Opuscula Math. 45, no. 6 (2025), 765–783
https://doi.org/10.7494/OpMath.2025.45.6.765 Opuscula Mathematica

ANISOTROPIC SINGULAR LOGISTIC EQUATIONS

João Pablo Pinheiro Da Silva, Giuseppe Failla,
Leszek Gasiński, and Nikolaos S. Papageorgiou

Communicated by Marek Galewski

Abstract. We consider a parametric Dirichlet problem driven by the anisotropic (p,q)-Laplacian and a reaction with a singular term and a superdiffusive logistic perturbation. We prove an existence and nonexistence theorem which is global with respect to the parameter $\lambda > 0$.

Keywords: anisotropic (p,q)-Laplacian, superdiffusive perturbation, anisotropic regularity, Hardy's inequality, strong comparison.

Mathematics Subject Classification: 35J60, 35J75.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following anisotropic Dirichlet problem:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = \lambda\big(u(z)^{-\eta(z)} + u(z)^{r(z)-1}\big) - f(z,u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ \lambda > 0. \end{cases} \tag{P_λ}$$

In this problem the reaction is singular, and the singular term is perturbed by a logistic term $x \longmapsto \lambda x^{r(z)-1} - f(z,x)$. Since $q(z) < p(z) < r(z)$ for all $z \in \overline{\Omega}$ and $f(z,\cdot)$ is $(r(z)-1)$-superlinear, the logistic perturbation is of the superdiffusive type. Our aim is to prove existence and nonexistence of positive solutions globally for the parameter $\lambda > 0$. Nonsingular anisotropic superdiffusive logistic equations driven by the $p(z)$-Laplacian with Robin boundary condition, were studied by Papageorgiou–Rădulescu–Tang [10], who proved existence and multiplicity of positive solutions globally in $\lambda > 0$ (a bifurcation-type theorem). To the best of our knowledge, there are no works on singular superdiffusive logistic equations driven by the $(p(z),q(z))$-differential operator. For isotropic equations, the problem was investigated by Papageorgiou–Winkert [7]. Our work extends to anisotropic equations that of [7] and in the process we relax some restrictive conditions imposed on $f(z,\cdot)$ in [7].

2. MATHEMATICAL BACKGROUND – HYPOTHESES

The analysis of (P_λ) requires the use of Lebesgue and Sobolev spaces with variable exponents. These are a particular case of generalized Orlicz spaces. For a detailed treatment of variable exponent spaces, we refer to the book of Diening–Harjulehto–Hästö–Růžička [1].

Let $L^0(\Omega)$ denote the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. We identify two such functions which differ on a Lebesgue-null set. If $\tau \in L^\infty(\Omega)$, we set

$$\tau_- = \operatorname*{ess\,inf}_{\Omega} \tau, \quad \tau_+ = \operatorname*{ess\,sup}_{\Omega} \tau.$$

We will consider variable exponents from the set

$$D = \{\tau \in L^\infty(\Omega) : \ 1 < \tau_-\}.$$

Given $\tau \in D$, the variable exponent Lebesgue space $L^{\tau(z)}(\Omega)$ is defined by

$$L^{\tau(z)}(\Omega) = \Big\{ u \in L^0(\Omega) : \ \varrho_\tau(u) = \int_\Omega |u|^{\tau(z)}\, dz < +\infty \Big\}.$$

The function $\varrho_\tau(\cdot)$ is known as the modular function corresponding to the variable exponent $\tau(\cdot)$. We equip $L^{\tau(z)}(\Omega)$ with the so-called "Luxemburg norm" defined by

$$\|u\|_{\tau(z)} = \inf \Big\{ \lambda > 0 : \ \int_\Omega \left(\frac{|u(z)|}{\lambda} \right)^{\tau(z)} dz \le 1 \Big\}.$$

With this norm $L^{\tau(z)}(\Omega)$ becomes a Banach space which is separable and reflexive (in fact uniformly convex).

Using $L^{\tau(z)}(\Omega)$, we can define the corresponding variable exponent Sobolev space $W^{1,\tau(z)}(\Omega)$ by

$$W^{1,\tau(z)}(\Omega) = \big\{ u \in L^{\tau(z)}(\Omega) : |Du| \in L^{\tau(z)}(\Omega) \big\},$$

with Du denoting the weak gradient of u. We equip $W^{1,\tau(z)}(\Omega)$ with the norm

$$\|u\|_{1,\tau(z)} = \|u\|_{\tau(z)} + \|Du\|_{\tau(z)},$$

where $\|Du\|_{\tau(z)} = \||Du|\|_{\tau(z)}$. Let

$$W_0^{1,\tau(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\tau(z)}}.$$

Both $W^{1,\tau(z)}(\Omega)$, $W_0^{1,\tau(z)}(\Omega)$ are separable and reflexive (in fact uniformly convex) Banach spaces. Moreover, if $\tau \in D \cap C(\overline{\Omega})$, then the Poincaré inequality is valid on $W_0^{1,\tau(z)}(\Omega)$, that is, there exists $\widehat{c} = \widehat{c}(\Omega) > 0$ such that

$$\|u\|_{\tau(z)} \le \widehat{c}\|Du\|_{\tau(z)} \quad \forall u \in W_0^{1,\tau(z)}(\Omega).$$

So on $W_0^{1,\tau(z)}(\Omega)$ we can consider the equivalent norm defined by

$$\|u\| = \|Du\|_{\tau(z)} \quad \forall u \in W_0^{1,\tau(z)}(\Omega).$$

Let $\tau \in D \cap C(\overline{\Omega})$ and define

$$\tau^*(z) = \begin{cases} \frac{N\tau(z)}{N-\tau(z)} & \text{if} \quad \tau(z) < N, \\ +\infty & \text{if} \quad N \le \tau(z). \end{cases}$$

In what follows, $\hookrightarrow$ denotes continuous embedding. Let $\tau, q \in D \cap C(\overline{\Omega})$. Then

$$\begin{aligned} &W_0^{1,\tau(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \quad \text{if } q(z) \le \tau^*(z) \text{ for all } z \in \overline{\Omega}, \\ &W_0^{1,\tau(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \quad \text{compactly if } q(z) < \tau^*(z) \text{ for all } z \in \overline{\Omega} \end{aligned}$$

(anisotropic Sobolev embedding theorem). There is a close relation between the modular functions $\varrho_p(\cdot)$ and the norm $\|\cdot\|$.

Proposition 2.1. *If* $\tau \in D \cap C(\overline{\Omega})$*, then*

(a) $\|u\| < 1$ *(*$= 1, > 0$*)* $\Longleftrightarrow \varrho_\tau(Du) < 1$ *(*$= 1, > 0$*)*,

(b) $\|u\| < 1 \Longrightarrow \|u\|^{\tau_+} \le \varrho_\tau(Du) \le \|u\|^{\tau_-}$,

(c) $\|u\| > 1 \Longrightarrow \|u\|^{\tau_-} \le \varrho_\tau(Du) \le \|u\|^{\tau_+}$,

(d) $\|u\| \to 0$ *(resp.* $\to +\infty$*)* $\Longleftrightarrow \varrho_\tau(Du) \to 0$ *(resp.* $\to +\infty$*)*.

With $\tau \in D \cap C(\overline{\Omega})$, let $A_\tau \colon W_0^{1,\tau(z)}(\Omega) \to W^{1,\tau(z)}(\Omega)^* = W^{-1,\tau'(z)}(\Omega)$ (with $\tau'(z) = \frac{\tau(z)}{\tau(z)-1}$ for all $z \in \overline{\Omega}$) be the nonlinear operator defined by

$$\langle A_\tau(u), h \rangle = \int_\Omega |Du|^{\tau(z)-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,\tau(z)}(\Omega).$$

From Fan–Zhang [3], we know that this operator has the following properties.

Proposition 2.2. $A_\tau(\cdot)$ *is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone as well) and of type* $(S)_+$*, that is, if* $u_n \xrightarrow{w} u$ *in* $W_0^{1,\tau(z)}(\Omega)$ *and* $\limsup_{n\to+\infty}\langle A_\tau(u_n), u_n - u\rangle \le 0$*, then* $u_n \to u$ *in* $W_0^{1,\tau(z)}(\Omega)$*.*

Let

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : \ u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0 \right\},$$

with $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$, where $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

A set $S \subseteq W_0^{1,p(z)}(\Omega)$ is said to be "downward directed", if for any $u, v \in S$, we can find $y \in S$ such that $y \leq u$, $y \leq v$.

If $u, v \in L^0(\Omega)$, then we write $u \prec v$ if and only if for all compact sets $K \subseteq \Omega$, we have $0 < c_K \leq v(z) - u(z)$ for a.a. $z \in K$. If $u, v \in C(\Omega)$ and $u(z) < v(z)$ for all $z \in \Omega$, then $u \prec v$.

If X is a Banach space and $\varphi \in C^1(X)$, then we define

$$K_\varphi = \{ u \in X : \ \varphi'(u) = 0 \}$$

(the critical set of φ).

If $u \in L^0(\Omega)$, then we define

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,\tau(z)}(\Omega)$, then $u^\pm \in W_0^{1,\tau(z)}(\Omega)$.

If $u, v \in L^0(\Omega)$ and $v(z) \leq u(z)$ for a.a. $z \in \Omega$, then

$$[v, u] = \{ y \in W_0^{1,\tau(z)}(\Omega) : \ v(z) \leq y(z) \leq u(z) \text{ for a.a. } z \in \Omega \}.$$

Our hypotheses on the data of (P_λ) are the following.

(H_0) $p, q \in C^{0,1}(\overline{\Omega})$, $\eta, r \in C(\overline{\Omega})$ and $0 < \eta(z) < 1 < q(z) < p(z) < r_- \leq r_+ < p^*_-$ for all $z \in \overline{\Omega}$.

(H_1) $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $0 \leq f(z, x) \leq c_0(1 + x^{\vartheta(z)-1})$ for a.a. $z \in \Omega$, all $x \geq 0$ with $\vartheta \in C(\overline{\Omega})$, $\vartheta(z) < p^*(z)$ for all $z \in \overline{\Omega}$ and $c_0 > 0$,

(ii) $\displaystyle\lim_{x \to +\infty} \frac{f(z, x)}{x^{r(z)-1}} = +\infty$ uniformly for a.a. $z \in \Omega$,

(iii) there exists $\delta > 0$ and $\mu \in C(\overline{\Omega})$ such that $\mu_+ < q_-$ and

$$\beta x^{\mu(z)-1} \leq f(z, x) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta,$$

with $\beta > 0$,

(iv) for every $\lambda > 0$ and every $\varrho > 0$, there exists $\widehat{\xi}_\varrho^\lambda > 0$ such that for a.a. $z \in \Omega$ the function

$$x \longmapsto \lambda x^{r(z)-1} - f(z, x) + \widehat{\xi}_\varrho^\lambda x^{p(z)-1}$$

is nondecreasing on $[0, \varrho]$.

Remark 2.3. Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Hypothesis H_1(iii) classifies the logistic perturbation of the singular term, as superdiffusive. In contrast to the isotropic work of Papageorgiou–Winkert [7], we do not assume that $f(z, \cdot)$ is nondecreasing. Moreover, the hypothesis near 0^+ (see hypothesis H_1(iii)) is less restrictive than the one used in [7].

Example 2.4. We highlight that our assumptions are not equivalent to the ones of Papageorgiou–Winkert [7]. In particular, consider the function

$$f(z, x) = \begin{cases} x^{\mu(z)-1} & \text{if } 0 \leq x \leq 1, \\ 2 - x & \text{if } 1 < x \leq 2, \\ (x-2)^{r(z)} & \text{if } x \geq 2, \end{cases}$$

with $\mu, r \in C(\overline{\Omega})$ such that $\mu_+ < q_-$ and $p(z) < r_- \leq r_+ < p^*_-$. Clearly, f satisfies hypotheses (H_1)(i),(ii),(iii). Hypothesis (H_1)(iv) is satisfied with $\hat{\xi}^\lambda_\rho > 0$ large enough in any $[0, \rho]$. However, $f(z, \cdot)$ does not satisfies the monotonicity assumption of Papageorgiou–Winkert [7, Assumption H].

3. AUXILIARY PROBLEM

In this section, we examine the following auxiliary nonsingular Dirichlet problem:

$$\begin{cases} -\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) = \lambda u(z)^{r(z)-1} - f(z, u(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ \lambda > 0. \end{cases} \tag{$\widehat{P}_\lambda$}$$

The solutions of this problem, will be used to show the existence of admissible parameters for problem (P_λ).

Let

$$\widehat{\mathcal{L}} = \big\{\lambda > 0 : \text{problem } (\widehat{P}_\lambda) \text{ has a solution}\big\}$$

and let $\widehat{S}_\lambda$ be the solution set of $(\widehat{P}_\lambda)$.

Proposition 3.1. *If hypotheses H_0 and H_1 hold, then we can find $\widehat{\lambda}_\infty > 0$ such that for all $\lambda > \widehat{\lambda}_\infty$ we have*

$$\lambda \in \widehat{\mathcal{L}} \text{ and } \emptyset \neq \widehat{S}_\lambda \subseteq \operatorname{int} C_+.$$

Moreover, there exists $\widehat{u}^*_\lambda \in \widehat{S}_\lambda$ such that

$$\widehat{u}^*_\lambda \leq \widehat{u} \quad \forall \widehat{u} \in \widehat{S}_\lambda.$$

Proof. Let $\widehat{\varphi}_\lambda\colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-energy functional for problem $(\widehat{P}_\lambda)$ defined by

$$\widehat{\varphi}_\lambda(u) = \int\limits_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int\limits_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz + \int\limits_\Omega F(z,u^+)\,dz - \int\limits_\Omega \frac{\lambda}{r(z)}(u^+)^{r(z)}\,dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

Hypotheses H_1(i),(ii) imply that given $M > \lambda$, we can find $c_1 = c_1(M) > 0$ such that

$$F(z,x) \geq \frac{M}{r(z)}x^{r(z)} - c_1 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.1}$$

If $u \in W_0^{1,p(z)}(\Omega)$ with $\|u\| \geq 1$, then

$$\begin{aligned}\widehat{\varphi}_\lambda(u) &\geq \frac{1}{p_+}\big(\varrho_p(Du) + \varrho_q(Du)\big) + \int\limits_\Omega \frac{M-\lambda}{r(z)}(u^+)^{r(z)}\,dz - c_1|\Omega|_N \\ &\geq \frac{1}{p_+}\|u\|^{p_-} - c_1|\Omega|_N\end{aligned}$$

(see (3.1), Proposition 2.1 and recall that $\|u\| \geq 1$), with $|\cdot|_N$ denoting the Lebesgue measure on $\mathbb{R}^N$, so $\widehat{\varphi}_\lambda$ is coercive.

Also, using the anisotropic Sobolev embedding theorem, we see that $\widehat{\varphi}_\lambda$ is sequentially weakly lower semicontinuous. The Weierstrass–Tonelli theorem, implies that we can find $u_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$\widehat{\varphi}_\lambda(u_\lambda) = \inf\{\widehat{\varphi}_\lambda(u) :\ u \in W_0^{1,p(z)}(\Omega)\}. \tag{3.2}$$

For $u \in C_0^1(\overline{\Omega})$ with $u(z) > 0$ for all $z \in \Omega$, we have

$$\widehat{\varphi}_\lambda(u) \leq \frac{1}{q_-}\big(\varrho_p(Du) + \varrho_q(Du)\big) + \int\limits_\Omega F(z,u)\,dz - \int\limits_\Omega \frac{\lambda}{r(z)}u^{r(z)}\,dz$$

and so we see that we can find $\widehat{\lambda}_\infty > 0$ such that if $\lambda > \widehat{\lambda}_\infty$, then

$$\widehat{\varphi}_\lambda(u) < 0,$$

so

$$\widehat{\varphi}_\lambda(u_\lambda) < 0 = \widehat{\varphi}_\lambda(0)$$

(see (3.2)), and thus $u_\lambda \neq 0$.

From (3.2), we have

$$\langle \widehat{\varphi}_\lambda'(u_\lambda), h\rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\langle V(u_\lambda), h\rangle = \int\limits_\Omega \big(\lambda (u_\lambda^+)^{r(z)-1} - f(z, u_\lambda^+)\big) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

with

$$V = A_p + A_q \colon W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega).$$

We choose the test function $h = -u_\lambda^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\varrho_p(Du_\lambda^-) \le 0,$$

so $u_\lambda \ge 0$, $u_\lambda \ne 0$ (see Proposition 2.1).

So, $\widehat{u}_\lambda \in W_0^{1,p(z)}(\Omega)$ (for $\lambda > \widehat{\lambda}_\infty$) is a nontrivial solution of problem $(\widehat{P}_\lambda)$. From Proposition A1 of Papageorgiou–Rădulescu–Zhang [11], we have $u_\lambda \in L^\infty(\Omega)$. Then the anisotropic global regularity theory of Fan [2] (extension of the corresponding isotropic global regularity theory of Lieberman [6]), implies that $\widehat{u}_\lambda \in C_+ \setminus \{0\}$. Let $\varrho = \|\widehat{u}_\lambda\|_\infty$ and let $\widehat{\xi}_\varrho^\lambda > 0$ be as postulated by hypothesis H_1(iv). We have

$$-\Delta_{p(z)} \widehat{u}_\lambda - \Delta_{q(z)} \widehat{u}_\lambda + \widehat{\xi}_\varrho^\lambda \widehat{u}_\lambda^{p(z)-1} \ge 0 \quad \text{in } \Omega,$$

so $\widehat{u}_\lambda \in \operatorname{int} C_+$ (see Zhang [12] and Papageorgiou–Rădulescu–Zhang [11, Proposition A2]).

We have proved that if $\lambda > \widehat{\lambda}_\infty$, then

$$\lambda \in \widehat{\mathcal{L}} \ne \emptyset \quad \text{and} \quad \emptyset \ne \widehat{S}_\lambda \subseteq \operatorname{int} C_+.$$

From Filippakis–Papageorgiou [4], we know that $\widehat{S}_\lambda$ is downward directed. So, invoking Theorem 5.109 of Hu–Papageorgiou [5, p. 308], we can find a decreasing sequence $\{\widehat{u}_n\}_{n\in\mathbb{N}} \subseteq \widehat{S}_\lambda$ such that

$$\inf \widehat{S}_\lambda = \inf_{n\in\mathbb{N}} \widehat{u}_n.$$

We have

$$\langle V(\widehat{u}_n), h\rangle = \int\limits_\Omega \big(\lambda \widehat{u}_n^{r(z)-1} - f(z, \widehat{u}_n)\big) \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),\ n \in \mathbb{N}, \tag{3.3}$$

$$0 \le \widehat{u}_n \le \widehat{u}_1, \quad \widehat{u}_n \in \operatorname{int} C_+ \quad \forall n \in \mathbb{N}. \tag{3.4}$$

In (3.3) we choose the test function $h = \widehat{u}_n \in W_0^{1,p(z)}(\Omega)$. From (3.4) and hypothesis H_1(i) it follows that the sequence $\{\widehat{u}_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded. So, passing to a subsequence if necessary, we may assume that

$$\widehat{u}_n \xrightarrow{w} \widehat{u}_\lambda^* \quad \text{in } W_0^{1,p(z)}(\Omega), \quad \widehat{u}_n \longrightarrow \widehat{u}_\lambda^* \quad \text{in } L^{r(z)}(\Omega). \tag{3.5}$$

In (3.3) we choose the test function $h = \widehat{u}_n - \widehat{u}_\lambda^* \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.5). Then

$$\lim_{n\to+\infty} \langle V(\widehat{u}_n), \widehat{u}_n - \widehat{u}_\lambda^*\rangle = 0,$$

so we obtain

$$\widehat{u}_n \longrightarrow \widehat{u}_\lambda^* \quad \text{in } W_0^{1,p(z)}(\Omega) \tag{3.6}$$

(see (3.5) and Proposition 2.2).

Suppose that $\widehat{u}_\lambda^* = 0$. Then on account of (3.6), we can find $n_0 \in \mathbb{N}$ such that $\|\widehat{u}_n\| \le 1$ for all $n \ge n_0$. In (3.3) we choose the test function $h = u_n \in W_0^{1,p(z)}(\Omega)$ and since $f \ge 0$, we obtain

$$\|\widehat{u}_n\|^{p_+} \le \lambda c_2 \|\widehat{u}_n\|^{r_-} \quad \forall n \ge n_0,$$

for some $c_2 > 0$ (recall that $\|\widehat{u}_n\| \le 1$ for all $n \ge n_0$), so

$$1 \le \lambda c_2 \|\widehat{u}_n\|^{r_- - p_+} \longrightarrow 0 \quad \text{as } n \to +\infty,$$

a contradiction. Therefore, $\widehat{u}_\lambda^* \ne 0$ and from (3.3) and (3.6), it follows that

$$\langle V(\widehat{u}_\lambda^*), h\rangle = \int_\Omega \big(\lambda (\widehat{u}_\lambda^*)^{r(z)-1} - f(z, \widehat{u}_\lambda^*)\big) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so $\widehat{u}_\lambda^* \in \widehat{S}_\lambda$ ($\lambda > \widehat{\lambda}_\infty$) and $\widehat{u}_\lambda^* = \inf \widehat{S}_\lambda$. □

For $\lambda > 0$ small the situation is different. We have nonexistence of solutions for $(\widehat{P}_\lambda)$.

Proposition 3.2. *If hypotheses H_0 and H_1 hold, then there exists $\widehat{\lambda}_0 > 0$ such that if $\lambda \in (0, \widehat{\lambda}_0)$, then problem $(\widehat{P}_\lambda)$ has no solution.*

Proof. Arguing by contradiction, suppose that we can find sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \to 0^+$ and $u_n \in \widehat{S}_{\lambda_n}$ for $n \in \mathbb{N}$. We have

$$\langle V(u_n), h\rangle = \int_\Omega \big(\lambda_n u_n^{r(z)-1} - f(z, u_n)\big) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),\ n \in \mathbb{N}.$$

Choosing the test function $h = u_n \in W_0^{1,p(z)}(\Omega)$, we obtain

$$\varrho_p(Du_n) \le \int_\Omega \big(\lambda_n u_n^{r(z)} - f(z, u_n) u_n\big) \, dz \quad \forall n \in \mathbb{N}. \tag{3.7}$$

From hypothesis H_1(ii), we see that given $\beta > \lambda_1$, we can find $M > 1$ large such that

$$\beta x^{r(z)} \le f(z, x) x \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M. \tag{3.8}$$

Recalling that $f \ge 0$ (see hypothesis H_1(i)) and using (3.8), we have

$$\begin{aligned} \int_\Omega \big(\lambda_n u_n^{r(z)} - f(z, u_n) u_n\big) \, dz &\le \int_{\{0 \le u_n \le M\}} \lambda_n u_n^{r(z)} \, dz + \int_{\{M < u_n\}} (\lambda_n - \beta) u_n^{r(z)} \, dz \\ &\le \int_{\{0 \le u_n \le M\}} \lambda_n u_n^{r(z)} \, dz \\ &\le \lambda_n M^{r_+} |\Omega|_N \quad \forall n \in \mathbb{N} \end{aligned} \tag{3.9}$$

(see (3.8) and since $\beta > \lambda_1 \geq \lambda_n$ and $M > 1$). We return to (3.7) and use (3.9). Then

$$\varrho_p(Du_n) \leq \lambda_n M^{r_+}|\Omega|_N \longrightarrow 0 \quad \text{as } n \to +\infty,$$

so

$$u_n \longrightarrow 0 \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{as } n \to +\infty \tag{3.10}$$

(see Proposition 2.1). So, we may assume that $\|u_n\| \leq 1$, $\|u_n\|_{r(z)} \leq 1$ (recall that $W_0^{1,p(z)}(\Omega) \hookrightarrow L^{r(z)}(\Omega)$). Then

$$\|u_n\|^{p_+} \leq \lambda_n \varrho_r(u_n) \leq \lambda_1 c_3 \|u_n\|^{r_-} \quad \forall n \in \mathbb{N},$$

for some $c_3 > 0$, so

$$1 \leq \lambda_1 c_3 \|u_n\|^{r_- - p_+} \longrightarrow 0 \quad \text{as } n \to +\infty$$

(see (3.10) and recall that $p_+ < r_-$), a contradiction. Therefore, there exists $\widehat{\lambda}_0 > 0$ such that for all $\lambda \in (0, \widehat{\lambda}_0)$, problem $(\widehat{P}_\lambda)$ has no solution. □

Remark 3.3. The above proposition implies that $\inf \widehat{\mathcal{L}} > 0$.

4. POSITIVE SOLUTIONS

We introduce the following two sets related to problem (P_λ):

$$\mathcal{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ has solution}\},$$
$$S_\lambda = \text{the set of solutions of } (P_\lambda).$$

Proposition 4.1. *If hypotheses H_0 and H_1 hold, then $\mathcal{L} \neq \emptyset$ and for every $\lambda \in \mathcal{L}$, $S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. Let $\lambda > \widehat{\lambda}_\infty$ and let $\widehat{u}_\lambda^* \in \operatorname{int} C_+$ be the minimal solution of problem $(\widehat{P}_\lambda)$ (see Proposition 3.1). We introduce the Carathéodory function $k_\lambda(z,x)$ defined by

$$k_\lambda(z,x) = \begin{cases} \lambda \widehat{u}_\lambda^*(z)^{-\eta(z)} + \lambda \widehat{u}_\lambda^*(z)^{r(z)-1} - f(z, \widehat{u}_\lambda^*(z)) & \text{if } \ x < \widehat{u}_\lambda^*(z), \\ \lambda x^{-\eta(z)} + \lambda x^{r(z)-1} - f(z,x) & \text{if } \ \widehat{u}_\lambda^*(z) \leq x. \end{cases} \tag{4.1}$$

We set $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\,ds$ and consider the C^1-functional $\psi_\lambda \colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega K_\lambda(z,u)\,dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

For $u \in W_0^{1,p(z)}(\Omega)$ with $\|u\| \geq 1$, we have

$$\begin{aligned}\psi_\lambda(u) \geq{}& \frac{1}{p_+}\|u\|^{p_-} - \lambda\varrho_r(\widehat{u}_\lambda^*) - \frac{\lambda}{r_-}\varrho_r(u^+)\\ &- \int\limits_{\{u<\widehat{u}_\lambda^*\}} \big(\lambda(\widehat{u}_\lambda^*)^{-\eta(z)} - f(z,u_\lambda^*)\big)\widehat{u}_\lambda^*\,dz\\ &- \int\limits_{\{\widehat{u}_\lambda^*\leq u\}} \big(\lambda(\widehat{u}_\lambda^*)^{1-\eta(z)} - f(z,\widehat{u}_\lambda^*)\widehat{u}_\lambda^*\big)\,dz\\ &- \frac{\lambda}{1-\eta_+}\int\limits_{\{\widehat{u}_\lambda^*\leq u\}} \big(u^{1-\eta(z)} - (\widehat{u}_\lambda^*)^{1-\eta(z)}\big)\,dz\\ &+ \int\limits_{\{u_\lambda^*\leq u\}} \big(F(z,u) - F(z,\widehat{u}_\lambda^*)\big)\,dz\\ \geq{}& \frac{1}{p_+}\|u\|^{p_-} - \frac{\lambda}{r_-}\varrho_r(u^+) - c_4(1+\|u\|) + \int\limits_{\{\widehat{u}_\lambda^*\leq u\}} F(z,u)\,dz,\end{aligned} \tag{4.2}$$

for some $c_4 > 0$ (see (4.1)).

Given $\zeta > \frac{r_+}{r_-}\lambda$, we can find $M \geq \|\widehat{u}_\lambda^*\|_\infty$ large such that

$$F(z,x) \geq \frac{\zeta}{r_+}x^{r(z)} \quad \text{for a.a. } z\in\Omega, \text{ all } x \geq M. \tag{4.3}$$

We have

$$\begin{aligned}\int\limits_{\{\widehat{u}_\lambda^*\leq u\}} F(z,u)\,dz &= \int\limits_{\{M\leq u\}} F(z,u)\,dz + \int\limits_{\{\widehat{u}_\lambda^*\leq u<M\}} F(z,u)\,dz\\ &\geq \frac{\zeta}{r_+}\int\limits_{\{M\leq u\}} u^{r(x)}\,dz - c_5\\ &\geq \frac{\zeta}{r_+}\varrho_r(u^+) - c_6\end{aligned} \tag{4.4}$$

for some $c_5, c_6 > 0$ (see (4.3) and hypothesis H_1(i)). We return to (4.2) and use (4.4). Then

$$\psi_\lambda(u) \geq \frac{1}{p_+}\|u\|^{p_-} + \left(\frac{\zeta}{r_+} - \frac{\lambda}{r_-}\right)\varrho_r(u^+) - c_4\|u\| - c_7$$

for some $c_7 > 0$. Since $\zeta > \frac{r_+}{r_-}\lambda$, we infer that ψ_λ is coercive.

Also, using the anisotropic Sobolev embedding theorem, we have that ψ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$\psi_\lambda(u_\lambda) = \inf\{\psi_\lambda(u):\ u \in W_0^{1,p(z)}(\Omega)\},$$

so

$$\langle \psi_\lambda'(u_\lambda), h\rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega)$$

and thus

$$\langle V(u_\lambda), h\rangle = \int_\Omega k_\lambda(z, u_\lambda) h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega).$$

We choose the test function $h = (\widehat{u}_\lambda^* - u_\lambda)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}\langle V(u_\lambda), (\widehat{u}_\lambda^* - u_\lambda)^+\rangle &= \int_\Omega \big(\lambda(u_\lambda^*)^{-\eta(z)} + \lambda(\widehat{u}_\lambda^*)^{r(z)-1} - f(z, \widehat{u}_\lambda^*)\big)(\widehat{u}_\lambda^* - u_\lambda)^+\, dz \\ &\geq \int_\Omega \big(\lambda(\widehat{u}_\lambda^*)^{r(z)-1} - f(z, \widehat{u}_\lambda^*)\big)(\widehat{u}_\lambda^* - u_\lambda)^+\, dz \\ &= \langle V(\widehat{u}_\lambda^*), (\widehat{u}_\lambda^* - u_\lambda)^+\rangle,\end{aligned}$$

(see (4.1) and Proposition 3.1), thus

$$\widehat{u}_\lambda^* \leq u_\lambda.$$

So, from (4.1), it follows that $u_\lambda \in S_\lambda$. We have proved that $(\widehat{\lambda}_\infty, \infty) \subseteq \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$.

Now let $u \in S_\lambda$. From Proposition A1 of Papageorgiou–Rădulescu–Zhang [11], we have that $u \in L^\infty(\Omega)$ and so the anisotropic regularity theory of Fan [2] implies that $u \in C_+ \setminus \{0\}$. Let $\varrho = \|u\|_\infty$ and let $\widehat{\xi}_\varrho^\lambda > 0$ be as postulated by hypothesis H_1(iv). Then

$$-\Delta_{p(z)} u - \Delta_{q(z)} u + \widehat{\xi}_\varrho^\lambda u^{p(z)-1} - \lambda u^{-\eta(z)} \geq 0 \quad \text{in } \Omega.$$

Then Theorem 5.9 of Hu–Papageorgiou [5, p. 311], implies that $u \in \operatorname{int} C_+$. So, we conclude that for all $\lambda \in \mathcal{L}$, $\emptyset \neq S_\lambda \subseteq C_+$. ☐

We show that $\mathcal{L}$ is a half-line in $(0, +\infty)$. Eventually we will show that $\mathcal{L}$ is a closed half-line in $(0, +\infty)$.

Proposition 4.2. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathcal{L}$ and $\mu > \lambda$, then $\mu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. We introduce the Carathéodory function $b_\mu(z, x)$ defined by

$$b_\mu(z,x) = \begin{cases} \mu u_\lambda(z)^{-\eta(z)} + \mu u_\lambda(z)^{r(z)-1} - f(z, u_\lambda(z)) & \text{if} \quad x < u_\lambda(z), \\ \mu x^{-\eta(z)} + \lambda x^{r(z)-1} - f(z,x) & \text{if} \quad u_\lambda(z) \leq x. \end{cases} \tag{4.5}$$

We set $B_\mu(z,x) = \int_0^x b_\mu(z,s)\, ds$ and consider the C^1-functional $\sigma_\mu \colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\sigma_\mu(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz - \int_\Omega B_\mu(z,u)\, dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

As before, hypothesis H_1(ii) implies that σ_μ is coercive.

Moreover, the anisotropic Sobolev embedding theory implies that σ_μ is sequentially weakly lower semicontinuous. Therefore, we can find $u_\mu \in W_0^{1,p(z)}(\Omega)$ such that

$$\sigma_\mu(u_\mu) = \inf\big\{\sigma_\mu(u) :\ u \in W_0^{1,p(z)}(\Omega)\big\},$$

so

$$\langle \sigma'_\mu(u_\mu), h\rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega)$$

and thus

$$\langle V(u_\mu), h\rangle = \int_\Omega b_\mu(z,u_\mu) h\, dz \quad \forall h \in W_0^{1,p(z)}(\Omega).$$

We choose the test function $h = (u_\lambda - u_\mu)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}\langle V(u_\mu), (u_\lambda - u_\mu)^+\rangle &= \int_\Omega \big(\mu u_\lambda^{-\eta(z)} + \mu u_\lambda^{r(z)-1} - f(z,u_\lambda)\big)(u_\lambda - u_\mu)^+\, dz \\ &\geq \int_\Omega \big(\lambda u_\lambda^{-\eta(z)} + \lambda u_\lambda^{r(z)-1} - f(z,u_\lambda)\big)(u_\lambda - u_\mu)^+\, dz \\ &= \langle V(u_\lambda), (u_\lambda - u_\mu)^+\rangle\end{aligned}$$

(see (4.5) and since $\lambda < \mu$, $u_\lambda \in S_\lambda$). Thus, $u_\lambda \leq u_\mu$ and so $u_\mu \in S_\mu$ (see (4.5)), hence $\mu \in \mathcal{L}$. ☐

Embedded in the above proof, is the following weak monotonicity property of the solution multifunction $\lambda \longmapsto S_\lambda$.

Corollary 4.3. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda$ and $\mu > \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu$ such that $u_\lambda \leq u_\mu$.*

In fact, we can improve this corollary and show that this monotonicity property is strict.

Proposition 4.4. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda$ and $\mu > \lambda$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu$ such that $u_\lambda - u_\mu \in \operatorname{int} C_+$.*

Proof. From Corollary 4.3, we know that $\mu \in \mathcal{L}$, and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $u_\lambda \leq u_\mu$.

Let $\varrho = \|u_\mu\|_\infty$ and let $\widehat{\xi}_\varrho^\mu > 0$ be as postulated by hypothesis H_1(iv). Then

$$\begin{aligned} &-\Delta_{p(z)} u_\lambda - \Delta_{q(z)} u_\lambda + \widehat{\xi}_\varrho^\mu u_\lambda^{p(z)-1} - \mu u_\lambda^{-\eta(z)} \\ &\qquad \leq \lambda u_\lambda^{r(z)-1} - f(z,u_\lambda) + \widehat{\xi}_\varrho^\mu u_\lambda^{p(z)-1} \\ &\qquad \leq \mu u_\lambda^{r(z)-1} - f(z,u_\lambda) + \widehat{\xi}_\varrho^\mu u_\lambda^{p(z)-1} \\ &\qquad \leq \mu u_\mu^{r(z)-1} - f(z,u_\mu) + \widehat{\xi}_\varrho^\mu u_\mu^{p(z)-1} \\ &\qquad = -\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + \widehat{\xi}_\varrho^\mu u_\mu^{p(z)-1} - \mu u_\mu^{-\eta(z)} \quad \text{in } \Omega. \end{aligned} \tag{4.6}$$

(since $\mu < \lambda$, $u_\lambda \leq u_\mu$ and see hypothesis H_1(iv)).

Note that since $u_\lambda \in \operatorname{int} C_+$, we have that

$$0 \prec (\mu - \lambda) u_\lambda^{r(z)-1}.$$

So, from (4.6) and Proposition 2.3 of Papageorgiou–Winkert [8], we have

$$u_\lambda - u_\mu \in \operatorname{int} C_+. \qquad \square$$

Let

$$\lambda_* = \inf \mathcal{L} \geq 0.$$

Proposition 4.5. *If hypotheses H_0 and H_1 hold, then $\lambda_* > 0$.*

Proof. Let $\widehat{\lambda}_0 > 0$ be as postulated in Proposition 3.2 and consider $\lambda \in (0, \widehat{\lambda}_0)$. Suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. We introduce the Carathéodory function $w_\lambda(z,x)$ defined by

$$w_\lambda(z,x) = \begin{cases} \lambda (x^+)^{r(z)-1} - f(z,x^+) & \text{if} \quad x \leq u_\lambda(z), \\ \lambda u_\lambda(z)^{r(z)-1} - f(z, u_\lambda(z)) & \text{if} \quad u_\lambda(z) < x. \end{cases} \tag{4.7}$$

We set $W_\lambda(z,x) = \int_0^x w_\lambda(z,s)\,ds$ and consider the C^1-functional $\widehat{\gamma}_\lambda \colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\gamma}_\lambda(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\,dz - \int_\Omega W_\lambda(z,u)\,dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

From (4.7), it is clear that $\widehat{\gamma}_\lambda$ is coercive. Also, $\widehat{\gamma}_\lambda$ is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$\widehat{\gamma}_\lambda(\widetilde{u}_\lambda) = \inf \left\{ \widehat{\gamma}_\lambda(u) : \ u \in W_0^{1,p(z)}(\Omega) \right\}. \tag{4.8}$$

Let $u \in \operatorname{int} C_+$. Since $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, using Proposition 2.86 of Hu–Papageorgiou [5, p. 90], we can find $t \in (0,1)$ small such that

$$tu \leq u_\lambda \quad \text{and} \quad tu(z) \in [0, \delta) \quad \forall z \in \overline{\Omega}, \tag{4.9}$$

with $\delta > 0$ as in hypothesis H_1(iii). We have

$$\widehat{\gamma}_\lambda(tu) \leq \frac{t^{q_-}}{q_-} \big(\varrho_p(Du) + \varrho_q(Du) \big) - \frac{\lambda t^{r_+}}{r^+} \varrho_r(u) - \frac{\beta t^{\mu_+}}{\mu_+} \varrho_\mu(u)$$

(see hypothesis H_1(iii) and recall that $0 < t < 1$).

Since $\mu_+ < q_- < r_+$, if we choose $t \in (0,1)$ even smaller if necessary, then

$$\widehat{\gamma}_\lambda(tu) < 0,$$

so

$$\widehat{\gamma}_\lambda(\widetilde{u}_\lambda) < 0 = \widehat{\gamma}_\lambda(0)$$

(see (4.8)) and thus

$$\widetilde{u}_\lambda \neq 0.$$

From (4.8), we have

$$\langle \widehat{\gamma}_\lambda'(\widetilde{u}_\lambda), h\rangle = 0 \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\langle V(\widetilde{u}_\lambda), h\rangle = \int_\Omega w_\lambda(z, \widetilde{u}_\lambda) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega).$$

We choose the test function $h = (\widetilde{u}_\lambda - u_\lambda)^+ \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned}\langle V(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u_\lambda)^+\rangle &= \int_\Omega \big(\lambda u_\lambda^{r(z)-1} - f(z, u_\lambda)\big)(\widetilde{u}_\lambda - u_\lambda)^+ \, dz \\ &\leq \int_\Omega \big(\lambda u_\lambda^{-\eta(z)} + \lambda u_\lambda^{r(z)-1} - f(z, u_\lambda)\big)(\widetilde{u}_\lambda - u_\lambda)^+ \, dz \\ &= \langle V(u_\lambda), (\widetilde{u}_\lambda - u_\lambda)^+\rangle\end{aligned}$$

(see (4.7) and since $u_\lambda \in S_\lambda$), so

$$\widetilde{u}_\lambda \leq u_\lambda$$

(see Proposition 2.2).

Also, choosing the test function $h = -\widetilde{u}_\lambda^- \in W_0^{1,p(z)}(\Omega)$, we obtain

$$0 \leq \widetilde{u}_\lambda, \quad \widetilde{u}_\lambda \neq 0.$$

Therefore, from (4.7), we see that

$$\widetilde{u}_\lambda \in \widehat{S}_\lambda,$$

a contradiction, since $0 < \lambda < \widehat{\lambda}_0$ (see Proposition 3.2). Therefore

$$0 < \widehat{\lambda}_0 \leq \lambda_*. \qquad \square$$

We want to check the admissibility of the critical parameter $\lambda_* > 0$. To this end, we will need the following auxiliary result.

Lemma 4.6. *If hypotheses H_0 and H_1 hold, then the minimal solution map $\lambda \longmapsto \widehat{u}_\lambda^*$ (see Proposition 3.1) is nondecreasing, $\widehat{u}_\lambda^* \leq u$ for all $u \in S_\lambda$ and $\mathcal{L} \subseteq \widehat{\mathcal{L}}$.*

Proof. Let $\mu, \lambda \in \mathcal{L}$ with $\lambda < \mu$. We introduce the Carathéodory function $l_\lambda(z, x)$ defined by

$$l_\lambda(z, x) = \begin{cases} \lambda (x^+)^{r(z)-1} - f(z, x^+) & \text{if} \quad x \leq \widehat{u}_\mu^*(z), \\ \lambda \widehat{u}_\mu^*(z)^{r(z)-1} - f(z, \widehat{u}_\mu^*(z)) & \text{if} \quad \widehat{u}_\mu^*(z) < x. \end{cases} \tag{4.10}$$

We set $L_\lambda(z,x) = \int_0^x l_\lambda(z,s)\,ds$ and consider the C^1-functional $j_\lambda\colon W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$j_\lambda(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega L_\lambda(z,u)\,dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

From (4.10), it is clear that j_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$j_\lambda(\widehat{u}_\lambda) = \inf\big\{j_\lambda(u) :\ u \in W_0^{1,p(z)}(\Omega)\big\}.$$

Moreover, as in the proof of Proposition 4.5, using hypothesis H_1(iii), we show that

$$j_\lambda(\widehat{u}_\lambda) < 0 = j_\lambda(0),$$

so

$$\widehat{u}_\lambda \neq 0.$$

In addition, on account of (4.10), we can see that

$$K_{j_\lambda} \subseteq [0, \widehat{u}^*_\mu]$$

and since $\widehat{u}_\lambda \in K_{j_\lambda}$, it follows that $\widehat{u}_\lambda \in \widehat{S}_\lambda$ (see (4.10)). Then

$$\widehat{u}^*_\lambda \leq \widehat{u}_\lambda \leq \widehat{u}^*_\mu,$$

so the map $\lambda \longmapsto \widehat{u}^*_\lambda$ is nondecreasing.

Let $u \in S_\lambda$. We have

$$-\Delta_{p(z)}u - \Delta_{q(z)}u \geq \lambda u^{r(z)-1} - f(z,u) \quad \text{in } \Omega. \tag{4.11}$$

So, if in previous argument we replace $\widehat{u}^*_\mu$ by u, then we produce $\widehat{u}_\lambda \in \widehat{S}_\lambda$ such that

$$\widehat{u}_\lambda \leq u,$$

thus $\widehat{u}^*_\lambda \leq u$ for all $u \in S_\lambda$ and $\mathcal{L} \subseteq \widehat{\mathcal{L}}$. □

Now we can show the admissibility of the critical parameter $\lambda_* > 0$.

Proposition 4.7. *If hypotheses H_0 and H_1 hold, then $\lambda_* \in \mathcal{L}$.*

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ be a sequence such that $\lambda_n \to \lambda_*^+$. Consider $u_n \in S_{\lambda_n}$, for $n \in \mathbb{N}$. From Lemma 4.6, we have

$$\widehat{u}^*_{\lambda_n} \leq u_n \quad \forall n \in \mathbb{N}. \tag{4.12}$$

If at each step, we truncate the reaction at $\widehat{u}^*_{\lambda_{n+1}}(z)$ (from below) and at $u_{\lambda_n}(z)$ (from above) and use the Weierstrass–Tonelli theorem, we can have a sequence $u_n \in S_{\lambda_n}$, for $n \in \mathbb{N}$, which is decreasing. We have

$$\langle V(u_n), h\rangle = \int_\Omega \big(\lambda_n u_n^{-\eta(z)} + \lambda_n u_n^{r(z)-1} - f(z,u_n)\big)h\,dz \quad \forall h \in W_0^{1,p(z)}(\Omega),\ n \in \mathbb{N}. \tag{4.13}$$

On account of hypotheses H_1(i),(ii), we have

$$\lambda_n x^{r(z)-1} - f(z,x) \le \lambda_1 x^{r(z)-1} - f(z,x) \le c_8 \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0, \tag{4.14}$$

for some $c_8 > 0$. In (4.13), we choose the test function $h = u_n \in W_0^{1,p(z)}(\Omega)$. Then

$$\varrho_p(Du_n) \le c_9 \|u_n\| \quad \forall n \in \mathbb{N},$$

for some $c_9 > 0$ (see (4.14)), so the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded (see Proposition 2.1). Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_* \quad \text{in } W_0^{1,p(z)}(\Omega), \quad u_n \longrightarrow u_* \quad \text{in } L^{\vartheta(z)}(\Omega). \tag{4.15}$$

Recall that $0 \le u_n \le u_1$ for all $n \in \mathbb{N}$. Hence,

$$\|u_n\|_\infty \le c_{10} \quad \forall n \in \mathbb{N}, \tag{4.16}$$

for some $c_{10} > 0$.

In (4.13), we choose the test function $h = u_n - u_* \in W^{1,p(z)}(\Omega)$. On account of (4.15) and (4.16), we have

$$\lim_{n\to+\infty} \int_\Omega \big(\lambda_n u_n^{r(z)-1} - f(z,u_n)\big)(u_n - u_*)\, dz = 0. \tag{4.17}$$

We know that $\widehat{\lambda}_* = \inf \widehat{\mathcal{L}} > 0$ (see Proposition 3.2) and arguing as in Papageorgiou–Rădulescu–Tang [10], we show that $\widehat{\lambda}_* \in \widehat{\mathcal{L}}$. From Lemma 4.6, we know that $\mathcal{L} \subseteq \widehat{\mathcal{L}}$ and so $\widehat{\lambda}_* \le \lambda_*$. Therefore, we can find $\widehat{u}^*_{\lambda_*} \in \operatorname{int} C_+$, minimal solution of $(\widehat{P}_{\lambda_*})$. Using that $\widehat{u}^*_{\lambda_*} \in \operatorname{int} C_+$, we can find $c_{11} > 0$ such that

$$c_{11} \widehat{d} \le \widehat{u}^*_{\lambda_*}, \tag{4.18}$$

where $\widehat{d}(z) = d(z, \partial\Omega)$ for all $z \in \Omega$.

Let $s \in (1, p_-)$. We have

$$\begin{aligned}\int_\Omega \left(\frac{u_n - u_*}{u_n^{\eta(z)}}\right)^s dz &= \int_\Omega \left(u_n^{1-\eta(z)} \frac{u_n - u_*}{u_n}\right)^s dz \\ &\le c_{11} \int_\Omega \left(\frac{u_n - u_*}{\hat{d}}\right)^s dz \le c_{12} \|u_n - u\|^s_{W_0^{1,s}(\Omega)},\end{aligned}$$

for some $c_{11}, c_{12} > 0$ (see (4.16), (4.12), use the fact that $\widehat{u}^*_{\lambda_*} \le \widehat{u}^*_{\lambda_n}$ for all $n \in \mathbb{N}$ and Hardy's inequality; see Papageorgiou–Winkert [9, p. 682]). It follows that

$$\left\{ \frac{u_n - u_*}{u_n^{\eta(\cdot)}} \right\}_{n \in \mathbb{N}} \subseteq L^s(\Omega) \text{ is bounded}$$

(see (4.15)), thus

$$\left\{ \frac{u_n - u_*}{u_n^{\eta(\cdot)}} \right\}_{n \in \mathbb{N}} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

From (4.15) and at least for a subsequence, we have

$$\frac{|(u_n - u_*)(z)|}{u_n(z)^{\eta(z)}} \leq \frac{|(u_n - u_*)(z)|}{\widehat{u}^*_{\lambda_*}(z)^{\eta(z)}} \leq \frac{|(u_n - u_*)(z)|}{(c_{11}\widehat{d})^{\eta(z)}} \longrightarrow 0 \quad \text{as } n \to +\infty$$

(see (4.12) and (4.18)). So, by Vitali's convergence theorem (see Papageorgiou–Winkert [9, p. 127]), we have

$$\int_\Omega \lambda_n \frac{u_n - u_*}{u_n^{\eta(z)}} \, dz \longrightarrow 0. \tag{4.19}$$

In (4.13), we choose the test function $h = u_n - u_* \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to +\infty$ and use (4.17) and (4.19). Then

$$\lim_{n \to +\infty} \langle V(u_n), u_n - u_* \rangle = 0,$$

so

$$u_n \longrightarrow u_* \quad \text{in } W_0^{1,p(z)}(\Omega) \tag{4.20}$$

(see Proposition 2.2), with $\widehat{u}^*_{\lambda_*} \leq u_*$.

So, from (4.13) and (4.20) in the limit as $n \to +\infty$, we obtain

$$\langle V(u_*), h \rangle = \int_\Omega \left(\lambda_* u_*^{-\eta(z)} + \lambda_* u_*^{r(z)-1} - f(z, u_*) \right) h \, dz \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so $u_* \in S_{\lambda_*}$ and $\lambda_* \in \mathcal{L}$. □

Summarizing, we can state the following exact existence and nonexistence theorem for problem (P_λ).

Theorem 4.8. *If hypotheses H_0 and H_1 hold, then there exists $\lambda_* > 0$ such that*

(a) *for all $\lambda \geq \lambda_*$ problem (P_λ) has at least one solution*

$$u_\lambda \in \operatorname{int} C_+,$$

(b) *for all $\lambda \in [0, \lambda_*)$ problem (P_λ) has no solution.*

Remark 4.9. It will be interesting to know if for $\lambda > \lambda_*$, we have multiplicity solutions (at least two solutions; a bifurcation type theorem). If there is no singular term, this can be done following the argument of Papageorgiou–Rădulescu–Tang [10].

Acknowledgements

The second-named author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This work was completed, while the second-named author was visiting the University of the National Education Commission of Krakow. He is grateful for the kind hospitality of the host university.

REFERENCES

[1] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Heidelberg, 2011.

[2] X.L. Fan, *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations **235** (2007), 397–417.

[3] X.L. Fan, Q.H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. **52** (2003), 1843–1852.

[4] M.E. Filippakis, N.S. Papageorgiou, *Multiple constant sign and nodal solutions for nonlinear elliptic equations with the p-Laplacian*, J. Differential Equations **245** (2008), 1883–1922.

[5] S. Hu, N.S. Papageorgiou, *Research Topics in Analysis, Vol. II. Applications*, Birkhäuser/Springer, Cham, 2024.

[6] G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), 311–361.

[7] N.S. Papageorgiou, P. Winkert, *Existence and nonexistence of positive solutions for singular (p,q)-equations with superdiffusive perturbation*, Results Math. **76** (2021), Paper no. 169.

[8] N.S. Papageorgiou, P. Winkert, *Positive solutions for singular anisotropic (p,q)-equations*, J. Geom. Anal. **31** (2021), 11849–11877.

[9] N.S. Papageorgiou, P. Winkert, *Applied Nonlinear Functional Analysis*, De Gruyter, Berlin, 2024.

[10] N.S. Papageorgiou, V. Rădulescu, X. Tang, *Anisotropic Robin problems with logistic reaction*, Z. Angew. Math. Phys. **72** (2021), Paper no. 94.

[11] N.S. Papageorgiou, V. Rădulescu, Y. Zhang, *Anisotropic singular double phase Dirichlet problems*, Discrete Contin. Dyn. Syst. Ser. S **14** (2021), 4465–4502.

[12] Q. Zhang, *A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions*, J. Math. Anal. Appl. **312** (2005), 24–32.

João Pablo Pinheiro Da Silva (corresponding author)
jpabloufpa@gmail.com
https://orcid.org/0000-0003-3002-8242

Universidade Federal do Pará
Departamento de Matemática
66075-110, Belém, PA, Brazil

Giuseppe Failla
giuseppe.failla@studenti.unime.it
https://orcid.org/0009-0007-7336-4410

Department of Mathematics and Computer Sciences
Physical Sciences and Earth Sciences (MIFT)
University of Messina
Viale Ferdinando Stagno d'Alcontres
98166, Messina, Italy

Leszek Gasiński
leszek.gasinski@uken.krakow.pl
https://orcid.org/0000-0001-8692-6442

University of the National Education Commission
Department of Mathematics
Podchorazych 2, 30-084 Cracow, Poland

Nikolaos S. Papageorgiou
npapg@math.ntua.gr
https://orcid.org/0000-0003-4800-1187

National Technical University
Department of Mathematics
Zografou Campus, Athens 15780, Greece

Center for Applied Mathematics
Yulin Normal University
Yulin 537000, P.R. China

University of Craiova
Department of Mathematics
200585 Craiova, Romania

Received: June 14, 2025.
Accepted: November 11, 2025.