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COMPARISON THEOREMS FOR OSCILLATION AND NON-OSCILLATION OF PERTURBED EULER TYPE EQUATIONS

Petr Hasil, Jiřina Šišoláková, and Michal Veselý

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Abstract. The aim of this paper is to present two comparison theorems. These results enable to describe the oscillation behavior of second order Euler type half-linear differential equations with perturbations in both terms using previously obtained oscillation and non-oscillation criteria. We point out that the comparison theorems are easy to use. This fact is also illustrated by a simple example. In addition, the number of perturbations is arbitrary and the last perturbations can be given by very general continuous functions. Note that the presented results are new even in the case of linear equations.

Keywords: comparison theorem, oscillation theory, non-oscillation, half-linear equation, Riccati equation, Prüfer angle.

Mathematics Subject Classification: 34C10, 34C15.

1. INTRODUCTION

In this paper, we present two comparison theorems about oscillatory and non-oscillatory solutions of perturbed Euler type half-linear differential equations with bounded coefficients. We recall that a solution of the considered equations is said to be non-oscillatory if there exists the greatest zero of the solution. In the other case, we say that the solution is oscillatory. At the same time, we recall that the Sturmian theory holds for general half-linear equations. Especially, the Sturm separation theorem (see, e.g., [11, Theorem 1.2.3]) gives the following implication. If a non-trivial solution of a half-linear equation is oscillatory, then all solutions of the equation are oscillatory. Next, we use the standard terminology concerning oscillatory and non-oscillatory equations. We say that a equation is oscillatory if all its solutions are oscillatory; and it is called non-oscillatory if all its non-trivial solutions are non-oscillatory.

In this paragraph, we collect a literature overview. The basic theory about the oscillation and non-oscillation of half-linear differential equations is presented, e.g., in [1, 11]. Concerning non-perturbed Euler type linear and half-linear differential equations, we refer to [13, 14, 20, 22, 26, 34, 39, 51]. For the corresponding discrete equations, see [19, 25, 33, 40, 41, 60] (and also [6, 21, 24, 45, 53, 61]) in the case of difference equations and to [28, 29, 31, 32, 49, 59] (and also [16, 36, 37, 50]) in the case of dynamic equations on time scales. The oscillation and non-oscillation of the studied perturbed Euler type half-linear differential equations are studied in many papers. We highlight at least [2–4, 7, 8, 12, 15, 17, 18, 23, 27, 30, 43, 44, 46–48, 58] (see also [5, 9, 10, 38, 42, 52]). In the discrete case, we point out [35, 62]. To complete the literature overview, we add the series of papers [54–57], where more general Euler type differential equations are analyzed (the considered function Φ in half-linear equations is replaced by more general functions in the second term).

The paper is organized as follows. In the next section, we mention the used notation together with basic used definitions, identities, and inequalities. In Section 3, we prepare tools for the announced comparison theorems (i.e., the half-linear Riccati equation and primarily the modified half-linear Prüfer transformation). In Section 4, we prove lemmas which we need to obtain our results and which form our technique. Finally, in the last section, we formulate and prove the comparison theorems (in fact, one theorem with two parts) with corollaries and an illustrative example.

2. PRELIMINARIES

In this section, we collect the used notation. Let p, q denote the conjugated real numbers, i.e., let $p, q > 1$ satisfy $p + q = pq$. For an arbitrarily given $p > 1$, we define $\Phi(x) := |x|^{p-1} \operatorname{sgn} x$. We consider the so-called iterated logarithms given by

$$\log_{(1)} t = \log t, \quad \log_{(k+1)} t = \log_{(k)}(\log t), \quad \operatorname{Log}_{(k)} t = \prod_{i=1}^{k} \log_{(i)} t \tag{2.1}$$

for all $k \in \mathbb{N}$ and all sufficiently large $t \in \mathbb{R}$, where log denotes the natural logarithm. Let $f : [1, \infty) \to [1, \infty)$ be a continuously differentiable function satisfying

$$\limsup_{t\to\infty} \frac{t\,|f'(t)|}{f(t)} < \infty, \quad \lim_{t\to\infty} \frac{f(t) \operatorname{Log}^2_{(n)} t}{t} = 0. \tag{2.2}$$

We will use the so-called half-linear sine and cosine functions. Let

$$\pi_p := \frac{2\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{q}\right)}{p\Gamma\left(\frac{1}{p} + \frac{1}{q}\right)} = \frac{2\pi}{p \sin\frac{\pi}{p}},$$

where Γ is the Euler gamma function. Then, the half-linear sine function $\sin_p$ is defined as the odd $2\pi_p$-periodic extension of the solution of the initial problem

$$[\Phi\left(x'(t)\right)]' + (p-1)\Phi(x(t)) = 0, \quad x(0) = 0, \quad x'(0) = 1 \tag{2.3}$$

and the half-linear cosine function $\cos_p$ is defined as the derivative of $\sin_p$. These functions satisfy the famous half-linear Pythagorean identity

$$|\sin_p y|^p + |\cos_p y|^p = 1, \quad y \in \mathbb{R}. \tag{2.4}$$

From (2.4), one can see

$$|\sin_p y|^p \le 1, \quad |\Phi(\cos_p y)\sin_p y| \le 1, \quad |\cos_p y|^p \le 1, \quad y \in \mathbb{R}. \tag{2.5}$$

The functions $\sin_p$ and $\Phi(\cos_p)$ are continuously differentiable and periodic (see, e.g., [10] or [11]). Thus, there exists $A > 0$ such that

$$\begin{aligned} \big|\, |\sin_p y|^p - |\sin_p z|^p \,\big| &\le A|y-z|, \\ |\Phi(\cos_p y)\sin_p y - \Phi(\cos_p z)\sin_p z| &\le A|y-z|, \\ \big|\, |\cos_p y|^p - |\cos_p z|^p \,\big| &\le A|y-z| \end{aligned} \tag{2.6}$$

for all $y, z \in \mathbb{R}$.

3. MODIFIED HALF-LINEAR PRÜFER ANGLE

In this section, we briefly derive the used modification of the Prüfer angle for the analyzed Euler type half-linear equations. Hence, let us consider equations in the form

$$\left[r^{-\frac{p}{q}}(t)\Phi\left(x'(t)\right)\right]' + \frac{s(t)}{t^p}\,\Phi(x(t)) = 0 \tag{3.1}$$

with continuous coefficients $r : [a, \infty) \to (0, \infty)$ and $s : [a, \infty) \to \mathbb{R}$. Putting

$$w(t) = r^{-\frac{p}{q}}(t)\Phi\left(\frac{x'(t)}{x(t)}\right),$$

where x is a non-trivial solution of Eq. (3.1), one can obtain the so-called Riccati equation

$$w'(t) + \frac{s(t)}{t^p} + (p-1)r(t)|w(t)|^q = 0. \tag{3.2}$$

Note that a solution w of Eq. (3.2) exists whenever $x(t) \neq 0$. Therefore, Eq. (3.1) is non-oscillatory if there exists a solution w of Eq. (3.2) on $[b, \infty)$ for some $b > a$. Using $v(t) = t^{p-1}w(t)$, one can transform Eq. (3.2) to the weighted Riccati equation

$$\begin{aligned} v'(t) &= \left[t^{p-1}w(t)\right]' \\ &= (p-1)t^{p-2}w(t) + t^{p-1}w'(t) \\ &= \frac{p-1}{t}\left[v(t) - \frac{s(t)}{p-1} - r(t)|v(t)|^q\right]. \end{aligned} \tag{3.3}$$

Finally, we apply the announced modified half-linear Prüfer transformation

$$x(t) = \rho(t)\sin_p \varphi(t), \quad x'(t) = \frac{r(t)\rho(t)}{t}\cos_p \varphi(t).$$

One can compute that (see also (2.3))

$$
\begin{aligned}
v(t) &= \Phi\left(\frac{\cos_p(\varphi(t))}{\sin_p(\varphi(t))}\right), \\
v'(t) &= (1-p)\left[1+\left|\frac{\cos_p(\varphi(t))}{\sin_p(\varphi(t))}\right|^p\right]\varphi'(t).
\end{aligned}
\tag{3.4}
$$

From (3.3) and (3.4), we obtain

$$
\begin{aligned}
&(1-p)\left[1+\left|\frac{\cos_p(\varphi(t))}{\sin_p(\varphi(t))}\right|^p\right]\varphi'(t) \\
&= \frac{p-1}{t}\left[\Phi\left(\frac{\cos_p(\varphi(t))}{\sin_p(\varphi(t))}\right) - \frac{s(t)}{p-1} - r(t)\left|\frac{\cos_p(\varphi(t))}{\sin_p(\varphi(t))}\right|^p\right].
\end{aligned}
$$

Thus (see (2.4)), the equation for the considered modified Prüfer angle has the form

$$
\varphi'(t) = \frac{1}{t}\left[r(t)|\cos_p(\varphi(t))|^p - \Phi(\cos_p(\varphi(t)))\sin_p(\varphi(t)) + s(t)\frac{|\sin_p(\varphi(t))|^p}{p-1}\right]. \tag{3.5}
$$

4. AUXILIARY RESULTS

Let $r_0 > 0$, $r_1, \ldots, r_n, s_0, s_1, \ldots, s_n$ be continuous bounded functions defined on a neighborhood of ∞. Let $m > 0$ be such that

$$
|r_i(t)| < m, \quad |s_i(t)| < m, \quad i \in \{0, 1, \ldots, n\}, \tag{4.1}
$$

for all considered large t. We consider the equation

$$
\left[\left(r_0(t) + \sum_{i=1}^{n} \frac{r_i(t)}{\operatorname{Log}^2_{(i)} t}\right)^{-\frac{p}{q}} \Phi(x'(t))\right]' + \left[s_0(t) + \sum_{i=1}^{n} \frac{s_i(t)}{\operatorname{Log}^2_{(i)} t}\right]\frac{\Phi(x(t))}{t^p} = 0. \tag{4.2}
$$

For Eq. (4.2), the corresponding equation for the modified Prüfer angle (3.5) takes the form

$$
\begin{aligned}
\varphi'(t) = \frac{1}{t}\Bigg[&\left(r_0(t) + \frac{r_1(t)}{\operatorname{Log}^2_{(1)} t} + \ldots + \frac{r_{n-1}(t)}{\operatorname{Log}^2_{(n-1)} t} + \frac{r_n(t)}{\operatorname{Log}^2_{(n)} t}\right)|\cos_p(\varphi(t))|^p \\
&- \Phi(\cos_p(\varphi(t)))\sin_p(\varphi(t)) \\
&+ \left(s_0(t) + \frac{s_1(t)}{\operatorname{Log}^2_{(1)} t} + \ldots + \frac{s_{n-1}(t)}{\operatorname{Log}^2_{(n-1)} t} + \frac{s_n(t)}{\operatorname{Log}^2_{(n)} t}\right)|\sin_p(\varphi(t))|^p \frac{1}{p-1}\Bigg].
\end{aligned}
\tag{4.3}
$$

We recall the Sturm–Picone half-linear comparison theorem and a well-known theorem which follow.

Theorem 4.1. *Let $\hat{r}, \tilde{r}, \hat{s}, \tilde{s}$ be continuous functions satisfying $\hat{r}(t) \geq \tilde{r}(t) > 0$ and $\tilde{s}(t) \geq \hat{s}(t)$ for all sufficiently large t. Let us consider the equations*

$$[\hat{r}(t)\Phi(x'(t))]' + \hat{s}(t)\Phi(x(t)) = 0, \tag{4.4}$$

$$[\tilde{r}(t)\Phi(x'(t))]' + \tilde{s}(t)\Phi(x(t)) = 0. \tag{4.5}$$

(i) *If Eq.* (4.5) *is non-oscillatory, then Eq.* (4.4) *is non-oscillatory.*
(ii) *If Eq.* (4.4) *is oscillatory, then Eq.* (4.5) *is oscillatory.*

Proof. See, e.g., [11, Theorem 1.2.4]. □

Theorem 4.2. *Any solution of Eq.* (4.3) *on a neighborhood of ∞ is bounded if and only if Eq.* (4.2) *is non-oscillatory.*

Proof. See, e.g., [12, Corollary 4.1]. □

Remark 4.3. Taking into account the right-hand side of Eq. (4.3) for $\sin_p(\varphi(t)) = 0$, one can see that the set of all values $\varphi(t)$ is unbounded if and only if $\lim_{t\to\infty} \varphi(t) = \infty$.

We need the following lemmas.

Lemma 4.4. *Let $f : [1, \infty) \to [1, \infty)$ be a continuously differentiable function satisfying* (2.2)*. If φ is a solution of Eq.* (4.3) *on $[T, \infty)$, where $\operatorname{Log}_{(n)} T > 1$, and*

$$\underline{\varphi}_f(t) := \frac{1}{f(t)} \int_t^{t+f(t)} \varphi(\tau)\,\mathrm{d}\tau, \quad t \in [T, \infty), \tag{4.6}$$

then

$$\sup_{\sigma\in[0,f(t)]} \left|\varphi(t+\sigma) - \underline{\varphi}_f(t)\right| \leq \frac{F(t)}{\operatorname{Log}^2_{(n)} t}, \quad t \in [T, \infty), \tag{4.7}$$

where $F : [T, \infty) \to (0, \infty)$ is a continuous function with the property that $\lim_{t\to\infty} F(t) = 0$.

Proof. Considering (4.6), for any $t \in [T, \infty)$, there exists $\varrho(t) \in [t, t + f(t)]$ with the property that $\underline{\varphi}_f(t) = \varphi(\varrho(t))$. For all $t \geq T$ and $\sigma \in [0, f(t)]$, we obtain (see (2.5) and (4.1))

$$
\begin{aligned}
&\left|\varphi(t+\sigma) - \underline{\varphi}_f(t)\right| = |\varphi(t+\sigma) - \varphi(\varrho(t))| \leq \int\limits_t^{t+f(t)} |\varphi'(\tau)| \,\mathrm{d}\tau \\
&\leq \frac{1}{t} \int\limits_t^{t+f(t)} \left(r_0(\tau) + \frac{|r_1(\tau)|}{\mathrm{Log}^2_{(1)} \tau} + \ldots + \frac{|r_{n-1}(\tau)|}{\mathrm{Log}^2_{(n-1)} \tau} + \frac{|r_n(\tau)|}{\mathrm{Log}^2_{(n)} \tau} \right) |\cos_p(\varphi(\tau))|^p \,\mathrm{d}\tau \\
&\quad + \frac{1}{t} \int\limits_t^{t+f(t)} |\Phi(\cos_p(\varphi(\tau))) \sin_p(\varphi(\tau))| \,\mathrm{d}\tau \\
&\quad + \frac{1}{t} \int\limits_t^{t+f(t)} \frac{|\sin_p(\varphi(\tau))|^p}{p-1} \left(|s_0(\tau)| + \frac{|s_1(\tau)|}{\mathrm{Log}^2_{(1)} \tau} + \ldots + \frac{|s_{n-1}(\tau)|}{\mathrm{Log}^2_{(n-1)} \tau} + \frac{|s_n(\tau)|}{\mathrm{Log}^2_{(n)} \tau} \right) \mathrm{d}\tau \\
&\leq \frac{1}{t} \left[f(t)m + \frac{f(t)m}{\mathrm{Log}^2_{(1)} t} + \ldots + \frac{f(t)m}{\mathrm{Log}^2_{(n-1)} t} + \frac{f(t)m}{\mathrm{Log}^2_{(n)} t} \right] + \frac{f(t)}{t} \\
&\quad + \frac{1}{(p-1)t} \left[f(t)m + \frac{f(t)m}{\mathrm{Log}^2_{(1)} t} + \ldots + \frac{f(t)m}{\mathrm{Log}^2_{(n-1)} t} + \frac{f(t)m}{\mathrm{Log}^2_{(n)} t} \right] \\
&\leq \frac{f(t)}{t} \left[(n+1)m + 1 + \frac{(n+1)m}{p-1} \right],
\end{aligned}
$$

i.e., it suffices to put

$$
F(t) := \frac{f(t) \,\mathrm{Log}^2_{(n)} t}{t} \left[(n+1)m + 1 + \frac{(n+1)m}{p-1} \right]
$$

for large t and consider (2.2). □

Remark 4.5. From (4.7), we immediately obtain

$$
\lim_{t \to \infty} \left|\varphi(t) - \underline{\varphi}_f(t)\right| = 0, \tag{4.8}
$$

where φ is a solution of Eq. (4.3) on $[T, \infty)$ and $\underline{\varphi}_f$ is defined in (4.6).

Lemma 4.6. *Let $f : [1,\infty) \to [1,\infty)$ be a continuously differentiable function satisfying* (2.2). *If φ is a solution of Eq.* (4.3) *on $[T,\infty)$, where $\operatorname{Log}_{(n)} T > 1$, and $\underline{\varphi}_f$ is defined in* (4.6)*, then*

$$\begin{aligned}
&\Bigg| \underline{\varphi}_f'(t) - \frac{1}{t}\Bigg[\frac{1}{f(t)}\int_t^{t+f(t)}\Bigg(r_0(\tau) + \frac{r_1(\tau)}{\operatorname{Log}^2_{(1)}\tau} + \ldots + \frac{r_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)}\tau} \\
&\qquad\qquad + \frac{r_n(\tau)}{\operatorname{Log}^2_{(n)} t}\Bigg)\,\mathrm{d}\tau \left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p \\
&- \Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right)\sin_p\left(\underline{\varphi}_f(t)\right) \\
&+ \frac{1}{f(t)}\int_t^{t+f(t)}\Bigg(s_0(\tau) + \frac{s_1(\tau)}{\operatorname{Log}^2_{(1)}\tau} + \ldots + \frac{s_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)}\tau} + \frac{s_n(\tau)}{\operatorname{Log}^2_{(n)} t}\Bigg)\,\mathrm{d}\tau \left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p \frac{1}{p-1}\Bigg]\Bigg| \\
&\le \frac{G(t)}{t\operatorname{Log}^2_{(n)} t}
\end{aligned} \tag{4.9}$$

for all $t \in (T,\infty)$, where $G : [T,\infty) \to (0,\infty)$ is a continuous function with the property that $\lim_{t\to\infty} G(t) = 0$.

Proof. For any $t > T$, we have (see Eq. (4.3) and also (4.6))

$$\begin{aligned}
\underline{\varphi}_f'(t) &= \left(\frac{1}{f(t)}\int_t^{t+f(t)}\varphi(\tau)\,\mathrm{d}\tau\right)' \\
&= -\frac{f'(t)}{f^2(t)}\int_t^{t+f(t)}\varphi(\tau)\,\mathrm{d}\tau + \frac{1}{f(t)}\left(\varphi\left(t+f(t)\right)\left[1+f'(t)\right] - \varphi(t)\right) \\
&= -\frac{f'(t)}{f^2(t)}\int_t^{t+f(t)}\varphi(\tau)\,\mathrm{d}\tau + \frac{f'(t)}{f(t)}\varphi\left(t+f(t)\right) + \frac{1}{f(t)}\int_t^{t+f(t)}\varphi'(\tau)\,\mathrm{d}\tau \\
&= \frac{f'(t)}{f(t)}\left(\varphi\left(t+f(t)\right) - \frac{1}{f(t)}\int_t^{t+f(t)}\varphi(\tau)\,\mathrm{d}\tau\right) \\
&\quad + \frac{1}{f(t)}\int_t^{t+f(t)}\frac{1}{\tau}\Bigg[\Bigg(r_0(\tau) + \frac{r_1(\tau)}{\operatorname{Log}^2_{(1)}\tau} + \ldots + \frac{r_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)}\tau} + \frac{r_n(\tau)}{\operatorname{Log}^2_{(n)}\tau}\Bigg)\left|\cos_p(\varphi(\tau))\right|^p \\
&\quad - \Phi(\cos_p(\varphi(\tau)))\sin_p(\varphi(\tau)) \\
&\quad + \Bigg(s_0(\tau) + \frac{s_1(\tau)}{\operatorname{Log}^2_{(1)}\tau} + \ldots + \frac{s_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)}\tau} + \frac{s_n(\tau)}{\operatorname{Log}^2_{(n)}\tau}\Bigg)\frac{\left|\sin_p(\varphi(\tau))\right|^p}{p-1}\Bigg]\,\mathrm{d}\tau.
\end{aligned} \tag{4.10}$$

From Lemma 4.4 (see (4.7) together with (4.6)), we obtain

$$\lim_{t\to\infty} \operatorname{Log}^2_{(n)} t \left| \varphi\left(t + f(t)\right) - \frac{1}{f(t)} \int\limits_t^{t+f(t)} \varphi(\tau)\,\mathrm{d}\tau \right| = 0.$$

Thus (see (2.2)), there exists a continuous function $H_1 : [T, \infty) \to (0, \infty)$ such that $\lim_{t\to\infty} H_1(t) = 0$ and

$$\left| \frac{f'(t)}{f(t)} \left(\varphi\left(t + f(t)\right) - \frac{1}{f(t)} \int\limits_t^{t+f(t)} \varphi(\tau)\,\mathrm{d}\tau \right) \right| \le \frac{H_1(t)}{t \operatorname{Log}^2_{(n)} t}, \quad t \in [T, \infty). \tag{4.11}$$

Evidently (see (2.5) and (4.1)),

$$\begin{aligned}
&\left| \frac{1}{f(t)} \int\limits_t^{t+f(t)} \frac{1}{\tau} \left[\left(r_0(\tau) + \frac{r_1(\tau)}{\operatorname{Log}^2_{(1)} \tau} + \ldots + \frac{r_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)} \tau} + \frac{r_n(\tau)}{\operatorname{Log}^2_{(n)} \tau} \right) |\cos_p(\varphi(\tau))|^p \right.\right. \\
&\quad - \Phi(\cos_p(\varphi(\tau))) \sin_p(\varphi(\tau)) \\
&\quad \left. + \left(s_0(\tau) + \frac{s_1(\tau)}{\operatorname{Log}^2_{(1)} \tau} + \ldots + \frac{s_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)} \tau} + \frac{s_n(\tau)}{\operatorname{Log}^2_{(n)} \tau} \right) \frac{|\sin_p(\varphi(\tau))|^p}{p-1} \right] \mathrm{d}\tau \\
&\quad - \frac{1}{f(t)} \int\limits_t^{t+f(t)} \frac{1}{t} \left[\left(r_0(\tau) + \frac{r_1(\tau)}{\operatorname{Log}^2_{(1)} \tau} + \ldots + \frac{r_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)} \tau} + \frac{r_n(\tau)}{\operatorname{Log}^2_{(n)} \tau} \right) |\cos_p(\varphi(\tau))|^p \right. \\
&\quad - \Phi(\cos_p(\varphi(\tau))) \sin_p(\varphi(\tau)) \\
&\quad \left.\left. + \left(s_0(\tau) + \frac{s_1(\tau)}{\operatorname{Log}^2_{(1)} \tau} + \ldots + \frac{s_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)} \tau} + \frac{s_n(\tau)}{\operatorname{Log}^2_{(n)} \tau} \right) \frac{|\sin_p(\varphi(\tau))|^p}{p-1} \right] \mathrm{d}\tau \right| \\
&\le \frac{1}{f(t)} \int\limits_t^{t+f(t)} \left(\frac{1}{t} - \frac{1}{t+f(t)} \right) \left[\left(r_0(\tau) + \frac{|r_1(\tau)|}{\operatorname{Log}^2_{(1)} T} + \ldots + \frac{|r_n(\tau)|}{\operatorname{Log}^2_{(n)} T} \right) \right. \\
&\quad \left. + 1 + \frac{1}{p-1} \left(|s_0(\tau)| + \frac{|s_1(\tau)|}{\operatorname{Log}^2_{(1)} T} + \ldots + \frac{|s_n(\tau)|}{\operatorname{Log}^2_{(n)} T} \right) \right] \mathrm{d}\tau \\
&\le \frac{f(t)}{t^2} \left[(n+1)m + 1 + \frac{(n+1)m}{p-1} \right], \quad t \in [T, \infty).
\end{aligned}$$

Hence (see (4.10) and (4.11) together with (2.2)), it suffices to consider

$$\begin{aligned}
&\frac{1}{f(t)} \int\limits_t^{t+f(t)} \frac{1}{t} \Bigg[\left(r_0(\tau) + \frac{r_1(\tau)}{\operatorname{Log}^2_{(1)} \tau} + \ldots + \frac{r_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)} \tau} + \frac{r_n(\tau)}{\operatorname{Log}^2_{(n)} \tau} \right) |\cos_p(\varphi(\tau))|^p \\
&\quad - \Phi(\cos_p(\varphi(\tau))) \sin_p(\varphi(\tau)) \\
&\quad + \left(s_0(\tau) + \frac{s_1(\tau)}{\operatorname{Log}^2_{(1)} \tau} + \ldots + \frac{s_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)} \tau} + \frac{s_n(\tau)}{\operatorname{Log}^2_{(n)} \tau} \right) \frac{|\sin_p(\varphi(\tau))|^p}{p-1} \Bigg] \mathrm{d}\tau
\end{aligned} \tag{4.12}$$

instead of $\underline{\varphi}_f'(t)$ of (4.9).

We have (see (2.6) and (4.7))

$$\begin{aligned}
&\left| \Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right) \sin_p\left(\underline{\varphi}_f(t)\right) - \frac{1}{f(t)} \int\limits_t^{t+f(t)} \Phi(\cos_p(\varphi(\tau))) \sin_p(\varphi(\tau))\, \mathrm{d}\tau \right| \\
&\le \frac{1}{f(t)} \int\limits_t^{t+f(t)} \left| \Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right) \sin_p\left(\underline{\varphi}_f(t)\right) - \Phi\left(\cos_p(\varphi(\tau))\right) \sin_p(\varphi(\tau)) \right| \mathrm{d}\tau \\
&\le \frac{1}{f(t)} \int\limits_t^{t+f(t)} A \left| \underline{\varphi}_f(t) - \varphi(\tau) \right| \mathrm{d}\tau \le \frac{AF(t)}{\operatorname{Log}^2_{(n)} t}, \quad t \in [T, \infty).
\end{aligned}$$

Considering (2.1) and (2.2), one can easily verify that the continuous function $H_2 : [T, \infty) \to (0, \infty)$ defined by

$$H_2(t) := 1 - \frac{\operatorname{Log}^2_{(n)} t}{\operatorname{Log}^2_{(n)}(t + f(t))}, \quad t \in [T, \infty),$$

satisfies $\lim_{t\to\infty} H_2(t) = 0$. Using (2.6), (4.1), and (4.7), for all considered t, we have

$$\begin{aligned}
&\left| \frac{1}{f(t)} \int\limits_t^{t+f(t)} r_0(\tau) \, |\cos_p(\varphi(\tau))|^p \, \mathrm{d}\tau - \frac{1}{f(t)} \int\limits_t^{t+f(t)} r_0(\tau)\, \mathrm{d}\tau \left| \cos_p \left(\underline{\varphi}_f(t)\right) \right|^p \right| \\
&\le \frac{m}{f(t)} \int\limits_t^{t+f(t)} A \left| \varphi(\tau) - \underline{\varphi}_f(t) \right| \mathrm{d}\tau \le \frac{mA\, F(t)}{\operatorname{Log}^2_{(n)} t}
\end{aligned}$$

and also

$$\begin{aligned}&\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{r_i(\tau)}{\operatorname{Log}^2_{(i)}\tau}\,|\cos_p(\varphi(\tau))|^p\,\mathrm{d}\tau-\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{r_i(\tau)}{\operatorname{Log}^2_{(i)}\tau}\,\mathrm{d}\tau\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\right|\\&\le\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{|r_i(\tau)|}{\operatorname{Log}^2_{(i)}T}\left||\cos_p(\varphi(\tau))|^p-\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\right|\mathrm{d}\tau\\&\le\frac{m}{f(t)}\int\limits_t^{t+f(t)}A\left|\varphi(\tau)-\underline{\varphi}_f(t)\right|\mathrm{d}\tau\le\frac{mA\;F(t)}{\operatorname{Log}^2_{(n)}t},\quad i\in\{1,\dots,n-1\},\end{aligned}$$

$$\begin{aligned}&\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{r_n(\tau)}{\operatorname{Log}^2_{(n)}\tau}\,|\cos_p(\varphi(\tau))|^p\,\mathrm{d}\tau-\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{r_n(\tau)}{\operatorname{Log}^2_{(n)}t}\,\mathrm{d}\tau\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\right|\\&\le\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{r_n(\tau)}{\operatorname{Log}^2_{(n)}\tau}\,|\cos_p(\varphi(\tau))|^p\,\mathrm{d}\tau-\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{r_n(\tau)}{\operatorname{Log}^2_{(n)}\tau}\,\mathrm{d}\tau\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\right|\\&\quad+\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{r_n(\tau)}{\operatorname{Log}^2_{(n)}\tau}\,\mathrm{d}\tau\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p-\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{r_n(\tau)}{\operatorname{Log}^2_{(n)}t}\,\mathrm{d}\tau\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\right|\\&\le\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{|r_n(\tau)|}{\operatorname{Log}^2_{(n)}T}\left||\cos_p(\varphi(\tau))|^p-\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\right|\mathrm{d}\tau\\&\quad+\frac{m}{f(t)}\int\limits_t^{t+f(t)}\left(\frac{1}{\operatorname{Log}^2_{(n)}t}-\frac{1}{\operatorname{Log}^2_{(n)}(t+f(t))}\right)\mathrm{d}\tau\\&\le\frac{m}{f(t)}\int\limits_t^{t+f(t)}A\left|\varphi(\tau)-\underline{\varphi}_f(t)\right|\mathrm{d}\tau+\frac{m}{f(t)}\int\limits_t^{t+f(t)}\frac{H_2(t)}{\operatorname{Log}^2_{(n)}t}\,\mathrm{d}\tau\le\frac{m\,(A\;F(t)+H_2(t))}{\operatorname{Log}^2_{(n)}t}.\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}&\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}s_0(\tau)\,|\sin_p(\varphi(\tau))|^p\frac{1}{p-1}\,\mathrm{d}\tau-\frac{1}{f(t)}\int\limits_t^{t+f(t)}s_0(\tau)\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\right|\\&\le\frac{m}{f(t)(p-1)}\int\limits_t^{t+f(t)}A\left|\varphi(\tau)-\underline{\varphi}_f(t)\right|\mathrm{d}\tau\le\frac{mA\;F(t)}{(p-1)\operatorname{Log}^2_{(n)}t},\end{aligned}$$

$$\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{s_i(\tau)}{\mathrm{Log}_{(i)}^2\tau}\ |\sin_p(\varphi(\tau))|^p\frac{1}{p-1}\,\mathrm{d}\tau-\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{s_i(\tau)}{\mathrm{Log}_{(i)}^2\tau}\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\right|$$

$$\le\frac{m}{f(t)(p-1)}\int\limits_t^{t+f(t)}A\left|\varphi(\tau)-\underline{\varphi}_f(t)\right|\mathrm{d}\tau\le\frac{mA\ F(t)}{(p-1)\,\mathrm{Log}_{(n)}^2 t},\quad i\in\{1,\dots,n-1\},$$

$$\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{s_n(\tau)}{\mathrm{Log}_{(n)}^2\tau}\ |\sin_p(\varphi(\tau))|^p\frac{1}{p-1}\,\mathrm{d}\tau-\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{s_n(\tau)}{\mathrm{Log}_{(n)}^2 t}\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\right|$$

$$\le\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{s_n(\tau)}{\mathrm{Log}_{(n)}^2\tau}\ |\sin_p(\varphi(\tau))|^p\frac{1}{p-1}\,\mathrm{d}\tau-\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{s_n(\tau)}{\mathrm{Log}_{(n)}^2\tau}\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\right|$$

$$+\left|\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{s_n(\tau)}{\mathrm{Log}_{(n)}^2\tau}\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\right.$$

$$\left.-\frac{1}{f(t)}\int\limits_t^{t+f(t)}\frac{s_n(\tau)}{\mathrm{Log}_{(n)}^2 t}\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\right|$$

$$\le\frac{m}{f(t)(p-1)}\int\limits_t^{t+f(t)}A\left|\varphi(\tau)-\underline{\varphi}_f(t)\right|\mathrm{d}\tau+\frac{m}{f(t)(p-1)}\int\limits_t^{t+f(t)}\frac{H_2(t)}{\mathrm{Log}_{(n)}^2 t}\,\mathrm{d}\tau$$

$$\le\frac{m\left(A\ F(t)+H_2(t)\right)}{(p-1)\,\mathrm{Log}_{(n)}^2 t}.$$

Since $\lim_{t\to\infty}F(t)=\lim_{t\to\infty}H_2(t)=0$, from (4.12) and the estimations above, we obtain (4.9). □

Lemma 4.7. *Let $f:[1,\infty)\to[1,\infty)$ be a continuously differentiable function satisfying* (2.2)*. Let η be a solution of the equation*

$$\begin{aligned}\eta'(t)=\frac{1}{t}\Bigg[&\frac{1}{f(t)}\int\limits_t^{t+f(t)}\left(r_0(\tau)+\frac{r_1(\tau)}{\mathrm{Log}_{(1)}^2\tau}+\ldots+\frac{r_{n-1}(\tau)}{\mathrm{Log}_{(n-1)}^2\tau}+\frac{r_n(\tau)}{\mathrm{Log}_{(n)}^2 t}\right)\mathrm{d}\tau\,|\cos_p(\eta(t))|^p\\&-\Phi\left(\cos_p(\eta(t))\right)\sin_p(\eta(t))\\&+\frac{1}{f(t)}\int\limits_t^{t+f(t)}\left(s_0(\tau)+\frac{s_1(\tau)}{\mathrm{Log}_{(1)}^2\tau}+\ldots+\frac{s_{n-1}(\tau)}{\mathrm{Log}_{(n-1)}^2\tau}+\frac{s_n(\tau)}{\mathrm{Log}_{(n)}^2 t}\right)\mathrm{d}\tau\,|\sin_p(\eta(t))|^p\frac{1}{p-1}\Bigg]\end{aligned}\tag{4.13}$$

on $[T,\infty)$, where $\mathrm{Log}_{(n)}T>1$ and functions $r_0>0$, $r_1,\dots,r_n,s_0,s_1,\dots,s_n$ are the coefficients of Eq. (4.2)*.*

(i) *If Eq.* (4.2) *is non-oscillatory, then*

$$\limsup_{t\to\infty} \eta(t) < \infty. \tag{4.14}$$

(ii) *If Eq.* (4.2) *is oscillatory, then*

$$\lim_{t\to\infty} \eta(t) = \infty. \tag{4.15}$$

Proof. The statement of the lemma follows from Theorem 4.2, Remark 4.3, and Lemmas 4.4 and 4.6. More precisely, in the first case, the non-oscillation of Eq. (4.2) gives the boundedness from above of any solution φ of Eq. (4.3) which is equivalent with the boundedness from above of $\underline{\varphi}_f$ given by (4.6) (see also (4.8)). Now, it suffices to compare (4.9) with (4.13). Analogously, one can proceed in the second (i.e., oscillatory) case. □

5. RESULTS

Our main result reads as follows.

Theorem 5.1. *Let* $f : [1,\infty) \to [1,\infty)$ *be a continuously differentiable function satisfying* (2.2). *Let* R *and* S *be continuous bounded functions defined on a neighborhood of* ∞. *Let us consider the equation*

$$\begin{aligned} &\left[\left(r_0(t) + \sum_{i=1}^{n-1} \frac{r_i(t)}{\operatorname{Log}^2_{(i)} t} + \frac{r_n(t) + R(t)}{\operatorname{Log}^2_{(n)} t}\right)^{-\frac{p}{q}} \Phi(x'(t))\right]' \\ &\quad + \left[s_0(t) + \sum_{i=1}^{n-1} \frac{s_i(t)}{\operatorname{Log}^2_{(i)} t} + \frac{s_n(t) + S(t)}{\operatorname{Log}^2_{(n)} t}\right] \frac{\Phi(x(t))}{t^p} = 0. \end{aligned} \tag{5.1}$$

(i) *Let Eq.* (4.2) *be non-oscillatory. If*

$$\limsup_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} R(\tau)\,\mathrm{d}\tau < 0, \quad \limsup_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} S(\tau)\,\mathrm{d}\tau < 0,$$

then Eq. (5.1) *is non-oscillatory as well.*

(ii) *Let Eq.* (4.2) *be oscillatory. If*

$$\liminf_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} R(\tau)\,\mathrm{d}\tau > 0, \quad \liminf_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} S(\tau)\,\mathrm{d}\tau > 0,$$

then Eq. (5.1) *is oscillatory as well.*

Proof. At first, we point out that Eq. (5.1) is a special case of Eq. (4.2), because the considered functions R, S are bounded. In fact, Eq. (5.1) is a modification of Eq. (4.2) for which we can use all results mentioned above. From Theorem 4.2, we know that Eq. (5.1) is non-oscillatory if and only if the corresponding Prüfer angle is bounded. From Remark 4.5 (see (4.8)), one can see that it suffices to analyze the finiteness of $\limsup_{t\to\infty} \underline{\varphi}_f(t)$ (see (4.6)) for a solution φ of the equation

$$\begin{aligned}\varphi'(t) = \frac{1}{t}\Bigg[&\left(r_0(t) + \frac{r_1(t)}{\mathrm{Log}^2_{(1)} t} + \ldots + \frac{r_{n-1}(t)}{\mathrm{Log}^2_{(n-1)} t} + \frac{r_n(t)+R(t)}{\mathrm{Log}^2_{(n)} t}\right) |\cos_p(\varphi(t))|^p \\ &- \Phi(\cos_p(\varphi(t)))\sin_p(\varphi(t)) \\ &+ \left(s_0(t) + \frac{s_1(t)}{\mathrm{Log}^2_{(1)} t} + \ldots + \frac{s_{n-1}(t)}{\mathrm{Log}^2_{(n-1)} t} + \frac{s_n(t)+S(t)}{\mathrm{Log}^2_{(n)} t}\right) |\sin_p(\varphi(t))|^p \frac{1}{p-1}\Bigg].\end{aligned} \tag{5.2}$$

Let φ be a solution of Eq. (5.2) on $[T,\infty)$, where $\mathrm{Log}_{(n)} T > 1$, and $\underline{\varphi}_f$ be defined in (4.6). We repeat that

$$\limsup_{t\to\infty} \underline{\varphi}_f(t) < \infty$$

if and only if Eq. (5.1) is non-oscillatory. We will apply Lemmas 4.6 and 4.7.

Part (i). Let $\varepsilon > 0$ be such that

$$\frac{1}{f(t)}\int_t^{t+f(t)} R(\tau)\,\mathrm{d}\tau < -\varepsilon, \quad \frac{1}{f(t)}\int_t^{t+f(t)} S(\tau)\,\mathrm{d}\tau < -\varepsilon \tag{5.3}$$

and

$$G(t) < \varepsilon \min\left\{1, \frac{1}{p-1}\right\} \tag{5.4}$$

for all large t. Let (5.3) and (5.4) be valid for all $t \in [T_1,\infty) \subset (T,\infty)$. For $t \in [T_1,\infty)$, we have (consider (2.4), (4.9), (5.3), and (5.4))

$$\begin{aligned}\underline{\varphi}_f'(t) \le \frac{1}{t}\Bigg[&\frac{1}{f(t)}\int_t^{t+f(t)}\left(r_0(\tau) + \frac{r_1(\tau)}{\mathrm{Log}^2_{(1)} \tau} + \ldots + \frac{r_{n-1}(\tau)}{\mathrm{Log}^2_{(n-1)} \tau}\right. \\ &\qquad\left. + \frac{r_n(\tau)+R(\tau)}{\mathrm{Log}^2_{(n)} t}\right)\mathrm{d}\tau \left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p \\ &- \Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right)\sin_p\left(\underline{\varphi}_f(t)\right) \\ &+ \frac{1}{f(t)}\int_t^{t+f(t)}\left(s_0(\tau) + \frac{s_1(\tau)}{\mathrm{Log}^2_{(1)} \tau} + \ldots + \frac{s_{n-1}(\tau)}{\mathrm{Log}^2_{(n-1)} \tau}\right. \\ &\qquad\left. + \frac{s_n(\tau)+S(\tau)}{\mathrm{Log}^2_{(n)} t}\right)\mathrm{d}\tau \left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p \frac{1}{p-1} + \frac{G(t)}{\mathrm{Log}^2_{(n)} t}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{t}\Bigg[\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(r_0(\tau)+\frac{r_1(\tau)}{\operatorname{Log}^2_{(1)}\tau}+\ldots+\frac{r_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)}\tau}\\
&\qquad\qquad +\frac{r_n(\tau)+R(\tau)+G(t)}{\operatorname{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\,\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\\
&\quad -\Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right)\sin_p\left(\underline{\varphi}_f(t)\right)\\
&\quad +\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(s_0(\tau)+\frac{s_1(\tau)}{\operatorname{Log}^2_{(1)}\tau}+\ldots+\frac{s_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)}\tau}\\
&\qquad\qquad +\frac{s_n(\tau)+S(\tau)+(p-1)G(t)}{\operatorname{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\,\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\Bigg]\\
&< \frac{1}{t}\Bigg[\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(r_0(\tau)+\frac{r_1(\tau)}{\operatorname{Log}^2_{(1)}\tau}+\ldots+\frac{r_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)}\tau}+\frac{r_n(\tau)}{\operatorname{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\,\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\\
&\quad -\Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right)\sin_p\left(\underline{\varphi}_f(t)\right)\\
&\quad +\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(s_0(\tau)+\frac{s_1(\tau)}{\operatorname{Log}^2_{(1)}\tau}+\ldots+\frac{s_{n-1}(\tau)}{\operatorname{Log}^2_{(n-1)}\tau}\\
&\qquad\qquad +\frac{s_n(\tau)}{\operatorname{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\,\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\Bigg].
\end{aligned}
$$

Applying the first part of Lemma 4.7 (see (4.14)), from the estimation above, we obtain

$$
\limsup_{t\to\infty}\underline{\varphi}_f(t)\le\limsup_{t\to\infty}\eta(t)+C<\infty,
$$

where η is a solution of Eq. (4.13) and C is a constant. Therefore, Eq. (5.1) is non-oscillatory because of Theorem 4.2 (and Remark 4.3) for Eq. (5.1).

Part (ii). Let $T_2 > T$ be such that

$$
\frac{1}{f(t)}\int\limits_t^{t+f(t)}R(\tau)\,\mathrm{d}\tau>\varepsilon,\qquad \frac{1}{f(t)}\int\limits_t^{t+f(t)}S(\tau)\,\mathrm{d}\tau>\varepsilon,\qquad t\in[T_2,\infty),\tag{5.5}
$$

and that (5.4) is true for all $t \in [T_2, \infty)$.

We have (see (2.4), (4.9), (5.4), and (5.5))

$$
\begin{aligned}
&\underline{\varphi}_f'(t)\\
&\geq \frac{1}{t}\Bigg[\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(r_0(\tau)+\frac{r_1(\tau)}{\mathrm{Log}^2_{(1)}\tau}+\ldots+\frac{r_{n-1}(\tau)}{\mathrm{Log}^2_{(n-1)}\tau}\\
&\qquad\qquad+\frac{r_n(\tau)+R(\tau)}{\mathrm{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\\
&\quad-\Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right)\sin_p\left(\underline{\varphi}_f(t)\right)\\
&\quad+\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(s_0(\tau)+\frac{s_1(\tau)}{\mathrm{Log}^2_{(1)}\tau}+\ldots+\frac{s_{n-1}(\tau)}{\mathrm{Log}^2_{(n-1)}\tau}\\
&\qquad\qquad+\frac{s_n(\tau)+S(\tau)}{\mathrm{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}-\frac{G(t)}{\mathrm{Log}^2_{(n)}t}\Bigg]\\
&=\frac{1}{t}\Bigg[\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(r_0(\tau)+\frac{r_1(\tau)}{\mathrm{Log}^2_{(1)}\tau}+\ldots+\frac{r_{n-1}(\tau)}{\mathrm{Log}^2_{(n-1)}\tau}\\
&\qquad\qquad+\frac{r_n(\tau)+R(\tau)-G(t)}{\mathrm{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\\
&\quad-\Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right)\sin_p\left(\underline{\varphi}_f(t)\right)\\
&\quad+\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(s_0(\tau)+\frac{s_1(\tau)}{\mathrm{Log}^2_{(1)}\tau}+\ldots+\frac{s_{n-1}(\tau)}{\mathrm{Log}^2_{(n-1)}\tau}\\
&\qquad\qquad+\frac{s_n(\tau)+S(\tau)-(p-1)G(t)}{\mathrm{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\Bigg]\\
&>\frac{1}{t}\Bigg[\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(r_0(\tau)+\frac{r_1(\tau)}{\mathrm{Log}^2_{(1)}\tau}+\ldots+\frac{r_{n-1}(\tau)}{\mathrm{Log}^2_{(n-1)}\tau}+\frac{r_n(\tau)}{\mathrm{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\left|\cos_p\left(\underline{\varphi}_f(t)\right)\right|^p\\
&\quad-\Phi\left(\cos_p\left(\underline{\varphi}_f(t)\right)\right)\sin_p\left(\underline{\varphi}_f(t)\right)\\
&\quad+\frac{1}{f(t)}\int\limits_t^{t+f(t)}\Bigg(s_0(\tau)+\frac{s_1(\tau)}{\mathrm{Log}^2_{(1)}\tau}+\ldots+\frac{s_{n-1}(\tau)}{\mathrm{Log}^2_{(n-1)}\tau}\\
&\qquad\qquad+\frac{s_n(\tau)}{\mathrm{Log}^2_{(n)}t}\Bigg)\,\mathrm{d}\tau\left|\sin_p\left(\underline{\varphi}_f(t)\right)\right|^p\frac{1}{p-1}\Bigg]
\end{aligned}
$$

for all $t\in[T_2,\infty)$.

Thus (see also (4.15)), we obtain

$$\lim_{t\to\infty} \underline{\varphi}_f(t) = \lim_{t\to\infty} \eta(t) = \infty$$

for any solution η of Eq. (4.13) on $[T_2, \infty)$. We have proved the oscillation of Eq. (5.1). The proof is complete. □

Remark 5.2. If R and S are negative in the first part of Theorem 5.1 or positive in its second part, then the statement of this theorem immediately follows from Theorem 4.1. The gist of Theorem 5.1 is given by the situation when the functions R and S change their sign.

Theorem 5.1 enables to improve many previously known oscillation results about the perturbed Euler type half-linear differential equations. To illustrate this fact, we mention the following corollaries together with one example. At first, we recall the mean values of continuous functions which are defined as follows.

Definition 5.3. Let a continuous function $g : [T, \infty) \to \mathbb{R}$ be such that the limit

$$\overline{g} := \lim_{t\to\infty} \frac{1}{t} \int_{\alpha}^{\alpha+t} g(s)\,\mathrm{d}s$$

is finite and exists uniformly with respect to $\alpha \in [T, \infty)$. The number $\overline{g}$ is called the mean value of g.

Corollary 5.4. *Let R and S be continuous bounded functions defined on a neighborhood of ∞ having mean values $\overline{R}$ and $\overline{S}$, respectively. Let us consider Eq.* (5.1)*.*

(i) *Let Eq.* (4.2) *be non-oscillatory. If $\overline{R} < 0, \overline{S} < 0$, then Eq.* (5.1) *is non-oscillatory as well.*
(ii) *Let Eq.* (4.2) *be oscillatory. If $\overline{R} > 0, \overline{S} > 0$, then Eq.* (5.1) *is oscillatory as well.*

Proof. Let $\delta > 0$ have the property that $|\overline{R}| - \delta, |\overline{S}| - \delta > 0$. The corollary follows from Theorem 5.1. Taking into account Definition 5.3, it suffices to put $f \equiv c$ for so large $c \geq 1$ that

$$\limsup_{t\to\infty} \left| \frac{1}{c} \int_t^{t+c} R(\tau)\,\mathrm{d}\tau - \overline{R} \right| < \delta, \quad \limsup_{t\to\infty} \left| \frac{1}{c} \int_t^{t+c} S(\tau)\,\mathrm{d}\tau - \overline{S} \right| < \delta.$$

Obviously, the constant function f satisfies (2.2). □

Example 5.5. For $k, n > 2$, in [46], there is proved that the equation

$$[\Phi(x'(t))]' + \left[q^{-p} + \frac{q^{1-p}}{2} \left(\sum_{i=1}^{n-1} \frac{1}{\operatorname{Log}_{(i)}^2 t} + \frac{k + \sin(2k \log_{n+1} t) - 2k \cos(2k \log_{n+1} t)}{\operatorname{Log}_{(n)}^2 t} \right) \right] \frac{\Phi(x(t))}{t^p} = 0$$

is oscillatory. Using the second part of Corollary 5.4, for any $\vartheta > 0$, one can simply show that the equation

$$\left[\left(1+\frac{\vartheta}{\operatorname{Log}^2_{(n)} t}\right)^{-\frac{p}{q}} \Phi(x'(t))\right]' + \left[q^{-p} + \frac{q^{1-p}}{2}\left(\sum_{i=1}^{n-1}\frac{1}{\operatorname{Log}^2_{(i)} t} + \frac{k+\sin\left(2k\log_{n+1} t\right) - 2k\cos\left(2k\log_{n+1} t\right) + \vartheta + \gamma(t)}{\operatorname{Log}^2_{(n)} t}\right)\right]\frac{\Phi(x(t))}{t^p} = 0$$

is also oscillatory for

$$\begin{aligned}
\gamma(t) &:= \sin t + \sin\left(\sqrt{2}t\right), \\
\gamma(t) &:= \sin_{p(1)} t + \sin_{p(2)} t, \quad p(1), p(2) > 1, \\
\gamma(t) &:= k\sin t + \frac{k+\sin t}{\log\left(t-\sqrt{t}-\sqrt[3]{t}\right)}, \\
\gamma(t) &:= \frac{(k+2)(t-2)}{(k-2)(t+2)} \arctan(\sin t).
\end{aligned}$$

Indeed, each one of these choices of γ is a continuous bounded function having zero mean value. Note that the oscillation of the considered modified equations does not follow from any previously known result (for all $\vartheta > 0$ and $k > 2$).

To the best of our knowledge, Corollary 5.4 is new even for linear equations (for concrete examples of linear equations for which the corollary gives new results, we refer to the equations in Example 5.5 for $p = 2$). For its importance, we formulate the corresponding two comparison theorems in the linear case in full.

Corollary 5.6. *Let $r_0 > 0$, $r_1, \ldots, r_n, s_0, s_1, \ldots, s_n$ be continuous bounded functions defined on a neighborhood of ∞ and let R and S be continuous bounded functions defined on a neighborhood of ∞ having mean values $\overline{R}$ and $\overline{S}$, respectively. Let us consider the linear equations*

$$\left[\left(r_0(t) + \sum_{i=1}^{n}\frac{r_i(t)}{\operatorname{Log}^2_{(i)} t}\right)^{-1} x'(t)\right]' + \left[s_0(t) + \sum_{i=1}^{n}\frac{s_i(t)}{\operatorname{Log}^2_{(i)} t}\right]\frac{x(t)}{t^2} = 0, \tag{5.6}$$

$$\left[\left(r_0(t) + \sum_{i=1}^{n-1}\frac{r_i(t)}{\operatorname{Log}^2_{(i)} t} + \frac{r_n(t)+R(t)}{\operatorname{Log}^2_{(n)} t}\right)^{-1} x'(t)\right]' + \left[s_0(t) + \sum_{i=1}^{n-1}\frac{s_i(t)}{\operatorname{Log}^2_{(i)} t} + \frac{s_n(t)+S(t)}{\operatorname{Log}^2_{(n)} t}\right]\frac{x(t)}{t^2} = 0. \tag{5.7}$$

(i) *Let Eq.* (5.6) *be non-oscillatory. If* $\overline{R} < 0, \overline{S} < 0$*, then Eq.* (5.7) *is non-oscillatory as well.*

(ii) *Let Eq.* (5.6) *be oscillatory. If* $\overline{R} > 0, \overline{S} > 0$*, then Eq.* (5.7) *is oscillatory as well.*

Proof. See Corollary 5.4 for $p = 2$. □

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Petr Hasil
https://orcid.org/0000-0002-8039-688X

Masaryk University
Faculty of Science
Department of Mathematics and Statistics
Kotlářská 2, CZ-611 37 Brno, Czech Republic

Jiřina Šišoláková
https://orcid.org/0000-0002-7487-5606

Masaryk University
Faculty of Science
Department of Mathematics and Statistics
Kotlářská 2, CZ-611 37 Brno, Czech Republic

Michal Veselý (corresponding author)
michal.vesely@mail.muni.cz
https://orcid.org/0000-0001-5306-7127

Masaryk University
Faculty of Science
Department of Mathematics and Statistics
Kotlářská 2, CZ-611 37 Brno, Czech Republic

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