an unknown function. Then, the problem consists in finding a function F that solves (1.1) in some sense. It is common to address this problem by means of the fundamental

solution of the operator Δ^m. A *fundamental solution* for Δ^m is a distribution K on $\mathbb{R}^n$ such that $\Delta^m K = \delta$ in the distributional sense, where δ is Dirac's delta at the origin. In this case, for every $m \in \mathbb{N}$ fixed, we have that

$$\Phi_m(x) = \begin{cases} C_1 \, |x|^{2m-n} \log |x|, & \text{if } n \text{ is even and } 2m - n \geq 0, \\ \\ C_2 \, |x|^{2m-n}, & \text{otherwise} \end{cases}$$

is a fundamental solution for Δ^m on $\mathbb{R}^n$ (see [11, 201–202]). That fundamental solution is not uniquely determined. Indeed, $\Phi_m + u$ with $\Delta^m u = 0$, it is other fundamental solution for Δ^m. These fundamental solutions are useful for producing solutions of the equation (1.1). For instance, if $m \geq 1$ and f is a C^∞-function with compact support, then $F = \Phi_m * f$ solves (1.1) in the classical sense. This formula also works for $m = 1$ when one assumes $f \in L^1(\mathbb{R}^n)$, and that $\int |f(x)| \log(|x|) dx < \infty$ in the case $n = 2$, (see [8, Theorem 2.21]). For $m \geq 1$ and $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$, A.P. Calderón proved that there exists a locally integrable function F what solves (1.1) in the distributional sense and $\|\partial^\alpha F\|_p \leq C\|f\|_p$ for all multi-index α such that $|\alpha| = 2m$, with C independent of f (see [3, Lemma 8]).

It is known that the Hardy spaces $H^p(\mathbb{R}^n)$ are good substitutes for Lebesgue spaces $L^p(\mathbb{R}^n)$ when $0 < p \leq 1$ (see [5, 19]). In this direction, A. Gatto, J. Jiménez and C. Segovia in [10], posed the problem (1.1) for $m \geq 1$ and $f \in H^p(\mathbb{R}^n)$, $0 < p \leq 1$. To solve it they introduce the Calderón–Hardy spaces $\mathcal{H}^p_{q,\gamma}(\mathbb{R}^n)$, $0 < p \leq 1 < q < \infty$ and $\gamma > 0$, and proved for $n(2m + n/q)^{-1} < p \leq 1$ that given $f \in H^p(\mathbb{R}^n)$ there exists a unique $F \in \mathcal{H}^p_{q,2m}(\mathbb{R}^n)$ that solves (1.1).

The underlying idea in [10] to address this problem is the following: given $f \in H^p(\mathbb{R}^n)$, there exists an atomic decomposition $f = \sum k_j a_j$, such that $\|f\|^p_{H^p(\mathbb{R}^n)} \sim \sum k_j^p$ (see [14]), then once defined the space $\mathcal{H}^p_{q,2m}(\mathbb{R}^n)$ (which is defined as a quotient space) together with its "norm" $\| \cdot \|_{\mathcal{H}^p_{q,2m}(\mathbb{R}^n)}$, they define $b_j = (a_j * \Phi_m)$ and consider the class $B_j \in \mathcal{H}^p_{q,2m}(\mathbb{R}^n)$ such that $b_j \in B_j$. Finally, for $n(2m+n/q)^{-1} < p \leq 1$, they prove that the series $\sum k_j B_j$ converges to F in $\mathcal{H}^p_{q,2m}(\mathbb{R}^n)$ and $\Delta^m F = f$. Moreover, Δ^m is a bijective mapping from $\mathcal{H}^p_{q,2m}(\mathbb{R}^n)$ onto $H^p(\mathbb{R}^n)$, with $\|F\|_{\mathcal{H}^p_{q,2m}(\mathbb{R}^n)} \sim \|\Delta^m F\|_{H^p(\mathbb{R}^n)}$.

In [4], R. Durán extended the definition and atomic decomposition of $\mathcal{H}^p_{q,2m}$ to the case of non-isotropic dilations on $\mathbb{R}^n$, solving an analogue problem to (1.1) for more general elliptic operators with symbols of the form $\xi_1^{2k_1} + \ldots + \xi_n^{2k_n}$, with $k_1, \ldots, k_n \in \mathbb{N}$.

The equation (1.1), for $f \in H^{p(\cdot)}(\mathbb{R}^n)$ and $f \in H^p(\mathbb{R}^n, w)$, was studied by the present author in [17] and [18] respectively, obtaining analogous results to those of Gatto, Jiménez and Segovia.

Recently, Z. Liu, Z. He and H. Mo in [15] extended the definition of Calderón–Hardy spaces to Orlicz setting. These new Orlicz Calderón–Hardy spaces can cover classical Calderón–Hardy spaces in [10]. As an application, they solved the equation (1.1) when $f \in H^\Phi(\mathbb{R}^n)$, where $H^\Phi(\mathbb{R}^n)$ are the Orlicz-Hardy spaces defined in [16].

On the other hand, it is well known that the Lie group "most commutative" among the non-commutative is the Heisenberg group, it plays an important role in several branches of mathematics (see [20]). So, one has the opportunity to ask whether certain standard results of Euclidean harmonic analysis can be adapted to the non-commutative setting of the Heisenberg group. Following this line, the purpose of this work is to pose and solve an analogous problem to (1.1) on the Heisenberg group with $m = 1$. More precisely, for $f \in H^p(\mathbb{H}^n)$, $0 < p \leq 1$, we consider the equation

$$\mathcal{L}F = f, \tag{1.2}$$

where $\mathcal{L}$ is the sub-Laplacian on $\mathbb{H}^n$. The solution obtained in [10], for the Euclidean case, suggests us that once defined the space $\mathcal{H}^p_{q,2}(\mathbb{H}^n)$ a representative for the solution $F \in \mathcal{H}^p_{q,2}(\mathbb{H}^n)$ of (1.2) should be $\sum k_j(a_j *_{\mathbb{H}^n} \Phi)$, where $\sum k_j a_j$ is an atomic decomposition for $f \in H^p(\mathbb{H}^n)$ (see [9]), and Φ is the fundamental solution of $\mathcal{L}$ obtained by G. Folland in [7]. We shall see that this argument works as well on $\mathbb{H}^n$, but taking into account certain non-trivial aspects inherent to the Heisenberg group.

Our main results are contained in Theorems 5.1 and 5.2 (see Section 5 below). The first of them states that if $Q = 2n + 2$, $1 < q < \frac{n+1}{n}$ and $Q(2 + \frac{Q}{q})^{-1} < p \leq 1$, then the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}^n$ is a bijective mapping from $\mathcal{H}^p_{q,2}(\mathbb{H}^n)$ onto $H^p(\mathbb{H}^n)$. Moreover, for every $G \in \mathcal{H}^p_{q,2}(\mathbb{H}^n)$, the quantities $\|\mathcal{L}G\|_{H^p(\mathbb{H}^n)}$ and $\|G\|_{\mathcal{H}^p_{q,2}(\mathbb{H}^n)}$ are comparable with implicit constants independent of G. In other words, for $Q(2 + \frac{Q}{q})^{-1} < p \leq 1$ and $f \in H^p(\mathbb{H}^n)$, the equation (1.2) has a unique solution in $\mathcal{H}^p_{q,2}(\mathbb{H}^n)$.

A key technical result needed to get Theorem 5.1 is Proposition 4.12 below. This establishes a pointwise inequality in $\mathbb{H}^n$ which can be inferred from Gatto, Jiménez and Segovia's approach, however its analogous in $\mathbb{R}^n$ is not explicitly stated in [10].

Although the fundamental solutions for the powers of the sub-Laplacian $\mathcal{L}^m$ are known for every integer $m \geq 2$ (see [1]), the problem in this case is much more complicated. For this reason we focus solely on the case $m = 1$.

Finally, our second result says that the case $0 < p \leq Q(2 + \frac{Q}{q})^{-1}$ is trivial. Indeed, we have that if $1 < q < \frac{n+1}{n}$ and $0 < p \leq Q(2 + \frac{Q}{q})^{-1}$, then $\mathcal{H}^p_{q,2}(\mathbb{H}^n) = \{0\}$.

This paper is organized as follows. In Section 2 we state the basics of the Heisenberg group. The definition and atomic decomposition of Hardy spaces on the Heisenberg group are presented in Section 3. We introduce the Calderón–Hardy spaces on the Heisenberg group and investigate their properties in Section 4. The key technical result mentioned above is also stated in Section 4. Finally, our main results are proved in Section 5.

We will use the following notation. The symbol $A \lesssim B$ stands for the inequality $A \leq cB$ for some constant c. We denote by $B(z_0, \delta)$ the ρ-ball centered at $z_0 \in \mathbb{H}^n$ with radius δ. Given $\beta > 0$ and a ρ-ball $B = B(z_0, \delta)$, we set $\beta B = B(z_0, \beta \delta)$. For a measurable subset $E \subseteq \mathbb{H}^n$ we denote by $|E|$ and χ_E the Haar measure of E and the characteristic function of E respectively. Given a real number $s \geq 0$, we write $\lfloor s \rfloor$ for the integer part of s.

Throughout this paper, C will denote a positive constant, not necessarily the same at each occurrence.

2. PRELIMINARIES

The Heisenberg group $\mathbb{H}^n$ can be identified with $\mathbb{R}^{2n} \times \mathbb{R}$ whose group law (noncommutative) is given by

$$(x,t) \cdot (y,s) = (x + y, t + s + x^t Jy),$$

where J is the $2n \times 2n$ skew-symmetric matrix given by

$$J = 2 \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

being I_n the $n \times n$ identity matrix.

The dilation group on $\mathbb{H}^n$ is defined by

$$r \cdot (x,t) = (rx, r^2 t), \quad r > 0.$$

With this structure we have that $e = (0,0)$ is the neutral element,

$$(x,t)^{-1} = (-x,-t)$$

is the inverse of (x,t), and

$$r \cdot ((x,t) \cdot (y,s)) = (r \cdot (x,y)) \cdot (r \cdot (y,s)).$$

The *Koranyi norm* on $\mathbb{H}^n$ is the function $\rho : \mathbb{H}^n \to [0,\infty)$ defined by

$$\rho(x,t) = \left(|x|^4 + t^2 \right)^{1/4}, \quad (x,t) \in \mathbb{H}^n, \tag{2.1}$$

where $|\cdot|$ is the usual Euclidean norm on $\mathbb{R}^{2n}$. It is easy to check that $|x| \leq \rho(x,t)$ and $|t| \leq \rho(x,t)^2$.

Let $z = (x,t)$ and $w = (y,s) \in \mathbb{H}^n$, the Koranyi norm satisfies the following properties:

$$\begin{aligned}
\rho(z) = 0 & \quad \text{if and only if } z = e, \\
\rho(z^{-1}) = \rho(z) & \quad \text{for all } z \in \mathbb{H}^n, \\
\rho(r \cdot z) = r\,\rho(z) & \quad \text{for all } z \in \mathbb{H}^n \text{ and all } r > 0, \\
\rho(z \cdot w) \leq \rho(z) + \rho(w) & \quad \text{for all } z, w \in \mathbb{H}^n, \\
|\rho(z) - \rho(w)| \leq \rho(z \cdot w) & \quad \text{for all } z, w \in \mathbb{H}^n.
\end{aligned}$$

Moreover, ρ is continuous on $\mathbb{H}^n$ and is smooth on $\mathbb{H}^n \setminus \{e\}$. The ρ-ball centered at $z_0 \in \mathbb{H}^n$ with radius $\delta > 0$ is defined by

$$B(z_0, \delta) := \{w \in \mathbb{H}^n : \rho(z_0^{-1} \cdot w) < \delta\}.$$

The topology in $\mathbb{H}^n$ induced by the ρ-balls coincides with the Euclidean topology of $\mathbb{R}^{2n} \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ (see [6, Proposition 3.1.37]). So, the borelian sets of $\mathbb{H}^n$ are identified

with those of $\mathbb{R}^{2n+1}$. The Haar measure in $\mathbb{H}^n$ is the Lebesgue measure of $\mathbb{R}^{2n+1}$, thus $L^p(\mathbb{H}^n) \equiv L^p(\mathbb{R}^{2n+1})$, for every $0 < p \leq \infty$. Moreover, for $f \in L^1(\mathbb{H}^n)$ and for $r > 0$ fixed, we have

$$\int_{\mathbb{H}^n} f(r \cdot z)\, dz = r^{-Q} \int_{\mathbb{H}^n} f(z)\, dz, \tag{2.2}$$

where $Q = 2n + 2$. The number $2n + 2$ is known as the *homogeneous dimension* of $\mathbb{H}^n$ (we observe that the *topological dimension* of $\mathbb{H}^n$ is $2n + 1$).

Let $|B(z_0, \delta)|$ be the Haar measure of the ρ-ball $B(z_0, \delta) \subset \mathbb{H}^n$. Then,

$$|B(z_0, \delta)| = c\delta^Q,$$

where $c = |B(e, 1)|$ and $Q = 2n + 2$. Given $\lambda > 0$, we put

$$\lambda B = \lambda B(z_0, \delta) = B(z_0, \lambda \delta).$$

So $|\lambda B| = \lambda^Q |B|$.

Remark 2.1. For any $z, z_0 \in \mathbb{H}^n$ and $\delta > 0$, we have

$$z_0 \cdot B(z, \delta) = B(z_0 \cdot z, \delta).$$

In particular, $B(z, \delta) = z \cdot B(e, \delta)$. It is also easy to check that $B(e, \delta) = \delta \cdot B(e, 1)$ for any $\delta > 0$.

Remark 2.2. If $f \in L^1(\mathbb{H}^n)$, then for every ρ-ball B and every $z_0 \in \mathbb{H}^n$, we have

$$\int_B f(w)\, dw = \int_{z_0^{-1} \cdot B} f(z_0 \cdot u)\, du.$$

The Hardy–Littlewood maximal operator M is defined by

$$Mf(z) = \sup_{B \ni z} |B|^{-1} \int_B |f(w)|\, dw,$$

where f is a locally integrable function on $\mathbb{H}^n$ and the supremum is taken over all the ρ-balls B containing z.

If f and g are measurable functions on $\mathbb{H}^n$, their convolution $f * g$ is defined by

$$(f * g)(z) := \int_{\mathbb{H}^n} f(w)g(w^{-1} \cdot z)\, dw,$$

when the integral is finite.

For every $i = 1, 2, \ldots, 2n + 1$, X_i denotes the left invariant vector field given by

$$X_i = \frac{\partial}{\partial x_i} + 2x_{i+n} \frac{\partial}{\partial t}, \quad i = 1, 2, \ldots, n,$$

$$X_{i+n} = \frac{\partial}{\partial x_{i+n}} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, 2, \ldots, n,$$

and

$$X_{2n+1} = \frac{\partial}{\partial t}.$$

Similarly, we define the right invariant vector fields $\{\widetilde{X}_i\}_{i=1}^{2n+1}$ by

$$\widetilde{X}_i = \frac{\partial}{\partial x_i} - 2x_{i+n}\frac{\partial}{\partial t}, \quad i = 1, 2, \ldots, n,$$

$$\widetilde{X}_{i+n} = \frac{\partial}{\partial x_{i+n}} + 2x_i\frac{\partial}{\partial t}, \quad i = 1, 2, \ldots, n,$$

and

$$\widetilde{X}_{2n+1} = \frac{\partial}{\partial t}.$$

The sub-Laplacian on $\mathbb{H}^n$, denoted by $\mathcal{L}$, is the counterpart of the Laplacain Δ on $\mathbb{R}^n$. The sub-Laplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = -\sum_{i=1}^{2n} X_i^2,$$

where X_i, $i = 1, \ldots, 2n$, are the left invariant vector fields defined above.

Given a multi-index $I = (i_1, i_2, \ldots, i_{2n}, i_{2n+1}) \in (\mathbb{N} \cup \{0\})^{2n+1}$, we set

$$|I| = i_1 + i_2 + \ldots + i_{2n} + i_{2n+1}, \quad d(I) = i_1 + i_2 + \ldots + i_{2n} + 2\,i_{2n+1}.$$

The amount $|I|$ is called the length of I and $d(I)$ the homogeneous degree of I. We adopt the following multi-index notation for higher order derivatives and for monomials on $\mathbb{H}^n$. If $I = (i_1, i_2, \ldots, i_{2n+1})$ is a multi-index, $X = \{X_i\}_{i=1}^{2n+1}$, $\widetilde{X} = \{\widetilde{X}_i\}_{i=1}^{2n+1}$, and $z = (x, t) = (x_1, \ldots, x_{2n}, t) \in \mathbb{H}^n$, we put

$$X^I := X_1^{i_1} X_2^{i_2} \cdots X_{2n+1}^{i_{2n+1}}, \quad \widetilde{X}^I := \widetilde{X}_1^{i_1} \widetilde{X}_2^{i_2} \cdots \widetilde{X}_{2n+1}^{i_{2n+1}},$$

and

$$z^I := x_1^{i_1} \cdots x_{2n}^{i_{2n}} \cdot t^{i_{2n+1}}.$$

A computation gives

$$X^I(f(r \cdot z)) = r^{d(I)}(X^I f)(r \cdot z), \quad \widetilde{X}^I(f(r \cdot z)) = r^{d(I)}(\widetilde{X}^I f)(r \cdot z)$$

and

$$(r \cdot z)^I = r^{d(I)} z^I.$$

So, the operators X^I and $\widetilde{X}^I$ and the monomials z^I are homogeneous of degree $d(I)$. In particular, the sub-Laplacian $\mathcal{L}$ is an operator homogeneous of degree 2. The operators X^I, $\widetilde{X}^I$, and $\mathcal{L}$ interact with the convolutions in the following way:

$$X^I(f * g) = f * (X^I g), \quad \widetilde{X}^I(f * g) = (\widetilde{X}^I f) * g, \quad (X^I f) * g = f * (\widetilde{X}^I g),$$

and

$$\mathcal{L}(f * g) = f * \mathcal{L}g.$$

Every polynomial p on $\mathbb{H}^n$ can be written as a unique finite linear combination of the monomials z^I, that is

$$p(z) = \sum_{I \in \mathbb{N}_0^n} c_I z^I, \qquad (2.3)$$

where all but finitely many of the coefficients $c_I \in \mathbb{C}$ vanish. The *homogeneous degree* of a polynomial p written as (2.3) is $\max\{d(I) : I \in \mathbb{N}_0^n \text{ with } c_I \neq 0\}$. Let $k \in \mathbb{N} \cup \{0\}$, with $\mathcal{P}_k$ we denote the subspace formed by all the polynomials of homogeneous degree at most k. So, every $p \in \mathcal{P}_k$ can be written as $p(z) = \sum_{d(I) \leq k} c_I\, z^I$, with $c_I \in \mathbb{C}$.

The Schwartz space $\mathcal{S}(\mathbb{H}^n)$ is defined as the collection of all $\phi \in C^\infty(\mathbb{H}^n)$ such that

$$\sup_{z \in \mathbb{H}^n} (1 + \rho(z))^N |(X^I \phi)(z)| < \infty,$$

for all $N \in \mathbb{N}_0$ and all $I \in (\mathbb{N}_0)^{2n+1}$. We topologize the space $\mathcal{S}(\mathbb{H}^n)$ with the following family of semi-norms:

$$\|\phi\|_{\mathcal{S}(\mathbb{H}^n), N} = \sum_{d(I) \leq N} \sup_{z \in \mathbb{H}^n} (1 + \rho(z))^N |(X^I \phi)(z)| \quad (N \in \mathbb{N}_0).$$

By $\mathcal{S}'(\mathbb{H}^n)$ we denote the dual space of $\mathcal{S}(\mathbb{H}^n)$.

A fundamental solution for the sub-Laplacian on $\mathbb{H}^n$ was obtained by G. Folland in [7]. More precisely, he proved the following result.

Theorem 2.3. $c_n\, \rho^{-2n}$ *is a fundamental solution for* $\mathcal{L}$ *with source at* 0, *where*

$$\rho(x, t) = (|x|^4 + t^2)^{1/4},$$

and

$$c_n = \left[n(n+2) \int_{\mathbb{H}^n} |x|^2 (\rho(x, t)^4 + 1)^{-(n+4)/2} dx dt \right]^{-1}.$$

In others words, for any $u \in \mathcal{S}(\mathbb{H}^n)$, $\big(\mathcal{L}u, c_n \rho^{-2n}\big) = u(0)$.

Lemma 2.4. *Let* $\alpha > 0$ *and* $\rho(x, t) = (|x|^4 + t^2)^{1/4}$, *then*

$$\left| \widetilde{X}^J \left(X^I \rho^{-\alpha} \right)(x, t) \right| \leq C \rho(x, t)^{-\alpha - d(I) - d(J)},$$

holds for all $(x, t) \neq e$ *and every pair of multi-indixes* I *and* J.

Proof. The proof follows from the homogeneity of the kernel $\rho^{-\alpha}$, i.e.

$$\rho(r \cdot (x, t))^{-\alpha} = r^{-\alpha} \rho(x, t)^{-\alpha},$$

and from the homogeneity of the operators $\widetilde{X}^J$ and X^I. $\qquad \square$

We conclude these preliminaries with the following supporting result.

Lemma 2.5. *Let* $0 < p < \infty$ *and let* $\mathcal{O}$ *be a measurable set of* $\mathbb{H}^n$ *such that* $|\mathcal{O}| < \infty$. *If* $h \in L^p(\mathbb{H}^n \setminus \mathcal{O})$, *then*

$$|\{z : |h(z)| < \varepsilon\}| > 0, \quad for \ all \ \varepsilon > 0.$$

Proof. Suppose that there exists $\varepsilon_0 > 0$ such that $|\{z : |h(z)| < \varepsilon_0\}| = 0$, so $|h(z)| \geq \varepsilon_0/2$ a.e. $z \in \mathbb{H}^n$, which implies that

$$\infty = |\mathcal{O}^c| = |\{z \in \mathcal{O}^c : |h(z)| \geq \varepsilon_0/2\}| \leq (2/\varepsilon_0)^p \|h\|_{L^p(\mathcal{O}^c)}^p,$$

contradicting the assumption that $h \in L^p(\mathbb{H}^n \setminus \mathcal{O})$. Then, the lemma follows. $\qquad\square$

3. HARDY SPACES ON THE HEISENBERG GROUP

In this section, we briefly recall the definition and the atomic decomposition of the Hardy spaces on the Heisenberg group (see [9]).

Given $N \in \mathbb{N}$, define

$$\mathcal{F}_N = \left\{\varphi \in \mathcal{S}(\mathbb{H}^n) : \|\varphi\|_{\mathcal{S}(\mathbb{H}^n),N} \leq 1\right\}.$$

For any $f \in \mathcal{S}'(\mathbb{H}^n)$, the grand maximal function of f is defined by

$$\mathcal{M}_N f(z) = \sup_{t>0} \sup_{\varphi \in \mathcal{F}_N} |(f * \varphi_t)(z)|,$$

where $\varphi_t(z) = t^{-2n-2}\varphi(t^{-1} \cdot z)$ with $t > 0$.

We put

$$N_p = \begin{cases} \lfloor Q(p^{-1} - 1)\rfloor + 1, & \text{if } 0 < p \leq 1, \\ 0, & \text{if } 1 < p \leq \infty. \end{cases}$$

The Hardy space $H^p(\mathbb{H}^n)$ is the set of all $f \in S'(\mathbb{H}^n)$ for which $\mathcal{M}_{N_p} f \in L^p(\mathbb{H}^n)$. In this case, we define

$$\|f\|_{H^p(\mathbb{H}^n)} = \left\|\mathcal{M}_{N_p} f\right\|_{L^p(\mathbb{H}^n)}.$$

For $p > 1$, it is well known that $H^p(\mathbb{H}^n) \equiv L^p(\mathbb{H}^n)$, and for $p = 1$, $H^1(\mathbb{H}^n) \subset L^1(\mathbb{H}^n)$. On the range $0 < p < 1$, the spaces $H^p(\mathbb{H}^n)$ and $L^p(\mathbb{H}^n)$ are not comparable.

Now, we introduce the definition of an atom in $\mathbb{H}^n$.

Definition 3.1. Let $0 < p \leq 1 < p_0 \leq \infty$. Fix an integer $N \geq N_p$. A measurable function $a(\cdot)$ on $\mathbb{H}^n$ is called an (p, p_0, N)-atom if there exists a ρ-ball B such that

(a_1) $\mathrm{supp}(a) \subset B$,

(a_2) $\|a\|_{L^{p_0}(\mathbb{H}^n)} \leq |B|^{\frac{1}{p_0} - \frac{1}{p}}$,

(a_3) $\int a(z) z^I \, dz = 0$ for all multi-index I such that $d(I) \leq N$.

A such atom is also called an atom centered at the ρ-ball B. We observe that every (p, p_0, N)-atom $a(\cdot)$ belongs to $H^p(\mathbb{H}^n)$. Moreover, there exists a universal constant $C > 0$ such that $\|a\|_{H^p(\mathbb{H}^n)} \leq C$ for all (p, p_0, N)-atom $a(\cdot)$.

Remark 3.2. It is easy to check that if $a(\cdot)$ is a (p, p_0, N)-atom centered at the ρ-ball $B(z_0, \delta)$, then the function $a_{z_0}(\cdot) := a(z_0 \cdot (\cdot))$ is a (p, p_0, N)-atom centered at the ρ-ball $B(e, \delta)$.

Definition 3.3. Let $0 < p \leq 1 < p_0 \leq \infty$ and let $N \geq N_p$ be fixed. The space $H^{p, p_0, N}_{atom}(\mathbb{H}^n)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{H}^n)$ such that it can be written as

$$f = \sum_{j=1}^{\infty} k_j a_j \tag{3.1}$$

in $\mathcal{S}'(\mathbb{H}^n)$, where $\{k_j\}_{j=1}^{\infty}$ is a sequence of non-negative numbers, the a_j's are (p, p_0, N)-atoms and $\sum_j k_j^p < \infty$. Then, one defines

$$\|f\|_{H^{p, p_0, N}_{atom}(\mathbb{H}^n)} := \inf \left\{ \sum_j k_j^p : f = \sum_{j=1}^{\infty} k_j a_j \right\},$$

where the infimum is taken over all admissible expressions as in (3.1).

For $0 < p \leq 1 < p_0 \leq \infty$ and $N \geq N_p$, Theorem 3.30 in [9] asserts that

$$H^{p, p_0, N}_{atom}(\mathbb{H}^n) = H^p(\mathbb{H}^n)$$

and the quantities $\|f\|_{H^{p, p_0, N}_{atom}(\mathbb{H}^n)}$ and $\|f\|_{H^p(\mathbb{H}^n)}$ are comparable. Moreover, if $f \in H^p(\mathbb{H}^n)$, then admits an atomic decomposition $f = \sum_{j=1}^{\infty} k_j a_j$ such that

$$\sum_j k_j^p \leq C \|f\|_{H^p(\mathbb{H}^n)}^p,$$

where C does not depend on f.

4. CALDERÓN–HARDY SPACES ON THE HEISENBERG GROUP

Let $L^q_{loc}(\mathbb{H}^n)$, $1 < q < \infty$, be the space of all measurable functions g on $\mathbb{H}^n$ that belong locally to L^q for compact sets of $\mathbb{H}^n$. We endowed $L^q_{loc}(\mathbb{H}^n)$ with the topology generated by the seminorms

$$|g|_{q, B} = \left(|B|^{-1} \int_B |g(w)|^q \, dw \right)^{1/q},$$

where B is a ρ-ball in $\mathbb{H}^n$ and $|B|$ denotes its Haar measure.

For $g \in L_{loc}^q(\mathbb{H}^n)$, we define a maximal function $\eta_{q,\gamma}(g; z)$ as

$$\eta_{q,\gamma}(g; z) = \sup_{r>0} r^{-\gamma} |g|_{q, B(z,r)},$$

where γ is a positive real number and $B(z,r)$ is the ρ-ball centered at z with radius r.

Let k a non-negative integer and $\mathcal{P}_k$ the subspace of $L_{loc}^q(\mathbb{H}^n)$ formed by all the polynomials of homogeneous degree at most k. We denote by E_k^q the quotient space of $L_{loc}^q(\mathbb{H}^n)$ by $\mathcal{P}_k$. If $G \in E_k^q$, we define the seminorm

$$\|G\|_{q,B} = \inf \{|g|_{q,B} : g \in G\}.$$

The family of all these seminorms induces on E_k^q the quotient topology.

Given a positive real number γ, we can write $\gamma = k + t$, where k is a non-negative integer and $0 < t \leq 1$. This decomposition is unique.

For $G \in E_k^q$, we define a maximal function $N_{q,\gamma}(G; z)$ as

$$N_{q,\gamma}(G; z) = \inf \{\eta_{q,\gamma}(g; z) : g \in G\}.$$

Lemma 4.1. *The maximal function $z \mapsto N_{q;\gamma}(G; z)$ associated with a class G in E_k^q is lower semicontinuous.*

Proof. It is easy to check that $\eta_{q,\gamma}(g; \cdot)$ is lower semicontinuous for every $g \in G$ (i.e. the set $\{z : \eta_{q,\gamma}(g; z) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$). Then, for $z_0 \in \mathbb{H}^n$, we have

$$N_{q;\gamma}(G; z_0) \leq \eta_{q,\gamma}(g; z_0) \leq \liminf_{z \to z_0} \eta_{q,\gamma}(g; z) \quad \text{for all } g \in G.$$

So,

$$N_{q;\gamma}(G; z_0) - \varepsilon < \liminf_{z \to z_0} \eta_{q,\gamma}(g; z) \quad \text{for all } \varepsilon > 0 \text{ and all } g \in G. \tag{4.1}$$

Suppose $\liminf_{z \to z_0} N_{q;\gamma}(G; z) < N_{q;\gamma}(G; z_0)$. Then, there exists $\varepsilon > 0$ such that

$$\liminf_{z \to z_0} N_{q;\gamma}(G; z) < N_{q;\gamma}(G; z_0) - \varepsilon.$$

Thus, there exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ there exist $z \in B(z_0, \delta) \setminus \{z_0\}$ and $g = g_z \in G$ such that

$$\eta_{q,\gamma}(g; z) \leq N_{q;\gamma}(G; z_0) - \varepsilon,$$

which contradicts (4.1). Hence, we have

$$N_{q;\gamma}(G; z_0) \leq \liminf_{z \to z_0} N_{q;\gamma}(G; z).$$

The lemma then follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 4.2. Let $0 < p < \infty$ be fixed. We say that an element $G \in E_k^q$ belongs to the Calderón–Hardy space $\mathcal{H}_{q,\gamma}^p(\mathbb{H}^n)$ if the maximal function $N_{q,\gamma}(G; \cdot) \in L^p(\mathbb{H}^n)$. The "norm" of G in $\mathcal{H}_{q,\gamma}^p(\mathbb{H}^n)$ is defined as

$$\|G\|_{\mathcal{H}_{q,\gamma}^p(\mathbb{H}^n)} = \|N_{q,\gamma}(G; \cdot)\|_{L^p(\mathbb{H}^n)}.$$

Lemma 4.3. *Let $G \in E_k^q$ with $N_{q,\gamma}(G; z_0) < \infty$, for some $z_0 \in \mathbb{H}^n$. Then:*

(i) *there exists a unique $g \in G$ such that $\eta_{q,\gamma}(g; z_0) < \infty$ and, therefore, $\eta_{q,\gamma}(g; z_0) = N_{q,\gamma}(G; z_0)$,*

(ii) *for any ρ-ball B, there is a constant c depending on z_0 and B such that if g is the unique representative of G given in* (i), *then*

$$\|G\|_{q,B} \leq |g|_{q,B} \leq c\, \eta_{q,\gamma}(g; z_0) = c\, N_{q,\gamma}(G; z_0).$$

The constant c can be chosen independently of z_0 provided that z_0 varies in a compact set.

Proof. Since every polynomial of homogeneous degree at most k can be centered at z_0, with z_0 being an arbitrary point of $\mathbb{H}^n$, by the formula that appears in [2, Section 5.2, p. 272]) for the Taylor polynomial of a smooth function, it follows that the argument used to prove [10, Lemma 3] works on $\mathbb{H}^n$ as well. □

Corollary 4.4. *If $\{G_j\}$ is a sequence of elements of E_k^q converging to G in $\mathcal{H}_{q,\gamma}^p(\mathbb{H}^n)$, then $\{G_j\}$ converges to G in E_k^q.*

Proof. For any ρ-ball B, by (ii) of Lemma 4.3, we have

$$\|G - G_j\|_{q,B} \leq c\, \|\chi_B\|_{L^p(\mathbb{H}^n)}^{-1} \|\chi_B\, N_{q,\gamma}(G - G_j; \cdot)\|_{L^p(\mathbb{H}^n)}$$
$$\leq c\, \|G - G_j\|_{\mathcal{H}_{q,\gamma}^p(\mathbb{H}^n)},$$

which proves the corollary. □

Lemma 4.5. *Let $\{G_j\}$ be a sequence in E_k^q such that for a given point $z_0 \in \mathbb{H}^n$, the series $\sum_j N_{q,\gamma}(G_j; z_0)$ is finite. Then:*

(i) *the series $\sum_j G_j$ converges in E_k^q to an element G and*

$$N_{q,\gamma}(G; z_0) \leq \sum_j N_{q,\gamma}(G_j; z_0),$$

(ii) *if g_j is the unique representative of G_j satisfying $\eta_{q,\gamma}(g_j; z_0) = N_{q,\gamma}(G_j; z_0)$, then $\sum_j g_j$ converges in $L_{loc}^q(\mathbb{H}^n)$ to a function g that is the unique representative of G satisfying $\eta_{q,\gamma}(g; z_0) = N_{q,\gamma}(G; z_0)$.*

Proof. The proof is similar to the one given in [10, Lemma 4]. □

Proposition 4.6. *The space $\mathcal{H}_{q,\gamma}^p(\mathbb{H}^n)$, $0 < p < \infty$, is complete.*

Proof. It is enough to show that $\mathcal{H}_{q,\gamma}^p$ has the Riesz–Fisher property: given any sequence $\{G_j\}$ in $\mathcal{H}_{q,\gamma}^p$ such that

$$\sum_j \|G_j\|_{\mathcal{H}_{q,\gamma}^p}^p < \infty,$$

the series $\sum_j G_j$ converges in $\mathcal{H}^p_{q,\gamma}$. Let $m \geq 1$ be fixed, then

$$\left\| \sum_{j=m}^k N_{q,\gamma}(G_j; \cdot) \right\|_{L^p}^p \leq \sum_{j=m}^k \|N_{q,\gamma}(G_j; \cdot)\|_{L^p}^p \leq \sum_{j=m}^\infty \|G_j\|_{\mathcal{H}^p_{q,\gamma}}^p =: \alpha_m < \infty,$$

for every $k \geq m$. Thus,

$$\int_{\mathbb{H}^n} \left(\alpha_m^{-1/p} \sum_{j=m}^k N_{q,\gamma}(G_j; z) \right)^p dz$$

$$\leq \int_{\mathbb{H}^n} \left(\left\| \sum_{j=m}^k N_{q,\gamma}(G_j; \cdot) \right\|_{L^p}^{-1} \sum_{j=m}^k N_{q,\gamma}(G_j; z) \right)^p dz = 1, \quad \forall k \geq m,$$

by applying Fatou's Lemma as $k \to \infty$, we obtain

$$\int_{\mathbb{H}^n} \left(\alpha_m^{-1/p} \sum_{j=m}^\infty N_{q,\gamma}(G_j; z) \right)^p dz \leq 1,$$

so

$$\left\| \sum_{j=m}^\infty N_{q,\gamma}(G_j; \cdot) \right\|_{L^p}^p \leq \alpha_m = \sum_{j=m}^\infty \|G_j\|_{\mathcal{H}^p_{q,\gamma}}^p < \infty, \quad \forall m \geq 1. \tag{4.2}$$

Taking $m = 1$ in (4.2), it follows that $\sum_j N_{q,\gamma}(G_j; z)$ is finite a.e. $z \in \mathbb{H}^n$. Then, by (i) of Lemma 4.5, the series $\sum_j G_j$ converges in E_k^q to an element G. Now

$$N_{q,\gamma}\left(G - \sum_{j=1}^k G_j; z \right) \leq \sum_{j=k+1}^\infty N_{q,\gamma}(G_j; z).$$

From this and (4.2) we get

$$\left\| G - \sum_{j=1}^k G_j \right\|_{\mathcal{H}^p_{q,\gamma}}^p \leq \sum_{j=k+1}^\infty \|G_j\|_{\mathcal{H}^p_{q,\gamma}}^p,$$

and since the right-hand side tends to 0 as $k \to \infty$, the series $\sum_j G_j$ converges to G in $\mathcal{H}^p_{q,\gamma}(\mathbb{H}^n)$. $\qquad \square$

Proposition 4.7. *If $g \in L^q_{loc}(\mathbb{H}^n)$, $1 < q < \infty$, and there is a point $z_0 \in \mathbb{H}^n$ such that $\eta_{q,\gamma}(g; z_0) < \infty$, then $g \in \mathcal{S}'(\mathbb{H}^n)$.*

Proof. We first assume that $z_0 = e = (0,0)$. Given $\varphi \in \mathcal{S}(\mathbb{H}^n)$ and $N > \gamma + Q$ (where $Q = 2n + 2$), we have that

$$|\varphi(w)| \leq \|\varphi\|_{\mathcal{S}(\mathbb{H}^n),N} (1 + \rho(w))^{-N}$$

for all $w \in \mathbb{H}^n$.

So, we obtain

$$\left| \int_{\mathbb{H}^n} g(w)\varphi(w)dw \right| \leq \|\varphi\|_{\mathcal{S}(\mathbb{H}^n),N} \int_{\rho(w)<1} |g(w)|(1+\rho(w))^{-N}dw$$

$$+ \|\varphi\|_{\mathcal{S}(\mathbb{H}^n),N} \sum_{j=0}^{\infty} \int_{2^j \leq \rho(w)<2^{j+1}} |g(w)|(1+\rho(w))^{-N}dw$$

$$\lesssim \|\varphi\|_{\mathcal{S}(\mathbb{H}^n),N}\, \eta_{q,\gamma}(g;e)$$

$$+ \|\varphi\|_{\mathcal{S}(\mathbb{H}^n),N}\, \eta_{q,\gamma}(g;e) \sum_{j=0}^{\infty} 2^{j(\gamma+Q-N)},$$

where in the last estimate we use the Jensen's inequality. Since $N > \gamma + Q$, it follows that $g \in \mathcal{S}'(\mathbb{H}^n)$. For the case $z_0 \neq e$ we apply the translation operator τ_{z_0} defined by $(\tau_{z_0}g)(z) = g(z_0^{-1} \cdot z)$ and use the fact that $\eta_{q,\gamma}(\tau_{z_0^{-1}}g;\,e) = \eta_{q,\gamma}(g;z_0)$ (see Remark 2.2). $\qquad\square$

Proposition 4.8. *Let $g \in L_{loc}^q \cap \mathcal{S}'(\mathbb{H}^n)$ and $f = \mathcal{L}g$ in $\mathcal{S}'(\mathbb{H}^n)$. If $\phi \in \mathcal{S}(\mathbb{H}^n)$ and $N > Q + 2$, then*

$$(M_\phi f)(z) := \sup \left\{ |(f * \phi_t)(w)| : \rho(w^{-1} \cdot z) < t, 0 < t < \infty \right\}$$

$$\leq C\|\phi\|_{\mathcal{S}(\mathbb{H}^n),N}\, \eta_{q,2}(g;z)$$

holds for all $z \in \mathbb{H}^n$.

Proof. Let $\rho(w^{-1} \cdot z) < t$, since $f = \mathcal{L}g$ in $\mathcal{S}'(\mathbb{H}^n)$ a computation gives

$$(f * \phi_t)(w) = t^{-2}(g * (\mathcal{L}\phi)_t)(w) = t^{-2} \int g(u)(\mathcal{L}\phi)_t(u^{-1} \cdot w)du.$$

Applying Remark 2.2 and (2.2), we get

$$(f * \phi_t)(w) = t^{-2} \int g(z \cdot tu)(\mathcal{L}\phi)(u^{-1} \cdot t^{-1}(z^{-1} \cdot w))du. \qquad (4.3)$$

Being $\rho(z^{-1} \cdot w) < t$, a computation gives

$$1 + \rho(u) \leq 2\left(1 + \rho(u^{-1} \cdot t^{-1}(z^{-1} \cdot w))\right). \qquad (4.4)$$

On the other hand, for $N > 2$, we have

$$\left|(\mathcal{L}\phi)(u^{-1} \cdot t^{-1}(z^{-1} \cdot w))\right| \left(1 + \rho(u^{-1} \cdot t^{-1}(z^{-1} \cdot w))\right)^N \leq \|\phi\|_{\mathcal{S}(\mathbb{H}^n),N}. \qquad (4.5)$$

Now, from (4.4) and (4.5), it follows that

$$\left|(\mathcal{L}\phi)(u^{-1} \cdot t^{-1}(z^{-1} \cdot w))\right| \leq 2^N\|\phi\|_{\mathcal{S}(\mathbb{H}^n),N}(1 + \rho(u))^{-N}, \qquad (4.6)$$

for $\rho(z^{-1} \cdot w) < t$.

Then, (4.3), (4.6) and (2.2) give

$$
\begin{aligned}
2^{-N}\|\phi\|_{\mathcal{S}(\mathbb{H}^n),N}^{-1}\,|(f*\phi_t)(w)| &\leq t^{-2}\int |g(z\cdot tu)|(1+\rho(u))^{-N}\,du \\
&= t^{-2}t^{-Q}\int |g(z\cdot u)|(1+\rho(t^{-1}u))^{-N}\,du \\
&\leq t^{-2}t^{-Q}\int_{\rho(u)<t} |g(z\cdot u)|(1+\rho(t^{-1}u))^{-N}\,du \\
&\quad + t^{-2}t^{-Q}\int_{2^j t\leq\rho(u)<2^{j+1}t} |g(z\cdot u)|\,\rho(t^{-1}u)^{-N}\,du \\
&\lesssim \left(1+\sum_{j=0}^{\infty} 2^{j(Q+2-N)}\right)\eta_{q,2}(g;z),
\end{aligned}
$$

for $\rho(z^{-1}\cdot w) < t$. Applying Jensen's inequality and taking $N > Q + 2$ in the last inequality the proposition follows. $\qquad\square$

Remark 4.9. We observe that if $G \in \mathcal{H}_{q,2}^p(\mathbb{H}^n)$, then $N_{q,2}(G;z_0) < \infty$, for some $z_0 \in \mathbb{H}^n$. By (i) in Lemma 4.3, there exists $g \in G$ such that $N_{q,2}(G;z_0) = \eta_{q,2}(g;z_0)$. From Proposition 4.7 it follows that $g \in \mathcal{S}'(\mathbb{H}^n)$. So $\mathcal{L}g$ is well defined in sense of distributions. On the other hand, since any two representatives of G differ in a polynomial of homogeneous degree at most 1, we get that $\mathcal{L}g$ is independent of the representative $g \in G$ chosen. Therefore, for $G \in \mathcal{H}_{q,2}^p(\mathbb{H}^n)$, we define $\mathcal{L}G$ as the distribution $\mathcal{L}g$, where g is any representative of G.

Theorem 4.10. *If $G \in \mathcal{H}_{q,2}^p(\mathbb{H}^n)$ and $\mathcal{L}G = 0$, then $G \equiv 0$.*

Proof. Let $G \in \mathcal{H}_{q,2}^p(\mathbb{H}^n)$ and $g \in G$ such that $\eta_{q,2}(g;z_0) = N_{q,2}(G;z_0) < \infty$ for some $z_0 \in \mathbb{H}^n \setminus \{e\}$. If $\mathcal{L}g = 0$, by Theorem 2 in [12], we have that g is a polynomial. To conclude the proof it is suffices to show that g is a polynomial of homogeneous degree less than or equal to 1. Suppose

$$
g(z) = \sum_{d(I)\leq k} c_I z^I,
$$

with $k \geq 2$.

Then, for $\delta \geq 2\rho(z_0)$, we obtain

$$[\eta_{q,2}(g;z_0)]^q \delta^{(2-k)q} \geq C\delta^{-Q-kq} \int\limits_{\rho(z_0^{-1}\cdot w)<\delta} \left| \sum_{d(I)\leq k} c_I\, w^I \right|^q dw$$

$$\geq C\delta^{-Q-kq} \int\limits_{\rho(w)<\delta/2} \left| \sum_{d(I)\leq k} c_I\, w^I \right|^q dw$$

$$= C2^{-Q-kq} \int\limits_{\rho(z)<1} \left| \sum_{d(I)=k} c_I\, z^I \right|^q dz + o_\delta(1).$$

Thus, if $k > 2$, letting $\delta \to \infty$, we have

$$\int\limits_{\rho(z)<1} \left| \sum_{d(I)=k} c_I\, z^I \right| dz = 0,$$

which implies that $c_I = 0$ for $d(I) = k$, contradicting the assumption that g is of homogeneous degree k. On the other hand, if $k = 2$, letting $\delta \to \infty$, we obtain that

$$\int\limits_{\rho(z)<1} \left| \sum_{d(I)=2} c_I\, z^I \right| dz \lesssim [\eta_{q,2}(g;z_0)]^q = [N_{q,2}(G;z_0)]^q.$$

Since $N_{q,2}(G;\cdot) \in L^p(\mathbb{H}^n)$, to apply Lemma 2.5 with $\mathcal{O} = \{z : N_{q,2}(G;z) > 1\}$ and $h = N_{q,2}(G;\cdot)$, the amount $N_{q,2}(G;z_0)$ can be taken arbitrarily small and so

$$\int\limits_{\rho(z)<1} \left| \sum_{d(I)=2} c_I\, z^I \right| dz = 0,$$

which contradicts that g is of homogeneous degree 2. Thus, g is a polynomial of homogeneous degree less than or equal to 1, as we wished to prove. $\qquad \square$

If a is a bounded function with compact support, its potential b, defined as

$$b(z) := \left(a \, * c_n\, \rho^{-2n} \right)(z) = c_n \int\limits_{\mathbb{H}^n} \rho(w^{-1}\cdot z)^{-2n} a(w)dw,$$

is a locally bounded function and, by Theorem 2.3, $\mathcal{L}b = a$ in the sense of distributions. For these potentials, we have the following result.

In the sequel, $Q = 2n+2$ and β is the constant in [9, Corollary 1.44], we observe that $\beta \geq 1$ (see [9, p. 29]).

Lemma 4.11. *Let $a(\cdot)$ be a (p, p_0, N)-atom centered at the ρ-ball $B(z_0, \delta)$ with $N \geq N_p$. If*

$$b(z) = \left(a * c_n \rho^{-2n}\right)(z),$$

then, for $\rho(z_0^{-1}z) \geq 2\beta^2\delta$ and every multi-index I there exists a positive constant C_I such that

$$\left|(X^I b)(z)\right| \leq C_I \, \delta^{2+Q} |B|^{-\frac{1}{p}} \rho(z_0^{-1} \cdot z)^{-Q-d(I)}$$

holds.

Proof. We fix a multi-index I, by the left invariance of the operator X^I and Remark 2.2, we have that

$$(X^I b)(z) = c_n \int\limits_{B(z_0, \delta)} \left(X^I \rho^{-2n}\right)(w^{-1} \cdot z) \, a(w) dw$$

$$= c_n \int\limits_{B(e, \delta)} \left(X^I \rho^{-2n}\right)(u^{-1} \cdot z_0^{-1} \cdot z) \, a(z_0 \cdot u) du,$$

for each $z \notin B(z_0, 2\beta^2\delta)$. By the condition a_3) of the atom $a(\cdot)$ and Remark 3.2, it follows for $z \notin B(z_0, 2\beta^2\delta)$ that

$$(X^I b)(z) = c_n \int\limits_{B(e, \delta)} \left[\left(X^I \rho^{-2n}\right)(u^{-1} \cdot z_0^{-1} \cdot z) - q(u^{-1})\right] a(z_0 \cdot u) \, du, \qquad (4.7)$$

where $u \mapsto q(u^{-1})$ is the right Taylor polynomial at e of homogeneous degree 1 of the function

$$u \mapsto \left(X^I \rho^{-2n}\right)(u^{-1} \cdot z_0^{-1} \cdot z).$$

Then by the right-invariant version of the Taylor inequality in [9, Corollary 1.44],

$$\left|\left(X^I \rho^{-2n}\right)(u^{-1} \cdot z_0^{-1} \cdot z) - q(u^{-1})\right|$$
$$\lesssim \rho(u)^2 \sup_{\rho(v) \leq \beta^2 \rho(u), d(J)=2} \left|\left(\widetilde{X}^J \left(X^I \rho^{-2n}\right)\right)(v \cdot z_0^{-1} \cdot z)\right|. \qquad (4.8)$$

Now, for $u \in B(e, \delta)$, $z_0^{-1} \cdot z \notin B(e, 2\beta^2\delta)$ and $\rho(v) \leq \beta^2 \rho(u)$, we obtain that $\rho(z_0^{-1} \cdot z) \geq 2\rho(v)$ and hence $\rho(v \cdot z_0^{-1} \cdot z) \geq \rho(z_0^{-1} \cdot z)/2$. Then (4.8) and Lemma 2.4 with $\alpha = 2n$ and $d(J) = 2$ allow us to get

$$\left|\left(X^I \rho^{-2n}\right)(u^{-1} \cdot z_0^{-1} \cdot z) - q(u^{-1})\right| \lesssim \delta^2 \rho(z_0^{-1} \cdot z)^{-2n-2-d(I)}.$$

This estimate, (4.7), and the conditions (a_1) and (a_2) of the atom $a(\cdot)$ lead to

$$\left|(X^I b)(z)\right| \lesssim \delta^2 \rho(z_0^{-1} \cdot z)^{-2n-2-d(I)} \|a\|_{L^1(\mathbb{H}^n)}$$
$$\lesssim \delta^2 \rho(z_0^{-1} \cdot z)^{-2n-2-d(I)} |B|^{1-\frac{1}{p_0}} \|a\|_{L^{p_0}(\mathbb{H}^n)}$$
$$\lesssim \delta^2 \rho(z_0^{-1} \cdot z)^{-2n-2-d(I)} |B|^{1-\frac{1}{p}}$$
$$\lesssim \delta^{2+Q} |B|^{-\frac{1}{p}} \rho(z_0^{-1} \cdot z)^{-Q-d(I)},$$

for $\rho(z_0^{-1} \cdot z) \geq 2\beta^2\delta$. This concludes the proof. $\square$

The following result is crucial to get Theorem 5.1.

Proposition 4.12. *Let $a(\cdot)$ be a (p, p_0, N)-atom centered at the ρ-ball $B = B(z_0, \delta)$. If $b(z) = (a * c_n \rho^{-2n})(z)$, then for all $z \in \mathbb{H}^n$*

$$N_{q,2}\left(\widetilde{b}; z\right) \lesssim |B|^{-1/p} \left[(M\chi_B)(z)\right]^{\frac{2+Q/q}{Q}} + \chi_{4\beta^2 B}(z)(Ma)(z)$$
$$+ \chi_{4\beta^2 B}(z) \sum_{d(I)=2} (T_I^* a)(z), \tag{4.9}$$

where $\widetilde{b}$ is the class of b in E_1^q, M is the Hardy–Littlewood maximal operator and

$$(T_I^* a)(z) = \sup_{\varepsilon > 0} \left| \int_{\rho(w^{-1} \cdot z) > \varepsilon} (X^I \rho^{-2n})(w^{-1} \cdot z) a(w) \, dw \right|.$$

Proof. For an atom $a(\cdot)$ satisfying the hypothesis of the proposition, we set

$$R(z, w) = b(z \cdot w) - \sum_{0 \leq d(I) \leq 1} (X^I b)(z) w^I$$

$$= b(z \cdot w) - \sum_{0 \leq d(I) \leq 1} \left[\int_{B(z_0, \delta)} (X^I c_n \rho^{-2n})(u^{-1} \cdot z) a(u) \, du \right] w^I,$$

where $w \mapsto \sum (X^I b)(z) w^I$ is the left Taylor polynomial of the function $w \mapsto b(z \cdot w)$ at $w = e$ of homogeneous degree 1 (see [2, p. 272]). We observe that if $I = (i_1, \ldots, i_{2n}, i_{2n+1})$ is a multi-index such that $d(I) \leq 1$, then $i_{2n+1} = 0$.

Next, we shall estimate $|R(z, w)|$ considering the cases

$$\rho(z_0^{-1} \cdot z) \geq 4\beta^2 \delta \quad \text{and} \quad \rho(z_0^{-1} \cdot z) < 4\beta^2 \delta$$

separately, and then we will obtain the estimate (4.9).

Case 1. $\rho(z_0^{-1} \cdot z) \geq 4\beta^2 \delta$.

For $\rho(z_0^{-1} \cdot z) \geq 4\beta^2 \delta$, $\rho(w) \leq \frac{1}{2\beta^2} \rho(z_0^{-1} \cdot z)$ and $\rho(u) \leq \beta^2 \rho(w)$, a computation gives $\rho(z_0^{-1} \cdot z \cdot u) \geq 2\beta^2 \delta$. Then, by the left-invariant Taylor inequality in [9, Corollary 1.44] and Lemma 4.11, we get

$$|R(z, w)| \lesssim \rho(w)^2 \sup_{\rho(u) \leq \beta^2 \rho(w), d(I) = 2} \left| (X^I b)(z \cdot u) \right|$$
$$\lesssim |B|^{-1/p} \left(\frac{\delta}{\rho(z_0^{-1} \cdot z)} \right)^{2+Q} \rho(w)^2. \tag{4.10}$$

Now, let $\rho(w) \geq \frac{1}{2\beta^2} \rho(z_0^{-1} \cdot z)$. We have

$$|R(z, w)| \leq |b(z \cdot w)| + \sum_{0 \leq d(I) \leq 1} |(X^I b)(z)| |w^I|.$$

Since $\rho(z_0^{-1} \cdot z) \geq 4\beta^2\delta$, by Lemma 4.11 and observing that $\rho(w)/\rho(z_0^{-1} \cdot z) > \frac{1}{2\beta^2}$, we have

$$|(X^I b)(z)||w^I| \lesssim |B|^{-1/p} \left(\frac{\delta}{\rho(z_0^{-1} \cdot z)} \right)^{2+Q} \rho(w)^2.$$

As for the other term, $|b(z \cdot w)|$, we consider separately the cases

$$\rho(z_0^{-1} \cdot z \cdot w) > 2\beta^2\delta \quad \text{and} \quad \rho(z_0^{-1} \cdot z \cdot w) \leq 2\beta^2\delta.$$

In the case $\rho(z_0^{-1} \cdot z \cdot w) > 2\beta^2\delta$, we apply Lemma 4.11 with $I = 0$, obtaining

$$|b(z \cdot w)| \lesssim |B|^{-1/p}\delta^{2+Q}\rho(z_0^{-1} \cdot z \cdot w)^{-Q}.$$

Then

$$|R(z,w)| \lesssim |B|^{-1/p}\delta^{2+Q}\rho(z_0^{-1} \cdot z \cdot w)^{-Q} + |B|^{-1/p} \left(\frac{\delta}{\rho(z_0^{-1} \cdot z)} \right)^{2+Q} \rho(w)^2 \quad (4.11)$$

holds if $\rho(z_0^{-1} \cdot z) > 4\beta^2\delta$, $\rho(w) \geq \frac{1}{2\beta^2}\rho(z_0^{-1} \cdot z)$ and $\rho(z_0^{-1} \cdot z \cdot w) > 2\beta^2\delta$.

For $\rho(z_0^{-1} \cdot z \cdot w) \leq 2\beta^2\delta$, we have

$$B(z_0,\delta) \subset \{u : \rho(u^{-1} \cdot z \cdot w) < (1 + 2\beta^2)\delta\} =: \Omega_\delta,$$

so it follows that

$$|b(z \cdot w)| = c_n \left| \int_{B(z_0,\delta)} \rho(u^{-1} \cdot z \cdot w)^{-2n} a(u) du \right|$$

$$\lesssim \|a\|_{L^{p_0}} \left(\int_{B(z_0,\delta)} \rho(u^{-1} \cdot z \cdot w)^{-2np_0'} du \right)^{1/p_0'}$$

$$\lesssim \|a\|_{L^{p_0}} \left(\int_{\Omega_\delta} \rho(u^{-1} \cdot z \cdot w)^{-2np_0'} du \right)^{1/p_0'}.$$

Since $a(\cdot)$ is an (p, p_0, N)-atom, we can choose $p_0 > Q/2$, and get

$$|b(z \cdot w)| \lesssim |B|^{-1/p}\delta^{Q/p_0} \left(\int_0^{(1+2\beta^2)\delta} r^{-2np_0'+Q-1} dr \right)^{1/p_0'}$$

$$\lesssim |B|^{-1/p}\delta^{Q/p_0}\delta^{-2n}\delta^{Q/p_0'} = |B|^{-1/p}\delta^2.$$

Since $\rho(z_0^{-1} \cdot z) \geq 4\beta^2\delta$, we conclude that

$$|R(z,w)| \lesssim |B|^{-1/p}\delta^2 + |B|^{-1/p} \left(\frac{\delta}{\rho(z_0^{-1} \cdot z)} \right)^{2+Q} \rho(w)^2, \quad (4.12)$$

for all $|\rho(w)| \geq \frac{1}{2\beta^2}\rho(z_0^{-1}z)$ and $\rho(z_0^{-1} \cdot z \cdot w) \leq 2\beta^2\delta$.

Let us the estimate

$$r^{-2}\left(|B(e,r)|^{-1}\int_{B(e,r)}|R(z,w)|^q dw\right)^{1/q}, \quad r>0.$$

For them, we split the domain of integration into three subsets:

$$D_1=\left\{w\in B(e,r):\rho(w)\leq\frac{1}{2\beta^2}\rho(z_0^{-1}\cdot z)\right\},$$

$$D_2=\left\{w\in B(e,r):\rho(w)\geq\frac{1}{2\beta^2}\rho(z_0^{-1}\cdot z),\rho(z_0^{-1}\cdot z\cdot w)>2\beta^2\delta\right\},$$

$$D_3=\left\{w\in B(e,r):\rho(w)\geq\frac{1}{2\beta^2}\rho(z_0^{-1}\cdot z),\rho(z_0^{-1}\cdot z\cdot w)\leq2\beta^2\delta\right\}.$$

According to the estimates obtained for $|R(z,w)|$ above, we use on D_1 the estimate (4.10), on D_2 the estimate (4.11) and on D_3 the estimate (4.12) to get

$$r^{-2}\left(|B(e,r)|^{-1}\int_{B(e,r)}|R(z,w)|^q dw\right)^{1/q}\lesssim|B|^{-1/p}\left(\frac{\delta}{\rho(z_0^{-1}\cdot z)}\right)^{2+Q/q}.$$

Thus,

$$N_{q,2}\left(\widetilde{b};z\right)\lesssim|B|^{-1/p}M(\chi_B)(z)^{\frac{2+Q/q}{Q}}, \tag{4.13}$$

if $\rho(z_0^{-1}\cdot z)\geq4\beta^2\delta$.

Case 2. $\rho(z_0^{-1}\cdot z)<4\beta^2\delta$.

We have

$$R(z,w)=c_n\int\left[\rho^{-2n}(u^{-1}\cdot z\cdot w)-\sum_{0\leq d(I)\leq1}(X^I\rho^{-2n})(u^{-1}\cdot z)w^I\right]a(u)du$$

$$=\int_{\rho(u^{-1}\cdot z)<2\beta^2\rho(w)}+\int_{\rho(u^{-1}\cdot z)\geq2\beta^2\rho(w)}=J_1(z,w)+J_2(z,w).$$

Assuming that $u\neq z\cdot w$ and $u\neq z$, we can write

$$U=\rho^{-2n}(u^{-1}\cdot z\cdot w)-\rho^{-2n}(u^{-1}\cdot z)-\sum_{d(I)=1}(X^I\rho^{-2n})(u^{-1}\cdot z)w^I.$$

By Lemma 2.4, we get

$$|U|\lesssim\rho(u^{-1}\cdot z\cdot w)^{-2n}+\rho(u^{-1}\cdot z)^{-2n}+\rho(w)\,\rho(u^{-1}\cdot z)^{-2n-1}.$$

Observing that $\rho(u^{-1} \cdot z) < 2\beta^2 \rho(w)$ implies $\rho(u^{-1} \cdot z \cdot w) < 3\beta^2 \rho(w)$, we obtain

$$
\begin{aligned}
|J_1(z,w)| &\leq \int\limits_{\rho(u^{-1} \cdot z) < 2\beta^2 \rho(w)} |U||a(u)|du \\
&\lesssim \int\limits_{\rho(u^{-1} \cdot z \cdot w) < 3\beta^2 \rho(w)} \rho(u^{-1} \cdot z \cdot w)^{-2n}|a(u)|du \\
&\quad + \int\limits_{\rho(u^{-1} \cdot z) < 2\beta^2 \rho(w)} \rho(u^{-1} \cdot z)^{-2n}|a(u)|du \\
&\quad + \rho(w) \int\limits_{\rho(u^{-1} \cdot z) < 2\beta^2 \rho(w)} \rho(u^{-1} \cdot z)^{-2n-1}|a(u)|du \\
&= \sum_{k=0}^{\infty} \int\limits_{3^{-k}\beta^2 \rho(w) \leq \rho(u^{-1} \cdot z \cdot w) < 3^{-(k-1)}\beta^2 \rho(w)} \rho(u^{-1} \cdot z \cdot w)^{-2n}|a(u)|du \\
&\quad + \sum_{k=0}^{\infty} \int\limits_{2^{-k}\beta^2 \rho(w) \leq \rho(u^{-1} \cdot z) < 2^{-(k-1)}\beta^2 \rho(w)} \rho(u^{-1} \cdot z)^{-2n}|a(u)|du \\
&\quad + \rho(w) \sum_{k=0}^{\infty} \int\limits_{2^{-k}\beta^2 \rho(w) \leq \rho(u^{-1} \cdot z) < 2^{-(k-1)}\beta^2 \rho(w)} \rho(u^{-1} \cdot z)^{-2n-1}|a(u)|du \\
&\lesssim \rho(w)^2 (Ma)(z).
\end{aligned}
$$

To estimate $J_2(z,w)$, we can write (see [2, p. 272], taking into account that $x^t J x = 0$ for all $x \in \mathbb{R}^{2n}$)

$$
\begin{aligned}
U &= \left[\rho^{-2n}(u^{-1} \cdot z \cdot w) - \sum_{d(I) \leq 2} (X^I \rho^{-2n})(u^{-1} \cdot z) \frac{w^I}{|I|!} \right] \\
&\quad + \sum_{d(I)=2} (X^I \rho^{-2n})(u^{-1} \cdot z) \frac{w^I}{|I|!} \\
&= U_1 + U_2.
\end{aligned}
$$

For $\rho(u^{-1} \cdot z) \geq 2\beta^2 \rho(w)$ and $\rho(\nu) \leq \beta^2 \rho(w)$, we have $\rho(u^{-1} \cdot z \cdot \nu) \geq \rho(u^{-1} \cdot z)/2$. Then, by the left-invariant Taylor inequality in [9, Corollary 1.44] and Lemma 2.4, we get

$$
\begin{aligned}
|U_1| &\lesssim \rho(w)^3 \sup_{\rho(\nu) \leq \beta^2 \rho(w), d(I)=3} \left| (X^I \rho^{-2n})(u^{-1} \cdot z \cdot \nu) \right| \\
&\lesssim \rho(w)^3 \rho(u^{-1} \cdot z)^{-2n-3}.
\end{aligned}
$$

Therefore,

$$|J_2(z,w)| \lesssim \rho(w)^3 \int_{\rho(u^{-1} \cdot z) \geq 2\beta^2 \rho(w)} \rho(u^{-1} \cdot z)^{-2n-3} |a(u)| du$$

$$+ \left| \int_{\rho(u^{-1} \cdot z) \geq 2\beta^2 \rho(w)} U_2 \, a(u) du \right|$$

$$\lesssim \rho(w)^2 \left((Ma)(z) + \sum_{d(I)=2} (T_I^* a)(z) \right),$$

where

$$(T_I^* a)(z) = \sup_{\varepsilon > 0} \left| \int_{\rho(u^{-1} \cdot z) > \varepsilon} (X^I \rho^{-2n})(u^{-1} \cdot z) a(u) \, du \right|.$$

Now, it is easy to check that

$$r^{-2} \left(|B(e,r)|^{-1} \int_{B(e,r)} |J_1(z,w)|^q dw \right)^{1/q} \lesssim (Ma)(z)$$

and

$$r^{-2} \left(|B(e,r)|^{-1} \int_{B(e,r)} |J_2(z,w)|^q dw \right)^{1/q} \lesssim (Ma)(z) + \sum_{d(I)=2} (T_I^* a)(z).$$

So

$$r^{-2} \left(|B(e,r)|^{-1} \int_{B(e,r)} |R(z,w)|^q dw \right)^{1/q} \lesssim (Ma)(z) + \sum_{d(I)=2} (T_I^* a)(z).$$

This estimate is global. In particular, we have that

$$N_{q,2} \left(\widetilde{b} \, ; z \right) \lesssim (Ma)(z) + \sum_{d(I)=2} (T_I^* a)(z), \tag{4.14}$$

for $\rho(z_0^{-1} \cdot z) < 4\beta^2 \delta$. Finally, the estimates (4.13) and (4.14) for $N_{q,2} \left(\widetilde{b} \, ; z \right)$ allow us to obtain (4.9). $\qquad \square$

5. MAIN RESULTS

We are now in a position to prove our main results.

Theorem 5.1. *Let $Q = 2n + 2$, $1 < q < \frac{n+1}{n}$ and $Q(2 + \frac{Q}{q})^{-1} < p \leq 1$. Then the sub-Laplacian $\mathcal{L}$ on $\mathbb{H}^n$ is a bijective mapping from $\mathcal{H}_{q,2}^p(\mathbb{H}^n)$ onto $H^p(\mathbb{H}^n)$. Moreover, there exist two positive constant c_1 and c_2 such that*

$$c_1 \|G\|_{\mathcal{H}_{q,2}^p(\mathbb{H}^n)} \leq \|\mathcal{L}G\|_{H^p(\mathbb{H}^n)} \leq c_2 \|G\|_{\mathcal{H}_{q,2}^p(\mathbb{H}^n)} \tag{5.1}$$

hold for all $G \in \mathcal{H}_{q,2}^p(\mathbb{H}^n)$.

Proof. The injectivity of the sublaplacion $\mathcal{L}$ in $\mathcal{H}_{q,2}^p(\mathbb{H}^n)$ was proved in Theorem 4.10.

Let $G \in \mathcal{H}_{q,2}^p(\mathbb{H}^n)$, since $N_{q,2}(G;z)$ is finite a.e. $z \in \mathbb{H}^n$, by (i) in Lemma 4.3 and Proposition 4.7 the unique representative g of G (which depends on z), satisfying $\eta_{q,2}(g;z) = N_{q,2}(G;z)$, is a function in $L_{loc}^q(\mathbb{H}^n) \cap \mathcal{S}'(\mathbb{H}^n)$. In particular, for a *commutative approximate identity*[1] ϕ, by Remark 4.9 and Proposition 4.8 we get

$$M_\phi(\mathcal{L}G)(z) \leq C_\phi \, N_{q,2}(G;z).$$

Then, this inequality and [9, Corollary 4.17] give $\mathcal{L}G \in H^p(\mathbb{H}^n)$ and

$$\|\mathcal{L}G\|_{H^p(\mathbb{H}^n)} \leq C \, \|G\|_{\mathcal{H}_{q,2}^p(\mathbb{H}^n)}. \tag{5.2}$$

This proves the continuity of sub-Laplacian $\mathcal{L}$ from $\mathcal{H}_{q,2}^p(\mathbb{H}^n)$ into $H^p(\mathbb{H}^n)$.

Now we shall see that the operator $\mathcal{L}$ is onto. Given $f \in H^p(\mathbb{H}^n)$, there exist a sequence of nonnegative numbers $\{k_j\}_{j=1}^\infty$ and a sequence of ρ-balls $\{B_j\}_{j=1}^\infty$ and (p, p_0, N)-atoms a_j supported on B_j, such that $f = \sum_{j=1}^\infty k_j a_j$ and

$$\sum_{j=1}^\infty k_j^p \lesssim \|f\|_{H^p(\mathbb{H}^n)}^p. \tag{5.3}$$

For each $j \in \mathbb{N}$, we put

$$b_j(z) = (a_j * c_n \rho^{-2n})(z) = \int_{\mathbb{H}^n} c_n \rho(w^{-1} \cdot z)^{-2n} a_j(w) dw.$$

From Proposition 4.12 we have

$$N_{q,2}\left(\widetilde{b}_j; z\right) \lesssim |B_j|^{-1/p} \left[(M\chi_{B_j})(z) \right]^{\frac{2+Q/q}{Q}} + \chi_{4\beta^2 B_j}(z)(Ma_j)(z)$$
$$+ \chi_{4\beta^2 B_j}(z) \sum_{d(I)=2} (T_I^* a_j)(z).$$

---

[1] A commutative approximate identity is a function $\phi \in \mathcal{S}(\mathbb{H}^n)$ such that $\int \phi(z)\,dz = 1$ and $\phi_s * \phi_t = \phi_t * \phi_s$ for all $s, t > 0$.

So it follows that

$$\sum_{j=1}^{\infty} k_j N_{q,2}\left(\widetilde{b}_j; z\right) \lesssim \sum_{j=1}^{\infty} k_j |B_j|^{-1/p} \left[(M\chi_{B_j})(z)\right]^{\frac{2+Q/q}{Q}}$$

$$+ \sum_{j=1}^{\infty} k_j \chi_{4\beta^2 B_j}(z)(Ma_j)(z)$$

$$+ \sum_{j=1}^{\infty} k_j \chi_{4\beta^2 B_j}(z) \sum_{d(I)=2} (T_I^* a_j)(z)$$

$$= I + II + III.$$

To study I, by hypothesis, we have that $0 < p \le 1$ and $(2 + Q/q)p > Q$. Then

$$\|I\|_{L^p(\mathbb{H}^n)} = \left\| \sum_{j=1}^{\infty} k_j |B_j|^{-1/p} M(\chi_{B_j})(\cdot)^{\frac{2+Q/q}{Q}} \right\|_{L^p(\mathbb{H}^n)}$$

$$= \left\| \left\{ \sum_{j=1}^{\infty} k_j |B_j|^{-1/p} M(\chi_{B_j})(\cdot)^{\frac{2+Q/q}{Q}} \right\}^{\frac{Q}{2+Q/q}} \right\|_{L^{\frac{2+Q/q}{Q}p}(\mathbb{H}^n)}^{\frac{2+Q/q}{Q}}$$

$$\lesssim \left\| \left\{ \sum_{j=1}^{\infty} k_j |B_j|^{-1/p} \chi_{B_j}(\cdot) \right\}^{\frac{Q}{2+Q/q}} \right\|_{L^{\frac{2+Q/q}{Q}p}(\mathbb{H}^n)}^{\frac{2+Q/q}{Q}}$$

$$= \left\| \sum_{j=1}^{\infty} k_j |B_j|^{-1/p} \chi_{B_j}(\cdot) \right\|_{L^p(\mathbb{H}^n)}$$

$$\lesssim \left(\sum_{j=1}^{\infty} k_j^p \right)^{1/p} \lesssim \|f\|_{H^p(\mathbb{H}^n)},$$

where the first inequality follows from [13, Theorem 1.2], the condition $0 < p \le 1$ gives the second inequality, and (5.3) gives the last one.

To study II, since $p \le 1$ we have that

$$\|II\|_{L^p(\mathbb{H}^n)}^p \lesssim \left\| \sum_j k_j \, \chi_{4\beta^2 B_j}(Ma_j)(\cdot) \right\|_{L^p(\mathbb{H}^n)}^p$$

$$\lesssim \sum_j k_j^p \int \chi_{4\beta^2 B_j}(z)(Ma_j)^p(z)\,dz,$$

applying Hölder's inequality with $\frac{p_0}{p}$, using that the maximal operator M is bounded on $L^{p_0}(\mathbb{H}^n)$ and that every $a_j(\cdot)$ is an (p, p_0, N)-atom, we get

$$\|II\|_{L^p(\mathbb{H}^n)}^p \lesssim \sum_j k_j^p |B_j|^{1-\frac{p}{p_0}} \left(\int (Ma_j)^{p_0}(z)\,dz \right)^{\frac{p}{p_0}}$$

$$\lesssim \sum_j k_j^p |B_j|^{1-\frac{p}{p_0}} \|a_j\|_{L^{p_0}(\mathbb{H}^n)}^p$$

$$\lesssim \sum_j k_j^p |B_j|^{1-\frac{p}{p_0}} |B_j|^{\frac{p}{p_0}-1}$$

$$= \sum_j k_j^p \lesssim \|f\|_{H^p(\mathbb{H}^n)}^p,$$

where the last inequality follows from (5.3)

To study III, by [7, Theorem 3] and [19, Corollary 2, p. 36] (see also [19, 2.5, p. 11]), we have, for every multi-index I with $d(I) = 2$, that the operator T_I^* is bounded on $L^{p_0}(\mathbb{H}^n)$ for each $1 < p_0 < \infty$. Proceeding as in the estimate of II, we get

$$\|III\|_{L^p(\mathbb{H}^n)} \lesssim \left(\sum_{j=1}^\infty k_j^p \right)^{1/p} \lesssim \|f\|_{H^p(\mathbb{H}^n)}.$$

Thus,

$$\left\| \sum_{j=1}^\infty k_j N_{q,2}\left(\widetilde{b}_j;\, \cdot \right) \right\|_{L^p(\mathbb{H}^n)} \lesssim \|f\|_{H^p(\mathbb{H}^n)}.$$

Then,

$$\sum_{j=1}^\infty k_j N_{q,2}\left(\widetilde{b}_j;\, z \right) < \infty \quad \text{a.e. } z \in \mathbb{H}^n \tag{5.4}$$

and

$$\left\| \sum_{j=M+1}^\infty k_j N_{q,2}\left(\widetilde{b}_j; \cdot \right) \right\|_{L^p(\mathbb{H}^n)} \to 0, \text{ as } M \to \infty. \tag{5.5}$$

From (5.4) and Lemma 4.5, there exists a function G such that $\sum\limits_{j=1}^\infty k_j \widetilde{b}_j = G$ in E_1^q and

$$N_{q,2}\left(\left(G - \sum_{j=1}^M k_j \widetilde{b}_j \right); z \right) \leq C \sum_{j=M+1}^\infty k_j N_{q,2}(\widetilde{b}_j; z).$$

This estimate together with (5.5) implies

$$\left\| G - \sum_{j=1}^M k_j \widetilde{b}_j \right\|_{\mathcal{H}_{q,2}^p(\mathbb{H}^n)} \to 0, \quad \text{as } M \to \infty.$$

By Proposition 4.6, we have that $G \in \mathcal{H}^p_{q,2}(\mathbb{H}^n)$ and $G = \sum_{j=1}^\infty k_j \widetilde{b}_j$ in $\mathcal{H}^p_{q,2}(\mathbb{H}^n)$. Since $\mathcal{L}$ is a continuous operator from $\mathcal{H}^p_{q,2}(\mathbb{H}^n)$ into $H^p(\mathbb{H}^n)$, we get

$$\mathcal{L}G = \sum_j k_j \mathcal{L}\widetilde{b}_j = \sum_j k_j a_j = f,$$

in $H^p(\mathbb{H}^n)$. This shows that $\mathcal{L}$ is onto $H^p(\mathbb{H}^n)$. Moreover,

$$\|G\|_{\mathcal{H}^p_{q,2}(\mathbb{H}^n)} = \left\| \sum_{j=1}^\infty k_j \widetilde{b}_j \right\|_{\mathcal{H}^p_{q,2}(\mathbb{H}^n)} \lesssim \left\| \sum_{j=1}^\infty k_j N_{q,2}(\widetilde{b}_j; \cdot) \right\|_{L^p(\mathbb{H}^n)} \tag{5.6}$$
$$\lesssim \|f\|_{H^p(\mathbb{H}^n)} = \|\mathcal{L}G\|_{H^p(\mathbb{H}^n)}.$$

Finally, (5.2) and (5.6) give (5.1), and the proof is thus complete. $\qquad\square$

Therefore, Theorem 5.1 allows us to conclude, for $Q(2 + Q/q)^{-1} < p \leq 1$, that the equation

$$\mathcal{L}F = f, \quad f \in H^p(\mathbb{H}^n)$$

has a unique solution in $\mathcal{H}^p_{q,2}(\mathbb{H}^n)$, namely, $F := \mathcal{L}^{-1}f$.

We shall now see that the case $0 < p \leq Q(2 + \frac{Q}{q})^{-1}$ is trivial.

Theorem 5.2. *If* $1 < q < \frac{n+1}{n}$ *and* $0 < p \leq Q(2 + \frac{Q}{q})^{-1}$, *then* $\mathcal{H}^p_{q,2}(\mathbb{H}^n) = \{0\}$.

Proof. Let $G \in \mathcal{H}^p_{q,2}(\mathbb{H}^n)$ and assume $G \neq 0$. Then there exists $g \in G$ that is not a polynomial of homogeneous degree less or equal to 1. It is easy to check that there exist a positive constant c and a ρ-ball $B = B(e, r)$ with $r > 1$ such that

$$\int_B |g(w) - P(w)|^q \, dw \geq c > 0,$$

for every $P \in \mathcal{P}_1$.

Let z be a point such that $\rho(z) > r$ and let $\delta = 2\rho(z)$. Then $B(e, r) \subset B(z, \delta)$. If $h \in G$, then $h = g - P$ for some $P \in \mathcal{P}_1$ and

$$\delta^{-2}|h|_{q, B(z, \delta)} \geq c\rho(z)^{-2-Q/q}.$$

So $N_{q,2}(G; z) \geq c\,\rho(z)^{-2-Q/q}$, for $\rho(z) > r$. Since $p \leq Q(2 + Q/q)^{-1}$, we have

$$\int_{\mathbb{H}^n} [N_{q,2}(G; z)]^p dz \geq c \int_{\rho(z) > r} \rho(z)^{-(2+Q/q)p} \, dz = \infty,$$

which gives a contradiction. Thus, $\mathcal{H}^p_{q,2}(\mathbb{H}^n) = \{0\}$, if $p \leq Q(2 + Q/q)^{-1}$. $\qquad\square$

REFERENCES

[1] C. Benson, A.H. Dooley, G. Ratcliff, *Fundamental solutions for the powers of the Heisenberg sub-Laplacian*, Ill. J. Math. **37** (1993), 455–476.

[2] A. Bonfiglioli, *Taylor formula for homogenous groups and applications*, Math. Z. **262** (2009), 255–279.

[3] A.P. Calderón, *Estimates for singular integral operators in terms of maximal functions*, Studia Math. **44** (1972), 563–582.

[4] R. Durán, *Parabolic maximal functions and potentials of distributions in H^p*, J. Math. Anal. Appl. **100** (1984), 130–154.

[5] C. Fefferman, E.M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

[6] V. Fischer, M. Ruzhansky, *Quantization on Nilpotent Lie groups*, Progress in Mathematics, vol. 314, Birkhäuser-Springer, 2016.

[7] G. Folland, *A fundamental solution for a subelliptic operator*, Bull. Am. Math. Soc. **79** (1973), 373–376.

[8] G. Folland, *Introduction to Partial Differential Equations*, 2nd edition, Princeton, NJ: Princeton University Press, 1995.

[9] G. Folland, E. Stein, *Hardy Spaces on Homogeneous Groups*, Mathematical Notes, vol. 28, Princeton Univ. Press, 1982.

[10] A.B. Gatto, J.G. Jiménez, C. Segovia, *On the solution of the equation $\Delta^m F = f$ for $f \in H^p$*, Conference on Harmonic Analysis in honor of Antoni Zygmund, vol. II, Wadsworth International Mathematics Series, 1983.

[11] I.M. Gel'fand, G.E. Shilov, *Generalized Functions: Properties and Operations, Vol. 1*, Academic Press, 1964

[12] D. Geller, *Liouville's theorem for homogeneous groups*, Commun. Partial Differ. Equations **8** (1983), 1665–1677.

[13] L. Grafakos, L. Liu, D. Yang, *Vector-valued singular integrals and maximal functions on spaces of homogeneous type*, Math. Scand. **104** (2009), 296–310.

[14] R. Latter, *A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms*, Studia Math. **62** (1978), 93–101.

[15] Z. Liu, Z. He, H. Mo, *Orlicz Calderón–Hardy spaces*, Front. Math (2025).

[16] E. Nakai, Y. Sawano, *Orlicz-Hardy spaces and their duals*, Sci. China Math. **57** (2014), 903–962.

[17] P. Rocha, *Calderón–Hardy spaces with variable exponents and the solution of the equation $\Delta^m F = f$ for $f \in H^{p(\cdot)}(\mathbb{R}^n)$*, Math. Inequal. Appl. **19** (2016), 1013–1030.

[18] P. Rocha, *Weighted Calderón–Hardy spaces*, Math. Bohem. **150** (2025), 187–205.

[19] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[20] S. Thangavelu, *Harmonic Analysis on the Heisenberg group*, Birkhäuser, 1998.

Pablo Rocha
pablo.rocha@uns.edu.ar

Universidad Nacional del Sur
Departamento de Matemática
Avenida Alem 1253. 2do Piso
8000 Bahía Blanca, Argentina