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A SHORT NOTE ON HARNACK INEQUALITY FOR k-HESSIAN EQUATIONS WITH NONLINEAR GRADIENT TERMS

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Abstract. In this short note we study a Harnack inequality for k-Hessian equations that involve nonlinear lower-order terms which depend on the solution and its gradient.

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1. INTRODUCTION

In this paper, we develop a Harnack inequality for non-positive solutions of k-Hessian equations whose lower-order terms are non-linear functions of the solution and its gradient. We follow the methods outlined in [9, Lemma 4.1], thus extending the Harnack inequality, [9, Theorem 4.2] given for non-positive solutions of

$$S_k(D^2u) = c,$$

where c is a non-negative constant. In [3], the authors show a Liouville property for k-convex functions $u \in C^3(\mathbb{R}^n)$ that satisfy $S_k(D^2u) = 0$ in $\mathbb{R}^n$, with $u(x) = \mathrm{o}(|x|)$ at infinity. Our main result provides Harnack inequality for non-positive, k-convex, and smooth solutions to a wide class of equations which includes

$$S_k(D^2u) = a(x)|u|^k,$$

where a is a non-negative C^1 function which, together with its gradient, is bounded.

We begin by recalling some basic definitions leading up to the operator S_k and some of its properties that are useful to our subsequent work. We refer the reader to the papers in the references for more properties.

Let us denote by $\mathcal{S}^{n\times n}$ the family of all $n \times n$ symmetric matrices with real entries, and an integer $1 \le k \le n$. Given $X \in \mathcal{S}^{n\times n}$ and an integer $1 \le k \le n$, consider the k-Hessian operator

$$S_k(X) := \sigma_k(\lambda(X)).$$

Here, σ_k represents the k-th elementary symmetric polynomial of $\lambda := (\lambda_1, \dots, \lambda_n)$; that is

$$\sigma_k(\lambda) := \sum_{1 \le i_1 < \dots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k},$$

and $\lambda(X)$ stands for the n-tuple of real eigenvalues $\lambda(X) = (\lambda_1, \dots, \lambda_n)$ of X. In particular, we note that

$$\sigma_1(\lambda) = \sum_{j=1}^{n} \lambda_j, \quad \text{and} \quad \sigma_n(\lambda) = \lambda_1 \lambda_2 \dots \lambda_n.$$

In connection with these elementary symmetric functions, we consider the following open and convex cones in $\mathbb{R}^n$, with vertex at the origin:

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \ \text{ for } j = 1, \dots, k\}.$$

We note that

$$\Gamma^+ \subset \Gamma_n \subset \Gamma_{n-1} \subset \dots \subset \Gamma_1,$$

where

$$\Gamma^+ := \{\lambda \in \mathbb{R}^n : \lambda_j > 0 \ \ j = 1, \dots, n\}.$$

In fact, it is known that $\Gamma_n = \Gamma^+$. The cone Γ_k can also be characterized as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : 0 < \sigma_k(\lambda) \le \sigma_k(\lambda + \mu) \ \ \forall\, \mu = (\mu_1, \dots, \mu_n), \ \ \mu_j \ge 0, \ \ j = 1, \dots, n\}.$$

A function $u \in C^2(\Omega)$ is said to be k-convex if and only if

$$D^2 u(x) \in \overline{\Gamma}_k, \quad \forall\, x \in \Omega.$$

We remark that 1-convex functions are subharmonic in Ω, in the sense that $D^2 u$ has a non-negative trace in Ω. Therefore, any k-convex function in Ω is subharmonic in Ω.

In an open connected set $\Omega \subset \mathbb{R}^n$, we take $X = D^2 u$, and consider the k-Hessian equation

$$S_k(D^2 u) = g(x, |u|) + h(x, |u|)|Du|^\alpha \quad \text{in } \Omega, \quad 0 \le \alpha \le 1, \quad 2 \le k \le n, \tag{1.1}$$

where $g, h : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are given C^1 functions, and will be required to meet further conditions to be specified later. Following [3, 9], we write

$$\sigma_{k;j}(\lambda) := \left. \sigma_k(\lambda) \right|_{\lambda_j = 0}.$$

In this notation, we see that

$$\frac{\partial \sigma_k(\lambda)}{\partial \lambda_j} = \sigma_{k-1;j}(\lambda).$$

Given a real symmetric matrix $X = (x_{ij})_{n \times n}$, we write

$$S_k^{ij}(X) := \frac{\sigma_k(\lambda(X))}{\partial x_{ij}}.$$

It is well-known that if $u \in C^2(\Omega)$ is k-convex, then the k-Hessian operator is degenerate elliptic at u in the sense that $(S_k^{ij}(X))$ is positive semi-definite with $X = D^2u$. The k-Hessian operator $S_k(D^2u)$ can also be written as

$$S_k(D^2u) = [D^2u]_k,$$

where for $X \in \mathcal{S}^{n\times n}$ we used $[X]_k$ to denote the sum of the k-th principal minors of X.

With the notation

$$\mathcal{S} := \sum_{i=1}^{n} S_k^{ii}, \tag{1.2}$$

we recall the following useful relation, see the identity [9, (ii) on p. 7],

$$\mathcal{S} = (n-k+1)S_{k-1}(X) \quad \text{for } X \in \mathbb{S}^{n\times n}. \tag{1.3}$$

Finally, we adopt the following notations: Given an open set $\Omega \subseteq \mathbb{R}^n$, we will use the notation $B_r(x)$ to denote a ball of radius $r > 0$, centered at x, and compactly contained in Ω. We will also write $\mathbb{R}_0^+$ for the set of non-negative real numbers. In this paper, we always suppose that $0 \le \alpha \le 1$.

2. THE SUB-LINEAR EQUATION FOR $2 \le k \le n$

In this section we will consider sub-linear equations (see [9, Section 6] or [3, p. 1031]) for the case $2 \le k \le n$.

In view of this we now suppose that the non-negative non-linearities g, h in Problem (1.1) satisfy the following conditions for some constants $C_0 > 0$, $0 \le \alpha \le 1$, and for all $(x,t) \in \Omega \times \mathbb{R}_0^+$:

(c-g) $|g_x(x,t)| \le C_0 t^k, \quad |g_t(x,t)| \le C_0 t^{k-1}$,
(c-h) $h(x,t) + |h_x(x,t)| \le C_0 t^{k-\alpha}, \quad |h_t(x,t)| \le C_0 t^{k-1-\alpha}$.

Note that if u is a non-positive k-convex solution of (1.1) in $B_r(x)$, then $-u$ is a non-negative superharmonic function and hence either $-u > 0$ in $B_r(x)$ or $-u \equiv 0$ in $B_r(x)$. In view of this, in the theorem below we will consider nowhere vanishing k-convex solutions of (1.1).

Theorem 2.1. *Suppose $g, h : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are C^1 functions that satisfy conditions* (c-g) *and* (c-h). *For any nowhere vanishing k-convex solution $u \in C^3(B_r(\tilde{x}))$, $0 < r \le 1$, of* (1.1), *the following estimate holds:*

$$|Du(\tilde{x})| \le C\frac{M}{r}. \tag{2.1}$$

Here, $M := 4 \sup_{B_r(\tilde{x})} |u|$ and C is a positive constant that depends on n, k, and C_0 only.

Proof. The proof is an adaptation of the one given in [9]. See also [3, 7]. Let

$$\varrho(x) = \left(1 - \frac{|x-\tilde{x}|^2}{r^2}\right)^+, \quad \text{and} \quad \varphi(t) = \frac{1}{(M-t)^{1/2}}, \quad t < M.$$

It is easy to see that

$$\frac{2}{\sqrt{5M}} \le \varphi(t) \le \frac{2}{\sqrt{3M}}, \quad \text{for } |t| < \frac{M}{4}. \tag{2.2}$$

We also note that

$$\varphi'' - \frac{2(\varphi')^2}{\varphi} \ge \frac{1}{16} M^{-5/2}. \tag{2.3}$$

Let us consider the following auxiliary function

$$G(x,\xi) = u_\xi(x)\varphi(u(x))\varrho(x), \quad (x,\xi) \in B_r(\tilde{x}) \times S^{n-1},$$

where $S^{n-1} \subset \mathbb{R}^n$ is the unit sphere. We note that $G(x,\xi) = 0$ for $|x - \tilde{x}| = r$ and $\xi \in S^{n-1}$. Suppose $G(x,\xi)$ attains its maximum at (x_0, ξ_0). We begin with some preliminaries. By rotating coordinates we can assume that $\xi_0 = \boldsymbol{e}_1 = (1, 0, \ldots, 0)$. Then it follows that $Du(x_0) = (u_1(x_0), 0, \ldots, 0)$. We must have $u_1(x_0) \ge 0$ (recall that $G = 0$ on the boundary of $B_r(\tilde{x})$). We also make a further rotation of coordinates so that $u_{ij} = 0$ at x_0 for $i, j \ge 2$, $i \ne j$. Let us write $\mu_i = u_{ii}$ for $1 \le i \le n$, and $\mu := (\mu_1, \ldots, \mu_n)$. As noted in [3], we have the following equation at x_0 and will be used later.

$$S_{k-1}(D^2u) = S_k^{11}(D^2u) + u_{11}S_{k-2;1}(\mu) - \sum_{i=2}^{n} u_{1i}^2 S_{k-3,1i}(\mu). \tag{2.4}$$

From now on we write $G(x) := G(x, \xi_0)$ for $x \in B_r(\tilde{x})$. Moreover, all the quantities will be evaluated at x_0. We have $G_i(x_0) = 0$ and $\{G_{ij}(x_0)\} \le 0$. Thus,

$$u_{1i} = -\frac{u_1}{\varphi\varrho}\left(u_i\varphi'\varrho + \varphi\varrho_i\right). \tag{2.5}$$

Moreover, we compute

$$\begin{aligned} G_{ij} = {} & u_{1ij}\varphi(u)\varrho + u_{1i}\varphi'(u)u_j\varrho + u_{1i}\varphi(u)\varrho_j + u_{1j}\varphi'(u)u_i\varrho + u_1\varphi''(u)u_iu_j\varrho \\ & + u_1\varphi'(u)u_{ij}\varrho + u_1\varphi'(u)u_i\varrho_j + u_{1j}\varphi(u)\varrho_i + u_1\varphi'(u)u_j\varrho_i + u_1\varphi(u)\varrho_{ij}. \end{aligned}$$

Therefore, we find

$$\begin{aligned} 0 \ge S_k^{ij}G_{ij} = {} & \varphi\varrho S_k^{ij}u_{ij1} + 2S_k^{ij}u_{1i}\left(u_j\varphi'\varrho + \varphi\varrho_j\right) + u_1\varphi''\varrho S_k^{ij}u_iu_j \\ & + u_1\varphi'\varrho S_k^{ij}u_{ij} + u_1\varphi S_k^{ij}\varrho_{ij} + u_1\varphi' S_k^{ij}\left(u_i\varrho_j + u_j\varrho_i\right). \end{aligned}$$

Let us compute

$$\begin{aligned} u_{1i}\left(u_j\varphi'\varrho + \varphi\varrho_j\right) &= -\frac{u_1}{\varphi\varrho}\left(u_i\varphi'\varrho + \varphi\varrho_i\right)\left(u_j\varphi'\varrho + \varphi\varrho_j\right) \\ &= -\frac{u_1}{\varphi\varrho}\left(u_iu_j(\varphi')^2\varrho^2 + u_i\varphi\varphi'\varrho\varrho_j + u_j\varphi\varphi'\varrho\varrho_i + \varphi^2\varrho_i\varrho_j\right) \\ &= -\varrho u_1\frac{(\varphi')^2}{\varphi}u_iu_j - u_1u_i\varphi'\varrho_j - u_1u_j\varphi'\varrho_i - u_1\frac{\varphi}{\varrho}\varrho_i\varrho_j. \end{aligned}$$

Hence,

$$0 \geq S_k^{ij} G_{ij} = \varphi \varrho S_k^{ij} u_{ij1} - 2\varrho u_1 \frac{(\varphi')^2}{\varphi} S_k^{ij} u_i u_j - 4u_1 \varphi' S_k^{ij} u_i \varrho_j - 2u_1 \frac{\varphi}{\varrho} S_k^{ij} \varrho_i \varrho_j$$
$$+ u_1 \varphi'' \varrho S_k^{ij} u_i u_j + u_1 \varphi' \varrho S_k^{ij} u_{ij} + u_1 \varphi S_k^{ij} \varrho_{ij} + 2u_1 \varphi' S_k^{ij} u_i \varrho_j.$$

On using (2.3), from the latter inequality we find

$$\begin{aligned} 0 \geq S_k^{ij} G_{ij} = \varphi \varrho S_k^{ij} u_{ij1} + \varrho u_1 \frac{1}{16} M^{-\frac{5}{2}} S_k^{ij} u_i u_j - 2u_1 \varphi' S_k^{ij} u_i \varrho_j \\ - 2u_1 \frac{\varphi}{\varrho} S_k^{ij} \varrho_i \varrho_j + u_1 \varphi S_k^{ij} \varrho_{ij}. \end{aligned} \tag{2.6}$$

Note that we have omitted the term $u_1 \varphi' \varrho S_k^{ij} u_{ij}$ because

$$u_1 \varphi' \varrho S_k^{ij} u_{ij} = k u_1 \varphi' \varrho \left(g(x, |u|) + h(x, |u|) |Du|^\alpha \right) \geq 0.$$

Now let us assume that

$$|Du(\tilde{x})| > \frac{20M}{r}, \tag{2.7}$$

for otherwise there is nothing left to show.

Let us first make the following observation. Since $G(x, \xi)$ takes its maximum in $B_r(\tilde{x}) \times S^{n-1}$ at (x_0, ξ_0) we have

$$Du(\tilde{x}) \cdot \xi \varphi(u(\tilde{x})) \leq u_1(x_0) \varphi(u(x_0)) \varrho(x_0), \quad \forall \xi \in S^{n-1}.$$

By considering $\xi := Du(\tilde{x})/|Du(\tilde{x})|$ in the above relation, and on recalling (2.2), we find

$$|Du(\tilde{x})| \leq 2u_1 \varrho. \tag{2.8}$$

Therefore,

$$u_1 \varrho \geq \frac{1}{2} |Du(\tilde{x})| \geq \frac{10M}{r}. \tag{2.9}$$

Using this and (2.5), let us show that

$$-\frac{3}{2} \frac{\varphi'}{\varphi} u_1^2 \leq u_{11} \leq -\frac{1}{2} \frac{\varphi'}{\varphi} u_1^2. \tag{2.10}$$

For this, we first note that $2r|\varrho_1| \leq 4$ and $4(M-u) \leq 5M$. These inequalities, together with $2\varphi'(M-u) = \varphi$, show that

$$2r|\varrho_1| \leq \frac{5M}{M-u} = 10M \frac{\varphi'}{\varphi}. \tag{2.11}$$

Note that by (2.9) we have $10M \leq r \varrho u_1$, which on using in (2.11), leads to

$$-\frac{\varphi'}{\varphi} u_1 \varrho \leq -2\varrho_1 \leq \frac{\varphi'}{\varphi} u_1 \varrho. \tag{2.12}$$

We now multiply all sides of (2.12) by $\frac{u_1}{2\rho}$, and then add $-\frac{\varphi'}{\varphi}u_1^2$ to obtain

$$-\frac{3}{2}\frac{\varphi'}{\varphi}u_1^2 \le -u_1^2\frac{\varphi'}{\varphi} - \frac{u_1\varrho_1}{\varrho} \le -\frac{1}{2}\frac{\varphi'}{\varphi}u_1^2. \tag{2.13}$$

On taking $i = 1$ in (2.5), we get

$$u_{11} = -u_1^2\frac{\varphi'}{\varphi} - \frac{u_1\varrho_1}{\varrho}. \tag{2.14}$$

Inserting (2.14) in (2.13) leads to the claimed inequality (2.10).

Note that (2.9) implies

$$\frac{1}{\varrho u_1} \le \frac{r}{M}. \tag{2.15}$$

Now we estimate the term $S_k^{ij}u_{ij1}$. For this we differentiate Equation (1.1) with respect to the first variable x_1. We find the following at x_0, on recalling that $u_j(x_0) = 0$ for $j \ge 2$,

$$\begin{aligned} S_k^{ij}u_{ij1} = {} & g_{x_1}(x,|u|) + (\operatorname{sgn} u)g_t(x,|u|)u_1 + h_{x_1}(x,|u|)u_1^\alpha + (\operatorname{sgn} u)h_t(x,|u|)u_1^{\alpha+1} \\ & + \alpha h(x,|u|)u_1^{\alpha-1}u_{11}. \end{aligned} \tag{2.16}$$

Using conditions (c-g) and (c-h), we estimate (2.16) as follows.

$$\begin{aligned} & S_k^{ij}u_{ij1} \ge -|S_k^{ij}u_{ij1}| \\ & \ge -|g_{x_1}(x,|u|)| - |g_t(x,|u|)|u_1 - |h_{x_1}(x,|u|)|u_1^\alpha - |h_t(x,|u|)|u_1^{\alpha+1} \\ & \quad - |u_{11}||h(x,|u|)|u_1^{\alpha-1} \\ & \ge -C_0\left(|u|^k + |u|^{k-1}u_1 + |u|^{k-\alpha}u_1^\alpha + |u|^{k-1-\alpha}u_1^{\alpha+1} - |u|^{k-\alpha}u_1^{\alpha-1}u_{11}\right) \\ & \ge -C_0\left(|u|^k + |u|^{k-1}u_1 + |u|^{k-\alpha}u_1^\alpha + |u|^{k-1-\alpha}u_1^{\alpha+1} + \frac{3}{2}|u|^{k-\alpha}u_1^{\alpha+1}\frac{\varphi'}{\varphi}\right), \end{aligned} \tag{2.17}$$

where the left inequality of (2.10) has been used to get (2.17). Let us show that

$$\frac{3}{2}|u|^{k-\alpha}u_1^{\alpha+1}\frac{\varphi'}{\varphi} \le |u|^{k-1-\alpha}u_1^{\alpha+1}.$$

Indeed, since $|u| \le M/4$, we have

$$\frac{3}{2}|u|\frac{\varphi'}{\varphi} = \frac{3|u|}{4(M-u)} \le 1.$$

In view of this remark, we can write (2.17) as

$$S_k^{ij}u_{ij1} \ge -2C_0\left(|u|^k + |u|^{k-1}u_1 + |u|^{k-\alpha}u_1^\alpha + |u|^{k-1-\alpha}u_1^{\alpha+1}\right). \tag{2.18}$$

Using (2.18) in (2.6) we find

$$\begin{aligned} 0 \geq S_k^{ij} G_{ij} \geq & -2C_0 \varphi \varrho \left(|u|^k + |u|^{k-1} u_1 + |u|^{k-\alpha} u_1^{\alpha} + |u|^{k-1-\alpha} u_1^{\alpha+1} \right) \\ & + \frac{1}{16} M^{-\frac{5}{2}} u_1 \varrho S_k^{ij} u_i u_j \\ & - 2u_1 \varphi' S_k^{ij} u_i \varrho_j - 2u_1 \frac{\varphi}{\varrho} S_k^{ij} \varrho_i \varrho_j + u_1 \varphi S_k^{ij} \varrho_{ij}. \end{aligned} \tag{2.19}$$

By Young's inequality, we have

$$2S_k^{ij} u_i \varrho_j \leq 2 \left(S_k^{ij} u_i u_j \right)^{1/2} \left(S_k^{ij} \varrho_i \varrho_j \right)^{1/2} \leq \varepsilon S_k^{ij} u_i u_j + \frac{1}{\varepsilon} S_k^{ij} \varrho_i \varrho_j. \tag{2.20}$$

With the choice of

$$\varepsilon := \frac{1}{32} M^{-5/2} \frac{\varrho}{\varphi'},$$

we now use inequality (2.20) in (2.19) to get,

$$\begin{aligned} 0 \geq & -2C_0 \varphi \varrho \left(|u|^k + |u|^{k-1} u_1 + |u|^{k-\alpha} u_1^{\alpha} + |u|^{k-\alpha-1} u_1^{\alpha+1} \right) \\ & + \frac{1}{32} M^{-5/2} u_1 \varrho S_k^{ij} u_i u_j \\ & - 32 M^{5/2} \frac{(\varphi')^2}{\varrho} u_1 S_k^{ij} \varrho_i \varrho_j - 2u_1 \frac{\varphi}{\varrho} S_k^{ij} \varrho_i \varrho_j + u_1 \varphi S_k^{ij} \varrho_{ij}. \end{aligned} \tag{2.21}$$

In subsequent calculations it will be convenient to denote any positive constant that depends at most on k, n, and C_0 by C, and its value may vary from line to line. Recalling that $u_i(x_0) = 0$ for $i = 2, \dots, n$, we find

$$S_k^{ij} u_i u_j = S_k^{11} u_1^2,$$

and

$$\frac{1}{32} M^{-5/2} u_1 \varrho S_k^{ij} u_i u_j = \frac{1}{32} M^{-5/2} u_1 \varrho S_k^{11} u_1^2. \tag{2.22}$$

Moreover, since

$$S_k^{ij} \varrho_i \varrho_j \leq \operatorname{tr}((S_k^{ij})) |\nabla \varrho|^2 \leq \frac{4}{r^2} \mathcal{S},$$

we find

$$-32 M^{5/2} \frac{(\varphi')^2}{\varrho} u_1 S_k^{ij} \varrho_i \varrho_j \geq -C M^{-\frac{1}{2}} \frac{u_1 \mathcal{S}}{\varrho r^2} \tag{2.23}$$

and

$$-2u_1 \frac{\varphi}{\varrho} S_k^{ij} \varrho_i \varrho_j \ \geq -C M^{-\frac{1}{2}} \frac{u_1 \mathcal{S}}{\varrho r^2}. \tag{2.24}$$

Finally, since

$$\varrho_{ij} = -\frac{\delta_{ij}}{r^2}, \quad \delta_{ij} = \text{Kronecker delta},$$

we find

$$u_1 \varphi S_k^{ij} \varrho_{ij} \geq -8 M^{-\frac{1}{2}} \frac{u_1 \mathcal{S}}{r^2} \ \geq -8 M^{-\frac{1}{2}} \frac{u_1 \mathcal{S}}{\varrho r^2}. \tag{2.25}$$

Using (2.22), (2.23), (2.24) and (2.25) in (2.21) we find

$$0 \geq -C\varphi\varrho\left(|u|^k + |u|^{k-1}u_1 + |u|^{k-\alpha}u_1^\alpha + |u|^{k-\alpha-1}u_1^{\alpha+1}\right) + \frac{1}{32}M^{-5/2}\varrho S_k^{11}u_1^3 - CM^{-\frac{1}{2}}\frac{u_1\mathcal{S}}{\varrho r^2}.$$

Multiplying both sides of this inequality by $32M^{5/2}\varrho^2$, we find

$$0 \geq -C\varphi\varrho^3\left(|u|^k + |u|^{k-1}u_1 + |u|^{k-\alpha}u_1^\alpha + |u|^{k-1-\alpha}u_1^{\alpha+1}\right)M^{5/2} + \varrho^3 S_k^{11}u_1^3 - C\frac{M^2}{r^2}\mathcal{S}\varrho u_1,$$

and thus

$$(\varrho u_1)^3 S_k^{11} \leq C\frac{M^2}{r^2}\mathcal{S}\varrho u_1 + C\varphi\varrho^3\left(|u|^k + |u|^{k-1}u_1 + |u|^{k-\alpha}u_1^\alpha + |u|^{k-1-\alpha}u_1^{\alpha+1}\right)M^{5/2}. \tag{2.26}$$

Multiplying both sides of (2.26) by $\mathcal{S}^{-1}(\varrho u_1)^{-2}$ and using (2.15) we obtain

$$\begin{aligned}
&\varrho u_1 S_k^{11}\mathcal{S}^{-1}\\
&\leq C\frac{M^2}{\varrho u_1 r^2} + C\varphi\varrho^3\left(\frac{|u|^k}{(\varrho u_1)^2} + \frac{|u|^{k-1}u_1}{(\varrho u_1)^2} + \frac{|u|^{k-\alpha}u_1^\alpha}{(\varrho u_1)^2} + \frac{|u|^{k-1-\alpha}u_1^{\alpha+1}}{(\varrho u_1)^2}\right)M^{5/2}\mathcal{S}^{-1}\\
&\leq C\frac{M^2}{\varrho u_1 r^2} + CM^2\left(\frac{|u|^k\varrho^3}{(\varrho u_1)^2} + \frac{|u|^{k-1}\varrho^2}{\varrho u_1} + \frac{|u|^{k-\alpha}\varrho^{3-\alpha}}{(\varrho u_1)^{2-\alpha}} + \frac{|u|^{k-1-\alpha}\varrho^{2-\alpha}}{(\varrho u_1)^{1-\alpha}}\right)\mathcal{S}^{-1}\\
&\leq C\frac{M}{r} + CM^2\left(|u|^k\left(\frac{r}{M}\right)^2 + |u|^{k-1}\left(\frac{r}{M}\right) + |u|^{k-\alpha}\left(\frac{r}{M}\right)^{2-\alpha} + |u|^{k-1-\alpha}\left(\frac{r}{M}\right)^{1-\alpha}\right)\mathcal{S}^{-1}\\
&\leq C\frac{M}{r} + C\frac{M}{r}\left(r^3\frac{|u|^k}{M} + r^2|u|^{k-1} + r^{3-\alpha}\frac{|u|^{k-\alpha}}{M^{1-\alpha}} + r^{2-\alpha}\frac{|u|^{k-1-\alpha}}{M^{-\alpha}}\right)\mathcal{S}^{-1},
\end{aligned} \tag{2.27}$$

where we have used the inequality $\varrho(x) \leq 1$ and $0 \leq \alpha \leq 1$. To find a bound for $\mathcal{S}^{-1}$, let $(\rho_1, \ldots, \rho_n)$ with $\rho_1 \geq \ldots \geq \rho_n$ be the eigenvalues of D^2u at x_0. Then, at x_0, we have

$$\rho_n \leq \zeta^T D^2 u\zeta,$$

for any unit vector $\eta \in \mathbb{R}^n$. Taking $\zeta = e_1$ we see that $e_1^T D^2u(x_0)e_1 = u_{11}$. Hence,

$$\rho_n \leq u_{11} \leq -\frac{\varphi'}{2\varphi}u_1^2 \leq -\frac{1}{4M}u_1^2. \tag{2.28}$$

Now we observe that at x_0

$$\begin{aligned}
0 \leq g(x,|u|) + h(x,|u|)|Du|^\alpha &= \sigma_k(\rho_1,\ldots,\rho_n)\\
&= \rho_n\sigma_{k-1}(\rho_1,\ldots,\rho_{n-1}) + \sigma_k(\rho_1,\ldots\rho_{n-1})\\
&\leq \rho_n\sigma_{k-1}(\rho_1,\ldots,\rho_{n-1}) + C\left[\sigma_{k-1}(\rho_1,\ldots,\rho_{n-1})\right]^{k/(k-1)},
\end{aligned}$$

where, for the last inequality, [9, (vi) on p. 7] has been used, with a positive constant C that depends on n and k only. Hence,

$$-\rho_n \leq C(\sigma_{k-1}(\rho_1, \ldots, \rho_{n-1}))^{1/(k-1)}. \tag{2.29}$$

From (2.28) and (2.29), we find

$$\tilde{S}_k^{nn} = \sigma_{k-1}(\rho_1, \ldots, \rho_{n-1}) \geq C|\rho_n|^{k-1} \geq C\frac{u_1^{2k-2}}{M^{k-1}}. \tag{2.30}$$

Note that, since $\mathcal{S}$ is invariant under rotation, we have

$$\mathcal{S} = S_k^{11} + \ldots + S_k^{nn} = \tilde{S}_k^{11} + \ldots + \tilde{S}_k^{nn}.$$

Therefore, (2.30) implies that

$$\mathcal{S} \geq C\frac{u_1^{2k-2}}{M^{k-1}}. \tag{2.31}$$

From (2.9), we have

$$u_1 \geq u_1\varrho \geq \frac{1}{2}|Du(\tilde{x})| \geq \frac{M}{r}.$$

Using this in (2.31), we find that

$$\mathcal{S} \geq C\frac{M^{k-1}}{r^{2k-2}}.$$

Employing this last inequality in (2.27), we obtain, on recalling that $0 < r \leq 1$,

$$\begin{aligned}
&\varrho u_1 S_k^{11}\mathcal{S}^{-1} \\
&\leq C\frac{M}{r} + C\frac{M}{r}\left(|u|^k\frac{r^3}{M} + r^2|u|^{k-1} + r^2|u|^{k-\alpha}\left(\frac{r}{M}\right)^{1-\alpha} + |u|^{k-\alpha-1}\left(\frac{r}{M}\right)^{2-\alpha}M^2\right)\frac{r^{2k-2}}{M^{k-1}} \\
&\leq C\frac{M}{r} + C\frac{M}{r}\left(r^{2k+1}\left(\frac{|u|}{M}\right)^k + r^{2k}\left(\frac{|u|}{M}\right)^{k-1} + r^{2k+1-\alpha}\left(\frac{|u|}{M}\right)^{k-\alpha} + r^{2k-\alpha}\left(\frac{|u|}{M}\right)^{k-\alpha-1}\right) \\
&\leq C\frac{M}{r},
\end{aligned} \tag{2.32}$$

where C is a positive constant that depends on n, k and C_0 only.

Recalling that $u_{11} < 0$ at x_0, we use (2.4) to find $S_k^{11} \geq S_{k-1}$. From (1.3), with $X = D^2u(x_0)$, we have $S_{k-1} = (n-k+1)^{-1}\mathcal{S}$. From these two relations, we obtain

$$S_k^{11}\mathcal{S}^{-1} \geq (n-k+1)^{-1}.$$

Using this in (2.32) we see that

$$\varrho u_1 \leq C\frac{M}{r}. \tag{2.33}$$

From (2.8) and (2.33) we find that

$$|Du(\tilde{x})| \leq C\frac{M}{r},$$

as was to be shown. □

Theorem 2.2. *Let $g, h : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy conditions* (c-g) *and* (c-h)*. Then there is a positive constant C that depends on n, k, and C_0 only such that for any non-positive k-convex solution $u \in C^3(B_R(0))$, $0 < R \leq 1$, of* (1.1) *we have*

$$\sup_{B_{R/2}(0)}(-u) \leq C \inf_{B_{R/2}(0)}(-u) \tag{2.34}$$

Proof. The Harnack inequality now follows from Theorem 2.1, [9, Lemma 4.1], and the same argument used to prove [9, Theorem 4.2]. In fact, by Theorem 2.1, and [9, Lemma 4.1] we have

$$\sup_{B_{R/2}(0)}(-u) \leq \frac{C}{|B_R(0)|}\int\limits_{B_R(0)}(-u). \tag{2.35}$$

Since u is subharmonic, we also have

$$\frac{1}{|B_R(0)|}\int\limits_{B_R(0)}(-u) \leq C \inf_{B_{R/2}(0)}(-u). \tag{2.36}$$

The conclusion follows from (2.35) and (2.36). □

3. THE SUPER-LINEAR EQUATION FOR THE CASE $2 \leq k \leq n$

It is also possible to obtain Harnack inequality for a family of smooth, non-positive and k-convex solutions of super-linear equations (see [9, Section 6] or [3, p. 1031]) that are bounded in the uniform metric of C^0. These class of equations include those of the form

$$S_k(D^2u) = |u|^p, \quad p > k.$$

For $p > k$ we assume conditions similar to (c-g) and (c-h) with p in place of k; that is we suppose the following hold in $\Omega \times \mathbb{R}_0^+$ for some $C_0 > 0$, $p > k$ and $0 \le \alpha \le 1$.

(p-g) $|g_x(x,t)| \le C_0 t^p, \quad |g_t(x,t)| \le C_0 t^{p-1}$,
(p-h) $h(x,t) + |h_x(x,t)| \le C_0 t^{p-\alpha}, \quad |h_t(x,t)| \le C_0 t^{p-1-\alpha}$.

Given a constant $L > 0$, and a subset $E \subset \Omega$, let us write $\mathcal{F}_L(E)$ for the family of all real-valued functions on E that are bounded by L.

Theorem 3.1. *Suppose $g, h : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are C^1 functions that satisfy conditions* (p-g) *and* (p-h)*. Given a constant $L > 0$, then for any nowhere vanishing k-convex solution $u \in C^3(B_r(\tilde{x})) \cap \mathcal{F}_L(B_1(\tilde{x}))$, $0 < r \le 1$, of* (1.1)*, the following estimate holds:*

$$|Du(\tilde{x})| \le C\frac{M}{r}. \tag{3.1}$$

Here, $M := 4\sup_{B_r(\tilde{x})}|u|$ and C is a positive constant that depends on n, k, p, L and C_0 only.

Proof. Consider first the case $\Lambda \le 1$, where $\Lambda := 4L$. The proof is the same as that of Theorem 2.1 until formula (2.32), which now becomes

$$\varrho u_1 S_k^{11}\mathcal{S}^{-1} \le C\frac{M}{r} + C\frac{M}{r}\left(|u|^p\frac{r^3}{M} + r^2|u|^{p-1} + r^2|u|^{p-\alpha}\left(\frac{r}{M}\right)^{1-\alpha} + |u|^{p-\alpha-1}\left(\frac{r}{M}\right)^{2-\alpha}M^2\right)\frac{r^{2k-2}}{M^{k-1}}.$$

Since $M \le 4\sup_{B_r(\tilde{x})}|u(x)| \le 1$, and $p > k$ we have $\frac{1}{M^{k-1}} \le \frac{1}{M^{p-1}}$. Hence,

$$\begin{aligned}\varrho u_1 S_k^{11}\mathcal{S}^{-1} &\le C\frac{M}{r} + C\frac{M}{r}\left(r^{2k+1}\left(\frac{|u|}{M}\right)^p + r^{2k}\left(\frac{|u|}{M}\right)^{p-1}\right.\\ &\qquad\left. + r^{2k+1-\alpha}\left(\frac{|u|}{M}\right)^{p-\alpha} + r^{2k-\alpha}\left(\frac{|u|}{M}\right)^{p-\alpha-1}\right)\\ &\le C\frac{M}{r}.\end{aligned} \tag{3.2}$$

From here on, the proof continues as in that of Theorem 2.1.

Now let $\Lambda > 1$. Putting $u = \Lambda v$, equation (1.1) implies

$$S_k(D^2v) = \Lambda^{-k}\left[g(x,\Lambda v) + h(x,\Lambda v)\Lambda^\alpha|Dv|^\alpha\right] \quad \text{in } B_r(\tilde{x}). \tag{3.3}$$

If we put $\tilde{g}(x,t) := \Lambda^{-k}g(x,\Lambda t)$ and $\tilde{h}(x,t) := \Lambda^{\alpha-k}h(x,\Lambda t)$, equation (3.3) reads as

$$S_k(D^2v) = \tilde{g}(x,v) + \tilde{h}(x,v)|Dv|^\alpha \quad \text{in } B_r(\tilde{x}).$$

From (p-g) and (p-h) we find

(Λ-g) $|\tilde{g}_x(x,t)| \le \Lambda^{p-k} C_0 t^p, \quad |\tilde{g}_t(x,t)| \le \Lambda^{p-k} C_0 t^{p-1}$,
(Λ-h) $\tilde{h}(x,t) + |\tilde{h}_x(x,t)| \le \Lambda^{p-k} C_0 t^{p-\alpha}, \quad |\tilde{h}_t(x,t)| \le \Lambda^{p-k} C_0 t^{p-\alpha-1}$.

Now the proof is the same as before and uses (Λ-g) and (Λ-h), instead of (p-h) and (p-g). One finds

$$|Dv(\tilde{x})| \le C\frac{M}{r}, \quad 0 < r \le 1, \tag{3.4}$$

where,

$$M := 4 \sup_{B_r(\tilde{x})} |v|,$$

and C is a positive constant that depends on n, k, p, L and C_0 only.

Since $u = \Lambda v$, (3.4) implies

$$|Du(\tilde{x})| \le C\frac{\tilde{M}}{r}, \quad 0 < r \le 1,$$

with

$$\tilde{M} := 4 \sup_{B_r(\tilde{x})} |u|.$$

The theorem is proved. □

Clearly, we also have the following theorem.

Theorem 3.2. *Suppose $g, h : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are C^1 functions that satisfy conditions* (p-g) *and* (p-h)*. Given a constant $L > 0$, then for any nowhere vanishing, non-positive, and k-convex solution $u \in C^3(B_R(0)) \cap \mathcal{F}_L(B_1(0))$, $0 < R \le 1$, of* (1.1)*, there is a positive constant C, that depends on n, k, p, L, and C_0 only, such that*

$$\sup_{B_{R/2}(0)} (-u) \le C \inf_{B_{R/2}(0)} (-u).$$

Proof. The proof uses (3.4) and is the same as that of Theorem 2.2. □

4. THE SUB/SUPER-LINEAR EQUATION FOR THE CASE $k = 1$

We have proved our previous results for $k \ge 2$. Now we discuss the case $k = 1$. Consider the equation

$$\Delta u = g(x, |u|) + h(x)|Du| \quad \text{in } \Omega, \tag{4.1}$$

where $g : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $h : \Omega \to \mathbb{R}_0^+$ are non-negative C^1 functions satisfying, for some constant $C_0 > 0$,

(c′-g) $|g_x(x,t)| \le C_0 t, \quad |g_t(x,t)| \le C_0, \quad t > 0$,
(c′-h) $h(x) + |h_x(x)| \le C_0$.

It is conceivable that the Harnack inequality for equation (4.1), under assumptions (c′-g) and (c′-h), has been addressed in the existing literature. However, in the absence of specific references known to us, we have elected to include a discussion of the Harnack inequality for non-negative solutions of (4.1).

Theorem 4.1. *Suppose $g : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $h : \Omega \to \mathbb{R}_0^+$ are non-negative C^1 functions that satisfy conditions* (c'-g) *and* (c'-h). *For any nowhere vanishing solution $u \in C^3(B_r(\tilde{x}))$, $0 < r \le 1$, of* (4.1), *the following estimate holds:*

$$|Du(\tilde{x})| \le C\frac{M}{r}. \tag{4.2}$$

Here, $M := 4\sup_{B_r(\tilde{x})} |u|$ and C is a positive constant that depends on n and C_0 only.

Proof. We use the same notations as in the proof of Theorem 2.1. Let us compute

$$\Delta u_1 = g_{x_1}(x,|u|) + (\text{sign } u)g_t(x,|u|)u_1 + h_{x_1}(x)u_1 + h(x)u_{11}. \tag{4.3}$$

Using (c′-g) and (c′-h) and the left inequality of (2.10) in (4.3), we find

$$\begin{aligned} |\Delta u_1| &\le C_0\big(|u| + 2u_1 + |u_{11}|\big) \\ &\le C_0\Big(|u| + 2u_1 + 3\frac{\varphi'}{2\varphi}u_1^2\Big). \end{aligned} \tag{4.4}$$

Since

$$3\frac{\varphi'}{2\varphi} = \frac{3}{4(M-u)} \le \frac{1}{M},$$

from (4.4) we find

$$|\Delta u_1| \le 2C_0\Big(|u| + u_1 + \frac{u_1^2}{M}\Big). \tag{4.5}$$

Using the estimate (4.5), we now proceed as in the proof Theorem 2.1, up to (2.26). Since in case of $k = 1$ we have $S_1^{11} = 1$ and $\mathcal{S} = n$, the inequality (2.26) becomes

$$(\varrho u_1)^3 \le C\frac{M^2}{r^2}\varrho u_1 + C\varphi\varrho^3\left(|u| + u_1 + \frac{u_1^2}{M}\right)M^{5/2}, \tag{4.6}$$

where the constant C depends on n and C_0 only. Multiplying both sides of (4.6) by $(\varrho u_1)^{-2}$ and using the estimate

$$\frac{1}{\varrho u_1} \le \frac{r}{M}$$

we obtain

$$\begin{aligned} \varrho u_1 &\le C\frac{M^2}{r^2}\frac{1}{\varrho u_1} + C\varphi\varrho^3\left(\frac{|u|}{(\varrho u_1)^2} + \frac{u_1}{(\varrho u_1)^2} + \frac{u_1^2}{M}\frac{1}{(\varrho u_1)^2}\right)M^{5/2} \\ &\le C\frac{M}{r} + CM^2\left(\frac{|u|\varrho^3}{(\varrho u_1)^2} + \frac{\varrho^2}{\varrho u_1} + \frac{\varrho}{M}\right) \\ &\le C\frac{M}{r} + CM^2\left(|u|\Big(\frac{r}{M}\Big)^2 + \frac{r}{M} + \frac{1}{M}\right) \\ &= C\frac{M}{r} + C\frac{M}{r}\left(r^3\frac{|u|}{M} + r^2 + r\right) \\ &\le C\frac{M}{r}, \end{aligned}$$

where we have used the inequalities $\varrho(x) \le 1$ and $0 \le r \le 1$.

We have found

$$\varrho u_1 \le C\frac{M}{r}. \tag{4.7}$$

From (2.8) and (4.7), it follows that

$$|Du(\tilde{x})| \le C\frac{M}{r},$$

as was to be shown. □

Theorem 4.2. *Let g and h be as in Theorem 4.1. Then there is a positive constant C that depends on n and C_0 only such that for any non-positive solution $u \in C^3(B_R(0))$, $0 < R \le 1$, of* (4.1) *we have*

$$\sup_{B_{R/2}(0)} (-u) \le C \inf_{B_{R/2}(0)} (-u). \tag{4.8}$$

Proof. The proof uses Theorem 4.1 and is the same as that of Theorem 2.2. □

Consider now Equation (4.1) with g and h satisfying, for some $p > 1$,

(p′-g) $|g_x(x,t)| \le C_0 t^p, \quad |g_t(x,t)| \le C_0 t^{p-1}, \quad t > 0,$
(p′-h) $h(x) + |h_x(x)| \le C_0 t^{p-1}$.

As in (4.3), we have

$$\Delta u_1 = g_{x_1}(x,|u|) + (\text{sign } u) g_t(x,|u|) u_1 + h_{x_1}(x) u_1 + h(x) u_{11}.$$

Using (p′-g) and (p′-h) and the left inequality in (2.10), from the latter equation we find

$$\begin{aligned} |\Delta u_1| &\le C_0\big(|u|^p + 2|u|^{p-1}u_1 + |u|^{p-1}|u_{11}|\big) \\ &\le C_0\Big(|u|^p + 2|u|^{p-1}u_1 + |u|^{p-1} 3\frac{\varphi'}{2\varphi} u_1^2\Big) \end{aligned} \tag{4.9}$$

Since

$$3\frac{\varphi'}{2\varphi} \le \frac{1}{M},$$

from (4.9) we find

$$|\Delta u_1| \le 2C_0\Big(|u|^p + |u|^{p-1}u_1 + |u|^{p-1}\frac{u_1^2}{M}\Big). \tag{4.10}$$

As in the previous Section, given a constant $L > 0$ and a subset $E \subset \Omega$, let us write $\mathcal{F}_L(E)$ for the family of all real-valued functions on E that are bounded by L.

Theorem 4.3. *Suppose $g : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $g : \Omega \to \mathbb{R}_0^+$ are C^1 functions that satisfy conditions* (p′-g) *and* (p′-h). *Given a constant $L > 0$ then, for any nowhere vanishing solution $u \in C^3(B_r(\tilde{x})) \cap \mathcal{F}_L(B_1(\tilde{x}))$, $0 < r \le 1$, of* (4.1), *the following estimate holds:*

$$|Du(\tilde{x})| \le C\frac{M}{r}. \tag{4.11}$$

Here, $M := 4 \ \sup_{B_r(\tilde{x})} |u|$ and C is a positive constant that depends on n, p, L and C_0 only.

Proof. Consider first the case $\Lambda \leq 1$, where $\Lambda := 4L$. For the proof we may start from the inequality (4.6) replacing estimate (4.5) by estimate (4.10). We find

$$\begin{aligned}(\varrho u_1)^3 &\leq C\frac{M^2}{r^2}\varrho u_1 + C\varphi\varrho^3\left(|u|^p + |u|^{p-1}u_1 + |u|^{p-1}\frac{u_1^2}{M}\right)M^{5/2}\\ &\leq C\frac{M^2}{r^2}\varrho u_1 + C\varrho^3 M\left(M|u|^p + M|u|^{p-1}u_1 + |u|^{p-1}u_1^2\right).\end{aligned}$$

Since $M \leq 4\sup_{B_r(\tilde{x})}|u(x)| \leq 1$, and $p > 1$ we have $1 \leq \frac{1}{M^{p-1}}$. Hence, on using (2.33) we find

$$\begin{aligned}\varrho u_1 &\leq C\frac{M^2}{r^2}\frac{1}{\varrho u_1} + CM\left(M|u|^p\frac{\varrho^3}{(\varrho u_1)^2} + M\varrho^3|u|^{p-1}u_1\frac{1}{(\varrho u_1)^2} + \varrho^3|u|^{p-1}u_1^2\frac{1}{(\varrho u_1)^2}\right)\frac{1}{M^{p-1}}\\ &\leq C\frac{M}{r} + CM\left(M|u|^p\frac{r^2}{M^2} + M|u|^{p-1}\frac{r}{M} + |u|^{p-1}\right)\frac{1}{M^{p-1}}\\ &= C\frac{M}{r} + C\frac{M}{r}\left(r^3\left(\frac{|u|}{M}\right)^p + r^2\left(\frac{|u|}{M}\right)^{p-1} + r\left(\frac{|u|}{M}\right)^{p-1}\right)\\ &\leq C\frac{M}{r}.\end{aligned} \tag{4.12}$$

If $\Lambda > 1$, one puts $u = \Lambda v$ and continues as in the proof of Theorem 3.1. One finds (4.12) again, but now C depends on n, C_0 and L. The theorem is proved. □

Clearly, we also have the following theorem.

Theorem 4.4. *Let g and h be as in Theorem 4.3. Given a constant $L > 0$, then for any nowhere vanishing, non-positive solution $u \in C^3(B_R(0)) \cap \mathcal{F}_L(B_1(0))$, $0 < R \leq 1$, of* (4.1)*, there is a positive constant C, that depends on n, p, L, and C_0 only, such that*

$$\sup_{B_{R/2}(0)}(-u) \leq C \inf_{B_{R/2}(0)}(-u).$$

Proof. The proof uses Theorem 4.3 and is the same as that of Theorem 2.2. □

5. A LIOUVILLE-TYPE THEOREM

In [3, 9], it is noted that any k convex solution $u \in C^3(\mathbb{R}^n)$ of $S_k(D^2u) = 0$ in $\mathbb{R}^n$ is a constant. The proof relies on the interior gradient estimate derived therein. Here, we prove a type of Liouville theorem for k-convex solutions $u \in C^2(\mathbb{R}^n)$ of $S_k(D^2u) \geq \omega(|x|)$, where $\omega(t)$ is a positive and non-decreasing continuous function. Our proof relies on a standard comparison principle, which we recall here for the reader's convenience. See [6] for a more general result.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and suppose that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are k-convex functions such that*

$$S_k(D^2u) \geq S_k(D^2v) \quad \text{in } \Omega.$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Now we state and prove the following Liouville theorem.

Theorem 5.2. *Suppose $u \in C^2(\mathbb{R}^n)$ is a k-convex solution of*

$$S_k(D^2u) \geq \omega(|x|) \quad \text{in } \mathbb{R}^n \tag{5.1}$$

for some continuous and non-decreasing function $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\omega(t) > 0$ for $t > 0$. Then

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{|x|} > 0.$$

Proof. Suppose the conclusion fails. Then $u(x) = o(|x|)$ as $|x| \to \infty$. We may assume that $u(0) \geq 0$; for otherwise, we can replace $u(x)$ by $u(x) - u(0)$. Given $\delta > 0$, we select $\rho = \rho(\delta) > 0$ such that

$$u(x) \leq u(0) + \delta, \quad x \in B_\rho(0). \tag{5.2}$$

Let us fix $\varepsilon_0 := \varepsilon_0(n, k, \rho, c) > 0$ small enough such[1] that

$$\binom{n-1}{k}\varepsilon_0^k \leq \rho^k \omega(\rho).$$

Let $0 < \varepsilon < \varepsilon_0$ be arbitrary. Since

$$\lim_{|x|\to\infty} \frac{u(x)}{|x| - \rho} = 0,$$

there is $R = R(\varepsilon) > \rho$ such that $u(x) \leq \varepsilon(|x| - \rho)$ for all $|x| \geq R$. Let

$$v_\varepsilon(x) := u(0) + \delta + \varepsilon(|x| - \rho), \quad |x| > \rho.$$

Then

$$S_k(D^2 v_\varepsilon) = \binom{n-1}{k}\varepsilon^k |x|^{-k} \leq \omega(|x|), \quad |x| \geq \rho.$$

For any fixed $r \geq R$, we note that $u(x) \leq v_\varepsilon(x)$ on the boundary $\partial(B_r(0) \setminus B_\rho(0))$. By the comparison principle, Theorem 5.1, we have

$$u(x) \leq v_\varepsilon(x), \quad \rho \leq |x| \leq r.$$

Letting $r \to \infty$, followed by sending $\varepsilon \to 0^+$, we obtain $u(x) \leq u(0) + \delta$ for all $|x| \geq \rho$. This, together with (5.2), gives $u(x) \leq u(0) + \delta$ in $\mathbb{R}^n$. Since δ is arbitrary we see that $u(x) \leq u(0)$. Since u is subharmonic, the Strong Maximum Principle shows that $u(x) = u(0)$ in $\mathbb{R}^n$. From (5.1), we get $\omega(|x|) \leq S_k(D^2u(0)) = 0$, contradicting the condition $\omega(|x|) > 0$. □

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1) Here $\binom{n-1}{k}$ is interpreted to be zero for $k = n$.

We wish to point out that gradient estimates analogous to those obtained in Sections 2, 3, and 4 can also be established for viscosity solutions of equation $\mathscr{P}(D^2u) = 0$, where $\mathscr{P}$ belongs to a related class of degenerate elliptic operators depending on the eigenvalues of the Hessian matrix D^2u. This class includes, in particular, the partial trace operators (see [4]), corresponding to the case

$$\mathscr{P}(D^2u) = \lambda_{i_1}(D^2u) + \ldots + \lambda_{i_k}(D^2u),$$

with $i_1 = 1$ and $i_k = n$. For details, we refer to [8].

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