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A COMPREHENSIVE REVIEW ON THE EXISTENCE OF NORMALIZED SOLUTIONS FOR FOUR CLASSES OF NONLINEAR ELLIPTIC EQUATIONS

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Abstract. This paper provides a comprehensive review of recent results concerning the existence of normalized solutions for four classes of nonlinear elliptic equations: Schrödinger equations, Schrödinger–Poisson equations, Kirchhoff equations, and Choquard equations.

Keywords: normalized solutions, Schrödinger equations, Schrödinger–Poisson equations, Kirchhoff equations, Choquard equations.

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1. INTRODUCTION

The Schrödinger equation is one of the most important equations in quantum mechanics. Since its introduction, the existence and various properties of its solutions have been a central focus in both mathematics and physics. Due to its significant applications in phenomena such as Bose-Einstein condensation, many mathematicians have conducted extensive research over the past few decades on the following nonlinear Schrödinger equation:

$$i\frac{\partial \Psi}{\partial t} - \Delta \Psi = g(|\Psi|)\Psi, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, $\Psi = \Psi(x,t)$ denotes the wave function, and $|\Psi(x,t)|^2$ represents the probability density of a single particle being at position x at time t. Therefore, it is natural to impose the normalization condition:

$$\int_{\mathbb{R}^N} |\Psi(x,t)|^2 \, \mathrm{d}x = 1.$$

This applies to a single-particle system. For a multi-particle system, the wave function is expressed as $\psi(x,t) = \sqrt{n}\Psi(x,t)$, and the corresponding normalization condition is determined by the total number of particles n in the system:

$$\int_{\mathbb{R}^N} |\psi(x,t)|^2 \,\mathrm{d}x = n.$$

However, in mathematical studies, the normalization condition is typically assumed to be any positive real number $c \in (0, +\infty)$. For further details, we refer the reader to [1] and the references therein.

In the study of equation (1.1), solutions of the form $\Psi(x,t) = e^{-i\lambda t}u(x)$ are known as standing wave solutions. Substituting this ansatz into equation (1.1) leads to the following elliptic equation:

$$-\Delta u + \lambda u = f(u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $f(u) = g(|u|)u$. A typical example of the nonlinear term is given by

$$f(t) = \sum_{i=1}^{m} a_i |t|^{p_i - 2} t,$$

with $2 < p_1 < p_2 < \cdots < p_m < 2^*$ and $a_i > 0$ for $i = 1, 2, \ldots, m$. Here, 2^* denotes the Sobolev critical exponent: when $N \geq 3$, $2^* = \frac{2N}{N-2}$; when $N = 1, 2$, $2^* = \infty$.

There have been numerous studies on equation (1.2). One of the main approaches involves analyzing the critical points of the following energy functional in a suitable function space:

$$\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \lambda |u|^2\right) \,\mathrm{d}x - \int_{\mathbb{R}^N} F(u) \,\mathrm{d}x,$$

where $F(t) = \int_0^t f(s)\,\mathrm{d}s$. In this setting, λ is treated as a fixed parameter. However, under the normalization constraint, if one aims to solve the constrained problem:

$$\begin{cases} -\Delta u + \lambda u = f(u), & x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2 \,\mathrm{d}x = c, \end{cases} \tag{1.3}$$

where $N \geq 1$, $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $c > 0$ is a given mass, and $\lambda \in \mathbb{R}$ will arise as a Lagrange multiplier depending on the solution $u \in H^1(\mathbb{R}^N)$ and be considered as an unknown in the problem, then the functional must be restricted to the L^2-sphere:

$$\mathcal{S}_c := \left\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c\right\},$$

and the critical points of the functional

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \,\mathrm{d}x$$

on this constraint manifold $\mathcal{S}_c$ are studied. In this case, $\lambda = \lambda_c$ becomes an unknown Lagrange multiplier, and the resulting pair (u_c, λ_c) is referred to as a normalized solution of equation (1.3). From a physical standpoint, such normalized solutions are of particular interest and relevance.

2. SCHÖDINGER EQUATIONS

For the L^2-constraint problem (1.3), if $\inf_{\mathcal{S}_c} \Phi > -\infty$ for any $c > 0$, then it is referred to as the L^2-subcritical case; if $\inf_{\mathcal{S}_c} \Phi = -\infty$ for all $c \in (0, +\infty)$, then it is referred to as the L^2-supercritical case; if there exists $c^* > 0$ such that $\inf_{\mathcal{S}_c} \Phi > -\infty$ for all $c \in (0, c^*]$ and $\inf_{\mathcal{S}_c} \Phi = -\infty$ for all $c \in (c^*, +\infty)$, then it is referred to as the L^2-critical case.

In particular, if $f(u) = |u|^{p-2}u$ with $2 < p \leq 2^*$, we have $F(u) = \frac{1}{p}|u|^p$. The cases $2 < p < 2 + \frac{4}{N}$, $p = 2 + \frac{4}{N}$, and $2 + \frac{4}{N} < p \leq 2^*$ correspond to the L^2-subcritical, L^2-critical, and L^2-supercritical cases, respectively.

In view of Kwong [31], if $2 < p < 2^*$, the following equation

$$-\Delta u + u = u^{p-1}, \quad u > 0, \quad x \in \mathbb{R}^N$$

has a unique positive solution $w_{N,p}$. Let $u(x) := a\, w_{N,p}(bx)$ with $a, b > 0$. Substituting u into (1.3), we obtain

$$\begin{cases} -ab^2 \Delta w_{N,p}(bx) + \lambda a\, w_{N,p}(bx) = a^{p-1} w_{N,p}^{p-1}(bx), \quad x \in \mathbb{R}^N, \\ a^2 \int\limits_{\mathbb{R}^N} w_{N,p}^2(bx)\, \mathrm{d}x = c. \end{cases}$$

This equation holds if and only if

$$\begin{cases} ab^2 = \lambda a \ = a^{p-1}, \\ \frac{a^2}{b^N} \int\limits_{\mathbb{R}^N} w_{N,p}^2\, \mathrm{d}x = c. \end{cases} \tag{2.1}$$

Clearly, (2.1) implies that if $p \neq 2 + \frac{4}{N}$, then (1.3) admits a unique positive normalized solution $u(x) := a\, w_{N,p}(bx)$ for some $a, b > 0$ and $\lambda > 0$; if $p = 2 + \frac{4}{N}$, then (1.3) admits a normalized solution if and only if $c = c_0 := \int_{\mathbb{R}^N} w_{N,p}^2\, \mathrm{d}x$.

2.1. L^2-SUBCRITICAL CASE

For the L^2-subcritical case, we know that $\inf_{\mathcal{S}_c} \Phi > -\infty$ for any $c > 0$. Therefore,

$$m(c) := \inf_{u \in \mathcal{S}_c} \Phi(u)$$

is well defined. If $m(c)$ is achieved, then (1.3) has at least one solution.

This type of problem was first studied in the works of Stuart [41, 42] by employing the bifurcation method. Later, using the minimizing method, Shibata [38] investigated the existence of solutions to (1.3) under the following assumptions:

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $f(0) = 0$,
(F2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$,
(F3) $\lim_{|t|\to\infty} \frac{f(t)}{|t|^{1+\frac{4}{N}}} = 0$,
(F4) there exists $s_0 > 0$ such that $F(s_0) > 0$.

It was proved that there exists $c_0 \geq 0$ such that

(i) if $c > c_0$, then $m(c)$ is achieved and (1.3) has at least one solution,
(ii) if $0 < c < c_0$, then $m(c)$ is not achieved.

In addition, if f also satisfies

(F5) $f(-t) = -f(t)$ for all $t \in \mathbb{R}$,

then for any $k \in \mathbb{N}$, there exists $c_k \geq 0$ such that for $c > c_k$, (1.3) has at least k solutions.

It is important to determine whether $c_0 = 0$. In fact, if

$$\lim_{|t|\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = +\infty,$$

then $c_0 = 0$; if

$$\lim_{|t|\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} < +\infty,$$

then $c_0 > 0$. See [28].

For the special case when $f(u) = |u|^{p-2}u$ with $2 < p < 2 + \frac{4}{N}$, Cazenave and Lions [10] proved that (1.3) admits a solution pair $(u, \lambda) \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \times (0, +\infty)$ using the minimizing method.

Jeanjean and Lu [26, 28] extended the results obtained by Shibata [38] by weakening condition (F3).

2.2. L^2-SUPERCRITICAL CASE

The first paper addressing the L^2-supercritical case is [24], where Jeanjean proved that when $N \geq 2$, for any $c > 0$, (1.3) admits a solution pair $(u_c, \lambda_c) \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \times (0, +\infty)$ under the following conditions:

(F6) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and f is odd,
(F7) there exist $\alpha, \beta \in \mathbb{R}$ satisfying $2 + \frac{4}{N} < \alpha \leq \beta < 2^*$ such that

$$0 < \alpha F(t) \leq f(t)t \leq \beta F(t), \quad \forall\, t \in \mathbb{R} \setminus \{0\}.$$

When $N \geq 1$, it was also shown that a ground state solution of (1.3) exists for any $c > 0$ if f satisfies additionally

(F8) the function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class $\mathcal{C}^1$ and

$$\tilde{F}'(t)t > \left(2 + \frac{4}{N}\right) F(t), \quad \forall\, t \in \mathbb{R} \setminus \{0\}.$$

As demonstrated in their work, it is relatively straightforward to verify that the functional Φ admits a (PS) sequence $\{u_n\}$ at the mountain pass level, owing to the easily verified mountain pass geometry. However, establishing the boundedness of such a sequence presents a significant challenge. To address this issue, Jeanjean [24] introduced a specialized (PS) sequence that satisfies the additional convergence condition $\mathcal{P}(u_n) \to 0$, where $\mathcal{P}$ denotes the Pohozaev-type functional defined by

$$\mathcal{P}(u) := \|\nabla u\|_2^2 - \frac{N}{2} \int\limits_{\mathbb{R}^N} [f(u)u - 2F(u)]\, \mathrm{d}x.$$

A key component of their approach involves the application of the Ekeland variational principle to an auxiliary functional, specifically a fibering map $\tilde{\Phi} : H^1(\mathbb{R}^N) \times \mathbb{R} \to \mathbb{R}$, defined as

$$\tilde{\Phi}(v, t) := \frac{e^{2t}}{2} \|\nabla v\|_2^2 - \frac{1}{e^{Nt}} \int\limits_{\mathbb{R}^N} F\left(e^{Nt/2} v\right) \mathrm{d}x.$$

Importantly, the mountain pass level of $\tilde{\Phi}$ over $\mathcal{S}_c \times \mathbb{R}$ corresponds precisely to that of Φ over $\mathcal{S}_c$.

Later, Bartsch and de Valeriola [2] proved that (1.3) actually possesses an unbounded sequence of pairs of radial solutions $(\pm u_n, \lambda_n)$ provided (F6) and (F7) hold.

It is noteworthy that condition (F7) is regarded as the natural L^2-constrained counterpart to the classical (AR) condition in unconstrained problems. Recently, Chen and Tang [13] weakened (F6) and (F7) to the following assumptions:

(F9) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and

$$\lim_{t \to 0} \frac{f(t)}{|t|^{1+\frac{4}{N}}} = 0, \quad \lim_{|t| \to +\infty} \frac{|f(t)|}{|t|^{2^*-1}} = 0,$$

(F10) there holds

$$\lim_{|t| \to +\infty} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = +\infty,$$

(F11) $0 < \left(2 + \frac{4}{N}\right) F(t) \le f(t)t < \frac{2N}{N-2} F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$,

(F12) there exists $\kappa > \frac{N}{2}$ such that

$$\limsup_{|t| \to +\infty} \frac{[f(t)t - 2F(t)]^\kappa}{t^{2\kappa}[Nf(t)t - (2N+4)F(t)]} < +\infty.$$

By replacing (F12) with the following assumption:

(F12′) there exist $\kappa > \frac{N}{2}$ and $\mathcal{C}_0 > 0$ such that

$$\left[\frac{f(t)t - 2F(t)}{t^2}\right]^\kappa \leq \mathcal{C}_0[Nf(t)t - (2N+4)F(t)], \quad \forall\, t \in \mathbb{R} \setminus \{0\},$$

Chen and Tang [13] proved that (1.3) admits a solution pair $(\bar{u}_c, \lambda_c) \in H^1_{\text{rad}}(\mathbb{R}^N) \times (0, +\infty)$ such that $\Phi(\bar{u}_c) = \inf_{\mathcal{K}_c} \Phi$, where

$$\mathcal{K}_c := \left\{u \in \mathcal{S}_c \cap H^1_{\text{rad}}(\mathbb{R}^N) : \Phi|'_{\mathcal{S}_c}(u) = 0\right\}.$$

In a different direction, Jeanjean and Lu [27] addressed the existence of ground state solutions of (1.3) and removed the requirement $f \in \mathcal{C}^1$ in (F8). They also relaxed (F7) to the following hypotheses:

(F13) $\frac{f(t)t-2F(t)}{|t|^{2+\frac{4}{N}}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$,
(F14) $0 < f(t)t < \frac{2N}{N-2}F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$.

They proved that for any $c > 0$,

(i) if f satisfies (F9), (F10), (F13), and (F14) for $N \geq 3$, then (1.3) admits a ground state solution,
(ii) if f is odd and satisfies (F9), (F10), (F13), and (F14) for $N \geq 5$, then (1.3) admits a positive ground state solution.

Condition (F13) has become a fundamental analytical tool in the study of L^2-constrained elliptic problems, serving as the natural counterpart to the classical Nehari-type strict monotonicity condition in unconstrained settings.

Bieganowski and Mederski [7] introduced a novel minimization scheme on the intersection of closed L^2-balls $\left\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 \leq c\right\}$ and the Pohozaev manifold in $H^1(\mathbb{R}^N)$, replacing the standard constraint sphere $\mathcal{S}_c$. This innovation allowed for the relaxation of (F14) to the following condition:

(F15) $0 < \left(2 + \frac{4}{N}\right) F(t) \preceq f(t)t \preceq \frac{2N}{N-2}F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$,

where $\preceq$ denotes the asymptotic non-strict inequality: for given functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $f_1(s) \preceq f_2(s)$ means that $f_1(s) \leq f_2(s)$ for all $s \in \mathbb{R}$, and for any $\gamma > 0$, there exists $|s| < \gamma$ such that $f_1(s) < f_2(s)$.

2.3. MIXED CASE

In this subsection, we consider the mixed case, which arises when the nonlinearity f includes L^2-subcritical, L^2-critical and L^2-supercritical growth terms. The specific cases

$$\begin{cases} -\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c, \end{cases}$$

and

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c, \end{cases} \tag{2.2}$$

have been studied by Soave [39, 40] and Jeanjean, Jendrej, Le and Visciglia [30], where $2 < q \leq 2 + \frac{4}{N} \leq p < 2^*$ and $\mu > 0$.

Solutions of (2.2) are critical points of the energy functional

$$\Phi_\mu(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} - \frac{\mu}{q}\|u\|_q^q, \quad \forall\, u \in H^1(\mathbb{R}^N), \tag{2.3}$$

restricted to the constraint $\mathcal{S}_c$, and Φ_μ is unbounded from below on $\mathcal{S}_c$ due to the presence of the Sobolev critical exponent. Define the Pohozaev manifold

$$\mathcal{M}_\mu(c) := \{u \in \mathcal{S}_c : \mathcal{P}_\mu(u) = 0\},$$

where the functional $\mathcal{P}_\mu$ is given by

$$\mathcal{P}_\mu(u) := \|\nabla u\|_2^2 - \|u\|_{2^*}^{2^*} - \mu\gamma_q\|u\|_q^q, \quad \forall\, u \in H^1(\mathbb{R}^N), \tag{2.4}$$

where $\gamma_q := \frac{N(q-2)}{2q}$. Soave [40] first studied the existence and nonexistence of normalized solutions for equations with mixed dispersion using the fibration method of Pohozaev, based on the decomposition of the Pohozaev manifold

$$\mathcal{M}_\mu(c) = \left\{u \in \mathcal{S}_c : \tilde{\phi}_u'(0) = 0\right\} = \mathcal{M}_\mu^-(c) \cup \mathcal{M}_\mu^0(c) \cup \mathcal{M}_\mu^+(c), \tag{2.5}$$

where $\tilde{\phi}_u(t) = \Phi_\mu\left(e^{Nt/2}u(e^t x)\right)$ for $u \in H^1(\mathbb{R}^N)$ and $t \in \mathbb{R}$, and

$$\mathcal{M}_\mu^\pm(c) := \left\{u \in \mathcal{M}_\mu(c) : \tilde{\phi}_u''(0) \gtrless 0\right\}, \quad \mathcal{M}_\mu^0(c) := \left\{u \in \mathcal{M}_\mu(c) : \tilde{\phi}_u''(0) = 0\right\}. \tag{2.6}$$

Soave [40] proved that there exists a constant $\alpha(N,q) > 0$ depending on N, q, and the best constant in the Gagliardo–Nirenberg inequality such that, if $\mu c^{\frac{(1-\gamma_q)q}{2}} < \alpha(N,q)$, then (2.2) admits a ground state solution $\tilde{u}$. Moreover,

(i) if $2 < q < 2 + \frac{4}{N}$, $\tilde{u}$ corresponds to a local minimizer satisfying $\inf_{\mathcal{M}_\mu^+(c)} \Phi_\mu = \Phi_\mu(\tilde{u}) < 0$,

(ii) if $2 + \frac{4}{N} \leq q < 2^*$, $\tilde{u}$ is a mountain-pass type solution with $0 < \inf_{\mathcal{M}_\mu^-(c)} \Phi_\mu = \Phi_\mu(\tilde{u}) < \frac{1}{N}\mathcal{S}^{\frac{N}{2}}$, where $\mathcal{S}$ denotes the best constant in the Sobolev inequality.

In that paper, Soave proposed the following open problem: Does $\Phi_\mu|_{\mathcal{S}_c}$ have a critical point of mountain-pass type in the L^2-subcritical case $2 < q < 2 + \frac{4}{N}$?

Jeanjean and Le [25] ($N \geq 4$) and Wei and Wu [45] ($N = 3$) gave a complete positive answer to this open problem.

By introducing the set

$$V(c) := \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 < \rho_0\right\}$$

with the property

$$m_\mu(c) := \inf_{u\in V(c)} \Phi_\mu(u) < 0 < \inf_{u\in\partial V(c)} \Phi_\mu(u),$$

where $\partial V(c) := \left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 = \rho_0\right\}$, and defining

$$\rho_0 := \left[\frac{2^*\mu\alpha_0\mathcal{C}_{N,q}^q\mathcal{S}^{2^*/2}}{q\alpha_2}\right]^{\frac{2}{\alpha_2+\alpha_0}} c_0^{\frac{\alpha_1}{\alpha_0+\alpha_2}},$$

$$c_0 := \left[\frac{2^*\alpha_0\mathcal{S}^{2^*/2}}{\alpha_0+\alpha_2}\left(\frac{q\alpha_2}{2^*\mu\alpha_0\mathcal{C}_{N,q}^q\mathcal{S}^{2^*/2}}\right)^{\frac{\alpha_2}{\alpha_0+\alpha_2}}\right]^{\frac{N}{2}},$$

with

$$\alpha_0 := 2 - \frac{N(q-2)}{2}, \quad \alpha_1 := \frac{2N-q(N-2)}{2}, \quad \alpha_2 := \frac{4}{N-2},$$

Jeanjean and Le [25] investigated the orbital stability of ground state solutions under the condition $2 < q < 2+\frac{4}{N}$ for any $c \in (0, c_0)$. This local minimizer structure enables the search for a solution at a mountain-pass level. In addition to the decomposition (2.5) similar to [40], Wei and Wu [45] complemented Soave's results using new energy estimates. Specifically:

(i) if $2 < q < 2+\frac{4}{N}$ and $\mu c^{\frac{(1-\gamma_q)q}{2}} < \alpha(N,q)$, they obtained a second solution $u_{\mu,c}^- \in \mathcal{M}_\mu^-(c)$ such that $\Phi_\mu(u_{\mu,c}^-) = \inf_{\mathcal{M}_\mu^-(c)} \Phi_\mu$ and $0 < \Phi_\mu(u_{\mu,c}^-) < \frac{1}{N}\mathcal{S}^{\frac{N}{2}}$,

(ii) if $q = 2+\frac{4}{N}$, there are no ground state solutions for $\mu c^{\frac{(1-\gamma_q)q}{2}} \geq \alpha(N,q)$,

(iii) if $2+\frac{4}{N} < q < 2^*$, the existence range $\mu c^{\frac{(1-\gamma_q)q}{2}} < \alpha(N,q)$ extends to all $c > 0$.

Instead of using the fibration argument of Pohozaev as in [40, 45], Jeanjean and Le [25] introduced a new set related to the mountain-pass level based on the decomposition

$$\mathcal{M}_\mu(c) = \widehat{\mathcal{M}}_\mu^-(c) \cup \widehat{\mathcal{M}}_\mu^+(c), \quad \text{with } \widehat{\mathcal{M}}_\mu^\pm(c) := \{u \in \mathcal{M}_\mu(c) : \Phi_\mu(u) \gtrless 0\},$$

and established its connection with $V(c)$. They proved the existence of a second solution $v_c \in \mathcal{S}_c$ of mountain-pass type satisfying $0 < \Phi_\mu(v_c) < m_\mu(c) + \frac{1}{N}\mathcal{S}^{\frac{N}{2}}$ for any $c \in (0, c_0)$ and $N \geq 4$. This second solution is not a ground state solution.

Chen and Tang [13] established novel critical point theorems on manifolds to construct bounded (PS) sequences in the context of mixed dispersion, offering a technically simpler alternative to Hirata and Tanaka [23] and the Ghoussoub minimax principle [21], which relies on topological arguments. Specifically, for the case $2 < q < 2+\frac{4}{N}$, they introduced a new mountain-pass geometry rooted in the local minimizer and constructed a sequence of test functions applicable across all spatial dimensions $N \geq 3$, thereby obtaining the strict upper bound through refined energy estimates.

For the remaining cases namely $2+\frac{4}{N}<q<2^*$ and $q=2+\frac{4}{N}$, they presented a unified framework that applies consistently to both L^2-subcritical and L^2-critical settings, as well as to dimensions $N=3$ and $N\geq 4$.

The following two theorems were obtained in Chen and Tang [13].

Theorem 2.1. *Let $N\geq 3$, $2<q<2+\frac{4}{N}$, $\mu>0$, and $c\in(0,c_0)$. Then (2.2) has a second solution pair $(u_c,\lambda_c)\in H^1_{\mathrm{rad}}(\mathbb{R}^N)\times(0,+\infty)$ such that*

$$0<\Phi_\mu(u_c)<m_\mu(c)+\frac{1}{N}\mathcal{S}^{\frac{N}{2}}.$$

Theorem 2.2. *Let $N\geq 3$, $c>0$, and $2+\frac{4}{N}\leq q<2^*$. Then (2.2) has a solution pair $(\bar{u}_c,\lambda_c)\in H^1(\mathbb{R}^N)\times(0,+\infty)$ such that*

$$\Phi_\mu(\bar{u}_c)=\inf_{\mathcal{M}_\mu(c)}\Phi_\mu$$

(i) *for any $\mu>0$ if $2+\frac{4}{N}<q<2^*$,*

(ii) *for any $0<\mu\leq\frac{1}{2\gamma_{\bar{q}}c^{2/N}\mathcal{C}^{\bar{q}}_{N,\bar{q}}}$ if $q=\bar{q}=2+\frac{4}{N}$.*

Chen and Tang [12] studied the following nonlinear Schrödinger equation with mixed dispersion and critical exponential growth:

$$\begin{cases}-\Delta u+\lambda u=\mu|u|^{p-2}u+\left(e^{u^2}-1-u^2\right)u, & x\in\mathbb{R}^2,\\ \int\limits_{\mathbb{R}^2}u^2\,\mathrm{d}x=c,\end{cases}\tag{2.7}$$

where $c>0$ is a given constant, and $\lambda\in\mathbb{R}$ is a Lagrange multiplier depending on the solution $u\in H^1(\mathbb{R}^2)$, not given a priori. By the Trudinger–Moser inequality, the functional

$$\Phi(u):=\frac{1}{2}\int\limits_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x-\frac{\mu}{p}\int\limits_{\mathbb{R}^2}|u|^p\,\mathrm{d}x-\frac{1}{2}\int\limits_{\mathbb{R}^2}\left(e^{u^2}-1-u^2-\frac{u^4}{2}\right)\mathrm{d}x$$

is of class $\mathcal{C}^1$ on $H^1(\mathbb{R}^2)$. A critical point of Φ restricted to the mass constraint

$$\mathcal{S}_c:=\left\{u\in H^1(\mathbb{R}^2):\|u\|_2^2=c\right\}$$

corresponds to a normalized solution of (2.7).

Define the L^2-Pohozaev functional

$$\mathcal{P}(u):=\|\nabla u\|_2^2-\frac{\mu(p-2)}{p}\|u\|_p^p-\int\limits_{\mathbb{R}^2}\left[\left(u^2-1\right)e^{u^2}+1-\frac{u^4}{2}\right]\mathrm{d}x,\quad\forall\,u\in H^1(\mathbb{R}^2),$$

and the L^2-Pohozaev manifold

$$\mathcal{M}(c):=\left\{u\in\mathcal{S}_c:\mathcal{P}(u)=0\right\}.$$

Chen and Tang [12] proved the following three theorems.

Theorem 2.3. *Let $2 < p < 4$. Then for any $\mu > 0$, there exist $c_0 = c_0(\mu) > 0$ and $s_0 = s_0(\mu) \in \left(0, \frac{\pi}{2}\right]$ such that, for any $c \in (0, c_0)$, the following statements hold:*

(i) *(2.7) has a ground state solution which is a minimizer of Φ in the set $\mathcal{S}_c \cap A_{s_0}$, where*

$$A_{s_0} := \left\{u \in H^1(\mathbb{R}^2) : \|\nabla u\|_2^2 < s_0\right\}.$$

Moreover, any ground state to (2.7) is a minimizer of Φ on $\mathcal{S}_c \cap A_{s_0}$, that is

$$\bar{u} \in \mathcal{K}(c) \quad \text{and} \quad \Phi(\bar{u}) = \inf_{\mathcal{K}(c)} \Phi \implies \bar{u} \in \mathcal{S}_c \cap A_{s_0} \quad \text{and} \quad \Phi(\bar{u}) = \inf_{\mathcal{S}_c \cap A_{s_0}} \Phi := m(c),$$

where

$$\mathcal{K}(c) := \left\{u \in \mathcal{S}_c : \ \Phi\big|'_{\mathcal{S}_c}(u) = 0\right\}.$$

(ii) *(2.7) has a second solution $u_c \in \mathcal{S}_c$ which satisfies*

$$0 < \Phi(u_c) < m(c) + 2\pi. \tag{2.8}$$

Theorem 2.4. *Let $p = 4$ and $c > 0$. Then for any $0 < \mu < \frac{2}{c\mathcal{C}_4^4}$, (2.7) has a ground state $\bar{u}_c \in H^1(\mathbb{R}^2)$ with some $\lambda_c > 0$, satisfying*

$$\Phi(\bar{u}_c) = \inf_{\mathcal{M}(c)} \Phi = \inf_{u \in \mathcal{S}_c} \max_{t>0} \Phi(tu_t), \tag{2.9}$$

where $u_t(x) := u(tx)$ for $t > 0$ and $x \in \mathbb{R}^2$ and $\mathcal{C}_4 > 0$ is a Gagliardo–Nirenberg constant.

Theorem 2.5. *Let $4 < p < \infty$ and $c > 0$. Then for any $\mu > 0$, (2.7) has a ground state $\bar{u}_c \in H^1(\mathbb{R}^2)$ with some $\lambda_c > 0$, satisfying (2.9).*

3. SCHÖDINGER–POISSON EQUATIONS

Consider the following nonlinear Schrödinger–Poisson equation

$$\begin{cases} -\Delta u + \lambda u + \mu \left(|x|^{-1} * u^2\right) u = |u|^{p-2}u, \quad x \in \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^3} u^2 \, \mathrm{d}x = c, \end{cases} \tag{3.1}$$

where $c > 0$ is a given constant, and $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier depending on the solution $u \in H^1(\mathbb{R}^3)$, and is not given a priori.

If $p \in (2, 3)$, Bellazzini and Siciliano [5] proved that minimizers exist provided $c > 0$ is sufficiently small, by establishing a strong subadditive inequality. Specifically, using techniques introduced in [8], the existence of minimizers for $p = \frac{8}{3}$ was proved in [37] for $c \in (0, c_0)$ with some suitable $c_0 > 0$. The case $p \in (3, 10/3)$ was considered in [6], where a minimizer was obtained for $c > 0$ sufficiently large. When $p \in \left[3, \frac{10}{3}\right]$, Jeanjean and Luo [29] established a threshold value of $c > 0$ separating existence and nonexistence of minimizers. Based on whether $\Phi|_{\mathcal{S}_c}$ is bounded from below, the value $p = \frac{10}{3}$ is referred to as the L^2-critical exponent for the Schrödinger–Poisson equation (3.1). If $p \in \left(\frac{10}{3}, 6\right)$, Φ is unbounded from below on $\mathcal{S}_c$. By constructing a bounded Palais–Smale sequence via a mountain-pass argument on $\mathcal{S}_c$, Bellazzini, Jeanjean, and Luo [4] found critical points of the functional Φ for $c > 0$ sufficiently small.

Later, Chen, Tang, and Yuan [17] considered the existence of solutions for the following nonlinear Schrödinger–Poisson equation with a general nonlinearity:

$$\begin{cases} -\Delta u + \lambda u + \mu \left(|x|^{-1} * u^2\right) u = f(u), \quad x \in \mathbb{R}^3, \\ \int\limits_{\mathbb{R}^3} u^2 \, \mathrm{d}x = c, \end{cases} \tag{3.2}$$

where $\mu > 0$, $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and the case $f(u) = |u|^{p-2}u$ with $p \in \left(3, \frac{10}{3}\right) \cup \left(\frac{10}{3}, 6\right)$ is included. Wang and Qian [43] proved that if f satisfies (F6)-(F8) with $N = 3$, then there exists $c_0 > 0$ such that for any $c \in (0, c_0)$, (3.2) admits a positive ground state $u \in \mathcal{S}_c$ and the associated Lagrange multiplier λ is positive.

Xie, Chen, and Shi [46] proved that if f satisfies (F9), (F10), (F13), and (F14) with $N = 3$, then there exists $c_0 > 0$ such that for any $c \in (0, c_0)$, (3.2) admits a ground state $u \in \mathcal{S}_c$.

Cingolani and Jeanjean [18] considered the following nonlinear Schrödinger–Poisson equation

$$\begin{cases} -\Delta u + \lambda u + \mu \left(\log|x| * u^2\right) u = a|u|^{p-2}u, \quad x \in \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} u^2 \, \mathrm{d}x = c, \end{cases} \tag{3.3}$$

where $c > 0$ is a given constant, and $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier depending on the solution $u \in X$, and is not given a priori. Here

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \int\limits_{\mathbb{R}^2} [1 + \ln(1 + |x|)]u^2 \, \mathrm{d}x < \infty \right\}$$

is equipped with the norm

$$\|u\|_X = \left(\int\limits_{\mathbb{R}^2} \left[|\nabla u|^2 + u^2 + \ln(1 + |x|)u^2\right] \mathrm{d}x \right)^{1/2}.$$

Normalized solutions to (3.3) can be obtained as critical points of the energy functional $\Phi : X \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2} \int\limits_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x + \frac{\mu}{4} \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \log|x - y| u^2(x) u^2(y) \, \mathrm{d}x \, \mathrm{d}y - \frac{a}{p} \int\limits_{\mathbb{R}^2} |u|^p \, \mathrm{d}x. \tag{3.4}$$

For a prescribed $c > 0$, a solution of (3.3) can be obtained as a critical point of Φ constrained to the sphere

$$\mathcal{S}_c = \left\{u \in X : \|u\|_2^2 = c\right\}.$$

Define the Pohozaev functional

$$\mathcal{P}(u) = \|\nabla u\|_2^2 - \frac{\mu c^2}{4} - \frac{a(p-2)}{p}\|u\|_p^p.$$

Let

$$m_c := \inf_{u \in \mathcal{S}_c} \Phi(u). \tag{3.5}$$

Cingolani and Jeanjean [18] proved that for any $c > 0$ and $\mu > 0$, the infimum m_c defined in (3.5) is achieved if one of the following three conditions holds:

(i) $a \leq 0$ and $p > 2$,
(ii) $a > 0$ and $p < 4$,
(iii) $a > 0$, $p = 4$, and $c < \frac{2}{aK_{GN}}$, where K_{GN} is the best constant in the Gagliardo–Nirenberg inequality.

Set

$$c_0 := 2\left[\frac{p(p-4)^{(p-2)/2}}{a(p-2)^{p/2}\mu^{(p-4)/2}K_{GN}}\right]^{\frac{1}{p-3}}.$$

For any $u \in \mathcal{S}_c$, define $g_u : (0, +\infty) \to \mathbb{R}$ by

$$g_u(t) := \frac{t^2}{2}\|\nabla u\|_2^2 + \frac{\mu}{4}I_0(u) - \frac{\mu c^2}{4}\log t - \frac{at^{p-2}}{p}\|u\|_p^p.$$

Clearly, g_u is $\mathcal{C}^2$ on $(0, +\infty)$, and we have

$$g_u'(t) = \frac{1}{t}\left[t^2\|\nabla u\|_2^2 - \frac{\mu c^2}{4} - \frac{at^{p-2}}{p}\|u\|_p^p\right] = \frac{1}{t}\mathcal{P}(tu_t), \quad \forall t > 0.$$

Define

$$\Lambda(c) := \{u \in \mathcal{S}_c : \mathcal{P}(u) = 0\} = \{u \in \mathcal{S}_c : g_u'(1) = 0\}.$$

Let

$$\Lambda^+(c) := \{u \in \mathcal{S}_c : g_u'(1) = 0,\ g_u''(1) > 0\},$$
$$\Lambda^-(c) := \{u \in \mathcal{S}_c : g_u'(1) = 0,\ g_u''(1) < 0\}$$

and

$$\Lambda^0(c) := \{u \in \mathcal{S}_c : g_u'(1) = 0,\ g_u''(1) = 0\}.$$

When $\mu > 0$, $a > 0$, and $p > 4$, Cingolani and Jeanjean [18] proved that for any $c < c_0$, $\Lambda^0(c) = \emptyset$, and there exist $u^+ \in \Lambda^+(c)$ and $u^- \in \Lambda^-(c)$ such that

$$\Phi(u^+) = \inf_{\Lambda^+(c)} \Phi, \quad \Phi(u^-) = \inf_{\Lambda^-(c)} \Phi.$$

Moreover, u^+ and u^- are critical points of Φ restricted to $\mathcal{S}_c$.

Chen and Tang [11] and Chen, Shi and Tang [16] introduced the natural constraint

$$E_{as} := X \cap H^1_{as},$$

where

$$H^1_{as} = \left\{ u \in H^1(\mathbb{R}^2) : u(x) = u(|x_1|, |x_2|),\ \forall\, x \in \mathbb{R}^2 \right\}$$

is endowed with the norm

$$\|u\|_X = \left(\int\limits_{\mathbb{R}^2} \left[|\nabla u|^2 + u^2 + \ln(2+|x|)u^2 \right] \mathrm{d}x \right)^{1/2}.$$

They demonstrated that the critical points of Φ constrained to E_{as} correspond to genuine critical points in X.

Chen, Rădulescu, and Tang [15] considered the following nonlinear Schrödinger equation with critical exponential growth

$$\begin{cases} -\Delta u + \lambda u + \mu \left(\log|x| * u^2 \right) u = \left(e^{u^2} - 1 - u^2 \right) u, \quad x \in \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} u^2 \,\mathrm{d}x = c, \end{cases} \tag{3.6}$$

where $c > 0$ is a given constant, and $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier depending on the solution $u \in X$, and is not given a priori.

The energy functional is

$$\begin{aligned} \Phi(u) = \frac{1}{2} \int\limits_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x &+ \frac{\mu}{4} \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} \log|x-y| u^2(x) u^2(y) \,\mathrm{d}x \,\mathrm{d}y \\ &- \frac{1}{2} \int\limits_{\mathbb{R}^2} \left(e^{u^2} - 1 - u^2 - \frac{u^4}{2} \right) \mathrm{d}x, \end{aligned}$$

and the constraint sphere is

$$\mathcal{S}_c = \left\{ u \in E_{as} : \|u\|_2^2 = c \right\}.$$

The Pohozaev functional is

$$\mathcal{P}(u) = \int\limits_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x - \frac{\mu c^2}{4} - \int\limits_{\mathbb{R}^2} \left[\left(u^2 - 1 \right) e^{u^2} + 1 - \frac{u^4}{2} \right] \mathrm{d}x.$$

Let $c_1 = c_1(\mu) > 0$ be the unique root of

$$\frac{19\pi}{72} = \frac{\mu \tau^2}{4} + \frac{2\tau^{3/2}}{3\sqrt{3\pi}} \sum_{k=0}^{\infty} \frac{(k+2)\left[4^{k+2}(k+1)+1\right]}{(k+1)(k+3)!} \left(\sqrt{\frac{\tau}{3\pi}} \right)^k.$$

Let $c_2 = c_2(\mu) > 0$ be the unique root of

$$c = \sqrt{\frac{2}{\mu}}\,\eta(c),$$

where $\eta(c) > 0$ is the unique root of

$$\frac{2c}{\pi}\sum_{k=3}^{\infty}\frac{(k-1)^2\left[4^{k-1}(k-2)+1\right]}{(k-2)k!}\left(\frac{\tau^2\sqrt{c}}{\pi}\right)^{k-2} + \frac{\tau^2\left(4\pi^2-3\pi\tau^2+\tau^4\right)}{2\pi\left(\pi-\tau^2\right)^3} = 1.$$

Chen, Rădulescu, and Tang [15] established the following two theorems.

Theorem 3.1. *For any $\mu > 0$ and any $c \in (0, c_1)$, (3.1) has a solution pair $(u_c, \lambda_c) \in \mathcal{S}_c \times \mathbb{R}$ such that*

$$u_c \in \mathcal{S}_c \cap A_{\pi/3}, \quad u_c > 0, \quad \Phi(u_c) = m(c) := \inf_{\mathcal{S}_c \cap A_{\pi/3}} \Phi.$$

Theorem 3.2. *Let $c_0 = \min\{c_1, c_2\}$. For any $\mu > 0$ and any $c \in (0, c_0)$, (3.1) has a second solution pair $(u_c, \lambda_c) \in \mathcal{S}_c \times \mathbb{R}$ such that*

$$0 < \Phi(u_c) < m(c) + 2\pi.$$

4. KIRCHHOFF EQUATIONS

Consider the following nonlinear Kirchhoff equation:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u + \lambda u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2\,\mathrm{d}x = c, \end{cases} \tag{4.1}$$

where $N = 1, 2, 3$, $a, b > 0$, and $c > 0$ are given constants, $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier depending on the solution $u \in H^1(\mathbb{R}^N)$, and is not given a priori, with $2 < p < 2^*$.

Normalized solutions to (4.1) can be obtained as critical points of the energy functional $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}\|u\|_p^p \tag{4.2}$$

restricted to the constraint sphere

$$\mathcal{S}_c := \left\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c\right\}. \tag{4.3}$$

To the best of our knowledge, the first work on normalized solutions for equation (4.1) was conducted by Ye [48–50], who systematically investigated this problem across a series of papers. Specifically, in [48], the author analyzed minimizers of the following minimization problem:

$$m(c) := \inf_{u \in \mathcal{S}_c} \Phi(u). \tag{4.4}$$

By employing a scaling technique combined with the concentration-compactness principle, it was established that there exists a threshold value $c_p^* \geq 0$ such that $m(c)$ is achieved if and only if: $c > c_p^*$ for $0 < p \leq 2 + \frac{4}{N}$, or $c \geq c_p^*$ for $2 + \frac{4}{N} < p < 2 + \frac{8}{N}$. Furthermore, no minimizers exist for (4.4) when $p \geq 2 + \frac{8}{N}$, and in fact, $m(c) = -\infty$ holds for $2 + \frac{8}{N} < p < 2^*$. Nevertheless, a mountain pass critical point for Φ constrained on $\mathcal{S}_c$ was identified in this regime. In [49], Ye considered the case $p = 2 + \frac{8}{N}$ and proved the existence of a mountain pass critical point on $\mathcal{S}_c$ when $c > c^*$. Moreover, for $0 < c < c^*$, the existence of minimizers was demonstrated by introducing a perturbation functional to Φ. In [50], the asymptotic behavior of critical points of Φ on $\mathcal{S}_c$ was examined for $p = 2 + \frac{8}{N}$. Similar perturbation techniques for $p = 2 + \frac{8}{N}$ were also explored in [52]. Utilizing refined scaling techniques and energy estimates, Zeng and Zhang [53] improved upon the results of [48], proving the existence and uniqueness of minimizers for $0 < p < 2 + \frac{8}{N}$, as well as the existence and uniqueness of mountain pass type critical points on the L^2-normalized manifold for $2 + \frac{8}{N} < p < 2^*$ or $p = 2 + \frac{8}{N}$ with $c > c^*$. Luo and Wang [35] further investigated the multiplicity of normalized solutions of (4.1) in dimension $N = 3$ for $\frac{14}{3} < p < 6$.

Cazenave [9] considered the number of positive normalized solutions to the following Schrödinger equation:

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c. \end{cases} \tag{4.5}$$

It was proved that if $2 < p < 2^*$ and $p \neq 2 + \frac{4}{N}$, then up to translations, there exists a unique positive normalized solution to (4.5) for any $c > 0$; and if $p = 2 + \frac{4}{N}$, there exists $c_0 > 0$ such that (4.5) admits a positive normalized solution if and only if $c = c_0$. A key ingredient in this result is the uniqueness of positive solutions for fixed frequency, established in [31]. Setting

$$k := a + b \int\limits_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x,$$

then (4.1) is equivalent to the following Schrödinger equation:

$$\begin{cases} -\Delta v + k^{-1}\lambda v = |v|^{p-2}v, \quad x \in \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} v^2 \, \mathrm{d}x = k^{\frac{2}{p-2}} c, \end{cases} \tag{4.6}$$

where k satisfies an algebraic equation. Based on this observation and the results in [9], Qi and Zou [36] investigated the exact number of positive normalized solutions for (4.1) and proved the following:

(1) For $2+\frac{4}{N}<p=\min\{2+\frac{8}{N},2^*\}$, define

$$c^* := \left[\frac{16}{N(p-2)}\right]^{\frac{4}{2N-p(N-2)}} \left[\frac{b}{N(p-2)-4}\right]^{\frac{N(p-2)-4}{2N-p(N-2)}} \cdot \left[\frac{a}{8-N(p-2)}\right]^{\frac{8-N(p-2)}{2N-p(N-2)}} \|Q_p\|_2^{\frac{4(p-2)}{2N-p(N-2)}},$$

where Q_p is the unique positive solution of

$$-\frac{N(p-2)}{4}\Delta u+\frac{2N-p(N-2)}{4}u=|u|^{p-2}u$$

satisfying $u(0)=\max_{\mathbb{R}^N} u$. Then:

(i) if $c>c_*$, (4.1) admits exactly two positive normalized solutions,
(ii) if $c=c_*$, (4.1) admits a unique positive normalized solution,
(iii) if $c<c_*$, (4.1) has no positive normalized solution.

(2) If $2<p<2+\frac{4}{N}$, then (4.1) admits a unique positive normalized solution.

For the case $2+\frac{8}{N}\le p<2^*$, they proved the following:

(1) For $p=2+\frac{8}{N}$, define

$$c_* := \left(\frac{b}{2}\right)^{\frac{4}{N-4}} \|Q_{2+\frac{8}{N}}\|_2^{\frac{4}{N-4}}.$$

Then:

(i) if $0<c\le c_*$, (4.1) has no positive normalized solution,
(ii) if $c>c_*$, (4.1) admits a unique positive normalized solution.

(2) If $2+\frac{8}{N}<p<2^*$, then (4.1) admits a unique positive normalized solution.

For the case $p=2+\frac{4}{N}$, define

$$c_* := a^{\frac{N}{2}} \|Q_{2+\frac{8}{N}}\|_2^2.$$

Then:

(i) if $0<c\le c_*$, (4.1) has no positive normalized solution,
(ii) if $c>c_*$, (4.1) admits a unique positive normalized solution.

Based on the above works, He, Lv, Zhang, and Zhong [22] investigated the existence of normalized solutions for the following Kirchhoff equation with general nonlinearity:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+\lambda u=f(u), \quad x\in\mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2\,\mathrm{d}x=c, \end{cases} \tag{4.7}$$

where $N=1,2,3$, $a,b>0$, and $c>0$ are given constants, and $\lambda\in\mathbb{R}$ arises as a Lagrange multiplier depending on the solution $u\in H^1(\mathbb{R}^N)$, and is not given a priori.

The energy functional $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\Phi(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x.$$

Under assumptions (F6), (F8), and the following condition:

(F16) there exist $\alpha, \beta \in \mathbb{R}$ satisfying

$$\begin{cases} 2 + \frac{8}{N} < \alpha \leq \beta < 2^* = \frac{2N}{N-2}, & N = 3, \\ 2 + \frac{8}{N} < \alpha \leq \beta < +\infty, & N = 1, 2, \end{cases}$$

such that

$$0 < \alpha F(t) \leq f(t)t \leq \beta F(t), \quad \forall\, t \in \mathbb{R} \setminus \{0\},$$

He *et al.* [22] proved that (4.7) admits a ground state solution $(u_c, \lambda_c) \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \times (0, +\infty)$.

Wang and Qian [44] proved that (4.7) admits a ground state solution for any $c > 0$, if $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ satisfies the following conditions:

(F17) $\lim_{t\to 0} \frac{f(t)}{|t|^{1+\frac{8}{N}}} = 0$,

(F18) $\lim_{|t|\to+\infty} \frac{|f(t)|}{|t|^5} = 0$ for $N = 3$, and $\lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\gamma t^2}} = 0$ for all $\gamma > 0$ for $N = 2$,

(F19) $\lim_{|t|\to+\infty} \frac{F(t)}{|t|^{2+\frac{8}{N}}} = +\infty$,

(F20) $\frac{f(t)t - 2F(t)}{|t|^{2+\frac{8}{N}}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$,

(F21) $f(t)t < 6F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$ when $N = 3$.

In particular, if f is odd, then (4.7) admits a positive ground state solution for any $c > 0$, and the associated Lagrange multiplier λ is positive.

Consider the following nonlinear Kirchhoff equation with Sobolev critical growth:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\right)\Delta u + \lambda u = u^5 + \mu|u|^{q-2}u, & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2\,\mathrm{d}x = c, \end{cases} \tag{4.8}$$

where $c > 0$ and $\mu > 0$ are given constants, and $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier depending on the solution $u \in H^1(\mathbb{R}^3)$, and is not given a priori.

The energy functional $\Phi : H^1(\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$\Phi(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x\right)^2 - \frac{1}{6}\int_{\mathbb{R}^3} u^6\,\mathrm{d}x - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q\,\mathrm{d}x, \tag{4.9}$$

and the Pohozaev functional $\mathcal{P} : H^1(\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$\mathcal{P}(u) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \|u\|_6^6 - \mu\gamma_q\|u\|_q^q. \tag{4.10}$$

When $\frac{14}{3} \leq q < 6$, Zhang and Han [51] proved that for any $c > 0$, (4.8) admits a solution $\tilde{u}$ such that

$$0 < \Phi(\tilde{u}) = \inf_{\mathcal{M}(c)} \Phi,$$

where $\mathcal{M}(c) := \{u \in \mathcal{S}_c : \mathcal{P}(u) = 0\}$.

When $2 < q < \frac{10}{3}$ or $\frac{14}{3} \le q < 6$, Li, Luo and Yang [32] proved that for any $c > 0$,

(i) if $2 < q < \frac{10}{3}$, then for $\mu \in (0, \mu_*)$, (4.8) admits a ground state solution $\tilde{u}$, and $\tilde{u}$ corresponds to a local minimizer satisfying

$$\inf_{\|\nabla u\|_2^2 < R_0(c,\mu)} \Phi = \Phi(\tilde{u}) < 0$$

for some $R_0(c, \mu) > 0$,

(ii) if $\frac{14}{3} < q < 6$, for $\mu \in (0, +\infty)$, (4.8) admits a solution of mountain-pass type $\tilde{u}$ with

$$0 < \Phi(\tilde{u}) < \Theta^* := \frac{ab\mathcal{S}^3}{4} + \frac{b^3\mathcal{S}^6}{24} + \frac{\left(4a\mathcal{S} + b^2\mathcal{S}^4\right)^{3/2}}{24}.$$

Define an auxiliary functional

$$\Psi(u) = \frac{(b^2\mathcal{S}^4 + 4a\mathcal{S})^{\frac{3}{2}}}{24}\left[\left(1 + \frac{4b}{b^2\mathcal{S}^3 + 4a}\|\nabla u\|_2^2\right)^{\frac{3}{2}} - 1\right] + \frac{b^2\mathcal{S}^3}{4}\|\nabla u\|_2^2 + \Phi(u),$$

and introduce the following constants:

$$s_0 := \left[\frac{(10-3q)a\mathcal{S}^3}{6-q}\right]^{\frac{1}{2}}, \tag{4.11}$$

$$c_0 := \left[\frac{4aq}{3(6-q)\mu\mathcal{C}_q^q}\left(\frac{4q}{3(10-3q)\mu\mathcal{C}_q^q\mathcal{S}^3}\right)^{\frac{3q-10}{3(6-q)}}\right]^{\frac{3}{2}}, \tag{4.12}$$

$$\begin{aligned} c_1 := \Bigg\{ \left[\frac{4q}{\mu(6-q)\mathcal{C}_q^q}\right]\Bigg[\left(\frac{a}{3} + \frac{b^2\mathcal{S}^3}{6} + \frac{b\mathcal{S}\sqrt{b^2\mathcal{S}^4 + 4(a+bs_0)\mathcal{S}}}{12}\right)s_0^{\frac{10-3q}{4}} \\ + \frac{b}{12}s_0^{\frac{14-3q}{4}}\Bigg]\Bigg\}^{\frac{4}{6-q}}, \end{aligned} \tag{4.13}$$

$$c_2 := \left(\frac{5a}{3\mu\mathcal{C}_{10/3}^{10/3}}\right)^{\frac{3}{2}}, \tag{4.14}$$

$$\begin{aligned} c_3 := \left[\frac{4q}{\mu(6-q)\mathcal{C}_q^q}\right]^{\frac{4}{6-q}}\left[\frac{b}{3(3q-10)}\right]^{\frac{3q-10}{6-q}} \\ \cdot\left[\frac{4}{14-3q}\left(\frac{a}{3} + \frac{b^2\mathcal{S}^3}{6} + \frac{b\mathcal{S}}{12}\sqrt{b^2\mathcal{S}^4 + 4a\mathcal{S}}\right)\right]^{\frac{14-3q}{6-q}}. \end{aligned} \tag{4.15}$$

Let

$$A_\rho := \left\{u \in H^1(\mathbb{R}^3) : \|\nabla u\|_2^2 < \rho\right\}.$$

Chen and Tang [14] established the following theorems.

Theorem 4.1. *Let* $2 < q < \frac{10}{3}$ *and* $c \in (0, c_0)$. *Then* (4.8) *has a couple solution* $(\tilde{u}_c, \lambda_c) \in (\mathcal{S}_c \cap H^1_{\mathrm{rad}}(\mathbb{R}^3)) \times (0, +\infty)$ *such that*

$$\tilde{u}_c \in \mathcal{S}_c \cap A_{s_0}, \quad u_c > 0, \quad \Phi(\tilde{u}_c) = m(c) := \inf_{\mathcal{S}_c \cap A_{s_0}} \Phi < 0. \tag{4.16}$$

Theorem 4.2. *Let* $2 < q < \frac{10}{3}$ *and* $c \in (0, c_*)$ *with* $c_* := \min\{c_0, c_1\}$. *Then* (4.8) *has a second couple solution* $(\hat{u}_c, \lambda_c) \in (\mathcal{S}_c \cap H^1_{\mathrm{rad}}(\mathbb{R}^3)) \times (0, +\infty)$ *such that*

$$0 < \Phi(\hat{u}_c) \le \inf_{\mathcal{S}_c \cap A_{s_0}} \Psi + \Theta^*. \tag{4.17}$$

Theorem 4.3. *Let* $\frac{10}{3} \le q < \frac{14}{3}$ *and* $c \in (0, \min\{c_2, c_3\})$. *Then* (4.8) *has a couple solution* $(\bar{u}_c, \lambda_c) \in (\mathcal{S}_c \cap H^1_{\mathrm{rad}}(\mathbb{R}^3)) \times (0, +\infty)$.

Theorem 4.4. *Let* $\frac{14}{3} \le q < 6$ *and* $c \in (0, +\infty)$. *Then* (4.8) *has a couple solution* $(\bar{u}_c, \lambda_c) \in H^1(\mathbb{R}^3) \times (0, +\infty)$ *such that*

$$\Phi(\bar{u}_c) = \inf_{\mathcal{M}(c)} \Phi. \tag{4.18}$$

5. CHOQUARD EQUATIONS

Li and Ye [33] studied the following semilinear Choquard equation:

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * F(u))\, f(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c, \end{cases} \tag{5.1}$$

where $N \ge 3$, $\alpha \in (0, N)$, $c > 0$, and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \pi^{\frac{N}{2}} \Gamma\left(\frac{\alpha}{2}\right) |x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Here, $f = F'$ satisfies the following assumptions:

(F22) $f(t) \equiv 0$ for $t \le 0$, and there exists $p \in \left(\frac{N+\alpha+2}{N}, \frac{N+\alpha}{N-2}\right)$ such that

$$\lim_{|t| \to 0} \frac{f(t)}{|t|^{p-2} t} = 0, \quad \lim_{|t| \to +\infty} \frac{F(t)}{|t|^p} = +\infty,$$

(F23) $\lim_{|t| \to +\infty} \frac{F(t)}{|t|^{\frac{N+\alpha}{N-2}}} = +\infty$,

(F24) there exists $\theta_1 \ge 1$ such that $\theta_1 \tilde{F}(t) \ge \tilde{F}(st)$ for all $s \in [0, 1]$ and $t \in \mathbb{R}$, where

$$\tilde{F}(t) := f(t)t - \frac{N + \alpha + 2}{N} F(t),$$

(F25) for all $t \in \mathbb{R}$,

$$\left[f(t)t - \frac{N+\alpha}{N}F(t)\right]' t \geq \frac{N+\alpha+2}{N}\left[f(t)t - \frac{N+\alpha}{N}F(t)\right],$$

(F26) there exist $\theta_2 \geq 1$ and $s_0 > 0$ such that

$$F(st) \leq \theta_2 |s|^{\frac{N+\alpha+2}{N}} F(t), \quad \forall\ |s| \leq s_0,\ t \in \mathbb{R},$$

(F27) $f(t)t < \frac{N+\alpha}{N-2}F(t)$ for all $t > 0$.

Under the above assumptions, they demonstrated that (5.1) admits a solution pair $(\bar{u}_c, \lambda_c) \in \mathcal{S}_c \times (0, +\infty)$ with $\bar{u}_c > 0$.

Under the following assumptions:

(F28) there exist $\beta_1, \beta_2 \in \mathbb{R}$ satisfying $\frac{N+\alpha+2}{N} < \beta_1 \leq \beta_2 < \frac{N+\alpha}{N-2}$ such that

$$0 < \beta_1 F(t) \leq f(t)t \leq \beta_2 F(t), \quad \forall\ t \in \mathbb{R} \setminus \{0\},$$

(F29) the function $\frac{f(t)t - \frac{N+\alpha}{N}F(t)}{|t|^{\frac{N+\alpha+2}{N}}}$ is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, +\infty)$,

Bartsch, Liu, and Liu [3] proved that (5.1) admits a solution $(\bar{u}_c, \lambda_c) \in H^1(\mathbb{R}^N) \times (0, +\infty)$ satisfying

$$\Phi(\bar{u}_c) = \inf\left\{\Phi(u) : u \in \mathcal{S}_c, \mathcal{P}(u) = 0\right\},$$

where the energy functional $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \left(I_\alpha * F(u)\right) F(u) \,\mathrm{d}x,$$

and the Pohozaev-type functional $\mathcal{P} : H^1(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$\mathcal{P}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \left(I_\alpha * F(u)\right) \left[N f(u)u - (N+\alpha)F(u)\right] \mathrm{d}x.$$

Cingolani and Tanaka [19] proved that there exists $c_0 > 0$ such that for any $c > c_0$, the problem (5.1) admits a radially symmetric solution if f satisfies the following assumptions:

(F30) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$,

(F31) there exists $C > 0$ such that for every $t \in \mathbb{R}$,

$$|f(t)t| \leq C\left(|t|^{\frac{N+\alpha}{N}} + |t|^{\frac{N+\alpha+2}{N}}\right),$$

(F32)

$$\lim_{t \to 0} \frac{F(t)}{t^{\frac{N+\alpha}{N}}} = 0, \quad \lim_{|t| \to \infty} \frac{F(t)}{|t|^{\frac{N+\alpha+2}{N}}} = 0,$$

(F33) there exists $s_0 > 0$ such that $F(s_0) > 0$.

Moreover, if

(F34) f is odd and $f > 0$ on $(0, +\infty)$,

then the solution is positive.

Cingolani and Tanaka [19] also proved that the above constant $c_0 = 0$ if the following condition holds:

$$\lim_{|t|\to 0} \frac{F(t)}{|t|^{\frac{N+\alpha+2}{N}}} = 0.$$

When $F(t) = |t|^p$ with $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, Ye [47] obtained the same results as above. In [47], the case $p = \frac{N+\alpha+2}{N}$ was also addressed using scaling invariance; the analysis is delicate, and we refer to [47] for details.

Li [34] generalized the results obtained in [39] to the following Choquard equation with upper critical exponent and a local nonlinear perturbation:

$$\begin{cases} -\Delta u + \lambda u = \left(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\right) |u|^{\frac{4+\alpha-N}{N-2}} u + \mu |u|^{q-2} u, & x \in \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c, \end{cases} \tag{5.2}$$

where $c > 0$, $N \geq 3$, $\mu > 0$, $\alpha \in (0, N)$, and $q \in (2, 2^*)$. The energy functional $\Phi_\mu : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated with (5.2) is defined by

$$\Phi_\mu(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{N-2}{2(N+\alpha)} \int\limits_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\right) |u|^{\frac{N+\alpha}{N-2}} \, \mathrm{d}x - \frac{\mu}{q}\|u\|_q^q, \tag{5.3}$$

and the Pohozaev functional is

$$\mathcal{P}_\mu(u) = \|\nabla u\|_2^2 - \int\limits_{\mathbb{R}^N} \left(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\right) |u|^{\frac{N+\alpha}{N-2}} \, \mathrm{d}x - \frac{\mu(q-2)N}{2q}\|u\|_q^q.$$

Define the Pohozaev manifold

$$\mathcal{M}_\mu(c) := \{u \in \mathcal{S}_c : \mathcal{P}_\mu(u) = 0\}.$$

Based on the fibration method of Pohozaev and the decomposition of the Pohozaev manifold

$$\mathcal{M}_\mu(c) = \left\{u \in \mathcal{S}_c : \tilde{\phi}_u'(0) = 0\right\} = \mathcal{M}_\mu^-(c) \cup \mathcal{M}_\mu^0(c) \cup \mathcal{M}_\mu^+(c),$$

where $\tilde{\phi}_u(t) := \Phi_\mu(e^{Nt/2}u(e^t x))$ for $u \in H^1(\mathbb{R}^N)$ and $t \in \mathbb{R}$, and

$$\begin{aligned} \mathcal{M}_\mu^+(c) &:= \left\{u \in \mathcal{M}_\mu(c) : \tilde{\phi}_u''(0) > 0\right\}, \\ \mathcal{M}_\mu^-(c) &:= \left\{u \in \mathcal{M}_\mu(c) : \tilde{\phi}_u''(0) < 0\right\}, \\ \mathcal{M}_\mu^0(c) &:= \left\{u \in \mathcal{M}_\mu(c) : \tilde{\phi}_u''(0) = 0\right\}, \end{aligned}$$

Li [34] proved that for $2 < q < 2 + \frac{4}{N}$ and $\mu c^{\frac{q-q\gamma_q}{2}} < K$, where K depends on N, q, and the best constant in the Gagliardo–Nirenberg inequality, the following hold:

(i) there exists a local minimizer u_c^+ of Φ_μ on the set $\left\{u \in \mathcal{S}_c : \|\nabla u\|_2^2 < \rho_0\right\}$ such that

$$\inf_{\mathcal{M}_\mu^+(c)} \Phi_\mu = \Phi_\mu(u_c^+) < 0;$$

(ii) there exists a second solution $u_c^- \in \mathcal{M}_\mu^-(c)$ to (5.2) such that

$$\Phi_\mu(u_c^-) = \inf_{\mathcal{M}_\mu^-(c)} \Phi_\mu.$$

Gao and He [20] generalized the results obtained in [39] to the following Choquard equation with a local nonlinear perturbation:

$$\begin{cases} -\Delta u + \lambda u = \left(I_\alpha * |u|^{\frac{N+\alpha}{N-2}}\right) |u|^{\frac{N+\alpha}{N-2}-2} u + f(u), \quad x \in \mathbb{R}^N, \\ \int\limits_{\mathbb{R}^N} u^2 \, \mathrm{d}x = c, \end{cases}$$

where f satisfies (F6)–(F8).

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