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PARAMETRIC FORMAL GEVREY ASYMPTOTIC EXPANSIONS IN TWO COMPLEX TIME VARIABLE PROBLEMS

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Abstract. The analytic and formal solutions to a family of singularly perturbed partial differential equations in the complex domain involving two complex time variables are considered. The analytic continuation properties of the solution of an auxiliary problem in the Borel plane overcomes the absence of adequate domains which would guarantee summability of the formal solution. Moreover, several exponential decay rates of the difference of analytic solutions with respect to the perturbation parameter at the origin are observed, leading to several asymptotic levels relating the analytic and the formal solution.

Keywords: singularly perturbed, formal solution, several complex variables, Cauchy problem.

Mathematics Subject Classification: 35C10, 35R10, 35C15, 35C20.

1. INTRODUCTION

The main aim of the present work is to describe the asymptotic relation existing between the analytic and the formal solutions of a family of singularly perturbed nonlinear partial differential equations in two complex time variables of the form

$$\begin{aligned} &Q(\partial_z)u(t_1,t_2,z,\epsilon)\\ &= \epsilon^{\Delta_0}(t_1^{k_1+1}\partial_{t_1})^{\delta_1}(t_2^{k_2+1}\partial_{t_2})^{\delta_2}R(\partial_z)u(t_1,t_2,z,\epsilon)\\ &\quad + P(t_1,t_2,\partial_{t_1},\partial_{t_2},z,\partial_z,\epsilon)u(t_1,t_2,z,\epsilon)\\ &\quad + \left(P_1(\epsilon,\partial_z)u(t_1,t_2,z,\epsilon)\right)\left(P_2(\epsilon,\partial_z)u(t_1,t_2,z,\epsilon)\right) + f(t_1,t_2,z,\epsilon), \end{aligned} \tag{1.1}$$

under initial data $u(t_1,0,z,\epsilon)\equiv u(0,t_2,z,\epsilon)\equiv 0$. In the previous equation ϵ acts as a small complex perturbation parameter. In addition to this, $Q(X),R(X)\in\mathbb{C}[X]$ and P_1,P_2 are polynomials in their second variable, with coefficients in the space of holomorphic functions on some neighborhood of the origin. $\Delta_0,k_1,k_2,\delta_1,\delta_2$ are nonnegative integers. The function $P(T_1, T_2, S_1, S_2, Z, S_3, \epsilon)$ turns out to be a polynomial in T_1, T_2, S_1, S_2, S_3 with coefficients being holomorphic and bounded functions on a horizontal strip, say H, w.r.t. the space variable Z, and some neighborhood of the origin in the perturbation parameter, say D. The forcing term f is a polynomial in its two first variables, with coefficients being holomorphic and bounded functions on $H \times D$. The precise shape and assumptions on the elements involved in the equation is described in detail in Section 2.

This work puts a step forward in the theory of analytic and asymptotic solutions to singularly perturbed partial differential equations in the complex domain, in several complex time variables. Different advances in this direction have been achieved in the last years. In [15], the two last authors observed a multilevel Gevrey asymptotic phenomenon with respect to the perturbation parameter relating the formal and the analytic solutions to a family of the form (1.1) where the leading operator

$$Q(\partial_z) - \epsilon^{\Delta_0}(t_1^{k_1+1}\partial_{t_1})^{\delta_1}(t_2^{k_2+1}\partial_{t_2})^{\delta_2}R(\partial_z) \tag{1.2}$$

is replaced by a product of operators in the form $Q_j(\partial_z) - \epsilon^{\Delta_j}(t_j^{k_j+1}\partial_{t_j})^{\delta_j}R_j(\partial_z)$, $j = 1, 2$. This leads to symmetric asymptotic properties of the solutions. Afterwards, the situation in which the role of the Borel time variables is asymmetric was initially considered in [12], observing a small division phenomena. In the present work, the configuration of the main problem does not allow us to follow the previous procedures developed in [12, 15] due to the shape of the operator (1.2).

It is worth mentioning the phenomena observed in the previous works from a geometric point of view. Indeed, the analytic solution of the main problem is constructed as a Laplace-like transform of a function defined in the Borel space. In the work [15], this function can be defined with respect to the Borel time variables on sets of the form $(D_1 \cup S_1) \times (D_2 \cup S_2)$. Here, S_j denotes an infinite sector with vertex at the origin, and D_j stands for a disc centered at the origin, $j = 1, 2$. In [12], the asymmetric behavior of the time variables causes that the function in the Borel space is only defined on sets of the form $(D_1 \cup S_1) \times S_2$, but not on sets of the form $S_1 \times (D_2 \cup S_2)$. In the present work, the auxiliary function in the Borel domain can not be defined on sets of the previous form: $(D_1 \cup S_1) \times (D_2 \cup S_2)$, nor $S_1 \times (D_2 \cup S_2)$, nor $(D_1 \cup S_1) \times S_2$, but only on sets of the form $S_1 \times S_2$. As a result, the deformation of the integration path defining the solution as a double Laplace-like operator is not available. This is essential in order to estimate the difference of two solutions which share a common domain in the perturbation parameter. Therefore, a strategy of a direct application of a Ramis–Sibuya theorem is no longer valid in the present situation. Alternatively, the analytic continuation of the auxiliary functions in the Borel space and an adequate splitting of the integration paths involved in the definition of the solutions will play the key point to achieve an asymptotic meaning of the solution. In the end, a fine structure related to the analytic solution is observed, involving two Gevrey orders, k_1 and k_2 which remain linked to the singular operators involved in the leading term of equation (1.1). This fine structure appears in the form of a decomposition of the formal and the analytic solution as the sum of terms involving different Gevrey asymptotic orders, appearing in Balser's decomposition approach to multisummability (see [18, Section 7.5]).

The appearance of a scheme involving several levels relating the analytic and the formal solutions to ordinary differential equations in the complex domain (under the action of a small perturbation complex parameter or not) has been a field of interest in the scientific community during the last three decades. See the classical references [1, 4, 5, 17, 21, 25, 28]. Regarding partial differential equations in the complex domain, the advances on multilevel solutions are much more limited. We refer to the works [3, 22–24, 26, 27] and the references therein, among many others. Multilevel results have also been recently extended to other more general functional equations in the last years, such as [9, 16, 23], in the context of moment partial differential equations.

The study of solutions to singularly perturbed partial differential equations involving several complex time variables has also been considered regarding boundary layer expansions, distinguishing outer and inner solutions, together with their asymptotic representation. This is the case of [13] in which a failure of the application of a Borel–Laplace procedure is also observed.

The main motivation for the present study consists on considering the nonlinear situation related to two previous research, namely [14] and [6]. In [14], the two last authors consider a linearized problem with respect to (1.1). In both works, the leading term coincides with that in (1.1), that we are about to study. However, in our new setting there is more freedom in the choice of the other terms in the linear part which not only allow powers of irregular operators in the time variables of the same nature as those in the distinguished term, i.e. of the form $(t_1^{k_1+1}\partial_{t_1})^{\delta_{\ell_1}}(t_2^{k_2+1}\partial_{t_2})^{\delta_{\ell_2}}$, but a wider variety of irregular operators $t_1^{\ell_1}\partial_{t_1}^{\ell_2}t_2^{\ell_3}\partial_{t_2}^{\ell_4}$, regarding the hypothesis (2.3) of the present work.

In that previous work, a small divisor phenomenon occurs so that the Borel–Laplace classical procedure in two time variables does not apply, and one has to search for solutions in the form

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_d}\omega(u,m,\epsilon)\exp\left(-\left(\frac{u}{\epsilon t_1}\right)^{k_1}-\left(\frac{u}{\epsilon t_2}\right)^{k_2}\right)\frac{du}{u}e^{izm}dm,$$

where L_d is an infinite ray with direction $d\in\mathbb{R}$, for some function ω. This approach leads to the construction of analytic solutions which adopt inner and outer asymptotic solutions as asymptotic representations.

On the other hand, in the work [6] one searches for analytic solutions of a linearized version of the equation under study in the form of a Fourier, truncated Laplace and Laplace transform of certain function

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1,\epsilon}}\int_{L_{d_2}}\omega(u_1,u_2,m,\epsilon)\exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1}-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right)\frac{du_2}{u_2}\frac{du_1}{u_1}e^{izm}dm,$$

for some $d_1,d_2\in\mathbb{R}$ and where L_{d_2} is an infinite ray with direction d_2, with $L_{d_1,\epsilon}$ is the segment $[0,h_1(\epsilon)e^{id_1}]$, for some holomorphic function $\epsilon\mapsto h_1(\epsilon)$. In that previous work, all the terms except from the leading term are of the form $(t_1^{k_1+1}\partial_{t_1})^{\delta_{\ell_1}}t_2^{d_{\ell_2}}\partial_{t_2}^{\delta_{\ell_2}}$, in contrast to the freedom acquired in the present work. One can additionally observe that the freedom acquired on the possible values of the parameters involved in the second time variable coincide in both works (see the second condition in (2.2).

The strategy of the present work is as follows. First, we search for solutions to the main problem in the form of a double Laplace and inverse Fourier transform of some auxiliary function. This allows us to substitute the main equation (1.1) by an auxiliary convolution problem in the Borel domain (see (3.6)). The solution to this auxiliary equation, say $(\tau_1, \tau_2, m, \epsilon) \mapsto \omega(\tau_1, \tau_2, m, \epsilon)$ is in principle defined on $S_1 \times S_2 \times \mathbb{R} \times D$, where $S_1 \times S_2$ stands for a product of unbounded sectors, and D is a punctured disc at the origin. However, it is proved (see Proposition 3.11) that such function can be analytically extended as follows:

- for all $\tau_1 \in S_1$ near the origin, $m \in \mathbb{R}$ and $\epsilon \in D$, the map $\tau_2 \mapsto \omega(\tau_1, \tau_2, m, \epsilon)$, defined on S_2, can be analytically continued to some neighborhood of the origin.
- for all $\tau_2 \in S_2$ near the origin, $m \in \mathbb{R}$ and $\epsilon \in D$, the map $\tau_1 \mapsto \omega(\tau_1, \tau_2, m, \epsilon)$, defined on S_1, can be analytically continued to some neighborhood of the origin.

The construction of the analytic solution $u(t_1, t_2, z, \epsilon)$ is achieved by means of a double Laplace transform with respect to (τ_1, τ_2) (see Theorem 3.12) following a classical procedure. It turns out to be holomorphic and bounded on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E}$, with $\mathcal{T}_j, \mathcal{E}$ being bounded sectors with vertex at the origin, and $H_{\beta'}$ a horizontal strip. Although the mentioned construction of the analytic solution $u(t_1, t_2, z, \epsilon)$ needs no analytic continuation of the auxiliary function $\omega(\tau_1, \tau_2, m, \epsilon)$ to be defined, such analytic continuation is indeed needed for setting the existence of an asymptotic expansion.

In order to attain such asymptotic expansion, we split the solution as the sum of three terms, J_1 (see Section 4.1), J_2 and J_3 (see Section 4.2). The term J_2 (resp. J_3) has null Gevrey asymptotic expansion of order $1/k_2$ (resp. $1/k_1$), see Proposition 4.7 and Proposition 4.8. On the other hand, a parametric Gevrey series expansion associated to J_1 is attained by completing J_1 into a set $(J_{1,p})_{0 \le p \le \varsigma - 1}$, each of them holomorphic on a finite sector $\mathcal{E}_p$, where $(\mathcal{E}_p)_{0 \le p \le \varsigma}$ describes a good covering (see Definition 4.1) and by describing the exponential decay of the difference of two consecutive maps (i.e. with nonempty intersection of their domains of definition) in the perturbation parameter (see Proposition 4.2).

The application of a multilevel Ramis–Sibuya Theorem (RS) to J_1 allows to conclude the main result of the present work (Theorem 4.9). More precisely, this result states the existence of a decomposition of the analytic solution of the main problem in the form

$$u(t_1, t_2, z, \epsilon) = a(t_1, t_2, z, \epsilon) + u_1(t_1, t_2, z, \epsilon) + u_2(t_1, t_2, z, \epsilon),$$

where $a(t_1, t_2, z, \epsilon)$ turns out to be a holomorphic function on some neighborhood of the origin with respect to the perturbation parameter ϵ, and with coefficients belonging to the Banach space of holomorphic and bounded functions on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'}$. Moreover, u_j are holomorphic and bounded functions on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E}$, for $j = 1, 2$. On the other hand, there exist formal power series in ϵ with coefficients in the Banach space of holomorphic and bounded functions on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'}$, say $\hat{u}_1(t_1, t_2, z, \epsilon)$ and $\hat{u}_2(t_1, t_2, z, \epsilon)$, which satisfy that u_j admits $\hat{u}_j$ as its common Gevrey asymptotic expansion of order $1/k_j$ with respect to ϵ on $\mathcal{E}$, $j = 1, 2$.

Throughout the paper, we use the following notation.

We write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For all $z_0 \in \mathbb{C}$ and $r > 0$, we denote the open disc centered at z_0 and radius r by $D(z_0, r)$.

Given two open and bounded sectors in the complex domain $\mathcal{E}$ and $\mathcal{T}$, with vertex at the origin, we say that T is a proper subsector of $\mathcal{E}$, and denote it by $T \prec \mathcal{E}$ whenever $\overline{T} \setminus \{0\} \subseteq \mathcal{E}$.

Given a complex Banach space $\mathbb{E}$ and a nonempty open set $U \subseteq \mathbb{C}$, we write $\mathcal{O}_b(U, \mathbb{E})$ for the set of holomorphic functions defined on U with values in $\mathbb{E}$. We simply write $\mathcal{O}_b(U)$ in the case that $\mathbb{E} = \mathbb{C}$. We write $\mathbb{E}\{\epsilon\}$ for the set of holomorphic functions with values on $\mathbb{E}$, defined on ϵ, which are convergent on some neighborhood of the origin.

2. STATEMENT OF THE MAIN PROBLEM

In this section, we establish the main Cauchy problem under study, providing the details on the elements involved in the problem.

Let k_1, k_2 be positive integer numbers. Let us assume that $k_1 > k_2 \geq 1$ and fix $\epsilon_0 > 0$. We also consider positive integers δ_1, δ_2 and a finite set $I \subseteq \mathbb{N}^4$. We assume $\delta_1 k_1 = \delta_2 k_2$, and define $\Delta_0 := k_1\delta_1 + k_2\delta_2$.

We assume that for every $\underline{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4) \in I$, one has ℓ_2 and ℓ_4 are positive integers such that

$$\ell_1 = \ell_2(k_1 + 1) + d_{k_1,\ell_1,\ell_2} \text{ and } \ell_3 = \ell_4(k_2 + 1) + d_{k_2,\ell_3,\ell_4}, \tag{2.1}$$

for some positive integers $d_{k_1,\ell_1,\ell_2}, d_{k_2,\ell_3,\ell_4}$. Moreover, for every $\underline{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4) \in I$ we fix an integer number $\Delta_{\underline{\ell}}$ such that

$$\Delta_{\underline{\ell}} \geq \ell_1 - \ell_2 + \ell_3 - \ell_4 + 1. \tag{2.2}$$

We also assume that

$$k_1 \leq \ell_1 - \ell_2 = \ell_3 - \ell_4 \leq \delta_1 k_1, \quad (\ell_1, \ell_2, \ell_3, \ell_4) \in I. \tag{2.3}$$

Let us fix polynomials $Q(X), R(X), R_{\underline{\ell}}(X) \in \mathbb{C}[X]$, for $\underline{\ell} \in I$. We assume that

$$\deg Q \geq \deg R \geq \deg(R_{\underline{\ell}}), \quad \underline{\ell} \in I, \tag{2.4}$$

and

$$R(im) \neq 0, \quad Q(im) \neq 0, \quad R_{\underline{\ell}}(im) \neq 0 \tag{2.5}$$

for every $m \in \mathbb{R}$ and all $\underline{\ell} \in I$. In addition to this, we assume the existence of an infinite sector $S_{Q,R}$, centered at the origin such that

$$\left\{ \frac{Q(im)}{R(im)} : m \in \mathbb{R} \right\} \subseteq S_{Q,R}. \tag{2.6}$$

Remark 2.1. An example of a situation in which the previous assumption holds is that in which one considers polynomials Q and R with positive coefficients with only powers which are divisible by 4, and where $S_{Q,R}$ is a sector containing the set of positive real numbers.

We also fix polynomials $P_1(\epsilon, X), P_2(\epsilon, X) \in \mathcal{O}_b(D(0,\epsilon_0))[X]$. We assume that

$$\deg R \geq \max\{\deg(P_1), \deg(P_2)\}. \tag{2.7}$$

In view of the assumptions made on P_1 and P_2, there exist $C_{P_1}, C_{P_2} > 0$ such that

$$|P_1(\epsilon, im)| \leq C_{P_1}(1+|m|)^{\deg(P_1)}, \quad |P_2(\epsilon, im)| \leq C_{P_2}(1+|m|)^{\deg(P_2)}, \tag{2.8}$$

for every $m \in \mathbb{R}$ and $\epsilon \in D(0,\epsilon_0)$. We also assume that C_{P_1} and C_{P_2} are small enough (see Proposition 3.9).

We consider the following nonlinear initial value problem

$$\begin{aligned} &Q(\partial_z)u(t_1,t_2,z,\epsilon) \\ &= \epsilon^{\Delta_0}(t_1^{k_1+1}\partial_{t_1})^{\delta_1}(t_2^{k_2+1}\partial_{t_2})^{\delta_2}R(\partial_z)u(t_1,t_2,z,\epsilon) \\ &\quad + \sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I} \epsilon^{\Delta_{\underline{\ell}}}c_{\underline{\ell}}(z,\epsilon)t_1^{\ell_1}\partial_{t_1}^{\ell_2}t_2^{\ell_3}\partial_{t_2}^{\ell_4}R_{\underline{\ell}}(\partial_z)u(t_1,t_2,z,\epsilon) \\ &\quad + \left(P_1(\epsilon,\partial_z)u(t_1,t_2,z,\epsilon)\right)\left(P_2(\epsilon,\partial_z)u(t_1,t_2,z,\epsilon)\right) + f(t_1,t_2,z,\epsilon), \end{aligned} \tag{2.9}$$

under vanishing initial data $u(0,t_2,z,\epsilon) = u(t_1,0,z,\epsilon) = 0$.

Remark 2.2. Observe all the previous assumptions made on the parameters involved in the problem are compatible. An example of equation satisfying the previous assumptions is the following:

$$\begin{aligned} &(\partial_z^4+1)u(t_1,t_2,z,\epsilon) \\ &= \epsilon^{12}(t_1^4\partial_{t_1})^2(t_2^3\partial_{t_2})^3 + \epsilon^{11}c_{(6,1,7,2)}(z,\epsilon)t_1^6\partial_{t_1}t_2^7\partial_{t_2}^2(-\partial_z^2+2)u(t_1,t_2,z,\epsilon) \\ &\quad + \left(P_1(\epsilon,\partial_z)u(t_1,t_2,z,\epsilon)\right)\left(P_2(\epsilon,\partial_z)u(t_1,t_2,z,\epsilon)\right) + f(t_1,t_2,z,\epsilon), \end{aligned}$$

with $\max\{\deg(P_1), \deg(P_2)\} \leq 2$. The functions f and $c_{(6,1,7,2)}$ are attained to the construction described below.

We choose $\beta > 0$ and fix $\mu > 0$ such that

$$\mu > \max\{\deg(P_1), \deg(P_2), \max_{\underline{\ell}\in I}\{\deg(R_{\underline{\ell}})\}\} + 1. \tag{2.10}$$

Let $0 < \beta' < \beta$ be fixed. We denote $H_{\beta'}$ the horizontal strip

$$H_{\beta'} = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \beta'\}.$$

For every $\underline{\ell} \in I$, the function $c_{\underline{\ell}}$ belongs to $\mathcal{O}_b(H_{\beta'} \times D(0,\epsilon_0))$. The function f turns out to be a polynomial in its two first variables, and a holomorphic and bounded function on $H_{\beta'} \times D(0,\epsilon_0)$ with respect to (z,ϵ). These functions are constructed as follows.

Given $\underline{\ell} \in I$, we choose a function $\mathbb{R} \times D(0, \epsilon_0) \ni (m, \epsilon) \mapsto C_{\underline{\ell}}(m, \epsilon)$ under the following assumptions.

– For every $\epsilon \in D(0, \epsilon_0)$, the function $\mathbb{R} \ni m \mapsto C_{\underline{\ell}}(m, \epsilon)$ is continuous on $\mathbb{R}$, and it satisfies there exists $K_{\underline{\ell}}(\epsilon) > 0$ with $|C_{\underline{\ell}}(m, \epsilon)| \leq K_{\underline{\ell}}(\epsilon) \frac{1}{(1+|m|)^\mu} e^{-\beta|m|}$, for $m \in \mathbb{R}$. Moreover, there exists $K > 0$ such that

$$\sup_{\underline{\ell} \in I} \sup_{\epsilon \in D(0,\epsilon_0)} K_{\underline{\ell}}(\epsilon) \leq K.$$

– For all $m \in \mathbb{R}$, the mapping $D(0, \epsilon_0) \ni \epsilon \mapsto C_{\underline{\ell}}(m, \epsilon)$ is a holomorphic function on $D(0, \epsilon_0)$.

In view of the previous assumptions, and the properties of inverse Fourier transform, one defines

$$c_{\underline{\ell}}(z, \epsilon) := \mathcal{F}^{-1} \left(m \mapsto C_{\underline{\ell}}(m, \epsilon) \right)(z)$$

which represents a holomorphic and bounded function on $H_{\beta'} \times D(0, \epsilon_0)$.

Remark 2.3. Observe that

$$\sup_{\underline{\ell} \in I} \sup_{\epsilon \in D(0,\epsilon_0)} \sup_{m \in \mathbb{R}} (1 + |m|)^\mu e^{\beta|m|} |C_{\underline{\ell}}(m, \epsilon)| \leq K. \tag{2.11}$$

For the construction of the forcing term, we make use of the following well-known property of Laplace transform.

Lemma 2.4. *Let k, n be positive integers. We also fix $d \in \mathbb{R}$. For every $T \in \mathbb{C}^\star$ with $\cos((d - \arg(T))k) > 0$, it holds that*

$$k \int_{L_d} u^{n-1} \exp\left(-\left(\frac{u}{T}\right)^k \right) du = T^n \Gamma\left(\frac{n}{k}\right),$$

with L_d being the integration path $[0, \infty) \ni r \mapsto re^{id}$. Here, $\Gamma(\cdot)$ stands for Gamma function.

The forcing term f is constructed as follows. Let $N_1, N_2 \subseteq \mathbb{N}$ be two nonempty finite subsets of positive integers. For every $(n_1, n_2) \in N_1 \times N_2$, let

$$\mathbb{R} \times D(0, \epsilon_0) \ni (m, \epsilon) \mapsto F_{n_1,n_2}(m, \epsilon)$$

be a function under the following assumptions.

– For every $\epsilon \in D(0, \epsilon_0)$, the function $m \mapsto F_{n_1,n_2}(m, \epsilon)$ is continuous on $\mathbb{R}$, and it satisfies there exists $K_{n_1,n_2}(\epsilon) > 0$ such that

$$|F_{n_1,n_2}(m, \epsilon)| \leq K_{n_1,n_2}(\epsilon) \frac{1}{(1 + |m|)^\mu} e^{-\beta|m|}, \quad m \in \mathbb{R}.$$

– For all $m \in \mathbb{R}$, the mapping $D(0, \epsilon_0) \ni \epsilon \mapsto F_{n_1,n_2}(m, \epsilon)$ is a holomorphic function. Moreover, there exists $\tilde{K} > 0$ such that

$$\sup_{(n_1,n_2) \in N_1 \times N_2} \sup_{\epsilon \in D(0,\epsilon_0)} K_{n_1,n_2}(\epsilon) \leq \tilde{K}.$$

Let us define the function Ψ on $\mathbb{C}^2 \times \mathbb{R} \times D(0,\epsilon_0)$ by

$$\Psi(\tau_1,\tau_2,m,\epsilon) := \sum_{(n_1,n_2)\in N_1\times N_2} F_{n_1,n_2}(m,\epsilon)\frac{\tau_1^{n_1}}{\Gamma\left(\frac{n_1}{k_1}\right)}\frac{\tau_2^{n_2}}{\Gamma\left(\frac{n_2}{k_2}\right)},$$

and consider its inverse Fourier and double Laplace transform, giving rise to the function

$$f(t_1,t_2,z,\epsilon)$$
$$:= \frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1}}\int_{L_{d_2}} \Psi(\tau_1,\tau_2,m,\epsilon)\exp\left(-\left(\frac{\tau_1}{\epsilon t_1}\right)^{k_1}-\left(\frac{\tau_2}{\epsilon t_2}\right)^{k_2}\right)e^{izm}\frac{d\tau_2}{\tau_2}\frac{d\tau_1}{\tau_1}dm.$$

Here, $d_1,d_2\in\mathbb{R}$ are arbitrary numbers, and L_{d_j} denotes the integration path $[0,\infty)e^{id_j}$, for $j=1,2$. Observe that, due to Cauchy Theorem and the fact that Ψ is a polynomial in its two first variables, the directions d_1 and d_2 can be arbitrarily chosen. In addition to that, in view of Lemma 2.4, it is straight to check that f determines a bounded holomorphic function with respect to (z,ϵ) on $H_{\beta'}\times D(0,\epsilon_0)$, and a polynomial with respect to its first two variables. In addition to this,

$$f(t_1,t_2,z,\epsilon)$$
$$= \sum_{(n_1,n_2)\in N_1\times N_2}\left[\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} F_{n_1,n_2}(m,\epsilon)e^{izm}dm\right](\epsilon t_1)^{n_1}(\epsilon t_2)^{n_2}$$
$$=: \sum_{(n_1,n_2)\in N_1\times N_2}\mathfrak{F}_{n_1,n_2}(z,\epsilon)(\epsilon t_1)^{n_1}(\epsilon t_2)^{n_2},\quad (t_1,t_2,z,\epsilon)\in\mathbb{C}^2\times H_{\beta'}\times D(0,\epsilon_0).$$

Remark 2.5. Observe that

$$\sup_{\epsilon\in D(0,\epsilon_0),(n_1,n_2)\in N_1\times N_2}\ \sup_{m\in\mathbb{R}}(1+|m|)^{\mu}e^{\beta|m|}|F_{n_1,n_2}(m,\epsilon)|\le\tilde{K}. \tag{2.12}$$

It is straightforward to check that for every $\epsilon\in D(0,\epsilon_0)$ and $\rho_1,\rho_2>0$, one has that $\Psi\in B_{(\beta,\mu,\rho_1,\rho_2)}$ (see Definition 5.1 from Subsection 5.1.1). Indeed,

$$\|\Psi(\tau_1,\tau_2,m,\epsilon)\|_{(\beta,\mu,\rho_1,\rho_2)}\le\tilde{K}\sum_{(n_1,n_2)\in N_1\times N_2}\frac{\rho_1^{n_1-1}}{\Gamma\left(\frac{n_1}{k_1}\right)}\frac{\rho_2^{n_2-1}}{\Gamma\left(\frac{n_2}{k_2}\right)}=:C_{\Psi}<\infty. \tag{2.13}$$

Observe this quantity can be diminish as desired as long as $\tilde{K}$ is small enough.

3. ANALYTIC SOLUTIONS TO THE MAIN PROBLEM

This section is devoted to the construction of analytic solutions to the main problem under study (2.9). First, we sketch the strategy to follow, and later we describe the solution by means of auxiliary problems solved by means of fixed point arguments involving certain operators defined on appropriate Banach spaces.

3.1. STRATEGY FOR THE CONSTRUCTION OF THE ANALYTIC SOLUTIONS TO (2.9)

We consider that the assumptions and constructions associated to the main problem (2.9), made in Section 2 hold, and we search for solutions of (2.9) in the form of an inverse Fourier and double Laplace transform, for every fixed value of the perturbation parameter ϵ. More precisely, we will search for solutions of (2.9) in the form

$$\begin{aligned} &u(t_1,t_2,z,\epsilon)\\ &= \frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1}}\int_{L_{d_2}} \omega(u_1,u_2,m,\epsilon)\exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1}-\left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right)e^{izm}\frac{du_2}{u_2}\frac{du_1}{u_1}dm \end{aligned} \tag{3.1}$$

for some $d_1, d_2 \in \mathbb{R}$ to be determined, and where L_{d_j} is the integration path $[0,\infty)e^{id_j}$, for $j = 1,2$.

We proceed by following several steps in the search of a solucion of (2.9) in the form (3.1). First, let us search for a function $u(t_1,t_2,z,\epsilon)$ showing a monomial behavior with respect to its first two variables, i.e. assume that $u(t_1,t_2,z,\epsilon) = U(\epsilon t_1,\epsilon t_2,z,\epsilon)$, for some function U defined on appropriate domains, to be specified. A direct inspection of the previous elements yields the next result.

Lemma 3.1. *From the formal point of view, $u(t_1,t_2,z,\epsilon)$ solves* (2.9) *whenever $U(T_1,T_2,z,\epsilon)$ provides a formal solution of*

$$\begin{aligned} &Q(\partial_z)U(T_1,T_2,z,\epsilon)\\ &= (T_1^{k_1+1}\partial_{T_1})^{\delta_1}(T_2^{k_2+1}\partial_{T_2})^{\delta_2}R(\partial_z)U(T_1,T_2,z,\epsilon)\\ &\quad+\sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I}\epsilon^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4}c_{\underline{\ell}}(z,\epsilon)T_1^{\ell_1}\partial_{T_1}^{\ell_2}T_2^{\ell_3}\partial_{T_2}^{\ell_4}R_{\underline{\ell}}(\partial_z)U(T_1,T_2,z,\epsilon)\\ &\quad+(P_1(\epsilon,\partial_z)U(T_1,T_2,z,\epsilon))\,(P_2(\epsilon,\partial_z)U(T_1,T_2,z,\epsilon))+F(T_1,T_2,z,\epsilon), \end{aligned} \tag{3.2}$$

where $F(T_1,T_2,z,\epsilon)$ is given by

$$\frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1}}\int_{L_{d_2}}\Psi(\tau_1,\tau_2,m,\epsilon)\exp\left(-\left(\frac{\tau_1}{T_1}\right)^{k_1}-\left(\frac{\tau_2}{T_2}\right)^{k_2}\right)e^{izm}\frac{d\tau_2}{\tau_2}\frac{d\tau_1}{\tau_1}dm. \tag{3.3}$$

As a second step, we rewrite the terms involved in the equation (3.2) by means of the next result.

Lemma 3.2 ([27, Formula (8.7)])**.** *Let m,k be positive integers. Then, for all $1 \le \ell \le m-1$, there exists a constant $A_{m,\ell} \in \mathbb{R}$ such that*

$$T^{m(k+1)}\partial_T^m = (T^{k+1}\partial_T)^m + \sum_{\ell=1}^{m-1}A_{m,\ell}T^{k(m-\ell)}(T^{k+1}\partial_T)^{\ell}.$$

Taking into account the previous Lemma, together with the assumption (2.1) made on the elements involved in the equation, one can write (3.2) in the form

$$
\begin{aligned}
&Q(\partial_z)U(T_1,T_2,z,\epsilon)\\
&= (T_1^{k_1+1}\partial_{T_1})^{\delta_1}(T_2^{k_2+1}\partial_{T_2})^{\delta_2}R(\partial_z)U(T_1,T_2,z,\epsilon)\\
&\quad + \sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I} \epsilon^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4} c_{\underline{\ell}}(z,\epsilon)\\
&\quad \times T_1^{d_{k_1,\ell_1,\ell_2}}\left[(T_1^{k_1+1}\partial_{T_1})^{\ell_2} + \sum_{h=1}^{\ell_2-1} A_{\ell_2,h}T_1^{k_1(\ell_2-h)}(T_1^{k_1+1}\partial_{T_1})^{h}\right]\\
&\quad \times T_2^{d_{k_2,\ell_3,\ell_4}}\left[(T_2^{k_2+1}\partial_{T_2})^{\ell_4} + \sum_{j=1}^{\ell_4-1} A_{\ell_4,j}T_2^{k_2(\ell_4-j)}(T_2^{k_2+1}\partial_{T_2})^{j}\right]R_{\underline{\ell}}(\partial_z)U(T_1,T_2,z,\epsilon)\\
&\quad + \left(P_1(\epsilon,\partial_z)U(T_1,T_2,z,\epsilon)\right)\left(P_2(\epsilon,\partial_z)U(T_1,T_2,z,\epsilon)\right) + F(T_1,T_2,z,\epsilon),
\end{aligned}
\tag{3.4}
$$

for $A_{\ell_2,h}, A_{\ell_4,j}$ for $1 \leq h \leq \ell_2 - 1$ and $1 \leq j \leq \ell_4 - 1$ determined in Lemma 3.2.

Our strategy of finding solutions of (2.9) in the form (3.1) lead us to search for solutions of (3.4) in the form

$$
\begin{aligned}
&U(T_1,T_2,z,\epsilon)\\
&= \frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1}}\int_{L_{d_2}} \omega(u_1,u_2,m,\epsilon)\exp\left(-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2}\right)e^{izm}\frac{du_2}{u_2}\frac{du_1}{u_1}dm,
\end{aligned}
\tag{3.5}
$$

defined on certain domains to be clarified.

The following result describes the action of some operators on an expression of the form (3.5).

Lemma 3.3. *Assume the expression of $U(T_1,T_2,z,\epsilon)$ is formally defined by (3.5). The following statements hold:*

(i) *For $j = 1, 2$, the expression $T_j^{k_j+1}\partial_{T_j}U(T_1,T_2,z,\epsilon)$ equals*

$$
\frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1}}\int_{L_{d_2}} k_ju_j^{k_j}\omega(u_1,u_2,m,\epsilon)\exp\left(-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2}\right)e^{izm}\frac{du_2}{u_2}\frac{du_1}{u_1}dm,
$$

(ii) *For every positive integer m_1, the expression $T_1^{m_1}U(T_1,T_2,z,\epsilon)$ equals*

$$
\begin{aligned}
&\frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1}}\int_{L_{d_2}}\left[\frac{u_1^{k_1}}{\Gamma\left(\frac{m_1}{k_1}\right)}\int_0^{u_1^{k_1}}(u_1^{k_1}-s_1)^{\frac{m_1}{k_1}-1}\omega(s_1^{\frac{1}{k_1}},u_2,m,\epsilon)\frac{ds_1}{s_1}\right]\\
&\times\exp\left(-\left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2}\right)e^{izm}\frac{du_2}{u_2}\frac{du_1}{u_1}dm,
\end{aligned}
$$

(iii) *For every positive integer* m_2*, the expression* $T_2^{m_2} U(T_1, T_2, z, \epsilon)$ *equals*

$$\frac{k_1 k_2}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_1}} \int_{L_{d_2}} \left[\frac{u_2^{k_2}}{\Gamma\left(\frac{m_2}{k_2}\right)} \int_0^{u_2^{k_2}} (u_2^{k_2} - s_2)^{\frac{m_2}{k_2} - 1} \omega(u_1, s_2^{\frac{1}{k_2}}, m, \epsilon) \frac{ds_2}{s_2} \right]$$
$$\times \exp\left(- \left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2} \right) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm,$$

(iv) *The expression* $(P_1(\epsilon, \partial_z) U(T_1, T_2, z, \epsilon)) (P_2(\epsilon, \partial_z) U(T_1, T_2, z, \epsilon))$ *equals*

$$\frac{k_1 k_2}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_1}} \int_{L_{d_2}} \Bigg[u_1^{k_1} u_2^{k_2} \int_0^{u_1^{k_1}} \int_0^{u_2^{k_2}} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} P_1(\epsilon, i(m - m_1))$$
$$\times \omega\left((u_1^{k_1} - s_1)^{1/k_1}, (u_2^{k_2} - s_2)^{1/k_2}, m - m_1, \epsilon \right)$$
$$\times P_2(\epsilon, i m_1) \omega\left(s_1^{1/k_1}, s_2^{1/k_2}, m_1, \epsilon \right)$$
$$\times \frac{1}{(u_1^{k_1} - s_1)} \frac{1}{s_1} \frac{1}{(u_2^{k_2} - s_2)} \frac{1}{s_2} dm_1 ds_2 ds_1 \Bigg]$$
$$\times \exp\left(- \left(\frac{u_1}{T_1}\right)^{k_1} - \left(\frac{u_2}{T_2}\right)^{k_2} \right) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm.$$

Proof. The expressions (i) is a direct consequence of the derivation under the integral symbol. (ii) and (iii) are derived from Fubini theorem. The formula (iv) is derived from Fubini's theorem and an application of the identities in Lemma 1 of [12]. □

For the sake of clarity, we simplify the expressions involving a convolution product, as in (iv) above, as follows. We write

$$f \star g(m) := \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} h(m - m_1) g(m_1) dm_1, \quad m \in \mathbb{R}.$$

The actions of the operators described in Lemma 3.3 and the expression of the auxiliary equation (3.4) allow us to reduce the problem of finding $U(T_1, T_2, z, \epsilon)$ to that of searching a function $\omega(\tau_1,\tau_2,m,\epsilon)$, defined in appropriate domains, which solves the following convolution equation in the Borel–Fourier plane:

$$
\begin{aligned}
&Q(im)\omega(\tau_1,\tau_2,m,\epsilon)\\
&=(k_1\tau_1^{k_1})^{\delta_1}(k_2\tau_2^{k_2})^{\delta_2}R(im)\omega(\tau_1,\tau_2,m,\epsilon)\\
&=\sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I}\epsilon^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4}C_{\underline{\ell}}(m,\epsilon)\star\big[\mathcal{A}_1(\tau_1,\tau_2,m,\epsilon)+\mathcal{A}_2(\tau_1,\tau_2,m,\epsilon)\\
&\quad+\mathcal{A}_3(\tau_1,\tau_2,m,\epsilon)+\mathcal{A}_4(\tau_1,\tau_2,m,\epsilon)\big]+\tau_1^{k_1}\tau_2^{k_2}\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(P_1(\epsilon,im)\\
&\qquad\times\omega\left((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m,\epsilon\right)\star(P_2(\epsilon,im)\omega(s_1^{1/k_1},s_2^{1/k_2},m,\epsilon))\\
&\qquad\times\frac{1}{(\tau_1^{k_1}-s_1)}\frac{1}{s_1}\frac{1}{(\tau_2^{k_2}-s_2)}\frac{1}{s_2}ds_2ds_1+\Psi(\tau_1,\tau_2,m,\epsilon),
\end{aligned}
\tag{3.6}
$$

where the expressions $\mathcal{A}_j$ for $1\le j\le 4$ are determined by

$$
\begin{aligned}
\mathcal{A}_1(\tau_1,\tau_2,m_1,\epsilon)&=\frac{\tau_1^{k_1}}{\Gamma\left(\frac{d_{k_1,\ell_1,\ell_2}}{k_1}\right)}\frac{\tau_2^{k_2}}{\Gamma\left(\frac{d_{k_2,\ell_3,\ell_4}}{k_2}\right)}\\
&\quad\times\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{\frac{d_{k_1,\ell_1,\ell_2}}{k_1}-1}(\tau_2^{k_2}-s_2)^{\frac{d_{k_2,\ell_3,\ell_4}}{k_2}-1}\\
&\quad\times(k_1s_1)^{\ell_2}(k_2s_2)^{\ell_4}R_{\underline{\ell}}(im_1)\omega(s_1^{1/k_1},s_2^{1/k_2},m_1,\epsilon)\frac{ds_2}{s_2}\frac{ds_1}{s_1},
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\mathcal{A}_2(\tau_1,\tau_2,m_1,\epsilon)&=\sum_{1\le h\le\ell_2-1}A_{\ell_2,h}\frac{\tau_1^{k_1}}{\Gamma\left(\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}\right)}\frac{\tau_2^{k_2}}{\Gamma\left(\frac{d_{k_2,\ell_3,\ell_4}}{k_2}\right)}\\
&\quad\times\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}-1}(\tau_2^{k_2}-s_2)^{\frac{d_{k_2,\ell_3,\ell_4}}{k_2}-1}\\
&\qquad\times(k_1s_1)^{h}(k_2s_2)^{\ell_4}R_{\underline{\ell}}(im_1)\\
&\qquad\times\omega(s_1^{1/k_1},s_2^{1/k_2},m_1,\epsilon)\frac{ds_2}{s_2}\frac{ds_1}{s_1},
\end{aligned}
\tag{3.8}
$$

$$\mathcal{A}_3(\tau_1,\tau_2,m_1,\epsilon) = \sum_{1\le q\le \ell_4-1} A_{\ell_4,q} \frac{\tau_1^{k_1}}{\Gamma\left(\frac{d_{k_1,\ell_1,\ell_2}}{k_1}\right)} \frac{\tau_2^{k_2}}{\Gamma\left(\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}\right)}$$
$$\times \int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{\frac{d_{k_1,\ell_1,\ell_2}}{k_1}-1}(\tau_2^{k_2}-s_2)^{\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}-1} \tag{3.9}$$
$$\times (k_1s_1)^{\ell_2}(k_2s_2)^q R_{\underline{\ell}}(im_1)$$
$$\times \omega(s_1^{1/k_1}, s_2^{1/k_2}, m_1,\epsilon)\frac{ds_2}{s_2}\frac{ds_1}{s_1},$$

$$\mathcal{A}_4(\tau_1,\tau_2,m_1,\epsilon)$$
$$= \sum_{1\le h\le \ell_2-1, 1\le q\le \ell_4-1} A_{\ell_2,h}A_{\ell_4,q} \frac{\tau_1^{k_1}}{\Gamma\left(\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}\right)} \frac{\tau_2^{k_2}}{\Gamma\left(\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}\right)}$$
$$\times \int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}-1}(\tau_2^{k_2}-s_2)^{\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}-1} \tag{3.10}$$
$$\times (k_1s_1)^h(k_2s_2)^q$$
$$\times R_{\underline{\ell}}(im_1)\omega(s_1^{1/k_1}, s_2^{1/k_2}, m_1,\epsilon)\frac{ds_2}{s_2}\frac{ds_1}{s_1}.$$

3.2. SOLUTION TO AN AUXILIARY PROBLEM. I

In this section, we preserve the assumptions made considering the main problem (2.9), in Section 2. We seek for a solution to the auxiliary problem (3.6) belonging to certain Banach spaces of functions, described in Section 5.1.1. This goal is attained by means of the estimation obtained in the next result.

For every $m\in\mathbb{R}$, we consider the polynomial

$$P_m(\tau_1,\tau_2) = Q(im) - R(im)(k_1\tau_1^{k_1})^{\delta_1}(k_2\tau_2^{k_2})^{\delta_2}. \tag{3.11}$$

Lemma 3.4. *There exist $\rho_1,\rho_2>0$ and $C_1>0$ such that*

$$|P_m(\tau_1,\tau_2)| \ge C_1|R(im)|, \quad (\tau_1,\tau_2,m)\in D(0,\rho_1)\times D(0,\rho_2)\times\mathbb{R}.$$

Proof. In view of the assumptions made in (2.4) and (2.5) on the polynomials Q,R, we arrive at the existence of $r_{Q,R}>0$ such that

$$\left|\frac{Q(im)}{R(im)}\right| \ge r_{Q,R}, \quad m\in\mathbb{R}.$$

Let us choose small enough $\rho_1,\rho_2>0$ such that $(k_1\rho_1^{k_1})^{\delta_1}(k_2\rho_2^{k_2})^{\delta_2} \le \frac{r_{Q,R}}{2}$. This entails that

$$\left|\frac{Q(im)}{R(im)} - (k_1\tau_1^{k_1})^{\delta_1}(k_2\tau_2^{k_2})^{\delta_2}\right| \ge \frac{r_{Q,R}}{2}, \quad m\in\mathbb{R}.$$

For all $m \in \mathbb{R}$ and for all $\tau_j \in D(0,\rho_j)$, $j = 1,2$, let us write

$$P_m(\tau_1,\tau_2) = R(im)\left[\frac{Q(im)}{R(im)} - (k_1\tau_1^{k_1})^{\delta_1}(k_2\tau_2^{k_2})^{\delta_2}\right], \tag{3.12}$$

which allow us to conclude the result for $C_1 = \frac{r_{Q,R}}{2}$. ☐

Proposition 3.5. *Let ρ_1, ρ_2 be prescribed as in Lemma 3.4. For every $\varpi > 0$ there exists $\varsigma_{F,1} > 0$ such that if $C_\Psi \leq \varsigma_{F,1}$, (see* (2.13) *for the value of C_Ψ) then the problem* (3.6) *admits a unique solution $\omega_{\rho_1,\rho_2}(\tau_1,\tau_2,m,\epsilon)$, continuous on $\mathbb{R}$ with respect to its third variable, and holomorphic with respect to (τ_1,τ_2,ϵ) on $D(0,\rho_1)\times D(0,\rho_2)\times D(0,\epsilon_0)$, such that*

$$|\,\omega\rho_1,\rho_2(\tau_1,\tau_2,m,\epsilon)| \leq \varpi\frac{1}{(1+|m|)^\mu}e^{-\beta|m|}\,|\tau_1\tau_2|\,, \tag{3.13}$$

for every $(\tau_1,\tau_2,m,\epsilon) \in D(0,\rho_1)\times D(0,\rho_2)\times\mathbb{R}\times D(0,\epsilon_0)$.

Proof. Let $\epsilon \in D(0,\epsilon_0)$ and consider the Banach space $B_{(\beta,\mu,\rho_1,\rho_2)}$ studied in Section 5.1.1. We fix $\varpi > 0$.

Let us denote by $\overline{B}(0,\varpi) \subseteq B_{(\beta,\mu,\rho_1,\rho_2)}$ the set of elements in $B_{(\beta,\mu,\rho_1,\rho_2)}$ whose norm is upper bounded by ϖ. We consider the operator

$$\begin{aligned}
&\mathcal{H}_\epsilon(\omega(\tau_1,\tau_2,m))\\
&:= \frac{1}{P_m(\tau_1,\tau_2)}\\
&\quad\times\Bigg[\sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I}\epsilon^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4}C_{\underline{\ell}}(m,\epsilon)\star\big[\mathcal{A}_1(\tau_1,\tau_2,m,\epsilon)+\mathcal{A}_2(\tau_1,\tau_2,m,\epsilon)\\
&\qquad+\mathcal{A}_3(\tau_1,\tau_2,m,\epsilon)+\mathcal{A}_4(\tau_1,\tau_2,m,\epsilon)\big]+\tau_1^{k_1}\tau_2^{k_2}\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(P_1(\epsilon,im)\\
&\qquad\times\omega\left((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m\right))\star(P_2(\epsilon,im)\omega(s_1^{1/k_1},s_2^{1/k_2},m))\\
&\qquad\times\frac{1}{(\tau_1^{k_1}-s_1)}\frac{1}{s_1}\frac{1}{(\tau_2^{k_2}-s_2)}\frac{1}{s_2}ds_2ds_1+\Psi(\tau_1,\tau_2,m,\epsilon)\Bigg],
\end{aligned}$$

where P_m is defined in (3.11), and $\mathcal{A}_j$ for $1 \leq j \leq 4$ are defined in (3.7)–(3.10), where the term $\omega(s_1^{1/k_1},s_2^{1/k_2},m_1,\epsilon)$ needs to be replaced by $\omega(s_1^{1/k_1},s_2^{1/k_2},m_1)$. Regarding the assumptions made in (2.5), there exists $C_R > 0$ such that

$$|R(im)| \geq C_R(1+|m|)^{\deg R}, \quad m \in \mathbb{R}.$$

Let $\omega(\tau_1,\tau_2,m) \in \overline{B}(0,\varpi)$. From the assumptions made in (2.7) and (2.10) together with (2.2), one can apply Lemma 3.4, Proposition 5.2, Proposition 5.3, Proposition 5.4, together with (2.11), one has that

$$\|\mathcal{H}_\epsilon(\omega(\tau_1,\tau_2,m))\|_{(\beta,\mu,\rho_1,\rho_2)} \leq \frac{1}{C_1C_R}\Bigg[\sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I} \epsilon_0^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4}\frac{1}{(2\pi)^{1/2}}K \\ \times \tilde{C}_1\left[\mathfrak{C}_1+\mathfrak{C}_2+\mathfrak{C}_3+\mathfrak{C}_4\right]\varpi + \tilde{C}_2\frac{1}{(2\pi)^{1/2}}\varpi^2 + C_\Psi\Bigg]$$

with $K>0$ being the constant involved in (2.11), $\tilde{C}_1$ and $\tilde{C}_2$ the positive constants appearing in Proposition 5.3 and Proposition 5.4, respectively, and with C_Ψ in (2.13), together with

$$\mathfrak{C}_1 = \frac{1}{\Gamma\left(\frac{d_{k_1,\ell_1,\ell_2}}{k_1}\right)\Gamma\left(\frac{d_{k_2,\ell_3,\ell_4}}{k_2}\right)}k_1^{\ell_2}k_2^{\ell_4},$$

$$\mathfrak{C}_2 = \sum_{1\leq h\leq \ell_2-1}|A_{\ell_2,h}|\frac{1}{\Gamma\left(\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}\right)\Gamma\left(\frac{d_{k_2,\ell_3,\ell_4}}{k_2}\right)}k_1^{h}k_2^{\ell_4},$$

$$\mathfrak{C}_3 = \sum_{1\leq q\leq \ell_4-1}|A_{\ell_4,q}|\frac{1}{\Gamma\left(\frac{d_{k_1,\ell_1,\ell_2}}{k_1}\right)\Gamma\left(\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}\right)}k_1^{\ell_2}k_2^{q},$$

$$\mathfrak{C}_4 = \sum_{1\leq h\leq \ell_2-1, 1\leq q\leq \ell_4-1}|A_{\ell_2,h}||A_{\ell_4,q}|\frac{1}{\Gamma\left(\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}\right)\Gamma\left(\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}\right)}k_1^{h}k_2^{q}.$$

It is worth remarking that the previous constants are linked to the conditions imposed on the parameters involved in the problem, (2.3) and (2.1).

Let us choose $\varsigma_{F,1}>0$, ρ_1,ρ_2 and $\epsilon_0>0$ small enough such that

$$\frac{1}{C_1C_R}\Bigg(\sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I}\epsilon_0^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4}\frac{K}{(2\pi)^{1/2}}\tilde{C}_1\left[\mathfrak{C}_1+\mathfrak{C}_2+\mathfrak{C}_3+\mathfrak{C}_4\right] \\ + 2\tilde{C}_2\frac{1}{(2\pi)^{1/2}}\varpi\Bigg) \leq \frac{1}{2}, \tag{3.14}$$

and $\frac{1}{C_1C_R}\varsigma_{F,1} \leq \frac{1}{2}\varpi$. Recall that $\tilde{C}_2$ can be taken close to 0 provided that ρ_1,ρ_2 are reduced, in view of Proposition 5.4.

The previous inequality entails that $\mathcal{H}_\epsilon(\omega(\tau_1,\tau_2,m)) \in \overline{B}(0,\varpi)$. On the other hand, given $\omega_1,\omega_2 \in \overline{B}(0,\varpi) \subseteq B_{(\beta,\mu,\rho_1,\rho_2)}$, it holds that

$$\begin{aligned}
&\mathcal{H}_\epsilon(\omega_1(\tau_1,\tau_2,m)) - \mathcal{H}_\epsilon(\omega_2(\tau_1,\tau_2,m))\\
&:= \frac{1}{P_m(\tau_1,\tau_2)}\Big[\sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I} \epsilon^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4} C_{\underline{\ell}}(m,\epsilon) \star \Big[\mathcal{A}_1^{\star}(\tau_1,\tau_2,m)\\
&\quad + \mathcal{A}_2^{\star}(\tau_1,\tau_2,m) + \mathcal{A}_3^{\star}(\tau_1,\tau_2,m) + \mathcal{A}_4^{\star}(\tau_1,\tau_2,m)\Big] + \tau_1^{k_1}\tau_2^{k_2}\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}\\
&\Big\{(P_1(\epsilon,im)\omega_1((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m)) \star (P_2(\epsilon,im)\omega_1(s_1^{1/k_1},s_2^{1/k_2},m))\\
&-(P_1(\epsilon,im)\omega_2((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m)) \star (P_2(\epsilon,im)\omega_2(s_1^{1/k_1},s_2^{1/k_2},m))\Big\}\\
&\quad\times \frac{1}{\tau_1^{k_1}-s_1}\frac{1}{s_1}\frac{1}{\tau_2^{k_2}-s_2}\frac{1}{s_2}ds_2ds_1\Big],
\end{aligned}$$

with

$$\begin{aligned}
&\mathcal{A}_1^{\star}(\tau_1,\tau_2,m_1)\\
&= \frac{\tau_1^{k_1}}{\Gamma\left(\frac{d_{k_1,\ell_1,\ell_2}}{k_1}\right)}\frac{\tau_2^{k_2}}{\Gamma\left(\frac{d_{k_2,\ell_3,\ell_4}}{k_2}\right)}\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{\frac{d_{k_1,\ell_1,\ell_2}}{k_1}-1}(\tau_2^{k_2}-s_2)^{\frac{d_{k_2,\ell_3,\ell_4}}{k_2}-1}\\
&\quad\times (k_1s_1)^{\ell_2}(k_2s_2)^{\ell_4}R_{\underline{\ell}}(im_1)(\omega_1(s_1^{1/k_1},s_2^{1/k_2},m_1)-\omega_2(s_1^{1/k_1},s_2^{1/k_2},m_1))\frac{ds_2}{s_2}\frac{ds_1}{s_1},
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\mathcal{A}_2^{\star}(\tau_1,\tau_2,m_1)\\
&= \sum_{1\le h\le \ell_2-1} A_{\ell_2,h}\frac{\tau_1^{k_1}}{\Gamma\left(\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}\right)}\frac{\tau_2^{k_2}}{\Gamma\left(\frac{d_{k_2,\ell_3,\ell_4}}{k_2}\right)}\\
&\quad\times\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}(\tau_1^{k_1}-s_1)^{\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}-1}(\tau_2^{k_2}-s_2)^{\frac{d_{k_2,\ell_3,\ell_4}}{k_2}-1}(k_1s_1)^h(k_2s_2)^{\ell_4}R_{\underline{\ell}}(im_1)\\
&\qquad\times(\omega_1(s_1^{1/k_1},s_2^{1/k_2},m_1)-\omega_2(s_1^{1/k_1},s_2^{1/k_2},m_1))\frac{ds_2}{s_2}\frac{ds_1}{s_1},
\end{aligned} \tag{3.16}$$

$$\mathcal{A}_3^{\star}(\tau_1,\tau_2,m_1)$$
$$= \sum_{1\leq q\leq \ell_4-1} A_{\ell_4,q} \frac{\tau_1^{k_1}}{\Gamma\left(\frac{d_{k_1,\ell_1,\ell_2}}{k_1}\right)} \frac{\tau_2^{k_2}}{\Gamma\left(\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}\right)}$$
$$\times \int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{\frac{d_{k_1,\ell_1,\ell_2}}{k_1}-1}(\tau_2^{k_2}-s_2)^{\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}-1}(k_1s_1)^{\ell_2}(k_2s_2)^qR_{\underline{\ell}}(im_1)$$
$$\times (\omega_1(s_1^{1/k_1},s_2^{1/k_2},m_1)-\omega_2(s_1^{1/k_1},s_2^{1/k_2},m_1))\frac{ds_2}{s_2}\frac{ds_1}{s_1}, \tag{3.17}$$

$$\mathcal{A}_4^{\star}(\tau_1,\tau_2,m_1)$$
$$= \sum_{1\leq h\leq \ell_2-1, 1\leq q\leq \ell_4-1} A_{\ell_2,h}A_{\ell_4,q} \frac{\tau_1^{k_1}}{\Gamma\left(\frac{k_1(\ell_2-h)+d_{k_1,\ell_1,\ell_2}}{k_1}\right)} \frac{\tau_2^{k_2}}{\Gamma\left(\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}\right)}$$
$$\times \int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{\frac{d_{k_1,\ell_1,\ell_2}+k_1(\ell_2-h)}{k_1}-1}(\tau_2^{k_2}-s_2)^{\frac{k_2(\ell_4-q)+d_{k_2,\ell_3,\ell_4}}{k_2}-1} \tag{3.18}$$
$$\times (k_1s_1)^h(k_2s_2)^qR_{\underline{\ell}}(im_1)$$
$$\times (\omega_1(s_1^{1/k_1},s_2^{1/k_2},m_1)-\omega_2(s_1^{1/k_1},s_2^{1/k_2},m_1))\frac{ds_2}{s_2}\frac{ds_1}{s_1}.$$

Finally, we check that the expression

$$P_1(\epsilon,i(m-m_1))\omega_1\big((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m-m_1\big)P_2(\epsilon,im_1)\omega_1(s_1^{1/k_1},s_2^{1/k_2},m_1)$$
$$-P_1(\epsilon,i(m-m_1))\omega_2\big((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m-m_1\big)P_2(\epsilon,im_1)\omega_2(s_1^{1/k_1},s_2^{1/k_2},m_1)$$

equals

$$P_1(\epsilon,i(m-m_1))\Big(\omega_1\big((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m-m_1\big)$$
$$-\omega_2\big((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m-m_1\big)\Big)$$
$$\times P_2(\epsilon,im_1)\omega_1(s_1^{1/k_1},s_2^{1/k_2},m_1)$$
$$+P_1(\epsilon,i(m-m_1))\omega_2((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m-m_1)$$
$$\times P_2(\epsilon,im_1)(\omega_1(s_1^{1/k_1},s_2^{1/k_2},m_1)-\omega_2(s_1^{1/k_1},s_2^{1/k_2},m_1)).$$

An analogous reasoning as before and (3.14) allow us to arrive at

$$
\begin{aligned}
&\|\mathcal{H}_\epsilon(\omega_1(\tau_1,\tau_2,m))-\mathcal{H}_\epsilon(\omega_2(\tau_1,\tau_2,m))\|_{(\beta,\mu,\rho_1,\rho_2)}\\
&\leq \frac{1}{C_1C_R}\Bigg[\sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I}\epsilon_0^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4}\frac{1}{(2\pi)^{1/2}}K\tilde{C}_1[\mathfrak{C}_1+\mathfrak{C}_2+\mathfrak{C}_3+\mathfrak{C}_4]\\
&\qquad\qquad \times \|\,\omega 1(\tau_1,\tau_2,m)-\omega_2(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2)}\\
&+\tilde{C}_2\frac{1}{(2\pi)^{1/2}}\|\,\omega 1(\tau_1,\tau_2,m)-\omega_2(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2)}\|\,\omega 1(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2)}\\
&+\tilde{C}_2\frac{1}{(2\pi)^{1/2}}\|\,\omega 2(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2)}\|\,\omega 1(\tau_1,\tau_2,m)-\omega_2(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2)}\Bigg]\\
&\leq \frac{1}{C_1C_R}\Bigg[\sum_{\underline{\ell}=(\ell_1,\ell_2,\ell_3,\ell_4)\in I}\epsilon_0^{\Delta_{\underline{\ell}}-\ell_1+\ell_2-\ell_3+\ell_4}\frac{1}{(2\pi)^{1/2}}K\tilde{C}_1[\mathfrak{C}_1+\mathfrak{C}_2+\mathfrak{C}_3+\mathfrak{C}_4]\\
&\qquad\qquad +2\tilde{C}_2\frac{1}{(2\pi)^{1/2}}\varpi\Bigg]\|\,\omega 1(\tau_1,\tau_2,m)-\omega_2(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2)}\\
&\leq \frac{1}{2}\|\,\omega 1(\tau_1,\tau_2,m)-\omega_2(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2)}.
\end{aligned}
$$

The classical contractive mapping theorem can be applied on

$$\mathcal{H}_\epsilon:\overline{B}(0,\varpi)\to\overline{B}(0,\varpi),$$

to arrive at the existence of a unique fixed point, say $\omega_\epsilon(\tau_1,\tau_2,m)$. This construction depends holomorphically on $\epsilon\in D(0,\epsilon_0)$. As a conclusion, one arrives at the existence of a function

$$(\tau_1,\tau_2,m,\epsilon)\mapsto\omega_{\rho_1,\rho_2}(\tau_1,\tau_2,m,\epsilon)=\omega_\epsilon(\tau_1,\tau_2,m),$$

which is, by construction, a solution of (3.6), and satisfies the estimates (3.13). □

3.3. SOLUTION TO AN AUXILIARY PROBLEM. II

In this section, we still preserve the assumptions made regarding the main problem (2.9), in Section 2 and search for solutions to the auxiliary problem (3.6) in the Banach space of Section 5.1.2.

We start by providing alternative lower bounds on P_m to those attained in Lemma 3.4.

Lemma 3.6. *There exist $d_1,d_2\in\mathbb{R}$ and $C_2>0$ such that*

$$|P_m(\tau_1,\tau_2)|\geq C_2|R(im)|(1+|k_1\tau_1^{k_1}|^{\delta_1}|k_2\tau_2^{k_2}|^{\delta_2}),$$

for every $(\tau_1,\tau_2,m)\in S_{d_1}\times S_{d_2}\times\mathbb{R}$. Here, P_m is the polynomial defined by (3.11), *and S_{d_j}, for $j=1,2$ stands for some infinite sector centered at the origin, and bisecting direction d_j.*

Proof. We recall from the proof of Lemma 3.4, together with assumption (2.6) the existence of $r_{Q,R} > 0$ such that

$$\nabla := \left\{ \frac{Q(im)}{R(im)} : m \in \mathbb{R} \right\} \subseteq S_{Q,R} \setminus D(0, r_{Q,R}). \tag{3.19}$$

Let $d_1, d_2 \in \mathbb{R}$ be chosen such that $d_{1,2} := \delta_1 k_1 d_1 + \delta_2 k_2 d_2 \notin \arg(\nabla)$, where $\arg(\nabla) = \{\arg(z) : z \in \nabla\}$. We define S_{d_1} (resp. S_{d_2}) an infinite sector with small opening, centered at the origin, and bisecting direction d_1 (resp. d_2). The infinite sector $S_{d_{1,2}}$ is defined accordingly. This entails in particular that $(k_1\tau_1^{k_1})^{\delta_1}(k_2\tau_2^{k_2})^{\delta_2} \notin \nabla$ for $(\tau_1, \tau_2) \in S_{d_1} \times S_{d_2}$ when the opening of S_{d_1}, S_{d_2} are close enough to 0. More precisely, for every $\xi \in S_{d_{1,2}}$ one guarantees the existence of a positive constant $C_2 > 0$ with

$$\left| \frac{Q(im)}{R(im)} - \xi \right| \geq C_2(1 + |\xi|).$$

In particular, one has that

$$\left| \frac{Q(im)}{R(im)} - (k_1\tau_1^{k_1})^{\delta_1}(k_2\tau_2^{k_2})^{\delta_2} \right| \geq C_2(1 + |k_1\tau_1^{k_1}|^{\delta_1}|k_2\tau_2^{k_2}|^{\delta_2}).$$

The result follows from the decomposition (3.12). □

The definition of Ψ, together with Lemma 3.6 and (2.5) yields the following result.

Proposition 3.7. *Let $\nu_1, \nu_2 > 0$ and choose $d_1, d_2 \in \mathbb{R}$ as in Lemma 3.6. Then, there exists $D_\Psi > 0$ such that*

$$\sup_{(\tau_1,\tau_2,m)\in S_{d_1}\times S_{d_2}\times\mathbb{R}} \left| \frac{\Psi(\tau_1, \tau_2, m, \epsilon)}{P_m(\tau_1, \tau_2)} \right| (1 + |m|)^\mu e^{\beta|m|} \frac{1 + |\tau_1|^{2k_1}}{|\tau_1|} \frac{1 + |\tau_2|^{2k_2}}{|\tau_2|} \times \exp\left(-\nu_1|\tau_1|^{k_1} - \nu_2|\tau_2|^{k_2}\right) \leq D_\Psi,$$

valid for all $\epsilon \in D(0, \epsilon_0)$.

Remark 3.8. In terms of the Banach space described in Section 5.1.2, the previous result can be read as follows: there exists $D_\Psi > 0$ such that for all $\epsilon \in D(0, \epsilon_0)$, the function $\frac{\Psi(\tau_1,\tau_2,m,\epsilon)}{P_m(\tau_1,\tau_2)}$ belongs to $E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$, and

$$\sup_{\epsilon\in D(0,\epsilon_0)} \left\| \frac{\Psi(\tau_1, \tau_2, m, \epsilon)}{P_m(\tau_1, \tau_2)} \right\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})} \leq D_\Psi.$$

The constant D_Ψ tends to 0 when $\tilde{K}$ approaches to 0.

Proposition 3.9. *For every $\varpi > 0$ there exists $\varsigma_{F,2} > 0$ such that if $D_\Psi \leq \varsigma_{F,2}$, provided that the constants C_{P_1}, C_{P_2} appearing in* (2.8) *are small enough, then the problem* (3.6) *admits a unique solution $\omega_{S_{d_1},S_{d_2}}(\tau_1, \tau_2, m, \epsilon)$, continuous on $\mathbb{R}$ with respect to* *its third variable, and holomorphic with respect to* $(\tau_1, \tau_2, \epsilon)$ *on* $S_{d_1} \times S_{d_2} \times D(0, \epsilon_0)$*, such that*

$$\begin{aligned} &|\,\omega S_{d_1}, S_{d_2}(\tau_1, \tau_2, m, \epsilon)| \\ &\leq \varpi \frac{1}{(1+|m|)^{\mu}} e^{-\beta|m|} \frac{|\tau_1|}{1+|\tau_1|^{2k_1}} \frac{|\tau_2|}{1+|\tau_2|^{2k_2}} \exp\left(\nu_1 |\tau_1|^{k_1} + \nu_2 |\tau_2|^{k_2}\right), \end{aligned} \tag{3.20}$$

for every $(\tau_1, \tau_2, m, \epsilon) \in S_{d_1} \times S_{d_2} \times \mathbb{R} \times D(0, \epsilon_0)$.

Proof. The proof of Proposition 3.9 follows the same line of arguments and similar notations as the one of Proposition 3.5 owing to the conditions (2.3). Indeed, an inequality for $\|\mathcal{H}_\epsilon(\omega(\tau_1, \tau_2, m))\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$ is reached similar to the one obtained, where C_Ψ is replaced by D_Ψ, and the constant C_1 is replaced by C_2, together with $K\tilde{C}_1$ and $\tilde{C}_2$ appearing in Proposition 5.8 and Proposition 5.9. Furthermore, similar inequalities to (3.14) and for the difference $\|\mathcal{H}_\epsilon(\omega_1) - \mathcal{H}_\epsilon(\omega_2)\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$ are achieved, provided that $\epsilon_0 > 0$ is taken close to 0 (bearing in mind the condition (2.2)) and that $C_{\rho_1}, C_{\rho_2} > 0$ are small enough, according to the remark after Proposition 5.8. □

3.4. SOLUTION TO AN AUXILIARY PROBLEM, III. ANALYTIC CONTINUATION

We maintain the assumptions made on the elements involved in the main equation (2.9), now searching for solutions in a third Banach space, and exploring its analytic continuation by means of the results obtained so far.

Taking into account the lower estimates obtained in Lemma 3.4, and an analogous line of arguments as those followed in Proposition 3.5, we arrive at the following result.

Proposition 3.10. *For every* $\varpi > 0$ *there exists* $\varsigma_{F,3} > 0$ *such that if* $C_\Psi \leq \varsigma_{F,3}$*, then the auxiliary problem* (3.6) *admits a unique solution* $\omega_{\rho_1,\rho_2,S_{d_1},S_{d_2}}(\tau_1, \tau_2, m, \epsilon)$*, continuous on* $\mathbb{R}$ *with respect to its third variable, and holomorphic with respect to* $(\tau_1, \tau_2, \epsilon)$ *on* $(S_{d_1} \cap D(0, \rho_1)) \times (S_{d_2} \cap D(0, \rho_2)) \times D(0, \epsilon_0)$*, such that*

$$|\,\omega_{\rho_1,\rho_2,S_{d_1},S_{d_2}}(\tau_1, \tau_2, m, \epsilon)| \leq \varpi \frac{1}{(1+|m|)^{\mu}} e^{-\beta|m|} |\tau_1 \tau_2|, \tag{3.21}$$

for every $(\tau_1, \tau_2, m, \epsilon) \in (S_{d_1} \cap D(0, \rho_1)) \times (S_{d_2} \cap D(0, \rho_2)) \times \mathbb{R} \times D(0, \epsilon_0)$.

The next result proves that the solutions of (3.6) obtained in Proposition 3.5 and Proposition 3.9 are related by analytic continuation, by means of the solution constructed in Proposition 3.10.

Proposition 3.11. *Let* $\varpi > 0$*. There exists* $\varsigma_F > 0$ *such that if* $C_\Psi \leq \varsigma_F$ *and* $D_\Psi \leq \varsigma_F$*, then the following statements hold:*

(i) *for every fixed* $\tau_1 \in S_{d_1} \cap D(0, \rho_1)$*,* $m \in \mathbb{R}$ *and* $\epsilon \in D(0, \epsilon_0)$*, the map* $\tau_2 \mapsto \omega_{S_{d_1},S_{d_2}}(\tau_1, \tau_2, m, \epsilon)$ *defined on* S_{d_2} *(see Proposition 3.9) has an analytic continuation on* $D(0, \rho_2)$*, which is* $\tau_2 \mapsto \omega_{\rho_1,\rho_2}(\tau_1, \tau_2, m, \epsilon)$*,*

(ii) *for every fixed* $\tau_2 \in S_{d_2} \cap D(0,\rho_2)$*,* $m \in \mathbb{R}$ *and* $\epsilon \in D(0,\epsilon_0)$*, the map* $\tau_1 \mapsto \omega_{S_{d_1},S_{d_2}}(\tau_1,\tau_2,m,\epsilon)$ *defined on* S_{d_1} *(see Proposition 3.9) has an analytic continuation on* $D(0,\rho_1)$*, which is* $\tau_1 \mapsto \omega_{\rho_1,\rho_2}(\tau_1,\tau_2,m,\epsilon)$*.*

Proof. We fix

$$\Delta := \max_{(\tau_1,\tau_2)\in D(0,\rho_1)\times D(0,\rho_2)} \frac{1}{1+|\tau_1|^{2k_1}} \frac{1}{1+|\tau_2|^{2k_1}} \exp(\nu_1 |\tau_1|^{k_1} + \nu_2 |\tau_2|^{k_2}).$$

We observe that $\Delta > 0$. Let $(\tau_1,\tau_2,m,\epsilon) \mapsto \omega_{S_{d_1},S_{d_2}}(\tau_1,\tau_2,m,\epsilon)$ be the function obtained in Proposition 3.9, which solves (3.6) and such that (3.20) holds for some given ϖ, if $D_\Psi \leq \varsigma_{F,2}$. Then, for $(\tau_1,\tau_2,m,\epsilon) \in (S_{d_1}\cap D(0,\rho_1))\times(S_{d_2}\cap D(0,\rho_2))\times\mathbb{R}\times D(0,\epsilon_0)$, one has that

$$\begin{aligned} &|\,\omega_{S_{d_1},S_{d_2}}(\tau_1,\tau_2,m,\epsilon)| \\ &\leq \varpi \frac{1}{(1+|m|)^\mu} e^{-\beta|m|} \frac{|\tau_1|}{1+|\tau_1|^{2k_1}} \frac{|\tau_2|}{1+|\tau_2|^{2k_2}} \exp\left(\nu_1 |\tau_1|^{k_1} + \nu_2 |\tau_2|^{k_2}\right) \\ &\leq \varpi\Delta \frac{1}{(1+|m|)^\mu} e^{-\beta|m|} |\tau_1| \, |\tau_2| \, . \end{aligned}$$

This entails that for all $\epsilon \in D(0,\epsilon_0)$, the function

$$(S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2)) \times \mathbb{R} \ni (\tau_1,\tau_2,m) \mapsto \omega_{S_{d_1},S_{d_2}}(\tau_1,\tau_2,m,\epsilon)$$

is the fixed point of the operator $\mathcal{H}_\epsilon$ when defined on the closed ball $\overline{B}(0,\Delta\varpi)$ of the Banach space $F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$ of Section 5.1.3. From unicity of the fixed point for $\mathcal{H}_\epsilon$ in such ball obtained in Proposition 3.10, $\omega_{\rho_1,\rho_2,S_{d_1},S_{d_2}}$, we conclude that

$$\omega_{S_{d_1},S_{d_2}}(\tau_1,\tau_2,m,\epsilon) = \omega_{\rho_1,\rho_2,S_{d_1},S_{d_2}}(\tau_1,\tau_2,m,\epsilon)$$

for every $(\tau_1,\tau_2,m) \in (S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2)) \times \mathbb{R}$. An analogous reasoning leads us to

$$\omega_{\rho_1,\rho_2}(\tau_1,\tau_2,m,\epsilon) = \omega_{\rho_1,\rho_2,S_{d_1},S_{d_2}}(\tau_1,\tau_2,m,\epsilon),$$

for every $(\tau_1,\tau_2,m) \in (S_{d_1}\cap D(0,\rho_1))\times(S_{d_2}\cap D(0,\rho_2))\times\mathbb{R}$, with ω_{ρ_1,ρ_2} being the fixed point of $\mathcal{H}_\epsilon$, defined on the ball $\overline{B}(0,\Delta\varsigma)$ of the Banach space $B_{(\beta,\mu,\rho_1,\rho_2)}$, obtained in Proposition 3.5.

The result follows from the variation of $\epsilon \in D(0,\epsilon_0)$. □

At this point, we can state the main result of the present section, summarizing all the previous results.

Theorem 3.12. *Under the assumptions of Section 2, we consider the Cauchy problem* (2.9)*. Let* $\rho_1,\rho_2 > 0$ *determined in Section 3.2, and* $d_1,d_2 \in \mathbb{R}$ *chosen in Section 3.3. Let* $\mathcal{E} \subseteq D(0,\epsilon_0)$ *and* $\mathcal{T}_1,\mathcal{T}_2 \subseteq D(0,r_{\mathcal{T}})$ *for some small enough* $r_{\mathcal{T}} > 0$ *be three bounded sectors with vertex at the origin, chosen in such a way that:*

(i) *there exists* $\Delta_1 > 0$ *with* $\cos(k_1(d_1 - \arg(\epsilon t_1))) > \Delta_1$*, for all* $\epsilon \in \mathcal{E}$ *and* $t_1 \in \mathcal{T}_1$*,*
(ii) *there exists* $\Delta_2 > 0$ *with* $\cos(k_2(d_2 - \arg(\epsilon t_2))) > \Delta_2$*, for all* $\epsilon \in \mathcal{E}$ *and* $t_2 \in \mathcal{T}_2$*.*

Then, provided that ϵ_0*, the quantity* $\tilde{K}$ *from* (2.12)*, together with the constants* $C_{P_1}, C_{P_2} > 0$ *from* (2.8) *are taken small enough, and for every* $0 < \beta' < \beta$*, the problem* (2.9)*, under null initial data* $u(0,t_2,z,\epsilon) = u(t_1,0,z,\epsilon) = 0$*, admits an analytic solution* $u_{d_1,d_2}(t_1,t_2,z,\epsilon) \in \mathcal{O}_b(\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E})$*.*

Proof. Let $\omega_{S_{d_1},S_{d_2}}(\tau_1,\tau_2,m,\epsilon)$ be the solution of the auxiliary equation (3.6), obtained in Proposition 3.9. Regarding (3.20), the choice of d_1, d_2 at the statement of the result guarantees that the function

$$\begin{aligned} u_{d_1,d_2}(t_1,t_2,z,\epsilon) = \frac{k_1k_2}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_1}} \int_{L_{d_2}} \omega_{S_{d_1},S_{d_2}}(u_1,u_2,m,\epsilon) \\ \times \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2} \right) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm, \end{aligned} \tag{3.22}$$

is well-defined, holomorphic and bounded on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E}$, provided that $r_{\mathcal{T}}, \epsilon_0 > 0$ are small enough, and $0 < \beta' < \beta$. Indeed, observe that for all $(t_1,t_2,z,\epsilon) \in \mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E}$, one has that

$$\begin{aligned} |u_{d_1,d_2}(t_1,t_2,z,\epsilon)| \le \varpi \frac{k_1k_2}{(2\pi)^{1/2}} \left[\int_{-\infty}^{\infty} \frac{1}{(1+|m|)^{\mu}} e^{-|m|(\beta - |\mathrm{Im}(z)|)} dm \right] \\ \times \prod_{j=1}^{2} \left(\int_0^{\infty} \frac{r_j}{1+r_j^{2k_j}} \exp\left(r_j^{k_j} (\nu_j - \frac{\Delta_j}{|\epsilon t_j|^{k_j}}) \right) dr_j \right), \end{aligned}$$

for some $\varpi > 0$, after the parametrization $u_j = r_j e^{id_j}$ for $r_j \in [0,\infty)$ and $j = 1,2$. Assuming that $\epsilon_0 r_{\mathcal{T}} < (\Delta_j/\nu_j)^{1/k_j}$, for $j = 1,2$, the previous integrals converge. □

4. PARAMETRIC GEVREY SERIES EXPANSION

In this section, we provide an asymptotic representation of the analytic solution, obtained in the previous section. We maintain the assumptions made on the elements in the construction of the main problem (2.9), stated in Section 2.

We split the integral representation of the solution to problem (2.9) obtained in Theorem 3.12 as the sum

$$u_{d_1,d_2}(t_1,t_2,z,\epsilon) = J_1(t_1,t_2,z,\epsilon) + J_2(t_1,t_2,z,\epsilon) + J_3(t_1,t_2,z,\epsilon), \tag{4.1}$$

with

$$\begin{aligned} &J_1(t_1,t_2,z,\epsilon) \\ &:= \frac{k_1k_2}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{d_1,\rho_1/2}} \int_{L_{d_2,\rho_2/2}} \omega_{S_{d_1},S_{d_2}}(u_1,u_2,m,\epsilon) G(u_1,u_2,t_1,t_2,\epsilon) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm, \end{aligned}$$

$$J_2(t_1,t_2,z,\epsilon) := \frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1,\rho_1/2}}\int_{L_{d_2,\rho_2/2,\infty}} \omega_{S_{d_1},S_{d_2}}(u_1,u_2,m,\epsilon)G(u_1,u_2,t_1,t_2,\epsilon)e^{izm}\frac{du_2}{u_2}\frac{du_1}{u_1}dm,$$

$$J_3(t_1,t_2,z,\epsilon) := \frac{k_1k_2}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\int_{L_{d_1,\rho_1/2,\infty}}\int_{L_{d_2}} \omega_{S_{d_1},S_{d_2}}(u_1,u_2,m,\epsilon)G(u_1,u_2,t_1,t_2,\epsilon)e^{izm}\frac{du_2}{u_2}\frac{du_1}{u_1}dm,$$

for

$$G(u_1,u_2,t_1,t_2,\epsilon) = \exp\left(-\left(\frac{u_1}{\epsilon t_1}\right)^{k_1} - \left(\frac{u_2}{\epsilon t_2}\right)^{k_2}\right),$$

and where

$$L_{d_1,\rho_1/2} = \left[0,\frac{\rho_1}{2}\right]e^{\sqrt{-1}d_1}, \qquad L_{d_2,\rho_2/2} = \left[0,\frac{\rho_2}{2}\right]e^{\sqrt{-1}d_2}, \tag{4.2}$$

$$L_{d_1,\rho_1/2,\infty} = \left[\frac{\rho_1}{2},\infty\right)e^{\sqrt{-1}d_1}, \quad L_{d_2,\rho_2/2,\infty} = \left[\frac{\rho_2}{2},\infty\right)e^{\sqrt{-1}d_2}. \tag{4.3}$$

Our objective is to study the asymptotic Gevrey related to each piece in the previous decomposition.

4.1. GEVREY EXPANSIONS FOR J_1

Let us recall the notion of a good covering, which will be essential in our reasoning.

Definition 4.1. Let $\varsigma \geq 2$ be an integer. Let $\underline{\mathcal{E}} = (\mathcal{E}_p)_{0\leq p\leq \varsigma-1}$ be a set of bounded sectors with vertex at the origin, $\mathcal{E}_p \subseteq D(0,\epsilon_0)$, for $0 \leq p \leq \varsigma-1$ such that $\mathcal{E}_p \cap \mathcal{E}_{p+1} \neq \emptyset$ for $0 \leq p \leq \varsigma-1$ (with the notation $\mathcal{E}_\varsigma := \mathcal{E}_0$), which are three by three disjoint, i.e. $\mathcal{E}_{p_1}\cap\mathcal{E}_{p_2}\cap\mathcal{E}_{p_3} = \emptyset$ for all $0 \leq p_1,p_2,p_3 \leq \varsigma-1$, with $p_1 \neq p_2 \neq p_3$ and $p_1 \neq p_3$. In addition to this, there exists a neighborhood of the origin $\mathcal{U}$ such that $\mathcal{U}\setminus\{0\} = \cup_{p=0}^{\varsigma-1}\mathcal{E}_p$. In this situation, we say the family $\underline{\mathcal{E}}$ determines a good covering in $\mathbb{C}^\star$.

Let us depart from given bounded open sectors $\mathcal{T}_1,\mathcal{T}_2,\mathcal{E}$ with vertex at the origin, and $\rho_1,\rho_2 > 0$ and $d_1,d_2 \in \mathbb{R}$, under the hypotheses of Theorem 3.12. We choose a good covering $\underline{\mathcal{E}} = (\mathcal{E}_p)_{0\leq p\leq\varsigma-1}$ such that $\mathcal{E}_0 := \mathcal{E}$. In addition to this, we choose the real numbers $\mathfrak{d}_p, \tilde{\mathfrak{d}}_p$, for $0 \leq p \leq \varsigma-1$ with $\mathfrak{d}_0 := d_1$, $\tilde{\mathfrak{d}}_0 = d_2$, in such a way that the following conditions hold:

(i) for all $0 \leq p \leq \varsigma-1$ there exists $\nabla_p > 0$ with $\cos(k_1(\mathfrak{d}_p - \arg(\epsilon t_1))) > \nabla_p$, for $\epsilon \in \mathcal{E}_p$, $t_1 \in \mathcal{T}_1$,

(ii) for all $0 \leq p \leq \varsigma-1$ there exists $\tilde{\nabla}_p > 0$ with $\cos(k_2(\tilde{\mathfrak{d}}_p - \arg(\epsilon t_2))) > \tilde{\nabla}_p$, for $\epsilon \in \mathcal{E}_p$, $t_2 \in \mathcal{T}_2$.

Observe that one can choose $\nabla_0 = \Delta_1$ and $\tilde{\nabla}_0 = \Delta_2$, where Δ_1,Δ_2 are the constants in Theorem 3.12.

For every $0 \leq p \leq \varsigma - 1$, we construct the function

$$J_{1,p}(t_1,t_2,z,\epsilon) := \frac{k_1k_2}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{L_{\mathfrak{d}_p,\rho_1/2}} \int_{L_{\tilde{\mathfrak{d}}_p,\rho_2/2}} \omega_{\rho_1,\rho_2}(u_1,u_2,m,\epsilon) G(u_1,u_2,t_1,t_2,\epsilon) e^{izm} \frac{du_2}{u_2} \frac{du_1}{u_1} dm, \tag{4.4}$$

with $L_{\mathfrak{d}_p,\rho_1/2} = [0,\rho_1/2]e^{i\mathfrak{d}_p}$, $L_{\tilde{\mathfrak{d}}_p,\rho_2/2} = [0,\rho_2/2]e^{i\tilde{\mathfrak{d}}_p}$, which turns out to be an analytic and bounded function on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E}_p$, for every $0 < \beta' < \beta$.

Observe that $J_{1,0}(t_1,t_2,z,\epsilon) = J_1(t_1,t_2,z,\epsilon)$ for all $\epsilon \in \mathcal{E} = \mathcal{E}_0$, $t_1 \in \mathcal{T}_1$, $t_2 \in \mathcal{T}_2$, $z \in H_{\beta'}$.

The next Proposition provides bounds for the differences of consecutive maps $J_{1,p}$. The proof of the next result is analogous to that of Theorem 1, [15], so we omit it.

Proposition 4.2. *Under the previous assumptions, the following statements hold for every $0 \leq p \leq \varsigma - 1$:*

Case 1. $\mathfrak{d}_p = \mathfrak{d}_{p+1}$*, and* $\tilde{\mathfrak{d}}_p \neq \tilde{\mathfrak{d}}_{p+1}$*. There exist* $C_{p,1}, C_{p,2} > 0$ *such that*

$$|J_{1,p+1}(t_1,t_2,z,\epsilon) - J_{1,p}(t_1,t_2,z,\epsilon)| \leq C_{p,1} \exp\left(-\frac{C_{p,2}}{|\epsilon|^{k_2}}\right),$$

for all $(t_1,t_2,z,\epsilon) \in \mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times (\mathcal{E}_p \cap \mathcal{E}_{p+1})$.

Case 2. $\mathfrak{d}_p \neq \mathfrak{d}_{p+1}$*, and* $\tilde{\mathfrak{d}}_p = \tilde{\mathfrak{d}}_{p+1}$*. There exist* $C_{p,3}, C_{p,4} > 0$ *such that*

$$|J_{1,p+1}(t_1,t_2,z,\epsilon) - J_{1,p}(t_1,t_2,z,\epsilon)| \leq C_{p,3} \exp\left(-\frac{C_{p,4}}{|\epsilon|^{k_1}}\right),$$

for all $(t_1,t_2,z,\epsilon) \in \mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times (\mathcal{E}_p \cap \mathcal{E}_{p+1})$.

Case 3. $\mathfrak{d}_p \neq \mathfrak{d}_{p+1}$*, and* $\tilde{\mathfrak{d}}_p \neq \tilde{\mathfrak{d}}_{p+1}$*. There exist* $C_{p,5}, C_{p,6} > 0$ *such that*

$$|J_{1,p+1}(t_1,t_2,z,\epsilon) - J_{1,p}(t_1,t_2,z,\epsilon)| \leq C_{p,5} \exp\left(-\frac{C_{p,6}}{|\epsilon|^{k_2}}\right),$$

for all $(t_1,t_2,z,\epsilon) \in \mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times (\mathcal{E}_p \cap \mathcal{E}_{p+1})$.

At this point, we can apply Theorem (RS). The classical Ramis–Sibuya Theorem can be found in detail in [8, Theorem XI-2-3] whereas the proof of the following generalization can be found in detail in [11, Theorem XI-2-3, pp. 63–65].

Theorem 4.3 (RS). *Let $0 < k_2 < k_1$. We also fix a complex Banach space $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$, and a good covering in $\mathbb{C}^\star$, $(\mathcal{E}_p)_{0 \leq p \leq \varsigma-1}$, for some integer $\varsigma \geq 2$ (see Definition 4.1 in Section 4). Given $0 \leq p \leq \varsigma - 1$, we assume $G_p \in \mathcal{O}_b(\mathcal{E}_p, \mathbb{E})$, and define $\Delta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$, for $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$, with the convention that $\mathcal{E}_\varsigma := \mathcal{E}_0$ and $G_\varsigma := G_0$. Assume moreover the existence of $I_1, I_2 \subseteq \{0, \dots, \varsigma-1\}$, such that $I_1, I_2 \neq \emptyset$, and $I_1 \cup I_2 = \{0, \dots, \varsigma - 1\}$, with $I_1 \cap I_2 = \emptyset$, satisfying the following properties:*

(i) *for every* $p \in I_1$, *there exists* $K_p, M_p > 0$ *such that* $\|\Delta_p(\epsilon)\|_{\mathbb{E}} \leq K_p \exp\left(-\frac{M_p}{|\epsilon|^{k_1}}\right)$ *for* $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$,

(ii) *for every* $p \in I_2$, *there exists* $\tilde{K}_p, \tilde{M}_p > 0$ *such that* $\|\Delta_p(\epsilon)\|_{\mathbb{E}} \leq \tilde{K}_p \exp\left(-\frac{\tilde{M}_p}{|\epsilon|^{k_2}}\right)$ *for* $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$.

Then, there exist $a(\epsilon) \in \mathbb{E}\{\epsilon\}$, *two formal power series* $\hat{G}^1, \hat{G}^2 \in \mathbb{E}[[\epsilon]]$, *and for all* $0 \leq p \leq \varsigma - 1$ *two functions* $G_p^1, G_p^2 \in \mathcal{O}_b(\mathcal{E}_p, \mathbb{E})$ *such that:*

(1) *for all* $0 \leq p \leq \varsigma - 1$, *the function* G_p *admits the decomposition*

$$G_p(\epsilon) = a_p(\epsilon) + G_p^1(\epsilon) + G_p^2(\epsilon), \quad \epsilon \in \mathcal{E}_p,$$

(2) *for* $j = 1, 2$, *and all* $0 \leq p \leq \varsigma - 1$, *the function* G_p^j *admits* $\hat{G}^j$ *as its Gevrey asymptotic expansion of order* $1/k_j$ *on* $\mathcal{E}_p$.

For $0 < \beta' < \beta$, we write $\mathbb{E}$ for the Banach space of bounded holomorphic functions defined on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'}$ equipped with the norm of the supremum, i.e. $\mathbb{E} := \mathcal{O}_b(\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'})$.

Proposition 4.4. *Under the previous assumptions, for every* $0 \leq p \leq \varsigma - 1$, *the solution* (4.4) *of* (2.9) *admits a splitting of the form*

$$J_{1,p}(t_1, t_2, z, \epsilon) = a(t_1, t_2, z, \epsilon) + J_{1,1,p}(t_1, t_2, z, \epsilon) + J_{1,2,p}(t_1, t_2, z, \epsilon),$$

where $a(t_1, t_2, z, \epsilon) \in \mathbb{E}\{\epsilon\}$, *and* $J_{1,j,p} \in \mathcal{O}_b(\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E}_p)$, *for all* $0 \leq p \leq \varsigma - 1$ *and* $j = 1, 2$. *Moreover, there exist two formal power series in* ϵ *with coefficients in* $\mathbb{E}$, *say* $\hat{J}_{1,j} \in \mathbb{E}[[\epsilon]]$, *for* $j = 1, 2$ *which satisfy that* $J_{1,j,p}$ *admits* $\hat{J}_{1,j}$ *as its common Gevrey asymptotic expansion of order* $1/k_j$ *with respect to* ϵ *on* $\mathcal{E}_p$, *for all* $0 \leq p \leq \varsigma - 1$.

Proof. Let us split the set $\{0, \dots, \varsigma - 1\}$ into the set I_1 of indices such that Case 1 or Case 3 of Proposition 4.2 hold, and $I_2 = \{0, \dots, \varsigma - 1\} \setminus I_1$ (i.e. the set of indices for which Case 2 of Proposition 4.2 holds). Multilevel Ramis–Sibuya Theorem (RS) can be applied to the functions $G_p : \mathcal{E}_p \to \mathcal{O}_b(\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'})$ defined by

$$G_p(\epsilon) := J_{1,p}(t_1, t_2, z, \epsilon), \quad \epsilon \in \mathcal{E}_p,$$

for $0 \leq p \leq \varsigma - 1$, and where $\mathbb{E}$ stands for the Banach space of holomorphic and bounded functions defined on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'}$ equipped with the norm of the supremum, for some fixed $0 < \beta' < \beta$. This is a consequence of the different exponential decays in the perturbation parameter, uniform on the rest of variables, showed in Proposition 4.2. □

The particularization of the previous result to the first index in the good covering allows us to conclude.

Corollary 4.5. *In the situation of Proposition 4.4, the analytic map* $J_1(t_1, t_2, z, \epsilon)$, *defined in* (4.1), *defined in* $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E}$ *admits a splitting of the form*

$$J_1(t_1, t_2, z, \epsilon) = a(t_1, t_2, z, \epsilon) + J_{1,1,0}(t_1, t_2, z, \epsilon) + J_{1,2,0}(t_1, t_2, z, \epsilon),$$

where $a(t_1,t_2,z,\epsilon) \in \mathbb{E}\{\epsilon\}$*, and* $J_{1,j,0} \in \mathcal{O}_b(\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E})$ *for* $j=1,2$*. Moreover, the formal power series* $\hat{J}_{1,j} \in \mathbb{E}[[\epsilon]]$*, for* $j=1,2$ *satisfy that* $J_{1,j,0}$ *admits* $\hat{J}_{1,j}$ *as its common Gevrey asymptotic expansion of order* $1/k_j$ *with respect to* ϵ *on* $\mathcal{E}$*.*

4.2. GEVREY BOUNDS FOR J_2 AND J_3

We recall the next lemma from [19], which will be crucial in the next two propositions.

Lemma 4.6 ([19, Lemma 14.1])**.** *Let* $k' \geq 1$ *be an integer number, and let* $M > 0$ *a real number. There exists* $C_{k'} > 0$ *only depending on* k' *such that the next inequality*

$$\left(\frac{1}{r}\right)^N \exp\left(-\frac{M}{r^{k'}}\right) \leq C_{k'} A_{k'}^N \left(\frac{N}{k'}\right)^{1/2} \Gamma\left(\frac{N}{k'}\right)$$

holds for all integer $N \geq 1$ *and any real number* $r > 0$*, and where* $A_{k'} = (1/M)^{1/k'}$*.*

The next result provides bounds for J_2.

Proposition 4.7. *There exist* $C_{J_2}, K_{J_2} > 0$ *such that*

$$|J_2(t_1,t_2,z,\epsilon)| \leq C_{J_2} K_{J_2}^N \left(\frac{N}{k_2}\right)^{1/2} \Gamma\left(\frac{N}{k_2}\right) |\epsilon|^N, \tag{4.5}$$

for all positive integer N*, all* $t_1 \in \mathcal{T}_1$*,* $t_2 \in \mathcal{T}_2$*,* $z \in H_{\beta'}$ *and* $\epsilon \in \mathcal{E}$*.*

Proof. In view of the estimates for $\omega_{S_{d_1},S_{d_2}}$ determined in (3.20) we deduce that

$$\begin{aligned} |J_2(t_1,t_2,z,\epsilon)| \leq {} & \frac{k_1k_2}{(2\pi)^{1/2}} \left(\int_{-\infty}^{\infty} (1+|m|)^{-\mu} e^{-\beta|m|} e^{\beta'|m|} dm \right) \varpi_{S_{d_1},S_{d_2}} \\ & \times \int_0^{\rho_1/2} \exp\left(\nu_1 r_1^{k_1}\right) \exp\left(-\frac{r_1^{k_1}}{|\epsilon t_1|^{k_1}}\Delta_1\right) dr_1 \\ & \times \int_{\rho_2/2}^{\infty} \exp\left(\nu_2 r_2^{k_2}\right) \exp\left(-\frac{r_2^{k_2}}{|\epsilon t_2|^{k_2}}\Delta_2\right) dr_2, \end{aligned} \tag{4.6}$$

where $\Delta_1, \Delta_2 > 0$ are the constants in Theorem 3.12, for all $t_1 \in \mathcal{T}_1$, $t_2 \in \mathcal{T}_2$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}$.

Observe that

$$\int_0^{\rho_1/2} \exp\left(\nu_1 r_1^{k_1}\right) \exp\left(-\frac{r_1^{k_1}}{|\epsilon t_1|^{k_1}}\Delta_1\right) dr_1 \leq \exp\left(\nu_1 \left(\frac{\rho_1}{2}\right)^{k_1}\right). \tag{4.7}$$

In addition, one has that

$$\begin{aligned}
&\int_{\rho_2/2}^{\infty} \exp\left(\nu_2 r_2^{k_2}\right) \exp\left(-\frac{r_2^{k_2}}{|\epsilon t_2|^{k_2}}\Delta_2\right) dr_2 \\
&= \int_{\rho_2/2}^{\infty} \exp\left(\nu_2 r_2^{k_2}\right) \exp\left(-\frac{r_2^{k_2}}{2|\epsilon t_2|^{k_2}}\Delta_2\right) \exp\left(-\frac{r_2^{k_2}}{2|\epsilon t_2|^{k_2}}\Delta_2\right) dr_2 \\
&\leq \exp\left(-\frac{(\rho_2/2)^{k_2}}{2|\epsilon|^{k_2} r_{\mathcal{T}_2}^{k_2}}\Delta_2\right) \left(\int_{\rho_2/2}^{\infty} \exp\left(\nu_2 r_2^{k_2}\right) \exp\left(-\frac{r_2^{k_2}}{2\epsilon_0^{k_2} r_{\mathcal{T}_2}^{k_2}}\Delta_2\right) dr_2\right),
\end{aligned} \tag{4.8}$$

the last integral appearing above being convergent provided that $r_{\mathcal{T}_2} > 0$ is chosen to be small enough. We apply Lemma 4.6 to conclude that

$$\exp\left(-\frac{(\rho_2/2)^{k_2}}{2|\epsilon|^{k_2} r_{\mathcal{T}_2}^{k_2}}\Delta_2\right) \leq C_{k_2} A_{k_2}^N \left(\frac{N}{k_2}\right)^{1/2} \Gamma\left(\frac{N}{k_2}\right) |\epsilon|^N, \tag{4.9}$$

for all positive integer $N \geq 1$, $\epsilon \in \mathcal{E}$, where C_{k_2} is a constant depending on k_2, and

$$A_{k_2} = \left(\frac{2r_{\mathcal{T}_2}^{k_2}}{\Delta_2(\rho_2/2)^{k_2}}\right)^{\frac{1}{k_2}}.$$

The estimates (4.5) are deduced from the inequalities (4.6), (4.7), (4.8), and (4.9). □

Regarding the estimates for J_3, and following an analogous reasoning as before, one arrives at the following result.

Proposition 4.8. *There exist $C_{J_3}, K_{J_3} > 0$ such that*

$$|J_3(t_1, t_2, z, \epsilon)| \leq C_{J_3} K_{J_3}^N \left(\frac{N}{k_1}\right)^{1/2} \Gamma\left(\frac{N}{k_1}\right) |\epsilon|^N, \tag{4.10}$$

for all positive integer N, all $t_1 \in \mathcal{T}_1$, $t_2 \in \mathcal{T}_2$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}$.

4.3. MAIN ASYMPTOTIC RESULT

As before, for $0 < \beta' < \beta$, we write $\mathbb{E}$ for the Banach space of bounded holomorphic functions defined on $\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'}$ equipped with the norm of the supremum, i.e. $\mathbb{E} := \mathcal{O}_b(\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'})$.

Theorem 4.9. *Under the previous assumptions, the solution* (3.22) *of* (2.9) *admits a splitting of the form*

$$u_{d_1,d_2}(t_1, t_2, z, \epsilon) = b(t_1, t_2, z, \epsilon) + u_{d_1,d_2,1}(t_1, t_2, z, \epsilon) + u_{d_1,d_2,2}(t_1, t_2, z, \epsilon),$$

where $b(t_1,t_2,z,\epsilon) \in \mathbb{E}\{\epsilon\}$*, and* $u_{d_1,d_2,j} \in \mathcal{O}_b(\mathcal{T}_1 \times \mathcal{T}_2 \times H_{\beta'} \times \mathcal{E})$*, for* $j=1,2$*. Moreover, there exist two formal power series in* ϵ *with coefficients in* $\mathbb{E}$*, say*

$$\hat{u}_j(t_1,t_2,z,\epsilon) = \sum_{k\geq 0} H_k^j(t_1,t_2,z)\epsilon^k \in \mathbb{E}[[\epsilon]],$$

for $j=1,2$ *which satisfy that* $u_{d_1,d_2,j}$ *admits* $\hat{u}_j$ *as its Gevrey asymptotic expansion of order* $1/k_j$ *with respect to* ϵ *on* $\mathcal{E}$*, which means that for all* $\mathcal{W} \prec \mathcal{E}$*, there exist* $C, A > 0$ *with*

$$\sup_{(t_1,t_2,z)\in\mathcal{T}_1\times\mathcal{T}_2\times H_{\beta'}} \left| u_{d_1,d_2,j}(t_1,t_2,z,\epsilon) - \sum_{p=0}^{N-1} H_p^j(t_1,t_2,z)\epsilon^p \right|$$
$$\leq CA^N\Gamma\left(1+\frac{N}{k_j}\right)|\epsilon|^N, \quad \epsilon \in \mathcal{W},$$

valid for all $N \geq 1$*.*

Proof. In view of the splitting in (4.1), the Gevrey expansions for J_1 determined in Proposition 4.4, together with the Gevrey bounds attained in Proposition 4.7 and Proposition 4.8, we set $b(t_1,t_2,z,\epsilon) = a(t_1,t_2,z,\epsilon)$ obtained in Proposition 4.4,

$$u_{d_1,d_2,1}(t_1,t_2,z,\epsilon) = J_{1,1,0}(t_1,t_2,z,\epsilon) + J_3(t_1,t_2,z,\epsilon),$$
$$u_{d_1,d_2,2}(t_1,t_2,z,\epsilon) = J_{1,2,0}(t_1,t_2,z,\epsilon) + J_2(t_1,t_2,z,\epsilon),$$

and $\hat{u}_j(t_1,t_2,z,\epsilon) = \hat{J}_{1,j}$, for $j=1,2$. The result of Theorem 4.9 is a straight consequence of Proposition 4.4, Proposition 4.7 and Proposition 4.8. □

5. APPENDIX

5.1. AUXILIARY BANACH SPACES OF ANALYTIC FUNCTIONS

In this section, we state the definition of some auxiliary Banach spaces of functions which allow us to provide some important properties of analytic continuation of the solution to the auxiliary problem (3.6) in the Borel–Fourier space.

5.1.1. First auxiliary Banach space

Let $\beta > 0$ and $\mu > 1$. Let us also fix $\rho_1, \rho_2 > 0$. In the whole section, we fix positive integers k_1, k_2.

Definition 5.1. Let us consider the set of continuous maps $(\tau_1,\tau_2,m) \mapsto h(\tau_1,\tau_2,m)$ defined on $D(0,\rho_1)\times D(0,\rho_2)\times\mathbb{R}$, holomorphic with respect to its first two variables on $D(0,\rho_1)\times D(0,\rho_2)$ and such that there exists $C>0$ (which depends on β,μ,ρ_1,ρ_2) with

$$|h(\tau_1,\tau_2,m)| \leq C\frac{1}{(1+|m|)^\mu}e^{-\beta|m|}|\tau_1\tau_2|,$$

for every $(\tau_1, \tau_2, m) \in D(0, \rho_1) \times D(0, \rho_2) \times \mathbb{R}$. Such set is denoted by $B_{(\beta,\mu,\rho_1,\rho_2)}$. Given h as before, we denote the minimum of such constant C by $\|h(\tau_1, \tau_2, m)\|_{(\beta,\mu,\rho_1,\rho_2)}$. The pair $(B_{(\beta,\mu,\rho_1,\rho_2)}, \|\cdot\|_{(\beta,\mu,\rho_1,\rho_2)})$ is a complex Banach space.

We state some properties associated to the previous Banach space, whose proof can straightly be adapted from those in the spaces considered in [15].

The first one is a direct consequence of its definition.

Proposition 5.2. *Let $(\tau_1, \tau_2, m) \mapsto b(\tau_1, \tau_2, m)$ be a continuous function defined on $D(0, \rho_1) \times D(0, \rho_2) \times \mathbb{R}$, holomorphic with respect to its first two variables on $D(0, \rho_1) \times D(0, \rho_2)$. Assume that*

$$C_b := \sup_{(\tau_1,\tau_2,m)\in D(0,\rho_1)\times D(0,\rho_2)\times\mathbb{R}} |b(\tau_1, \tau_2, m)|$$

is finite. Then, for every $h \in B_{(\beta,\mu,\rho_1,\rho_2)}$, the function

$$(\tau_1, \tau_2, m) \mapsto b(\tau_1, \tau_2, m)h(\tau_1, \tau_2, m)$$

belongs to $B_{(\beta,\mu,\rho_1,\rho_2)}$, and it holds that

$$\|b(\tau_1, \tau_2, m)h(\tau_1, \tau_2, m)\|_{(\beta,\mu,\rho_1,\rho_2)} \le C_b \|h(\tau_1, \tau_2, m)\|_{(\beta,\mu,\rho_1,\rho_2)}.$$

Proposition 5.3. *Let $a(\tau_1, \tau_2, m)$ be a continuous function defined on $D(0, \rho_1) \times D(0, \rho_2) \times \mathbb{R}$, holomorphic with respect to its two first variables on $D(0, \rho_1) \times D(0, \rho_2)$. We assume this function satisfies there exists $\gamma_1 \ge 0$ with*

$$|a(\tau_1, \tau_2, m)| \le \frac{C_1}{(1+|m|)^{\gamma_1}}, \quad (\tau_1, \tau_2, m) \in D(0, \rho_1) \times D(0, \rho_2) \times \mathbb{R}.$$

We choose a mapping $m \mapsto h(m, \epsilon)$ such that for every $m \in \mathbb{R}$, the function $D(0, \epsilon_0) \ni \epsilon \mapsto h(m, \epsilon)$ is holomorphic on $D(0, \epsilon_0)$ and there exists $K > 0$ with

$$\sup_{\epsilon\in D(0,\epsilon_0)} \sup_{m\in\mathbb{R}} (1 + |m|)^{\mu} e^{\beta|m|} |h(m, \epsilon)| \le K.$$

We consider a polynomial $P(X) \in \mathbb{C}[X]$. We assume that $\gamma_1 \ge \deg P$, $\mu > \deg P+1$. Let us also fix $\sigma_j > -1$ for $j = 1, \dots, 6$ with $k_1\sigma_1+\sigma_3+\sigma_5+\frac{1}{k_1} \ge 0$, $k_2\sigma_2+\sigma_4+\sigma_6+\frac{1}{k_2} \ge 0$. Then, for every $f \in B_{(\beta,\mu,\rho_1,\rho_2)}$ the function

$$\begin{aligned}\mathcal{B}_1(f) := {} & a(\tau_1, \tau_2, m) \\ & \times \int_{-\infty}^{\infty} h(m - m_1, \epsilon)\tau_1^{\sigma_1 k_1} \tau_2^{\sigma_2 k_2} \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1} - s_1)^{\sigma_3} (\tau_2^{k_2} - s_2)^{\sigma_4} s_1^{\sigma_5} s_2^{\sigma_6} P(im_1) \\ & \times f(s_1^{1/k_1}, s_2^{1/k_2}, m_1) ds_2 ds_1 dm_1\end{aligned}$$

belongs to $B_{(\beta,\mu,\rho_1,\rho_2)}$. In addition to this, there exists $\tilde{C}_1 > 0$ with

$$\|\mathcal{B}_1(f)\|_{(\beta,\mu,\rho_1,\rho_2)} \le K\tilde{C}_1 \|f\|_{(\beta,\mu,\rho_1,\rho_2)}.$$

Proposition 5.4. *Let $a(\tau_1,\tau_2,m)$ be a continuous function defined on $D(0,\rho_1)\times D(0,\rho_2)\times\mathbb{R}$, holomorphic with respect to its two first variables on $D(0,\rho_1)\times D(0,\rho_2)$. We assume this function satisfies there exist $\gamma_1\ge 0$ and $C_1>0$ with*

$$|a(\tau_1,\tau_2,m)|\le \frac{C_1}{(1+|m|)^{\gamma_1}},\quad (\tau_1,\tau_2,m)\in D(0,\rho_1)\times D(0,\rho_2)\times\mathbb{R}.$$

Let $P_1(\epsilon,X),P_2(\epsilon,X)\in\mathcal{O}_b(D(0,\epsilon_0))[X]$ be polynomials with coefficients in the set of holomorphic and bounded functions on $D(0,\epsilon_0)$. We assume that $\gamma_1\ge\max\{\deg(P_1),\deg(P_2)\}$. Let us choose μ such that $\mu>\max\{\deg(P_1),\deg(P_2)\}+1$. For every $f,g\in B_{(\beta,\mu,\rho_1,\rho_2)}$, the function

$$\begin{aligned}\mathcal{B}_2(f,g):=&\,a(\tau_1,\tau_2,m)\\ &\times\tau_1^{k_1}\tau_2^{k_2}\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}\int_{-\infty}^{\infty}P_1(\epsilon,i(m-m_1))f((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m-m_1)\\ &\times P_2(\epsilon,im_1)g(s_1^{1/k_1},s_2^{1/k_2},m_1)\frac{1}{\tau_1^{k_1}-s_1}\frac{1}{s_1}\frac{1}{\tau_2^{k_2}-s_2}\frac{1}{s_2}dm_1ds_2ds_1\end{aligned}$$

belongs to $B_{(\beta,\mu,\rho_1,\rho_2)}$. In addition to this, there exists $\tilde{C}_2>0$ such that

$$\left\|\mathcal{B}_2(f,g)\right\|_{(\beta,\mu,\rho_1,\rho_2)}\le\tilde{C}_2\left\|f\right\|_{(\beta,\mu,\rho_1,\rho_2)}\left\|g\right\|_{(\beta,\mu,\rho_1,\rho_2)}.$$

Remark 5.5. $\tilde{C}_2$ approaches 0 when the quantities ρ_1 or ρ_2 are reduced.

5.1.2. Second auxiliary Banach space

Let $\beta>0$ and $\mu>1$. We fix $\nu_1,\nu_2>0$. Let S_{d_j} be an infinite sector of positive opening, with vertex at the origin and bisecting direction $d_j\in\mathbb{R}$, for $j=1,2$. k_1,k_2 are positive integers.

Definition 5.6. Let us denote by $E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$ the set of all continuous maps $(\tau_1,\tau_2,m)\mapsto h(\tau_1,\tau_2,m)$ on $S_{d_1}\times S_{d_2}\times\mathbb{R}$, which are holomorphic on its two first variables on $S_{d_1}\times S_{d_2}$ and such that there exists $C>0$ (depending on $\beta,\mu,S_{d_1},S_{d_2}$) such that

$$|h(\tau_1,\tau_2,m)|\le C\frac{1}{(1+|m|)^{\mu}}e^{-\beta|m|}\frac{|\tau_1|}{1+|\tau_1|^{2k_1}}\frac{|\tau_2|}{1+|\tau_2|^{2k_2}}\exp\left(\nu_1|\tau_1|^{k_1}+\nu_2|\tau_2|^{k_2}\right),$$

for every $(\tau_1,\tau_2,m)\in S_{d_1}\times S_{d_2}\times\mathbb{R}$. Given such h, the minimum of the constant C above is denoted by $\|h(\tau_1,\tau_2,m)\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$. The pair $(E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})},\|\cdot\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})})$ turns out to be a complex Banach space.

As in the previous section, one can state some properties associated to the application of certain operators on the previous Banach space. We omit their proofs, directly adapted from those in [15].

Proposition 5.7. *Let $(\tau_1,\tau_2,m) \mapsto b(\tau_1,\tau_2,m)$ be a continuous function defined on $S_{d_1} \times S_{d_2} \times \mathbb{R}$, holomorphic with respect to its first two variables on $S_{d_1} \times S_{d_2}$. Assume that*

$$C_b := \sup_{(\tau_1,\tau_2,m)\in S_{d_1}\times S_{d_2}\times\mathbb{R}} |b(\tau_1,\tau_2,m)|$$

is finite. Then, for every $h \in E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$, the function

$$(\tau_1,\tau_2,m) \mapsto b(\tau_1,\tau_2,m)h(\tau_1,\tau_2,m)$$

belongs to $E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$, and it holds that

$$\|b(\tau_1,\tau_2,m)h(\tau_1,\tau_2,m)\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})} \le C_b \|h(\tau_1,\tau_2,m)\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}.$$

Proposition 5.8. *Let $a(\tau_1,\tau_2,m)$ be a continuous function defined on $S_{d_1} \times S_{d_2} \times \mathbb{R}$, holomorphic with respect to its two first variables on $S_{d_1} \times S_{d_2}$. Assume there exist $\gamma_1,\delta_1,\delta_2 \ge 0$ and $C_1 > 0$ such that*

$$|a(\tau_1,\tau_2,m)| \le \frac{C_1}{(1+|m|)^{\gamma_1}(1+|\tau_1|^{\delta_1 k_1}|\tau_2|^{\delta_2 k_2})}, \quad (\tau_1,\tau_2,m) \in S_{d_1} \times S_{d_2} \times \mathbb{R}.$$

Let $m \mapsto h(m,\epsilon)$ be a function such that for every $m \in \mathbb{R}$, the function $D(0,\epsilon_0) \ni \epsilon \mapsto h(m,\epsilon)$ is holomorphic on $D(0,\epsilon_0)$ and there exists $K > 0$ with

$$\sup_{\epsilon\in D(0,\epsilon_0)} \sup_{m\in\mathbb{R}} (1+|m|)^{\mu} e^{\beta|m|} |h(m,\epsilon)| \le K.$$

We consider a polynomial $P(X) \in \mathbb{C}[X]$. We assume that $\gamma_1 \ge \deg P$, $\mu > \deg P + 1$. Let us also fix $\sigma_j > -1$ for $j = 1,\dots,6$. We assume that

$$k_1(\sigma_1+\sigma_3+\sigma_5+1) = k_2(\sigma_2+\sigma_4+\sigma_6+1), \quad \delta_1 k_1 = \delta_2 k_2, \quad \sigma_1+\sigma_3+\sigma_5+1 \le \delta_1. \tag{5.1}$$

Then, for every $f \in E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$ the function

$$\begin{aligned}\mathcal{B}_1(f) &:= a(\tau_1,\tau_2,m) \\ &\times \int_{-\infty}^{\infty} h(m-m_1,\epsilon)\tau_1^{\sigma_1 k_1}\tau_2^{\sigma_2 k_2} \int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{\sigma_3}(\tau_2^{k_2}-s_2)^{\sigma_4} s_1^{\sigma_5} s_2^{\sigma_6} P(im_1) \\ &\times f(s_1^{1/k_1}, s_2^{1/k_2}, m_1)\, ds_2 ds_1 dm_1\end{aligned}$$

belongs to $E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$. Moreover, there exists $\tilde{C}_1 > 0$ with

$$\|\mathcal{B}_1(f)\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})} \le K\tilde{C}_1 \|f\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}.$$

The proof of the following result follows analogous lines as that of Proposition 5.4.

Proposition 5.9. *Let $a(\tau_1,\tau_2,m)$ be a continuous function defined on $S_{d_1}\times S_{d_2}\times\mathbb{R}$, holomorphic with respect to its two first variables on $S_{d_1}\times S_{d_2}$. We assume such function satisfies there exist $\gamma_1,\delta_1,\delta_2\ge 0$ and $C_1>0$ with*

$$|a(\tau_1,\tau_2,m)|\le\frac{C_1}{(1+|m|)^{\gamma_1}(1+|\tau_1|^{\delta_1k_1}|\tau_2|^{\delta_2k_2})},\quad(\tau_1,\tau_2,m)\in S_{d_1}\times S_{d_2}\times\mathbb{R}.$$

We also assume that $\delta_1k_1=\delta_2k_2\ge 1$. Let $P_1(\epsilon,X),P_2(\epsilon,X)\in\mathcal{O}_b(D(0,\epsilon_0))[X]$ be polynomials with coefficients being holomorphic and bounded functions on $D(0,\epsilon_0)$. We assume that $\gamma_1\ge\max\{\deg(P_1),\deg(P_2)\}$, and choose μ such that $\mu>\max\{\deg(P_1),\deg(P_2)\}+1$. For every $f,g\in E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$, the function

$$\begin{aligned}\mathcal{B}_2(f,g):=&\,a(\tau_1,\tau_2,m)\tau_1^{k_1}\tau_2^{k_2}\\&\times\int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}\int_{-\infty}^{\infty}P_1(\epsilon,i(m-m_1))f((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m-m_1)\\&\times P_2(\epsilon,im_1)g(s_1^{1/k_1},s_2^{1/k_2},m_1)\frac{1}{\tau_1^{k_1}-s_1}\frac{1}{s_1}\frac{1}{\tau_2^{k_2}-s_2}\frac{1}{s_2}dm_1ds_2ds_1\end{aligned}$$

belongs to $E_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}$. In addition to this, there exists $\tilde{C}_2>0$ such that

$$\|\mathcal{B}_2(f,g)\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}\le\tilde{C}_2\,\|f\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}\,\|g\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}\,.$$

Remark 5.10. The constant $\tilde{C}_2$ approaches to 0 provided that the quantities $C_{P_1},C_{P_2}>0$ are small.

5.1.3. Third auxiliary Banach space

Let $\beta>0$ and $\mu>1$. We fix $\rho_1,\rho_2>0$, and $\nu_1,\nu_2>0$. Let S_{d_j} be an infinite sector of some positive opening, with vertex at the origin, and bisecting direction $d_j\in\mathbb{R}$, for $j=1,2$. As in the previous sections, we fix positive integers k_1,k_2.

Definition 5.11. Let us denote by $F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$ the set of all continuous maps $(\tau_1,\tau_2,m)\mapsto h(\tau_1,\tau_2,m)$ on $(S_{d_1}\cap D(0,\rho_1))\times(S_{d_2}\cap D(0,\rho_2))\times\mathbb{R}$, which are holomorphic on its two first variables on $(S_{d_1}\cap D(0,\rho_1))\times(S_{d_2}\cap D(0,\rho_2))$ and such that there exists $C>0$ (depending on $\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2}$) such that

$$|h(\tau_1,\tau_2,m)|\le C\frac{1}{(1+|m|)^{\mu}}e^{-\beta|m|}\,|\tau_1\tau_2|\,,$$

for every $(\tau_1,\tau_2,m)\in(S_{d_1}\cap D(0,\rho_1))\times(S_{d_2}\cap D(0,\rho_2))\times\mathbb{R}$. For such h, the minimum of such constant C is denoted by $\|h(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$. The pair $(F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})},\|\cdot\|_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})})$ is a complex Banach space.

Analogous results as those in the two previous sections regarding the action of certain operators acting on functions belonging to this Banach space can be stated. We omit their proof which follow analogous arguments as before.

Proposition 5.12. *Let $(\tau_1,\tau_2,m) \mapsto b(\tau_1,\tau_2,m)$ be a continuous function defined on $(S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2)) \times \mathbb{R}$, holomorphic with respect to its first two variables on $(S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2))$. Assume that*

$$C_b := \sup_{(\tau_1,\tau_2,m)\in(S_{d_1}\cap D(0,\rho_1))\times(S_{d_2}\cap D(0,\rho_2))\times\mathbb{R}} |b(\tau_1,\tau_2,m)|$$

is finite. Then, for every $h \in F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$, the function

$$(\tau_1,\tau_2,m) \mapsto b(\tau_1,\tau_2,m)h(\tau_1,\tau_2,m)$$

belongs to $F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$, and it holds that

$$\|b(\tau_1,\tau_2,m)h(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})} \leq C_b \|h(\tau_1,\tau_2,m)\|_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}.$$

Proposition 5.13. *Let $a(\tau_1,\tau_2,m)$ be a continuous function defined on $(S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2)) \times \mathbb{R}$, holomorphic with respect to its two first variables on $(S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2))$. Assume there exist $\gamma_1 \geq 0$ and $C_1 > 0$ such that*

$$|a(\tau_1,\tau_2,m)| \leq \frac{C_1}{(1+|m|)^{\gamma_1}}, \quad (\tau_1,\tau_2,m) \in (S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2)) \times \mathbb{R}.$$

Let $m \mapsto h(m,\epsilon)$ be a function such that for every $m \in \mathbb{R}$, the function $D(0,\epsilon_0) \ni \epsilon \mapsto h(m,\epsilon)$ is holomorphic on $D(0,\epsilon_0)$ and there exists $K > 0$ with

$$\sup_{\epsilon\in D(0,\epsilon_0)} \sup_{m\in\mathbb{R}} (1+|m|)^{\mu} e^{\beta|m|} |h(m,\epsilon)| \leq K.$$

We consider a polynomial $P(X) \in \mathbb{C}[X]$. We assume that $\gamma_1 \geq \deg P$, $\mu > \deg P + 1$. Let us also fix $\sigma_j > -1$ for $j = 1,\dots,6$, such that $k_1\sigma_1 + \sigma_3 + \sigma_5 + \frac{1}{k_1} \geq 0$ and $k_2\sigma_2 + \sigma_4 + \sigma_6 + \frac{1}{k_2} \geq 0$. Then, for every $f \in F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$ the function

$$\begin{aligned}\mathcal{B}_1(f) &:= a(\tau_1,\tau_2,m) \\ &\times \int_{-\infty}^{\infty} h(m-m_1,\epsilon)\tau_1^{\sigma_1 k_1}\tau_2^{\sigma_2 k_2} \int_0^{\tau_1^{k_1}} \int_0^{\tau_2^{k_2}} (\tau_1^{k_1}-s_1)^{\sigma_3}(\tau_2^{k_2}-s_2)^{\sigma_4} s_1^{\sigma_5} s_2^{\sigma_6} P(im_1) \\ &\times f(s_1^{1/k_1}, s_2^{1/k_2}, m_1) ds_2 ds_1 dm_1\end{aligned}$$

belongs to $F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$. Moreover, there exists $\tilde{C}_1 > 0$ with

$$\|\mathcal{B}_1(f)\|_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})} \leq K\tilde{C}_1 \|f\|_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}.$$

Proposition 5.14. *Let $a(\tau_1,\tau_2,m)$ be a continuous function defined on $(S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2)) \times \mathbb{R}$, holomorphic with respect to its two first variables on $(S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2))$. We assume such function satisfies there exists $\gamma_1 \geq 0$ and $C_1 > 0$ with*

$$|a(\tau_1,\tau_2,m)| \leq \frac{C_1}{(1+|m|)^{\gamma_1}}, \quad (\tau_1,\tau_2,m) \in (S_{d_1} \cap D(0,\rho_1)) \times (S_{d_2} \cap D(0,\rho_2)) \times \mathbb{R}.$$

Let $P_1(\epsilon,X), P_2(\epsilon,X) \in \mathcal{O}_b(D(0,\epsilon_0))[X]$ be polynomials with coefficients being holomorphic and bounded functions on $D(0,\epsilon_0)$. We assume that $\gamma_1 \geq \max\{\deg(P_1),\deg(P_2)\}$, and choose μ such that $\mu > \max\{\deg(P_1),\deg(P_2)\}+1$. For every $f,g \in F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$, the function

$$\begin{aligned}\mathcal{B}_2(f,g) := {} & a(\tau_1,\tau_2,m)\tau_1^{k_1}\tau_2^{k_2} \\ & \times \int_0^{\tau_1^{k_1}}\int_0^{\tau_2^{k_2}}\int_{-\infty}^{\infty} P_1(\epsilon,i(m-m_1))f((\tau_1^{k_1}-s_1)^{1/k_1},(\tau_2^{k_2}-s_2)^{1/k_2},m-m_1) \\ & \qquad \times P_2(\epsilon,im_1)g(s_1^{1/k_1},s_2^{1/k_2},m_1)\frac{1}{\tau_1^{k_1}-s_1}\frac{1}{s_1}\frac{1}{\tau_2^{k_2}-s_2}\frac{1}{s_2} \\ & \qquad \times dm_1 ds_2 ds_1\end{aligned}$$

belongs to $F_{(\beta,\mu,\rho_1,\rho_2,S_{d_1},S_{d_2})}$. In addition to this, there exists $\tilde{C}_2 > 0$ such that

$$\|\mathcal{B}_2(f,g)\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})} \leq \tilde{C}_2 \|f\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})} \|g\|_{(\beta,\mu,\nu_1,\nu_2,S_{d_1},S_{d_2})}.$$

Remark 5.15. The constant $\tilde{C}_2$ approaches 0 when the quantities ρ_1 or ρ_2 are reduced.

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