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WINTNER-TYPE ASYMPTOTIC BEHAVIOR OF LINEAR DIFFERENTIAL SYSTEMS WITH A PROPORTIONAL DERIVATIVE CONTROLLER

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Abstract. This study investigated the asymptotic behavior of linear differential systems incorporating a proportional derivative-type (PD) differential operator. Building on the classical asymptotic convergence property of Wintner, a generalized Wintner-type asymptotic result was established for such systems. The proposed framework encompasses a wide class of time-varying coefficient matrices and extends classical asymptotic theory to equations governed by PD operators. An illustrative example is presented to demonstrate the applicability of the proposed theorem.

Keywords: Wintner-type, asymptotic behavior, linear differential systems, proportional-derivative controller.

Mathematics Subject Classification: 34D05, 26A24.

1. INTRODUCTION

In this study, we investigate the asymptotic behavior of solutions to linear differential systems that involve a derivative operator based on proportional-derivative (PD) control, as defined by Anderson and Ulness [7] below.

Definition 1.1. Let α be a fixed real number in the interval $[0,1]$. Let two continuous functions $\kappa_0 : [0,1]\times\mathbb{R}\to[0,\infty)$ and $\kappa_1 : [0,1]\times\mathbb{R}\to[0,\infty)$ satisfy

$$\begin{cases}\kappa_0(0,t)=0, & \kappa_0(1,t)=1,\\ \kappa_1(0,t)=1, & \kappa_1(1,t)=0\end{cases} \quad\text{and}\quad \begin{cases}\kappa_0(\alpha,t)\neq 0, & \alpha\in(0,1],\\ \kappa_1(\alpha,t)\neq 0, & \alpha\in[0,1).\end{cases} \tag{1.1}$$

Then, the differential operator D^α is defined as follows:

$$D^\alpha f(t)=\kappa_1(\alpha,t)f(t)+\kappa_0(\alpha,t)\frac{d}{dt}f(t), \tag{1.2}$$

where κ_0 and κ_1 satisfy (1.1).

For further details on the properties of the PD-type derivative operator D^α, see [4, 5, 7]. The condition (1.1) for the PD-type derivative operator D^α defined by Anderson and Ulness [7] is originally expressed in terms of limits. However, since the functions κ_0 and κ_1 are assumed to be continuous on $[0,1]\times\mathbb{R}$, the corresponding values in (1.1) can be used directly, without expressing them as limits. The PD-type derivative operator D^α possesses linearity, and a symbolic and algebraic framework of calculus generalizing ordinary differential and integral calculus has already been established by Anderson and Ulness [7]. The fundamental properties of this calculus based on D^α are presented in Section 5 for reference. In addition, some remarks concerning the PD-type derivative operator D^α are provided below.

Remark 1.2. From condition (1.1), we have

$$D^0 f(t) = f(t) \quad \text{and} \quad D^1 f(t) = f'(t).$$

Furthermore, for arbitrary $\alpha \in [0,1]$ and $\beta \in [0,1]$, $D^\alpha D^\beta \neq D^\beta D^\alpha$ in general. However, if the two continuous functions, κ_0 and κ_1, are constant, then $D^\alpha D^\beta = D^\beta D^\alpha$.

Remark 1.3. Notably, the domain and range of the PD-type derivative operator D^α may differ from those of the classical derivative. For instance, even when the classical derivative $f'(t)$ does not exist over the entire domain, the operator $D^\alpha f(t)$ can remain well-defined for all $t \geq 0$. As an illustrative example, consider

$$f(t) = t^\alpha + t^r, \quad 0 < \alpha \leq r < 1.$$

In this case, the ordinary derivative $f'(t) = \alpha t^{\alpha-1} + r t^{r-1}$ is not defined at $t = 0$ when $0 < \alpha \leq r < 1$, whereas the PD-type derivative $D^\alpha f(t)$ remains finite and continuous on $[0,\infty)$. For instance, by setting

$$\kappa_1(\alpha,t) = (1-\alpha)t^\alpha, \quad \kappa_0(\alpha,t) = \alpha t^{1-\alpha},$$

and using the linearity of D^α (see Section 5), the following equation is obtained:

$$D^\alpha f(t) = (D^\alpha t^\alpha) + (D^\alpha t^r) = (1-\alpha)(t^{2\alpha} + t^{\alpha+r}) + \alpha^2 + \alpha r t^{r-\alpha},$$

which shows that $D^\alpha f(t)$ is well-defined at $t = 0$ even when $f'(t)$ is not. The analysis of such singular cases is beyond the scope of this study and requires further research.

Numerous studies discuss Definition 1.1 in connection with newly introduced differential operators, such as fractional and conformable derivatives (see, e.g., [11]). Fractional derivatives, dating back to Newton and Leibniz, are widely used in engineering, physics, economics, and other sciences [24]. Traditional definitions, such as those of Riemann–Liouville and Caputo derivatives, do not fully satisfy the properties of ordinary derivatives [22, 26], motivating the development of new definitions that preserve classical differential laws. Among these, the conformable derivative introduced by Khalil *et al.* [22] defines

$$\begin{aligned} T^\alpha f(t) &= \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0, \quad \alpha \in (0,1], \\ T^\alpha f(0) &= \lim_{t\to 0^+} T^\alpha f(t), \end{aligned}$$

and retains many properties of ordinary derivatives. Subsequent studies [1, 2, 15, 21, 32] have extended, compared, and unified various conformable derivatives, showing that they can often be reduced to Khalil *et al.*'s form via appropriate transformations. Research on differential equations using the conformable derivative introduced by Khalil *et al.* has also been actively conducted (see [8, 19, 25, 28]). If f is differentiable, $T^\alpha f(t) = t^{1-\alpha} f'(t)$ holds. Numerous researchers have proposed various extensions to Khalil *et al.*'s definition (see [9, Section 6, p. 150]). However, Khalil *et al.*'s definition does not reduce to the identity operator as $\alpha \to 0$. To address this limitation, Anderson and Ulness [7] introduced Definition 1.1. In fact, Definition 1.1 can partially encompass Khalil *et al.*'s definition; refer to the introduction of [20] for further details. In addition, Gao and Chi [15] conducted comparative numerical experiments with fractional differential equations and confirmed that, by adjusting the parameter α in the present operator, the weights of the proportional and derivative terms can be controlled, thereby partially reproducing the approximate behavior of fractional differential operators. In their study, they specifically set $\kappa_1(\alpha, t) = 1 - \alpha$ and $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$.

Recent research in fractional calculus [23, 29] has focused on the analysis of boundary value problems for mixed ordinary-fractional differential equations with non-autonomous variable order, as well as hybrid differential equations (HDEs) using the conformable fractal-fractional derivative (CFFD). These studies establish the existence of solutions via Krasnoselskii's fixed-point theorem and assess stability using Ulam–Hyers and U-H Rassias approaches, providing a solid theoretical framework. Numerical and concrete examples are also employed to validate and illustrate the practical relevance of the theoretical results. Such research outcomes suggest potential applications in engineering and applied sciences, particularly in PD-operator controlled systems like robotics (see [3, 13]). However, the qualitative analysis of differential equations involving the operator defined in Definition 1.1 remains at an early stage of development.

This study focuses on the asymptotic behavior of solutions to the system

$$D^\alpha \mathbf{x} = A(t)\mathbf{x}, \quad t \geq t_0 \geq 0, \tag{1.3}$$

where $\alpha \in (0, 1]$, $\mathbf{x}(t)$ is a two-dimensional vector-valued function and $A(t)$ is a 2×2 matrix-valued function whose components are continuous on $[t_0, \infty)$. For notational convenience, we define

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad D^\alpha \mathbf{x}(t) = \begin{pmatrix} D^\alpha x(t) \\ D^\alpha y(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}.$$

The uniqueness of solutions to the initial value problem associated with System (1.3) has already been established (see, e.g., [6, p. 118]). Particularly, when $\alpha = 1$, System (1.3) reduces to the classical two-dimensional linear differential system

$$\mathbf{x}' = A(t)\mathbf{x}. \tag{1.4}$$

Moreover, if $\alpha(t) \equiv \delta(t) \equiv 0$ and $\beta(t) > 0$, then System (1.3) yields a Sturm–Liouville-type equation:

$$D^\alpha \left[\frac{1}{\beta(t)} D^\alpha x \right] - \gamma(t)x = 0. \tag{1.5}$$

Recent studies have addressed the qualitative theory behind Equation (1.5) (e.g., [5, 6, 12, 17, 18, 20]). The relationship between (1.3) and the definition proposed by Khalil *et al.* is summarized below as a remark.

Remark 1.4. When $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$, System (1.3) can be transformed into the following form:

$$\hat{D}^\alpha \mathbf{y}(s) = \tilde{A}(s)\mathbf{y}(s), \quad s \geq s_0 = \alpha^{-1/\alpha} t_0, \tag{1.6}$$

where $\hat{D}^\alpha \mathbf{y}$ is defined by

$$\hat{D}^\alpha \mathbf{y}(s) := \kappa_1\big(\alpha, \alpha^{1/\alpha} s\big)\mathbf{y}(s) + s^{1-\alpha}\frac{d\mathbf{y}}{ds}, \quad \mathbf{y}(s) = \begin{pmatrix} u(s) \\ v(s) \end{pmatrix},$$

with the derivative taken componentwise, and the transformed matrix $\tilde{A}(s)$ is given by

$$\tilde{A}(s) = A(\alpha^{1/\alpha} s).$$

Indeed, letting $\mathbf{x}$ be a solution of (1.3) and setting

$$s = \alpha^{-1/\alpha} t, \quad \mathbf{y}(s) = \mathbf{x}(t),$$

we obtain

$$\hat{D}^\alpha \mathbf{y}(s) = \kappa_1(\alpha, t)\mathbf{x}(t) + s^{1-\alpha}\mathbf{x}'(t)\frac{ds}{dt} = D^\alpha \mathbf{x}(t),$$

so that the transformed system satisfies the form shown in (1.6). In particular, for $\alpha \in (0, 1)$, if the nonzero condition on κ_1 in condition (1.1) of Definition 1.1 is relaxed, that is, if $\kappa_1(\alpha, t) \equiv 0$ is assumed, then System (1.6) reduces to a system corresponding to a unique case of the definition provided by Khalil *et al.* In this case,

$$\hat{D}^\alpha \mathbf{y}(s) = s^{1-\alpha}\frac{d\mathbf{y}}{ds},$$

which corresponds to a special case of the PD-type operator. Therefore, investigating the properties of solutions to (1.3) with $\kappa_0(\alpha, t) = \alpha t^{1-\alpha}$ also provides insights into the properties of solutions for linear systems defined via the PD-type differential operator.

To describe the asymptotic behavior of System (1.4), a classical convergence result obtained by Wintner was employed (see [30, 31]). Although originally established for nonlinear systems, it has been widely applied to linear systems and serves as a fundamental tool in the study of global asymptotic and oscillatory behavior of solutions (see, e.g., [10, 14, 16, 27]).

For clarity, the result of Wintner [30, 31] is restated in the context of linear systems as follows:

Theorem 1.5 ([30, 31])**.** *Let h be a continuous function that satisfies the following conditions for any $t \geq t_0$:*

(i) $h(t) \geq 0$ *for* $t \geq t_0$*,*
(ii) $|A(t)\mathbf{x}(t)| \leq h(t)|\mathbf{x}(t)|$ *for any solution* $\mathbf{x}$ *of* (1.4)*,*
(iii) $\lim_{t\to\infty} \int_{t_0}^{t} h(s)ds$ *exists as a finite number.*

Then, for any solution $\mathbf{x}$ of System (1.4) *defined on* $[t_0, \infty)$*, there exist constants c_1 and c_2 such that*

$$\lim_{t\to\infty} \mathbf{x}(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The significance of Theorem 1.5 is as follows: Under the assumptions of Theorem 1.5, the matrix function $A(t)$ approaches the zero matrix as t becomes sufficiently large. Consequently, from System (1.4), the derivative $\mathbf{x}'$ approaches the zero vector, and the solution $\mathbf{x}$ approaches a constant vector.

This study aims to extend and adapt the classical result, Theorem 1.5, to make it applicable to System (1.3), which involves a PD control-type differential operator. In particular, by comparing the asymptotic behavior of the global solutions to Systems (1.3) and (1.4), which share the same coefficient matrix $A(t)$, we examine how the PD control-type differential operator affects the global asymptotic behavior.

The remainder of the study is organized as follows:

Section 2 presents Theorem 1.5, adapted for System (1.3). The proof of the main result demonstrates that, under certain integrability and boundedness conditions on the coefficient matrix, the global solution of System (1.3) ultimately converges. Section 3 presents an example illustrating the main theorem. Section 4 summarizes the results of this study, and Section 5 presents the basic differential properties of the operator D^α to facilitate understanding of the computations.

2. MAIN THEOREM AND ITS PROOF

The following lemma is used to prove the main result:

Lemma 2.1. *The following equalities and inequalities hold:*

(i) $\mathbf{x}(t) \cdot \mathbf{y}(t) \leq |\mathbf{x}(t) \cdot \mathbf{y}(t)| \leq |\mathbf{x}(t)||\mathbf{y}(t)|$,
(ii) $D^\alpha|\mathbf{x}(t)|^2 = 2\mathbf{x}(t) \cdot (D^\alpha \mathbf{x}(t)) - \kappa_1(\alpha, t)|\mathbf{x}(t)|^2$.

Proof. The proof of (i) is straightforward. Here, we only prove (ii). By Theorems 1.5 (i) and (iii),

$$\begin{aligned}
D^\alpha|\mathbf{x}(t)|^2 &= D^\alpha(x^2(t)+y^2(t)) \\
&= D^\alpha x^2(t) + D^\alpha y^2(t) \\
&= 2x(t)(D^\alpha x(t)) - \kappa_1(\alpha,t)x^2(t) + 2y(t)D^\alpha y^2(t) - \kappa_1(\alpha,t)y^2(t) \\
&= 2\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \cdot \begin{pmatrix} D^\alpha x(t) \\ D^\alpha y(t) \end{pmatrix} - \kappa_1(\alpha,t)(x^2(t)+y^2(t)) \\
&= 2\mathbf{x}(t)\cdot(D^\alpha \mathbf{x}(t)) - \kappa_1(\alpha,t)|\mathbf{x}(t)|^2.
\end{aligned}$$

□

The main result of this study is as follows:

Theorem 2.2. *Let h be a continuous function defined for all $t \geq t_0$ that satisfies the following conditions:*

(i) $2h(t) - \kappa_1(\alpha,t) \geq 0$ *for* $t \geq t_0$,
(ii) $|A(t)\mathbf{x}(t)| \leq h(t)|\mathbf{x}(t)|$ *for any solution* $\mathbf{x}$ *of* (1.3),
(iii) $\lim_{t\to\infty} \int_{t_0}^{t} (2h(s) - \kappa_1(\alpha,s))d_\alpha s$ *exists as a finite number,*
(iv) $\lim_{t\to\infty} \int_{t_0}^{t} h(s)e_0(t,s)d_\alpha s$ *exists as a finite number.*

Then, for every solution $\mathbf{x}$ *of System* (1.3) *defined on* $[t_0,\infty)$, *there exist constants* c_1 *and* c_2 *such that*

$$\lim_{t\to\infty} \mathbf{x}(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Here, $e_0(t,s) = e^{-\int_s^t \kappa_1(\alpha,\tau)d_\alpha \tau}$, *where* $d_\alpha s = ds/\kappa_0(\alpha,s)$ *(for details, see Section 5).*

Theorem 2.2 shows that every solution of the system converges to a finite constant vector as $t \to \infty$. The key steps include:

1. Construct an energy function: Define

$$H(t) = e_{2h}(t,t_0) = \exp\left(\int_{t_0}^{t} (2h(s) - \kappa_1(\alpha,s))\, d_\alpha s\right),$$

which is positive and bounded.
2. Monotonicity: The quantity $|\mathbf{x}|^2/H(t)$ is α-decreasing, ensuring that $\mathbf{x}$ remains bounded.
3. PD-type integral transformation: The weighted integral with $e_0(t,s)$ applied to the derivative converges.
4. Convergence: As the integral converges, each component of $\mathbf{x}$ has a finite limit.

The key idea is that, by interpreting $H(t)$ as an energy function, its decrease over time guarantees the boundedness and convergence of all solutions. The detailed proof of Theorem 2.2 is presented below.

Proof. From Lemma 2.1(i) and Assumption (ii) of Theorem 2.2, we obtain

$$\mathbf{x}(t) \cdot (D^\alpha \mathbf{x}(t)) \leq |\mathbf{x}(t)||D^\alpha \mathbf{x}(t)| = |\mathbf{x}(t)||A(t)\mathbf{x}(t)| \leq h(t)|\mathbf{x}(t)|^2. \tag{2.1}$$

We define function $H(t)$ as

$$\begin{aligned} H(t) := e_{2h}(t, t_0) &= \exp\left(\int_{t_0}^{t} (2h(s) - \kappa_1(\alpha, s)) d_\alpha s\right) \\ &= \exp\left(\int_{t_0}^{t} \frac{2h(s) - \kappa_1(\alpha, s)}{\kappa_0(\alpha, s)} ds\right). \end{aligned}$$

For any $t \geq t_0$, Assumptions (i) and (iii) of Theorem 2.2 yield

$$0 < H(t) \leq \exp\left(\int_{t_0}^{\infty} (2h(t) - \kappa_1(\alpha, t)) d_\alpha t\right) = L \tag{2.2}$$

for some constant $L > 0$. Because

$$D^\alpha H(t) = 2h(t)H(t),$$

we apply Theorem 5.3(iv), Lemma 2.1(ii), and (2.1) to compute

$$\begin{aligned} D^\alpha\left(\frac{|\mathbf{x}(t)|^2}{H(t)}\right) &= \frac{H(t)(D^\alpha|\mathbf{x}(t)|^2) - |\mathbf{x}(t)|^2(D^\alpha H(t))}{H^2(t)} + \frac{|\mathbf{x}(t)|^2}{H(t)}\kappa_1(\alpha, t) \\ &= \frac{2\mathbf{x}(t) \cdot (D^\alpha \mathbf{x}(t)) - \kappa_1(\alpha, t)|\mathbf{x}(t)|^2 - 2|\mathbf{x}(t)|^2 h(t)H(t)}{H^2(t)} \\ &\quad + \frac{|\mathbf{x}(t)|^2}{H(t)}\kappa_1(\alpha, t) \\ &= \frac{2}{H(t)}(\mathbf{x}(t) \cdot (D^\alpha \mathbf{x}(t)) - h(t)|\mathbf{x}(t)|^2) \leq 0. \end{aligned}$$

Thus, $|\mathbf{x}|^2/H(t)$ is α-decreasing (Definition 5.4 and Theorem 5.5). Using the inequality

$$0 < e_0(t, t_0) = e^{-\int_{t_0}^{t} \frac{\kappa_1(\alpha,s)}{\kappa_0(\alpha,s)} ds} \leq 1,$$

we obtain

$$\frac{|\mathbf{x}(t)|^2}{H(t)} \leq \frac{|\mathbf{x}(t_0)|^2}{H(t_0)} e_0(t, t_0) \leq |\mathbf{x}(t_0)|^2.$$

Hence, from (2.2), it follows that

$$|\mathbf{x}(t)|^2 \leq |\mathbf{x}(t_0)|^2 H(t) \leq |\mathbf{x}(t_0)|^2 L = M,$$

for some constant $M > 0$. Therefore,

$$|D^\alpha \mathbf{x}(t)| = |A(t)\mathbf{x}(t)| \leq h(t)|\mathbf{x}(t)| \leq \sqrt{M} h(t).$$

However, noting that

$$|D^\alpha \mathbf{x}(t)| = \sqrt{(D^\alpha x(t))^2 + (D^\alpha y(t))^2} \geq \sqrt{(D^\alpha x(t))^2} = |D^\alpha x(t)|,$$

we obtain

$$|D^\alpha x(t)| \leq \sqrt{M} h(t). \tag{2.3}$$

Multiplying both sides of (2.3) by $e_0(\tau, t)$ and integrating from t_0 to t based on the α-integral yields

$$\int_{t_0}^{t} |D^\alpha x(s)| e_0(t,s) d_\alpha s \leq \sqrt{M} \int_{t_0}^{t} h(s) e_0(t,s) d_\alpha s. \tag{2.4}$$

By letting $t \to \infty$ in (2.4), Assumption (iv) of Theorem 2.2 ensures that

$$\lim_{t\to\infty} \int_{t_0}^{t} |D^\alpha x(s)| e_0(t,s) d_\alpha s = \lim_{t\to\infty} \int_{t_0}^{t} \left| \frac{(D^\alpha x(s)) e_0(t,s)}{\kappa_0(\alpha, s)} \right| ds$$

converges. Therefore, by using absolute integrability and Theorem 5.3(vi), we have

$$\lim_{t\to\infty} \int_{t_0}^{t} (D^\alpha x(s)) e_0(t,s) d_\alpha s = \lim_{t\to\infty} x(t) - \lim_{t\to\infty} x(t_0) e_0(t, t_0),$$

and the integral converges. Hence, the limit

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \int_{t_0}^{t} (D^\alpha x(s)) e_0(t,s) d_\alpha s + \lim_{t\to\infty} x(t_0) e_0(t, t_0)$$

exists. Similarly, using the inequality

$$|D^\alpha \mathbf{x}(t)| = \sqrt{(D^\alpha x(t))^2 + (D^\alpha y(t))^2} \geq \sqrt{(D^\alpha y(t))^2} = |D^\alpha y(t)|,$$

we show that $\lim_{t\to\infty} y(t)$ also exists. This completes the proof of Theorem 2.2. □

Remark 2.3. Consider Theorem 2.2 with $\alpha = 1$. From (1.1), Condition (i) of Theorem 2.2 is reduced to Condition (i) of Theorem 1.5. Moreover, noting from (1.1) that $d_\alpha s = ds/\kappa_0(\alpha, s) = ds$ and $e_0(t,s) = 1$, Condition (iii) of Theorem 2.2 becomes

$$\lim_{t\to\infty} \int_{t_0}^{t} 2h(s) ds \text{ exists as a finite number,}$$

whereas Condition (iv) becomes

$$\lim_{t\to\infty}\int_{t_0}^{t} h(s)ds \text{ exists as a finite number.}$$

Thus, Theorem 2.2 includes Theorem 1.5 as a special case.

Remark 2.4. The meaning of Theorem 2.2 is as follows: If $A(t)$ satisfies the conditions of Theorem 2.2, then for sufficiently large t, $A(t)$ is sufficiently close to the zero matrix. Consequently, from System (1.3), $\mathbf{x}'$ becomes sufficiently close to $\mathbf{0}$, and $\mathbf{x}$ approaches a constant vector. The details of each condition are as follows: Conditions (i) and (iii) are required to ensure the boundedness of the energy function $H(t)$, while condition (iv) is used to guarantee that $\mathbf{x}$ converges to a constant vector. Condition (ii) is identical to condition (ii) in Theorem 1.5.

3. ILLUSTRATIVE EXAMPLE AND NUMERICAL SIMULATION

Consider the following system:

$$\begin{aligned} &D^{\alpha}\mathbf{x} = A(t)\mathbf{x}, \quad t \geq t_0 > 0, \\ &A(t) = \begin{pmatrix} e^{lt} & e^{lt} \\ -e^{lt} & e^{lt} \end{pmatrix}, \end{aligned} \tag{3.1}$$

where l is a real number, $\kappa_0(\alpha,t) = \alpha$, and $\kappa_1(\alpha,t) = 1-\alpha$. When $\alpha = 1$, if $l < 0$, then every solution $\mathbf{x}$ of System (3.1) converges to a certain constant vector as $t \to \infty$. To verify this, we define a function $h(t) = \sqrt{2}e^{lt}$ to show that the conditions of Theorem 1.5 are satisfied. We consider System (3.1) with $1 > \alpha > 0$. In this case, if $l < 0$ and $l \neq -(1-\alpha)/\alpha$, then any solution $\mathbf{x}$ of System (3.1) converges to a constant vector. This can be verified by applying Theorem 2.2 and setting

$$h(t) = \sqrt{2}e^{lt} + (1-\alpha)/2.$$

Then, function h satisfies Condition (i) of Theorem 2.2. Furthermore, as

$$|A(t)\mathbf{x}(t)| = \sqrt{2}e^{l}\sqrt{x^2+y^2} \leq \left(\sqrt{2}e^{lt} + \frac{1-\alpha}{2}\right)\mathbf{x}(t),$$

Condition (ii) is also satisfied. Moreover, when $l < 0$,

$$\lim_{t\to\infty}\int_{t_0}^{t}(2h(s)-\kappa_1(\alpha,s))d_{\alpha}s = \lim_{t\to\infty}\frac{2\sqrt{2}}{\alpha}\int_{0}^{t} e^{ls}ds = -\frac{2\sqrt{2}}{\alpha l},$$

which implies that Condition (iii) is satisfied. Furthermore, based on the definitions of κ_0 and κ_1, we obtain

$$e_0(t,s) = \frac{e^{\frac{1-\alpha}{\alpha}s}}{e^{\frac{1-\alpha}{\alpha}t}},$$

such that

$$\lim_{t\to\infty}\int\limits_{t_0}^{t} h(s)e_0(t,s)d_\alpha s$$

$$= \lim_{t\to\infty}\frac{1}{\alpha e^{\frac{1-\alpha}{\alpha}t}}\int\limits_{0}^{t}\left(\sqrt{2}e^{\left(l+\frac{1-\alpha}{\alpha}\right)s}+\frac{1-\alpha}{2}e^{\frac{1-\alpha}{\alpha}s}\right)ds$$

$$= \lim_{t\to\infty}\left(\frac{\sqrt{2}}{\alpha\left(l+\frac{1-\alpha}{\alpha}\right)}e^{lt}-\frac{\sqrt{2}}{\alpha\left(l+\frac{1-\alpha}{\alpha}\right)e^{\frac{1-\alpha}{\alpha}t}}+\frac{1}{2}-\frac{1}{2e^{\frac{1-\alpha}{\alpha}t}}\right)=\frac{1}{2},$$

which verifies Condition (iv).

Remark 3.1. The solution trajectories of System (3.1) with a PD-type derivative operator are compared with those of a system with a classical derivative operator. Figure 1 shows two sets of solution trajectories for System (3.1) with different values of α: $\alpha = 1$ and $\alpha = 0.5$, both with $l = -0.4$. In both cases, the trajectories begin from the same four initial points: $(2, 1)$, $(2, -1)$, $(-2, 1)$, and $(-2, -1)$. When $\alpha = 1$, the trajectories do not converge to the origin but instead approach certain constant vectors away from the origin. Conversely, when $\alpha = 0.5$, corresponding to the system with the PD control-type differential operator, the trajectories clearly converge to the origin. This illustrates that the presence of the PD control operator significantly changes the asymptotic behavior of the solutions.

4. CONCLUSION AND FUTURE WORK

This study extends Theorem 1.5 to System (1.3), which involves a PD-type differential operator (1.1), and establishes sufficient conditions for solutions to converge to constant vectors. Through an illustrative example, convergence to stable steady states under

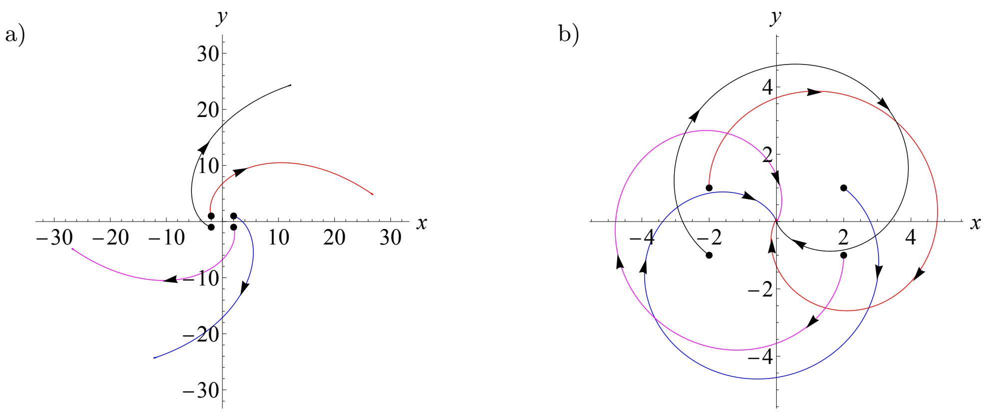

Fig. 1. The solution of System (3.1) with $l = -0.4$: a) $\alpha = 1$; b) $\alpha = 0.5$

certain parameter conditions was confirmed. It is conjectured that for $1 > \alpha > 0$, $l < 0$, and $l \neq -(1-\alpha)/\alpha$, any solution $\mathbf{x}$ of System (3.1) converges to the origin.

The following issues are highlighted as important directions for future research:

1. Investigation of whether the present methods and results can be extended to nonlinear systems or to time-varying coefficient systems beyond those considered in this study.
2. Examination of the stability and robustness properties of PD-controlled systems beyond asymptotic convergence.
3. Identification of practical systems or control applications where the PD-type differential operator framework has been, or could be, implemented effectively.

5. APPENDIX: BASIC PROPERTIES OF CALCULUS FOR D^α

The calculus background for the PD-type operator (1.2) is outlined in [5, 7].

Theorem 5.1 ([5, 7])**.** *Let $\alpha \in (0,1]$, point $s,t \in \mathbb{R}$ with $s \leq t$, and the function $\phi : [s,t] \to \mathbb{R}$ be continuous. Furthermore, let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy* (1.1)*, with ϕ/κ_0 and κ_1/κ_0 being Riemann-integrable on $[s,t]$. Next, the exponential function regarding D^α in* (1.2) *is defined as*

$$e_\phi(t,s) := e^{\int_s^t \frac{\phi(\tau)-\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau}, \quad e_0(t,s) = e^{-\int_s^t \frac{\kappa_1(\alpha,\tau)}{\kappa_0(\alpha,\tau)} d\tau}, \tag{5.1}$$

and

$$D^\alpha e_\phi(t,s) = \phi(t) e_\phi(t,s), \quad D^\alpha e_0(t,s) = 0.$$

Definition 5.2 ([5, 7])**.** Let $\alpha \in (0,1]$ and $t_0 \in \mathbb{R}$. The antiderivative is defined as

$$\int D^\alpha f(t) d_\alpha t = f(t) + c e_0(t,t_0), \quad c \in \mathbb{R}.$$

Similarly, the integral of f over $[a,b]$ is defined as

$$\int_a^t f(s) e_0(t,s) d_\alpha s := \int_a^t \frac{f(s) e_0(t,s)}{\kappa_0(\alpha,s)} ds, \quad d_\alpha s := \frac{1}{\kappa_0(\alpha,s)} ds.$$

Theorem 5.3 ([5, 7]). *Let the differential operator D^α be expressed as* (1.2) *with $\alpha \in [0,1]$. Let function $\phi : [s,t] \to \mathbb{R}$ be continuous. Let $\kappa_0, \kappa_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ be continuous and satisfy* (1.1)*. Assume that functions f and g are differentiable as needed. Then,*

(i) $D^\alpha[kf(t) + lg(t)] = k(D^\alpha f(t)) + l(D^\alpha g(t))$ *for all* $k, l \in \mathbb{R}$,

(ii) $D^\alpha k = k\kappa_1(\alpha, t)$ *for all* $k \in \mathbb{R}$,

(iii) $D^\alpha[f(t)g(t)] = g(t)(D^\alpha f(t)) + f(t)(D^\alpha g(t)) - f(t)g(t)\kappa_1(\alpha, t)$,

(iv) $D^\alpha[f(t)/g(t)] = \dfrac{g(t)(D^\alpha f(t)) - f(t)(D^\alpha g(t))}{g^2(t)} + \dfrac{f(t)}{g(t)}\kappa_1(\alpha, t)$,

(v) *for $\alpha \in (0,1]$ and exponential function e_0 given in* (5.1)*, we have*

$$D^\alpha \left[\int_a^t f(s) e_0(t,s) d_\alpha s \right] = f(t),$$

(vi)

$$\int_a^t (D^\alpha g(s)) e_0(t,s) d_\alpha s = g(s) e_0(t,s) \Big|_{s=a}^{t} = g(t) - g(a) e_0(t,a).$$

Definition 5.4 ([5]). Let $\alpha \in [0,1]$ and $I \subseteq \mathbb{R}$. A function f is α-increasing on interval I if $e_0(t_1,t_2)f(t_2) \geq f(t_1)$, whenever $t_2 > t_1$, $t_1, t_2 \in I$, and it is strictly α-increasing if $e_0(t_1,t_2)f(t_2) > f(t_1)$, whenever $t_2 > t_1$, $t_1, t_2 \in I$. A function f is α-decreasing on interval I if $f(t_2) \leq e_0(t_2,t_1)f(t_1)$, whenever $t_2 > t_1$, $t_1, t_2 \in I$, and it is strictly α-decreasing if $f(t_2) < e_0(t_2,t_1)f(t_1)$, whenever $t_2 > t_1$, $t_1, t_2 \in I$.

Theorem 5.5 ([5]). *Let $\alpha \in [0,1]$ and $I \subseteq \mathbb{R}$. Suppose that $D^\alpha f$ exists on some interval I. If $D^\alpha f(t) \geq 0$ for all $t \in I$, then f is α-increasing on I. If $D^\alpha f(t) \leq 0$ for all $t \in I$, then f is α-decreasing on I.*

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