Opuscula Math. 45, no. 6 (2025), 723–738
https://doi.org/10.7494/OpMath.2025.45.6.723 Opuscula Mathematica

A NOTE ON DISTANCE LABELING OF GRAPHS

Carl Johan Casselgren and Anders Henricsson

Communicated by Dalibor Frorček

Abstract. We study labelings of connected graphs G using labels $1, \ldots, |V(G)|$ encoding the distances between vertices in G. Following Lennerstad and Eriksson [Electron. J. Graph Theory Appl. 6 (2018), 152–165], we say that a graph G which has a labeling c satisfying that $d(u,v) < d(x,y) \Rightarrow c(u,v) \leq c(x,y)$, where $c(u,v) = |c(u) - c(v)|$, is a *list graph*. We show that these graphs are very restricted in nature and enjoy very strong structural properties. Relaxing this condition, we say that a vertex u in a graph G with a labeling c is *list-distance consistent* if $d(u,v) \leq d(u,w)$ holds for all vertices v, w satisfying that $c(u,w) = c(u,v) + 1$. The maximum k such that G has a labeling where k vertices are list-distance consistent is the *list-distance consistency* $\operatorname{ldc}(G)$ of G; if $\operatorname{ldc}(G) = |V(G)|$, then G is a *local list graph*. We prove a structural theorem characterizing local list graphs implying that they are a quite restricted family of graphs; this answers a question of Lennerstad. Furthermore, we investigate the parameter $\operatorname{ldc}(G)$ for various classes of graphs. In particular, we prove that for all k, n satisfying $4 \leq k \leq n$ there is a graph G with n vertices and $\operatorname{ldc}(G) = k$, and demonstrate that there are large classes of graphs G satisfying $\operatorname{ldc}(G) = 1$. Indeed, we prove that almost every graph have this property, which implies that graphs G satisfying $\operatorname{ldc}(G) > 1$ are in a sense quite rare (let alone local list graphs). We also suggest further variations on the topic of list graphs for future research.

Keywords: graph labeling, distance labeling.

Mathematics Subject Classification: 05C78.

1. INTRODUCTION

We only consider connected graphs, and assume that all our graphs are connected unless otherwise stated. A *(vertex) labeling* of a graph G is a bijection $c : V(G) \to \{1, \ldots, |V(G)|\}$. Labelings of graphs is a huge area of research, and an extensive number of variants of labelings satisfying various constraints have been studied, see e.g. [1].

In this paper we study a new variant of distance labeling introduced in [4]: the aim is to find a labeling c of the vertices of a graph G such that the "label distance" $c(u,v) = |c(u) - c(v)|$ deviates as little as possible from the ordinary distance $d(u,v)$

between u and v in G. Indeed, a labeling c of G is *distance consistent* if it minimizes the sum

$$l(G) = \sum_{u,v \in V(G)} (c(u,v) - d(u,v))^2.$$

The authors of [4] refer to this sum as the *ampleness* of a graph. In [4], they argue that this kind of distance labelings can provide insight into the structure of distances between nodes in a graph, which is useful in many applications involving large networks (cf. [4]).

Different version of labelings encoding distances in graphs have been studied before, see e.g. [2, 5, 6], but in these papers the numbers $1, \ldots, |V(G)|$ are not used for marking the vertices, so they are not labelings in our sense. Similarly, "labelings" encoding adjacency between vertices in a graph using binary codes have been considered in several papers, see e.g. [7], but the idea of encoding distances in a graph using the labels $1, \ldots, |V(G)|$ appears to be a quite new approach first studied in [4].

A general motivation for studying such general graph "labelings" encoding distances in graphs is that most representations of graphs (e.g. using the adjacency matrix) are global in nature and a priori gives very little *structural* information about the graphs. Thus, while very basic properties, such as adjacencies, can easily be retrieved, there is a need for efficient encoding of more elaborate properties of graphs. As regards distances in a graph, one natural approach (see e.g. [2]) is to study the minimal length of the labels needed for encoding the distances in a graph with given accuracy. The main purpose of this note is to investigate on a qualitative level how far a simpler approach as the one suggested in [4] may take us towards such a goal. After presenting our results, we elaborate further on this question in the last section.

In [4], the authors calculate the ampleness for some basic families of graphs, and it is clear that it is quite large for many natural families of graphs. They also suggest a different approach based on so-called *list graphs*: a graph is a *list graph* if there is a labeling such that for all vertices u_1, u_2, v_1, v_2 it holds that if $d(u_1, v_1) < d(u_2, v_2)$, then $c(u_1, v_1) \leq c(u_2, v_2)$. The idea is thus that even if the ampleness is large, the labeling should in a sense reflect structural distance properties of the graph in question.

Lennerstad and Eriksson [4] proved some basic properties of list graphs: they are path-hamiltonian, chordal and claw-free, where a graph is *chordal* if every induced cycle has length three, it is *claw-free* if no induced subgraph is isomorphic to $K_{1,3}$, and it is *path-hamiltonian* if it has a hamiltonian path. Note that if we replace the second inequality in the condition by a strict inequality, then we get a condition which is only satisfied by paths and complete graphs, as noted by [4]. Indeed, in [4], this latter variant is referred to as the *strong list condition*, and the variant with non-strict inequality as the *weak list condition*. As we shall see in this paper, the family of graphs satisfying the weak version is richer than paths and complete graphs, although yet very restricted.

A graph G is a *local list graph* if there is a labeling c such that $d(u,v) \leq d(u,w)$ for all vertices $u, v, w \in V(G)$ satisfying that $c(u,w) = c(u,v) + 1$. This notion was suggested by Lennerstad [personal communication] as a relaxation of the idea of a list graph.

The main purpose of this note is to investigate these labeling schemes and also a further weakening of a local list graph. We observe that one can easily prove that both list graphs and local list graphs are very restricted families of graphs. In light of this fact, we introdce a further weakening of list graphs: we say that a vertex u in a graph G with a labeling c is *list-distance consistent* if $d(u,v) \leq d(u,w)$ for all vertices v, w satisfying that $c(u,w) = c(u,v) + 1$. The maximum k such that G has a labeling where k vertices are list-distance consistent is the *list-distance consistency* $\mathrm{ldc}(G)$ of G; so $\mathrm{ldc}(G) = |V(G)|$ if and only if G is a local list graph. If $\mathrm{ldc}(G) \geq k$, then G is *k-list-distance consistent.*

We study list-distance consistency of various classes of graphs. In particular, we prove that for all integers $k \geq 1, n \geq 4$ satisfying $1 \leq k \leq n$, there is a graph G with n vertices and $\mathrm{ldc}(G) = k$, and that there are large classes of graphs G satisfying $\mathrm{ldc}(G) = 1$. Indeed we show that almost all graphs (for increasing number of vertices) have list-distance consistency 1. Thus even the property of having $\mathrm{ldc}(G) > 1$, which is much weaker than being a list graph or a local list graph, is quite uncommon. This suggests that for these labeling schemes to be of any practical use, one should aim for weaker properties than having large list-distance consistency. We conclude the paper by suggesting further variations on the topic of list graphs which be believe could be promising for future research.

2. PROPERTIES OF LIST AND LOCAL LIST GRAPHS

In this section we shall prove strong structural properties of list graphs and local list graphs. As mentioned above, in [4] it is proved that if G is a list graph then it is chordal, path-hamiltonian and claw-free. The authors also proved that for all n and $m \in \{n-1, \ldots, n(n-1)/2\}$, there is a list graph with n vertices and m edges.

First we prove that a list graph is in fact a union of cliques all of which have size k or $k+1$. Moreover, the cliques lie along a hamiltonian path, where two consecutive cliques intersect in a clique of size at least $k-1$. This gives a much stronger structural description of list graphs than the one obtained in [4].

Proposition 2.1. *If a graph G is a list graph, then it has a hamiltonian path P, such that if two vertices at distance $k \geq 2$ along P are adjacent, then all pairs of vertices at distance less than k along P are adjacent.*

Proof. Suppose that G is an n-vertex graph with a list labeling c. Then the vertex labeled i must be adjacent to the vertex labeled $i+1$, $i = 1, \ldots, n-1$ (otherwise the condition of being a list labeling would be violated). This implies that G has a hamiltonian path P where the vertices are labeled in increasing order $1, \ldots, n$. Now, if there is an edge between two vertices at distance $k > 1$ along P, then all edges between vertices at distance less than k (along the path) are present; this is a direct consequence of the list condition above. □

We remark that it is possible to prove a stronger characterization of list graphs; indeed the "longest" chords on P have to be rather "evenly distributed" along P

to avoid breaking the list condition. However, since this condition is somewhat technical, and the preceding proposition already yields a quite strong structural characterization, we choose not to pursue this line of investigation here.

Note that although all list graphs are path-hamiltonian, they need not be hamiltonian. Further, powers of paths are list graphs, but not all list graphs are powers of paths, since the graph obtained from a path by adding some suitable edges with distinct endponts is also a list graph. (For instance, take a path and add only one edge between two vertices at distance 2 on the path.) Nevertheless, the condition of being a list graph is of course very strong. A weaker condition was suggested by Lennerstad [personal communication]: a labeling of a graph is a *local list labeling* if $d(u,w) \leq d(u,v)$ holds for all vertices u, v, w where $c(u,v) = c(u,w)+1$. As mentioned above, a graph with such a labeling is called *a local list graph*. Thus, in a local list graph G all vertices are list-distance consistent; that is, $\mathrm{ldc}(G) = |V(G)|$.

We denote by $N_G(v)$ the set of neighbors of a vertex v in a graph G, and more generally, for a positive integer k, by $N^k_G(v)$ the set of vertices at distance at most k from v in G; so $v \in N^k_G(v)$, but $v \notin N_G(v)$. The following basic property of list-distance consistent vertices will be used repeatedly throughout the paper.

Lemma 2.2. *Let G be a graph, c a labeling of G, and k a positive integer. If v is a list-distance consistent vertex of G (under c), then $c\left(N^k_G(v)\right)$ is a set of consecutive integers.*

Proof. Suppose, for a contradiction, that there is some integer $j \notin c(N^k_G(v))$, and let $s, t > 0$ be the smallest positive integers such that $\{j-s, j+t\} \subseteq c(N^k_G(v))$. Let $u \in V(G)$ be the vertex such that $c(u) = j$; then by assumption, $d(u,v) > k$. On the other hand,

$$d(v, c^{-1}(j-s)) \leq k \quad \text{and} \quad d(v, c^{-1}(j+t)) \leq k.$$

Now, if $c(v) < j$, then the latter statement contradicts that v is list-distance consistent, and if $c(v) > j$, then the former statement gives the required contradiction. □

Using the preceding lemma it is now easy to see that every local list graph has a hamiltonian path, which resolves a question of Lennerstad [personal communication].

Proposition 2.3. *Every local list graph has a hamiltonian path.*

Proof. Suppose that G is a n-vertex graph with a local list labeling c. Lemma 2.2 implies that for any i, $1 \leq i < n$, the vertex with label i is adjacent to the vertex with label $i+1$, so G has a hamiltonian path. □

Next, we prove a structural characterization of local list graphs, similar to the characterization of list graphs above. For a path P containing the vertices u and v, we denote by uPv the subpath of P from u to v.

Proposition 2.4. *A graph is a local list graph iff it has a hamiltonian path P such that if two vertices u and v at distance $k \geq 2$ along P are adjacent, then all possible edges between vertices on uPv are present in G.*

Proof. Given that a graph G has a hamiltonian path P satisfying the condition in the proposition, label the vertices in increasing order along P. It is straightforward that this yields a required labeling.

Conversely, assume G is a local list graph with a local list labeling c. From the preceding proposition, G has a hamiltonian path $P = v_1v_2 \dots v_n$ where $c(v_i) = i$, for $i = 1, \dots, n$. Suppose that u and v are adjacent and at distance $k \geq 2$ along P. Now, it follows directly from the local list condition that any two vertices lying in-between u and v on P are adjacent. □

Note that this proposition in fact implies that G is a union of cliques arranged along a hamiltonian path. Moreover, the intersection of two distinct such cliques forms a clique of smaller size, but in contrast to the characterization of list graphs above, the "intersection cliques" may be arbitrarily small and only consist of one single vertex. Moreover, unlike the case of list graphs, the cliques along the hamiltonian path P can be in a lof of different sizes. Consequently, the family of local list graphs is much larger than list graphs.

3. LIST-DISTANCE CONSISTENT LABELINGS

Recall that a labeling c is *k-list-distance consistent* if there exist k vertices u that satisfies the local list condition, i.e. that $d(u, w) \leq d(u, v)$ holds for all vertices v, w where $c(u, v) = c(u, w) + 1$. In this section we derive some basic properties of list-distance consistent labelings.

All graphs are 1-list-distance consistent. Moreover, for any vertex v of a graph G, there is a labeling such that v is list-distance consistent, obtained by labeling v by 1, and thereafter labeling all other vertices of G so that vertices with greater distance to v get larger labels. Thus, we have the following.

Proposition 3.1. *Every graph is* 1*-list-distance consistent. Moreover, any vertex of G can be selected to satisfy the local list condition.*

The coming proposition can be used for obtaining infinite families of graphs with small list-distance consistency. The following lemma will prove useful; the proof is a simple verification, so we omit it.

Lemma 3.2. *Let c be a labeling of a graph G, and let c' be a labeling defined by setting $c'(v) = |V(G)| + 1 - c(v)$ for all vertices $v \in V(G)$. Then a vertex u is list-distance consistent with respect to c if and only if u is list-distance consistent with respect to c'.*

We shall use this lemma in many places throughout the paper without explicitly referring to it.

Proposition 3.3. *Let G be a graph with a cut-vertex v such that $G - v$ has at least three components, every component of $G - v$ has at least two vertices, and v has exactly one neighbor in every component of $G - v$. Then*

$$\mathrm{ldc}(G) \leq \max\{|V(F)| : F \text{ is a component of } G - v\} + 1.$$

Moreover, if $G - v$ has at least four components, then

$$\mathrm{ldc}(G) \leq \max\{|V(F)| : F \text{ is a component of } G - v\}.$$

Proof. Let F be a component of $G-v$, and assume that at least one vertex of F is list-distance consistent with respect to a labeling c. Let u be such a vertex in F, and assume that $c(v)=j$. By Lemma 3.2, we may assume that $c(u)<j$. Then Lemma 2.2 implies that all labels in $\{c(u),\dots,j-1\}$ appear on vertices in F under c.

Let J be a component of $G-v-V(F)$ and assume that $x \in V(J)$ is list-distance consistent. Then, since all the labels in $\{c(u),\dots,j-1\}$ appear on vertices in F under c, Lemma 2.2 yields that $c(x)>j$. As before, we deduce that all labels in $\{j+1,\dots,c(x)\}$ must appear on vertices in J.

We claim that there are two vertices in F labeled $j-1$ and $j-2$, respectively, and two vertices in J labeled $j+1$ and $j+2$, respectively. If none of u and x is adjacent to v, then this is clear. On the other hand, if u is adjacent to v, then it follows from Lemma 2.2 that $c\left(N^1_G(v)\right)=\{j-s,\dots,j\}$, where $s \geq 2$ becasue $|V(F)| \geq 2$. Thus there are vertices in F labeled $j-1$ and $j-2$, respectively. The argument for J is analogous.

Consider a vertex y in $G-v-V(F)-V(J)$ that is adjacent to v. Then we must have that $c(y)>j+2$ or $c(y)<j-2$. However, since v has only one neighbor in each component of $G-v$, in the first case this contradicts that u is list-distance consistent, and in the second case it contradicts that x is list-distance consistent.

We conclude that no vertex in $G-v-V(F)$ is list-distance consistent. This proves the first part of the proposition.

Now, suppose that every vertex of F is list-distance consistent and that $G-v$ consist of at least four components. Since all vertices in F are list-distance consistent, and $u \in V(F)$ with $c(u)<j$, the vertex labeled $j-1$ must be contained in F, and also be adjacent to v. Moreover, no vertex with a label greater than j can be contained in F because $c^{-1}(j-1)$ is the only neighbor of v in F. Thus the vertices in $N_G(v) \setminus V(F)$ must be labeled $j+1,\dots,j+s$, for some integer $s \geq 3$. Since v has distance at least 2 to the vertex labeled $j-2$, it cannot be list-distance consistent, as required. □

Note that the first part of the proposition is sharp by the example of three disjoint triangles, where exactly one vertex of each triangle is joined to a vertex v; we shall see in the next section that the second part is sharp as well.

We conclude this section by determining the list-distance consistency of two simple families of graphs. The following observation will prove useful.

Lemma 3.4. *If u is a list-distance consistent vertex of a graph, then one of the vertices at maximum distance from u must be labeled 1 or $|V(G)|$.*

Proposition 3.5. *For a cycle C_n, we have*

$$\operatorname{ldc}(C_n)=\begin{cases}2, & \text{if } n \text{ is even},\\ 3, & \text{if } n \text{ is odd}.\end{cases}$$

Proof. In an even cycle, there is only one vertex at maximum distance from a given vertex u. Thus it follows from Lemma 3.4 that $\operatorname{ldc}(C_n) \leq 2$ if n is even.

Next, assume n is even and label the vertices of C_n in increasing order so that the ith vertex is labeled i. Then both the vertex labeled $n/2$ and the one labeled $n/2+1$ are list-distance consistent, so $\operatorname{ldc}(C_n) \geq 2$.

Next, let us consider the case when n is odd, say $n = 2k + 1$. Again, we label the vertices in increasing order along C_n, so that the ith vertex is labeled i. Then the vertices labeled $k, k+1, k+2$ are list-distance consistent. Hence, $\mathrm{ldc}(G) \geq 3$.

Assume for a contradiction that there is a 4-list distance consistent labeling c of G. Then by Lemma 3.4, the vertices x, y at maximum distance from the vertex labeled 1 must be list-distance consistent, as are also the vertices u, v at maximum distance from the vertex labeled $2k + 1$. Since u, v, x, y are by assumption all distinct, the vertices $c^{-1}(1)$ and $c^{-1}(2k + 1)$ are not adjacent.

Now assume that $c(x) > 2$. Since x is list-distance consistent, the vertex at distance k from x that is not labeled 1 must be labeled 2, since it cannot be labeled $2k + 1$. However, the same holds for the vertex at distance k from y that is not labeled 1. So both neighbors of $c^{-1}(1)$ must be labeled 2, which is a contradiction. Thus, $\mathrm{ldc}(G) \leq 3$. □

A *wheel* is a graph obtained from the cycle C_n $(n \geq 3)$ by adding a new vertex u and joining it to every vertex of C_n by an edge. Denote by W_{n+1} the wheel obtained from a cycle C_n by adding one additional vertex, the so-called *central vertex* of W_{n+1}.

Proposition 3.6. *For a wheel W_n, we have*

$$\mathrm{ldc}(W_n) = \begin{cases} 4, & \text{if } n = 4, \\ 3, & \text{if } n = 5, \\ 4, & \text{if } n = 6, \\ 5, & \text{if } n \geq 7. \end{cases}$$

Proof. The graph W_4 is isomorphic to K_4, so $\mathrm{ldc}(W_4) = 4$.

Assume now that $n \geq 5$. Let w be the central vertex of W_n, and consider a labeling c of W_n. Then w is clearly list-distance consistent, since it has distance 1 to every other vertex. Moreover, since $d(x, w) = 1$ for any vertex x in $W_n - w$, $c(x, w) \leq 2$ if x is list-distance consistent. This implies that at most 4 vertices in $W_n - w$ are list-distance consistent.

If $n \geq 7$, then if we label the vertices of W_n consecutively $1, 2, \ldots, n$ along the cycle $W_n - w$, except that w is labeled $\lfloor \frac{n}{2} \rfloor$, then the vertices with labels in the set

$$\{\lfloor n/2 \rfloor - 2, \lfloor n/2 \rfloor - 1, \lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2\}$$

are all list-distance consistent. In conclusion, $\mathrm{ldc}(W_n) = 5$ if $n \geq 7$.

Let us now consider the case when $n = 5$; let c be a labeling of W_n. By Lemma 3.2, we may assume that some vertex distinct from w is labeled 5. If $c(w) = 1$, then Lemma 3.4 implies that at most one vertex in $W_n - w$ is list-distance consistent. Otherwise, if $c(w) \neq 1$, then at most two vertices of $W_n - w$ are list-distance consistent. Indeed, the labeling from the preceding paragraph is such a labeling where exactly two vertices in $W_n - w$ are list-distance consistent.

If $n = 6$, then the argument in the preceding paragraph implies that at most three vertices of $W_n - w$ are list-distance consistent. Again, the labeling described above yields 3 list-distance consistent vertices except w. □

4. BIPARTITE GRAPHS

In this section we consider the list distance consistency of bipartite graphs. Let us start by considering trees. It is not difficult to verify that a tree T has list-distance consistency $|V(T)|$ if and only if T is a path. Moreover, every tree has list-distance consistency at least 2, since we can label some leaf v by 1, its neighbor by 2, and then label all other vertices with increasing labels according to the distance to the leaf v. Thus, it is natural to ask if there is some tree with list-distance consistency 2. Indeed, we have the following consequence of Proposition 3.3.

Corollary 4.1. *The tree obtained from $K_{1,4}$ by joining each pendant vertex to a new vertex by an edge has list-distance consistency* 2.

Next, we shall determine the list-distance consistency of stars $K_{1,n}$ and double stars $S_{k,l}$, that is, trees obtained from $K_{1,k}$ and $K_{1,l}$ by joining the central vertices in these graphs by an edge.

We shall need the following easy observation.

Lemma 4.2. *If xy is a cut-edge of a graph G and there is a labeling c such that both x and y are list-distance consistent, then $c(x,y) = 1$.*

The following lemma will also prove useful.

Lemma 4.3. *If two leaves x_1, x_2 of a graph G, adjacent to the same vertex v, are list-distance consistent, then no vertex distinct from v, x_1, x_2 is list-distance consistent.*

Proof. Suppose that c is a labeling where v, x_1, x_2 and some additional vertex $u \neq v$ is list-distance consistent. Then $\{c(x_1), c(x_2)\} = \{c(v) - 1, c(v) + 1\}$. But this implies that u cannot be list-distance consistent, because $d(u, x_1) = d(u, x_2) > d(u, v)$. □

Proposition 4.4. *For stars and double-stars, we have the following.*

(i) *If $k \geq 3$, then* $\mathrm{ldc}(K_{1,k}) = 3$*,*
(ii) $\mathrm{ldc}(S_{2,2}) = 4$*, and*
(iii) *if* $\max\{k, l\} \geq 3$*, then* $\mathrm{ldc}(S_{k,l}) = 3$*.*

Proof. Let us first prove part (i). Suppose that v is the central vertex of $K_{1,k}$ and consider a labeling c of this graph. Then any vertex which is list-distance consistent must be labeled $c(v) - 1$ or $c(v) + 1$. Hence $\mathrm{ldc}(K_{1,k}) \leq 3$, and any labeling c such that $c(v) \in \{2, \ldots, k\}$ shows that $\mathrm{ldc}(K_{1,k}) \geq 3$.

As for part (ii), it follows from Lemma 4.3 that $\mathrm{ldc}(S_{2,2}) \leq 4$. A 4-list-distance consistent labeling of $S_{2,2}$ can be obtained by assigning the labels $2, 3, 4, 5$ to the vertices of a path of length 4 in $S_{2,2}$ (in that order), and then labeling the vertex adjacent to the one labeled 3 by 1, and the other one by 6.

We now prove part (iii). Let us first show that $\mathrm{ldc}(S_{k,l}) \geq 3$. This is easily seen by labeling one central vertex u_1 by 2, and labeling two of its neighbors 1 and 3. All other pendant neighbors of u_1 are labeled using the remaining positive integers not exceeding the degree $d(u_1)$. Next, the other central vertex of $S_{k,l}$ is labeled $d(u_1) + 1$, and the remaining hitherto unlabeled vertices are labeled arbitrarily. This labeling yields three list-distance consistent vertices.

Next, we show that $\mathrm{ldc}(S_{k,l}) \le 3$. It follows from Lemma 4.3, that $\mathrm{ldc}(S_{k,l}) \le 3$ unless the two central vertices u_1, u_2 of $S_{k,l}$ as well as two leaves x_1, x_2 are all list distance-constistent, where $u_1x_1 \in E(S_{k,l})$ and $u_2x_2 \in E(S_{k,l})$. So assuming that $\mathrm{ldc}(S_{k,l}) \ge 4$, Lemma 4.2 implies that x_1, u_1, u_2, x_2 must be labeled by four consecutive labels (in that order).

Now, since $\max\{k, l\} \ge 3$, we may assume that u_1 has two neighbors y_1, y_2 distinct from x_1. Then either $c(u_1, y_1) \ge 3$ or $c(u_1, y_2) \ge 3$; suppose the latter condition holds. However, since u_1 and y_2 are adjacent, $d(u_1, x_2) = 2$ and $c(u_1, x_2) = 2$, this is a contradiction. We conclude that $\mathrm{ldc}(S_{k,l}) \le 3$. □

Next, we shall prove that for each integer $n \ge 3$, and each $3 \le k \le n$, there is a tree T with n vertices and list distance consistency k.

Theorem 4.5. *For each pair of integers k, n such that $3 \le k \le n$, there is a tree T on n vertices with $\mathrm{ldc}(T) = k$.*

Proof. The case $k = 3$ is dealt with in Proposition 4.4, and the case $k = n$ is covered by paths, so let us assume that $3 < k < n$. We consider some different cases.

Case 1. $k = n - 1$.

We can construct a tree T with n vertices and $\mathrm{ldc}(T) = k$ by adding an additional vertex v_{k+1} to a path $P_k = v_1v_2 \dots v_k$, and joining v_{k+1} to v_{k-1} by an edge. The labeling $c(v_j) = j$ is k-list distance consistent, but T is not $(k+1)$-list distance consistent, since it is not a path.

Case 2. $\frac{n+2}{2} \le k \le n - 2$.

Let $T_{n,k}$ be the tree obtained from the path $P = v_1v_2 \dots v_k$ and $P' = v_{k+1}v_{k+2} \dots v_n$ by joining v_{k-1} and v_{k+1} by an edge. We define a labeling c by setting $c(v_i) = i$, $i = 1, \dots, n$. Then the vertices $v_1, \dots, v_k$ are all list-distance consistent, so $\mathrm{ldc}(T_{n,k}) \ge k$. Next, we prove that $\mathrm{ldc}(T_{n,k}) \le k$.

Assume, for a contradiction, that there is a $(k+1)$-list-distance consistent labeling c' of $T_{n,k}$. Then since $n \le 2k - 2$, there are list-distance consistent vertices $v_a \in V(P - v_k)$ and $v_b \in V(P')$. Without loss of generality, we assume that $c'(v_b) > c'(v_a)$. Since v_b is list-distance consistent, this means that $c'(v_j) < c'(v_a)$ for every $j < a$. In fact, since a vertex at maximum distance from v_b must be labeled 1, we must have that $c'(v_i) = i$, for $i = 1, \dots, a$. Similarly, since v_a is list-distance consistent, $c'(v_i) = i$, for $i = a + 1, \dots, k - 1$.

Furthermore, since v_a is list-distance consistent under c',

$$\{c'(v_k), c'(v_{k+1})\} = \{k, k + 1\}.$$

However, this implies that v_b cannot be list-distance consistent, a contradiction.

Case 3. $k < \frac{n+2}{2}$.

We define a tree $F_{n,k}$ by taking a copy of $T_{2k-2,k}$, defined as in the preceding case, and adding $n - (2k - 2)$ new vertices $v_{2k-1}, v_{2k}, \dots, v_n$ and joining every such vertex to v_{2k-3} by an edge.

We define a labeling c of $F_{n,k}$ by setting $c(i) = i$. Then, as in the preceding case, the vertices $v_1, \dots, v_k$ are list-distance consistent, so $\mathrm{ldc}(F_{n,k}) \ge k$. Let us prove that $\mathrm{ldc}(F_{n,k}) \le k$.

Assume, for a contradiction, that there is a $(k+1)$-list distance consistent labeling c' of $F_{n,k}$. It follows from Lemma 4.3 that at most one of the leaves $v_{2k-2}, \dots, v_n$ is list-consistent, because $k+1 \geq 3$. Without loss of generality, we assume that none of these leaves except possibly v_n is list-distance consistent. Moreover, we denote the path from v_{k+1} to v_n in $F_{n,k}$ by P', and set $P = v_1 \dots v_{k-1}$.

We now proceed as in the preceding case. As $\mathrm{ldc}(F_{n,k}) \geq k+1$, there is at least one list-distance consistent vertex $v_a \in V(P)$ and one list-distance consistent vertex $v_b \in V(P')$. Moreover, assuming $c'(v_b) > c'(v_a)$, we deduce that must have that $c'(v_i) = i$, $i = 1, \dots, k-1$, as in the preceding case. Again, we get a contradiction to the fact that v_b is list-distance consistent by considering the possible values of c' for the vertices v_k and v_{k+1}. □

Next, let us consider complete bipartite graphs. The family $K_{1,n}$ was considered above. For complete bipartite graphs where one part has at least two vertices we have the following.

Proposition 4.6. *For complete bipartite graphs where one part has at most* 3 *vertices, we have the following.*

(i) *If* $n \geq 2$*, then* $\mathrm{ldc}(K_{2,n}) = 2$.
(ii) $\mathrm{ldc}(K_{3,3}) = 2$.

Proof. First we prove part (i). We may assume that $n \geq 3$, since the case of cycles was considered above. By labeling the vertices of $K_{2,n}$ so that the vertices in the smallest part have labels 1 and $n+2$, we get that $\mathrm{ldc}(K_{2,n}) \geq 2$.

Next, we show that $\mathrm{ldc}(K_{2,n}) \leq 2$. Let X and Y be the parts of $K_{2,n}$, where $|X| = 2$. Let c' be a labeling of $K_{2,n}$. If Y contains a list-distance consistent vertex u and $c'(u) \notin \{1, n+2\}$, then the vertices in X are labeled $c(u)+1$ and $c(u)-1$, so no other vertex of $K_{2,n}$ can be list-distance consistent. On the other hand, if $c'(u) \in \{1, n+2\}$, say $c'(u) = 1$, then the vertices in X are labeled 2 and 3, so no other vertex in $K_{2,n}$ is list-distance consistent. We conclude that $\mathrm{ldc}(K_{2,n}) \leq 2$.

Next, we prove part (ii). Let X and Y be parts of $K_{3,3}$, and consider a labeling where the vertices in X are labeled $1, 2, 4$, respectively, and the vertices in Y are labeled $3, 5, 6$, respectively. Then the vertices labeled 3 and 4 are list-distance consistent, so $\mathrm{ldc}(K_{3,3}) \geq 2$.

To prove that $\mathrm{ldc}(K_{3,3}) \leq 2$, we consider an arbitrary labeling c' of $K_{3,3}$, and assume, for a contradiction, that there are three list-distance consistent vertices u, v, w under c'. If these three vertices are contained in the same partite set, say $\{u, v, w\} \subseteq X$, then two of these vertices must be labeled 1 and 6. Moreover, since the vertex labeled 1 is list-distance consistent, the third vertex of X is labeled 5, but this contradicts that the vertex labeled 6 is list-distance consistent.

Suppose now instead that exactly two vertices from $\{u, v, w\}$ are contained in the same set, say $\{u, v\} \subseteq X$. Without loss of generality, we assume that $c'(w) \leq 3$. Then the vertex labeled 6 is contained in Y.

If $c'(w) = 3$, then $c'(Y) = \{3, 5, 6\}$ or $c'(Y) = \{1, 3, 6\}$. In both cases, two vertices of X are labeled by consecutive integers, so there cannot be two list-distance consistent vertices in X. On the other if $c'(w) \leq 2$, then the vertices in Y are labeled $c'(w), 5, 6$, so, again, there cannot be two list-distance consistent vertices in X. □

The next result on complete bipartite graphs with parts of "sufficiently" large size concludes our investigation of complete bipartite graphs.

Theorem 4.7. *If* $\max\{m, n\} \geq 4$ *and* $\min\{m, n\} \geq 3$*, then* $\mathrm{ldc}(K_{m,n}) = 1$.

Proof. Let X and Y be the parts of $K_{m,n}$, and assume that $u \in X$ is list-distance consistent under some labeling c. We shall prove by contradiction that there is no other list-distance consistent vertex in $K_{m,n}$. If $c(u) = 1$ or $c(u) = m + n$, then it is easy to check that no other vertex can be list-distance consistent. Hence, we assume that $1 < c(u) < m + n$.

Suppose first that there is another vertex $v \in X$ that is list-distance consistent, where $c(v) > c(u)$. Then all labels in the set $\{c(u) + 1, \ldots, c(v) - 1\}$ must be assigned to vertices in Y. Moreover, those must be the only labels appearing on vertices in Y, since otherwise the list-distance condition fails. Hence, the label $c(u) - 1$ appears on some vertex in X, which contradicts that u is list-distance consistent, because $|Y| \geq 3$.

Suppose now that v is list-distance consistent and $v \in Y$. Without loss of generality, we assume that $|X| \geq 4$, and that $c(v) > c(u)$. Then $c(v) = c(u) + 1$, since otherwise the vertices with labels in $\{c(u) + 1, \ldots, c(v) - 1\}$ must appear in both X and Y, which is not possible. Moreover, some vertex $y \in Y$ must be labeled $c(u) - 1$, because $c(u), c(v) > 1$, and u is list-distance consistent. Now, since v is list-distance consistent, no label smaller than $c(u) - 1$ can appear on a vertex in X. Thus, since $|X| \geq 4$, the labels $c(v) + 1, c(v) + 2, c(v) + 3$ all appear on vertices in X. However, since $d(v, y) = 2$ and $c(v, y) = 2$, this is a contradiction. □

From the above results on trees and complete bipartite graphs we deduce the following.

Corollary 4.8. *For every pair of integers* $n \geq 4, k \geq 1$ *such that* $1 \leq k \leq n$*, there is a graph* G *with* n *vertices and* $\mathrm{ldc}(G) = k$.

Our next result shows that there are infinite families of bipartite graphs with small edge density that have list-distance consistency 1.

We define a family of bipartite graphs $\mathcal{B}(n)$ as follows. Let $V_1, \ldots, V_n$ be sets of vertices such that $|V_i| \geq 4$. Next, we add an edge between every pair of vertices that are contained in different sets V_i and V_{i+1}, $1 \leq i \leq n - 1$. Every graph constructed according to this process is included in the set $\mathcal{B}(n)$.

Proposition 4.9. *If* $n \geq 2$ *and* $G \in \mathcal{B}(n)$*, then* $\mathrm{ldc}(G) = 1$.

Proof. By Theorem 4.7, we may assume that $n \geq 3$. So assume that this holds and let $u \in V_j$ be a vertex that is list-distance consistent under some labeling c. Consider the set of neighbors $N_G(u)$ of u and their labels under c. Since u is list-distance consistent and every set V_i contains at least 4 vertices, by Lemma 2.2, the set $c(N_G^1(u))$ contains at least five consecutive integers. Since $|V_j| \geq 4$, as in the proof of Theorem 4.7, it is easy to see that this implies that no other vertex in V_j is list-distance consistent.

Similarly, it is easy to check that no vertex in $N_G(u)$ is list-distance consistent. So suppose that there is some vertex v in $V(G) \setminus (V_j \cup N_G(u))$ that is list-distance consistent under c, say $v \in V_r$ and assume $c(v) > c(u)$. If some vertex in V_r is labeled $c(v) - 1$ or $c(v) + 1$, then v is clearly not list-distance consistent, so assume that this is not the case.

Now, since v is list-distance consistent, the set $c(N_G^1(v))$ contains at least five consecutive integers. Since u is also list-distance consistent, and each V_i has size at least 4, it follows from Lemma 2.2 that the vertices in $N_G^1(v)$ cannot be contained in three different sets V_{r-1}, V_r, V_{r+1}. Thus $v \in V_1$ or $v \in V_n$, say $v \in V_n$. Similarly, since v is list-distance consistent, and $c(N_G^1(u))$ also is a set of consecutive integers, we must have that $u \in V_1$.

Since v is list-distance consistent, by Lemma 2.2, no vertex with a label smaller than $c(u)$ can be contained in $V_2 \cup \ldots \cup V_{n-1}$. However, since u is list-distance consistent, this means that we must have $c(u) = 1$, and a similar argument shows that $c(v) = |V(G)|$. Again, since u is list-distance consistent, a vertex in V_n or V_1 is labeled $|V(G)| - 1$, which contradicts that v is list distance-consistent. □

5. GRAPHS WITH CHROMATIC NUMBER AT LEAST 3

In this section we consider graphs with chromatic number greater than 2. We start our investigation with the *Petersen graph* which can be defined by letting the vertex set be the set of all 2-subsets of a set of five elements, and where two vertices are adjacent if and only if the corresponding 2-subsets are disjoint.

Proposition 5.1. *For the Petersen graph* G_p*, we have* $\mathrm{ldc}(G_p) = 2$.

Proof. Consider the labeling of the Petersen graph depicted in Figure 1. Since the labels of the closed neighborhoods of the vertices labeled 5 and 3, respectively, are sets of consecutive integers, and the diameter of the Petersen graph is 2, it follows that the vertices labeled 5 and 3 are list-distance consistent. This shows that $\mathrm{ldc}(G_p) \geq 2$.

Next, we prove that $\mathrm{ldc}(G_p) \leq 2$ by contradiction. Assume that $\mathrm{ldc}(G_p) \geq 3$ so that there are three vertices v_1, v_2, v_3 which are list-distance consistent under a labeling c. Hence, it follows from Lemma 2.2 that for $i = 1, 2, 3$, $c(N_{G_p}^1(v_i))$ is a set of consecutive integers. We consider two cases.

Case 1. $\{v_1, v_2, v_3\}$ *is independent.*

Any two nonadjacent vertices in G_p has exactly one common neighbor. Thus since all sets $c(N_{G_p}^1(v_i))$ are intervals of integers, all three vertices v_1, v_2, v_3 cannot have one common neighbor in G_p. So for all $1 \leq i < j \leq 3$, there is exactly one common neighbor of v_i and v_j. This implies that there are three 4-subsets A_1, A_2, A_3 of consecutive integers from $\{1, \ldots, 10\}$, such that $|A_i \cap A_j| = 1$ if $i \neq j$, which is clearly not possible.

Case 2. $\{v_1, v_2, v_3\}$ *is not independent.*

Suppose that v_1 and v_2 are adjacent. Since v_1 and v_2 have no common neighbors, they are assigned consecutive labels. So there is an integer $j \in \{3, 4, 5, 6, 7\}$, such that $\{c(v_1), c(v_2)\} = \{j, j+1\}$, and the four vertices in $N_G(v_1) \cup N_G(v_2) \setminus \{v_1, v_2\}$ must be assigned labels $j-2, j-1, j+2, j+3$ for some $j \in \{3, 4, 5, 6, 7\}$. This implies that each vertex in $V(G_p) \setminus \{v_1, v_2\}$ has one neighbor x with $c(x) < j$ and one neighbor y with $c(y) > j+1$, so no other vertex in G_p can be list-distance consistent.

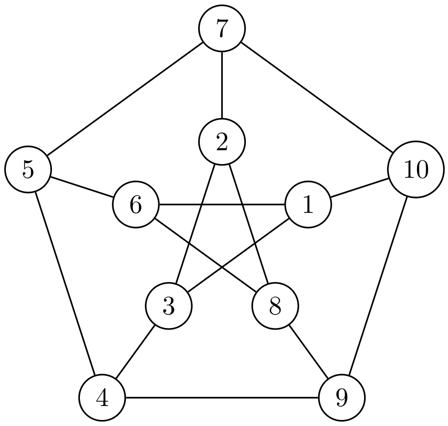

Fig. 1. A labeling of the Petersen graph where two vertices are list-distance consistent

□

Next, we have the following, which gives a family of graphs with large edge density and chromatic number with list-distance consistency 1.

Proposition 5.2. *If* G *is a complete multipartite graph where each part has size at least* 4*, then* $\mathrm{ldc}(G) = 1$*.*

The proof of this proposition is along the same line as the proofs of Theorem 4.7 and Proposition 4.9, so we omit it.

We have seen that the are many graphs which have list-distance consistency greater than 1. However, for increasing n, the set of graphs with list-distance consistency at least 2 is small; indeed, our final result shows that almost every graph has list-distance consistency 1.

We denote by $\mathcal{G}(n, 1/2)$ the probability space of "random graphs" on n vertices where every possible edge appears independently with probability 1/2.

Theorem 5.3. *If* G *is a random graph distributed as* $\mathcal{G}(n, 1/2)$*, then* $\mathbb{P}[\mathrm{ldc}(G) = 1] \to 1$ *as* $n \to \infty$*.*

To prove this theorem we shall use some well-known facts on random graphs. We say that an event A_n occurs *with high probability* if $\mathbb{P}[A_n] \to 1$ as $n \to \infty$.

Lemma 5.4. *If G is a random graph distributed as $\mathcal{G}(n, 1/2)$, then with high probability G has diameter* 2.

This lemma is well-known and easy to prove, so we omit the proof.

Lemma 5.5. *If G is as in the preceding lemma and $\varepsilon > 0$, then with high probability all vertex degrees lie in the interval $[(1/2-\varepsilon)n, (1/2+\varepsilon)n]$.*

This is a variant of standard lemmas for bounding the degrees in a random graph, and it follows easily from the fact that the degree of a vertex is binomially distributed and thus concentrated around its mean (e.g. by applying a standard Chernoff bound).

Equipped with these lemmas, we can now prove Theorem 5.3.

Proof of Theorem 5.3. By Lemma 5.5, every vertex has degree roughly $n/2$, so if u is list-distance consistent under some labeling c, then it follows from Lemma 2.2 that $c(N_G^1(u))$ is a set of roughly $n/2$ consecutive integers from $\{1, \dots, n\}$. Moreover, by Lemma 5.4 all other vertices of G are at distance 2 from u, and they are assigned all other labels from $\{1, \dots, n\} \setminus c(N_G^1(u))$.

Now, we can arbitrarily pick a set A of say $\lceil n/100 \rceil$ vertices in $N_G^1(u)$ and a set B of $\lceil n/100 \rceil$ vertices in the complement of $N_G^1(u)$ so that if $a \in A$ and $b \in B$, then $c(a, b) > 6n/10$.

Now, let $v \neq u$ be a vertex in G. If v is list distance consistent, then $c(N_G^1(v))$ should be a set of roughly $n/2$ consecutive integers. However, the probability that v has no neighbor in A is $(1/2)^{\lceil n/100 \rceil}$, which also precisely equals the probability that v has no neighbor in B.

Hence, the expected number of vertices (distinct from u) with no neighbors in A tends to 0 as $n \to \infty$, and similarly for B. It now follows from Markov's inequality that with high probability every vertex distinct from u has at least one neighbor in A and at least one neighbor in B. Hence, with high probability no vertex $v \neq u$ satisfies that $c(N_G^1(v))$ is a set of consesutive integers. In conclusion, with probability tending to 1 as $n \to \infty$, it holds that $\mathrm{ldc}(G) = 1$. □

6. CONCLUDING REMARKS

In this short note, we have studied qualitative aspects of a distance labeling model suggested in [4]. As it turns out, many graphs have quite small list-distance consistency. Indeed, almost every graph satisfies that $\mathrm{ldc}(G) = 1$, and, moreover, list and local list graphs are very restricted families of graphs. In light of these facts, the usefulness of the model from [4] is somewhat in doubt.

Nevertheless, there is an ever increasing need for efficient methods for analyzing global properties of large complex networks, and even though the model suggested in [4] apparently has its shortcomings, we believe that it could still have some merit as a starting point for further investigation of qualitative aspects of distance labeling problems. There are many comprehendable variations on the basic model suggested in [4]; let us here just point to one or two such directions which we believe could be interesting for further research.

We say that a labeling $c : V(G) \to \{1, \ldots, |V(G)|\}$ is (k, ℓ)-*list-distance consistent* if at least k vertices u satisfy that $d(u, v) \leq d(u, w)$ for all but ℓ vertices $v, w \in V(G)$ satisfying that $c(u, w) = c(u, v) + 1$. The idea is thus that the list-distance condition should hold for as many vertices as possible if we may exclude ℓ vertices. A study of such labelings, for various graphs as well as for different values of k and ℓ, could lead to further insights into qualitative properties of simple labeling schemes for the problem of approximately encoding distances in graphs and large complex networks. Moreover, it would be interesting to analyze how different the vertices in a graph can perform in regards to this parameter. For instance, informally speaking, if a large number of vertices in a graph are list-distance consistent, how "far" from being list-distance consistent can the remaining vertices be? More generally, how much can the "closeness" of being list-distance consistent vary among the vertices in a graph?

Another direction for future research is to determine for which families of graphs the model from [4] performs "reasonably well". For instance, we proved that the family of graphs G that satisfy $\mathrm{ldc}(G) = |V(G)|$ is quite restricted, but what does a "typical graph" G with list-distance consistency at least $|V(G)|/2$ look like? A (partial) answer to this question would potentially imply that list-distance consistent labelings are useful for encoding distances in graphs from certain families. Theorem 4.5 and the characterization in Proposition 2.4 may be considered first steps in this direction.

Acknowledgements

This paper is partially based on the thesis [3] *written by the second author under the supervision of the first author.*

Casselgren thanks Håkan Lennerstad for introducing him to this topic and for stimulating discussions on this graph labeling model. The authors thank a referee for helpful suggestions.

REFERENCES

[1] J.A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. (2022), DS6.

[2] S.P.C. Gavoille, D. Peleg, R. Raz, *Distance labeling in graphs*, J. Algorithms **54** (2001), 85–112.

[3] A. Henricsson, *Distance consistent labellings and the local list number*, Bachelor thesis, Linköping University, 2023.

[4] H. Lennerstad, M. Eriksson, *List graphs and distance-consistent node labelings*, Electron. J. Graph Theory Appl. **6** (2018), 152–165.

[5] D.D.-F. Liu, X. Zhu, *Multilevel distance labelings for paths and cycles*, SIAM J. Disc. Math. **19** (2005), 610–621.

[6] D. Peleg, *Proximity-preserving labeling schemes and their applications*, Lecture Notes in Computer Science **1665** (1999), 30–41.

[7] M. Thorup, S. Alstrup, H. Kaplan, U. Zwick, *Adjacency labeling schemes and induced universal graphs*, SIAM J. Disc. Math. **33** (2019), 116–137.

Carl Johan Casselgren (corresponding author)
carl.johan.casselgren@liu.se

Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden

Anders Henricsson
anders@ahenricsson.se

Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden

Received: September 10, 2025.
Revised: October 18, 2025.
Accepted: October 19, 2025.