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NORMALIZED SOLUTIONS FOR CRITICAL SCHRÖDINGER EQUATIONS INVOLVING $(2,q)$-LAPLACIAN

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Abstract. In this paper, we consider the following critical Schrödinger equation involving $(2,q)$-Laplacian:

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + \mu |u|^{\gamma-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

where $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ is the q-Laplacian operator, $\mu, a > 0$, $\lambda \in \mathbb{R}$, $\gamma \in (2, 2^*)$, $q \in (\frac{2N}{N+2}, 2)$ and $N \geq 3$. The meaningful and interesting phenomenon is the simultaneous occurrence of $(2,q)$-Laplacian and critical nonlinearity in the above equation. In order to obtain existence of multiple normalized solutions for such equation, we need to make a detailed estimate. More precisely, for the L^2-subcritical case, we use the truncation technique, concentration-compactness principle and the genus theory to get the existence of multiple normalized solutions. For the L^2-supercritical case, we obtain a couple of normalized solution for the above equation by a fiber map and concentration-compactness principle.

Keywords: Schrödinger equation, $(2,q)$-Laplacian, variational methods, critical growth, normalized solutions.

Mathematics Subject Classification: 35J20, 35R03, 46E35.

1. INTRODUCTION AND MAIN RESULT

In this paper, we are interested in the following critical Schrödinger equations involving $(2,q)$-Laplacian:

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + \mu |u|^{\gamma-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.1}$$

where $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ is q-Laplacian with $q \in (\frac{2N}{N+2}, 2)$, $\lambda \in \mathbb{R}$, $\mu, a > 0$, $\gamma \in (2, 2^*)$, $N \geq 3$.

At the beginning of this paper, we first give the features and novelties of equation (1.1) as follows:

(a) The appearance of two differential operators with different growth, which makes a double phase associated energy occur.
(b) Equation (1.1) combines the effects generated by general nonlinearity, critical nonlinearity and $(2, q)$-Laplacian.
(c) Since the presence of critical nonlinearity and the unboundedness of the domain, the loss of compactness for the Palais–Smale sequences shall occur, which makes the research of such equation (1.1) more meaningful and interesting.

It is well known that equation (1.1) is derived from the general reaction-diffusion system

$$\partial_t u - \Delta_p u - \Delta_q u = f(x, u), \tag{1.2}$$

where the function u can be used to describe a concentration, (p, q)-Laplacian corresponds to the diffusion asdiv$[(|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u] = \Delta_p u + \Delta_q u$, whereas the nonlinearity $f(x, u)$ is the reaction and relates to sources and loss processes. Such system also motivated by numerous models arising in physics and related sciences, such as biophysics, chemical reaction an exhibits the Lavrentiev gap phenomenon, which arises in the context of variational problems characterized by non-standard (p, q) growth behavior, please refer to [13, 53] and reference. That is the reason why a great number of scholars focus on the study of this topic.

The study of such operators was initially proposed by Zhikov [51], who introduced these classes to model strongly anisotropic materials [52]. Please see the outstanding work initiated by Marcellini [35–37], where the regularity and existence of solutions to elliptic equations with non-uniform growth conditions were extensively analyzed. These representative achievements have brought about a lot of inspirations and ideas to the scholars who are engaged in the research of this topic. In general, there are two methods to dealing with equation (1.1), depending on the properties of the frequency parameter λ. One common method treats λ as a fixed constant and research for nontrivial solutions by analyzing the associated energy functional $I_\lambda : X_{rad}(\mathbb{R}^N) \to \mathbb{R}$ defined as follows:

$$\begin{aligned} I_\lambda(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{q}\int\limits_{\mathbb{R}^N} |\nabla u|^q dx - \int\limits_{\mathbb{R}^N} \lambda |u|^2 dx \\ - \frac{\mu}{\gamma}\int\limits_{\mathbb{R}^N} |u|^\gamma dx - \frac{1}{2^*}\int\limits_{\mathbb{R}^N} |u|^{2^*} dx. \end{aligned}$$

In this case, there are some interesting results, but we merely present some results regarding this topic. For example, Ambrosio and Repovš [4] considered the following Schrödinger equations involving (p, q)-Laplacian and the nonnegative potential:

$$-\Delta_p u - \Delta_q u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $\varepsilon > 0$ is small, $1 < p < q < N$, the potential $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies the global Rabinowitz condition and the nonlinearity f satisfies Sobolev subcritical growth.

By the Ljusternik–Schnirelmann category theory, the authors obtained the existence, multiplicity and concentration of solutions for equation (1.3). After that, Ambrosio [2] extended the results of [4] to fractional Choquard equations. For the critcal case, Ambrosio and Rădulescu [3] considered the following critical Schrödinger equations involving (p, q)-Laplacian:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)(u^{p-1} + u^{q-1}) = f(u) + \gamma u^{q^*-1} & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where the parameter $\varepsilon > 0$ is small, $q^* = \frac{Nq}{N-q}$ is the critical Sobolev exponent, $V(x)$ is the continuous potential and the continuous nonlinearity f satisfies Sobolev subcritical growth. With the aid of minimax theorems, penalization technique and Ljusternik–Schnirelmann category theory, the authors obtained the multiplicity of concentrating solutions for equation (1.4). In addition, the multiplicity of solutions for a supercritical version of equation (1.4) is obtained by a truncation argument with a Moser-type iteration. By asymptotic estimates and the Mountain Pass Theorem, Cui and Yang in [14] explored the existence of solutions for fractional (p, q)-Laplacian equations involving critical Hardy potentials in the bounded domain. Very recently, Zhang *et al.* [49] considered the existence and multiplicity of multi-bump solutions for (N, q)-Lapalcian equations with exponential critical growth by the variational methods and Morse iteration technique. For more interesting results, please refer to [16, 17, 27, 29–31, 40]. Xiang, Ma and Yang in [46] studied the existence of normalized solutions to the following nonlocal double phase problems driving by the discrete fractional (p, q)-Laplacian:

$$\begin{cases} (-\Delta_{\mathbb{D}})^{\alpha}_p u(k) + \mu(-\Delta_{\mathbb{D}})^{\beta} q u(k) + \omega(k)|u(k)|^{p-2}u(k) & \\ = \lambda |u(k)|^{q-2}u(k) + h(k)|u(k)|^{r-2}u(k) & \text{for } |k| \in \mathbb{Z}, \\ \sum_{k\in\mathbb{Z}} |u(k)|^q = \rho^q > 0, & \\ u(k) \to 0 & \text{as } |k| \to \infty, \end{cases} \tag{1.5}$$

where $\alpha, \beta \in (0, 1)$, $\omega : \mathbb{Z} \to (0, \infty)$, $1 < p \le q < \infty$, $\lambda, \mu \in \mathbb{R}$, $h \in \ell^{\frac{q}{q-r}}(\mathbb{Z})$ if $1 < r < q$, $h \in \ell^\infty(\mathbb{Z})$ if $r > q$ and $(-\Delta_{\mathbb{D}})^s_k$ ($s = \alpha$ or $\beta, \kappa = p$ or q) is the discrete fractional κ-Laplacian. They used variational techniques to investigate the existence of non-negative normalized homoclinic solutions when the nonlinear term is subject to sublinear or superlinear growth conditions. Notably, they established the compactness of the relevant energy functional of the problem in the absence of weights. In [42], Sanhaji, Dakkak and Moussaoui demonstrated the existence and uniqueness of the first eigencurve for a homogeneous Neumann problem with singular weights corresponding to the equation

$$\begin{cases} -\Delta_p u = \alpha m_1(x)|u|^{p-2}u + \beta m_2(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. Subsequently, they established numerous properties of this eigencurve, including continuity, variational characterization, asymptotic behavior, concavity, and differentiability.

The other one is to regard the frequency λ as an unknown quantity to eqaution (1.1). In this situation, it is natural to prescribe the value of the mass so that λ can be interpreted as a Lagrange multiplier. From a physical perspective, the scholars focus on the solutions satisfying

$$\int_{\mathbb{R}^N} |u|^2 dx = a^2 \quad \text{for } a > 0$$

for a priori given a. Such solutions can reveal more clearly the dynamical properties, such as orbital stability or instability, and can describe attractive Bose–Einstein condensates. In addition, this type of solution is usually called prescribed L^2-norm solutions or normalized solutions in mathematics.

In recent years, there are some scholars exploring equation (1.1), about the existence, multiplicity, and asymptotic characteristics of normalized solutions under various conditions through a range of methodologies. For example, as $q = 2$, Jeanjean [18] first considered the following nonlinear elliptic equations:

$$\begin{cases} -\Delta u = \lambda u + f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2. \end{cases} \tag{1.7}$$

By a minimax procedure, the author showed that for each $a > 0$, the existence of multiple normalized solutions for equation (1.7). Cazenave and Lions [7] demonstrated the orbital stability of certain standing wave solutions within nonlinear Schrödinger equations, including those derived from models such as laser beams, time-dependent Hartree equations and Pekar–Choquard time-dependent equations. Soave [44] studied critical Schrödinger equations as follows:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{q-2} u + |u|^{2^*-2} u \ \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.8}$$

where $N \geq 3$ and $|u|^{q-2}u$ satisfies L^2-mass supercritical growth. They proposed new criteria for the existence of global solutions and finite time blow-up in the associated dispersive equation to obtain the existence and properties of solutions for equation (1.8). Jeanjean and Le [19] studied Schrödinger equations, and proved the existence of standing waves solutions that are not ground states but lie at a mountain-passlevel of the energy functional. Moreover, these solutions are unstable in the sense that they blow up in finite time. For further related results, refer [12, 28, 39].

For $q \neq 2$, Baldelli *et al.* [5] explored the following critical (p,q)-Laplacian equation:

$$-\Delta_p u - \Delta_q u = \lambda V(x)|u|^{k-2}u + f(u) + K(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N. \tag{1.9}$$

Using the variational methods and concentration compactness principle [33], they obtained the existence of multiple solutions in the whole space. In particular, under suitable conditions on f, they proved existence of infinitely many weak solutions with negative energy when λ belongs to a certain interval. Recently, Chen *et al.* [9] used Ekeland's variational principle to study the existence of multiple normalized solutions for the following $(2,q)$-Laplacian equation:

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + h(\varepsilon x)f(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.10}$$

where $2 < q < N, \varepsilon > 0, a > 0$. The parameter $\lambda \in \mathbb{R}$ serves as an unknown Lagrange multiplier, h is a continuous positive function and f is also continuous and satisfies L^2 -subcritical growth. They divided the problem into autonomous case and nonautonomous case, then they found that when ε is sufficiently small, the number of normalized solutions is at least the number of global maximum points of h. In [11], Chen and Qin studied the existence of ground state and mountain-pass solutions for the quasilinear equation:

$$-\Delta_N u + V(x)|u|^{N-2}u = f(u) \quad \text{in } \mathbb{R}^N, \quad N \geq 2, \tag{1.11}$$

they used variational methods to establish unified results for periodic, radial and asymptotically constant potentials $V(x)$, with $f(u)$ having critical exponential growth. They obtained new compactness lemmas in $W^{1,N}(\mathbb{R}^N)$ generalizing radial case results and a path construction controlling mountain-pass levels to restore compactness.

When focusing on the study of normalized solutions for the critical Schrödinger equation involving $(2,q)$-Laplacian in $\mathbb{R}^N$, we note that the existing results on this problem remain relatively limited. Motivated by previous results, we proves the existence of multiple normalized solutions for equation (1.1) in this paper. To some extent, we generalize and supplement some previous results.

Now, we present the main results of this paper. For the L^2-subcritical case, we establish the existence of multiple normalized solutions to equation (1.1).

Theorem 1.1. *Assume that $q \in (\frac{2N}{N+2}, 2)$ and $\gamma \in (2, 2+\frac{4}{N})$. Then, for $k \in \mathbb{N}$, there exists a constant $\alpha > 0$ independent of k and $\mu_k := \mu(k)$ such that equation* (1.1) *possesses at least k couples $(u_i, \lambda_i) \in X \times \mathbb{R}$ of distinct weak solutions for $\mu \geq \mu_k$ and*

$$0 < a < \min\left\{ \left(\frac{\alpha}{\mu K}\right)^{\frac{2}{2N-\gamma(N-2)}}, \left(\frac{\gamma}{2\mu K}\right)^{\frac{2}{2\gamma - N\gamma + 2N}} \theta_1^{\frac{4-N\gamma+2N}{2\gamma-N\gamma+2N}} \right\}$$

with $\int_{\mathbb{R}^N} |u_i|^2 \, \mathrm{d}x = a^2, \lambda_i < 0$ for all $i = 1, \ldots, k$. Here, the Sobolev space X is given in Section 2, and the constant K is given in (3.1) *of Section 3.*

Remark 1.2. In L^2-subcritical case, as $\gamma \in (2+\frac{4}{N}, q+\frac{2q}{N})$, the energy functional is unbound from below on $S(a)$ which leads to the non-existence of the $(PS)_c$ sequence. Therefore, the range $2 < \gamma \in (2, 2+\frac{4}{N})$ is suitable.

For the L^2-supercritical case, we establish the existence and multiplicity of normalized solution for equation (1.1).

Theorem 1.3. *Assume that* $q \in (\frac{2N}{N+2}, 2)$ *and* $\gamma \in (q+\frac{2q}{N}, 2^*)$*. Then there exists* $\mu^* = \mu^*(a) > 0$ *such that equation* (1.1) *admits a couple* $(u_\alpha, \lambda_\alpha) \in X \times \mathbb{R}$ *of weak radical solution with* $\int_{\mathbb{R}^3} |u|^p = a^p$ *and* $\lambda_a < 0$.

Remark 1.4. The proofs of Theorems 1.1 and 1.3 are derived through the application of appropriate variational methods. It is evident that we shall encounter the following difficulties.

(i) We use a truncation methord for the L^2-subcritical case to guarantee that the truncated energy functional is bounded from below and coercive. For the L^2-supercritical case, due to the functional $\mathcal{I}$ exhibits the mountain-pass structure on $S(a)$, it ensures the existence of a Palais–Smale sequence. However, the functional $\mathcal{I}$ corresponding to equation (1.1) is unbounded below on

$$S(a) = \{u \in X_{rad}(\mathbb{R}^N) : |u|_2 = a\} \tag{1.12}$$

which leads the Palais–Smale sequence is also unbounded. To address this challenge, we inspired by Jeanjean in [18], and introduce the auxiliary energy functional defined by

$$\tilde{\mathcal{I}} : S(a) \times \mathbb{R} \to \mathbb{R}, (u, \vartheta) \mapsto \mathcal{I}(\varphi(u, \vartheta)),$$

where $\varphi(u, \vartheta)(x) = e^{\frac{N\vartheta}{2}} u\left(e^\vartheta x\right)$. Both $\mathcal{I}$ and $\tilde{\mathcal{I}}$ satisfies the mountain pass geometry on the manifold $S(a)$. Their mountain pass levels are equivalent.

(ii) Compared with Jeanjean [18], it is more difficult to prove compactness in the entire space when the critical nonlinearity in equation (1.1) appears. To address this issue, we employ the concentration-compactness principles from [8, 34]. Furthermore, the occurrence of a non-local term requires the development of new methods to deal with.

(iii) The solution space is no more a Hilbert space since the $(2, q)$-Laplacian operator is non-linear and different from the classical Laplacian $-\Delta$. Consequently, the standard tools relying on Hilbert space structure are not applicable. Motivated by the work of Baldelli *et al.* [5] and consider a suitable concentration compactness principle.

The structure of the paper is as follows: We provide the variational setting and provide preliminary lemmas in Section 2. And the purpose of Section 3 is using concentration-compactness principle, the truncation technique and the genus theory to demonstrate Theorem 1.1. In Section 4, the auxiliary energy functional and concentration compactness principle can be used to prove Theorem 1.3.

To some extent, the results of this paper extend those of Cai and Rădulescu [6], Chen *et al.* [9], Li and Zou [26], and Xiao *et al.* [47].

2. PRELIMINARIES

In this section, we briefly review the definitions and list some basic properties of the workspace. Let $X := H^1(\mathbb{R}^N) \cap D^{1,q}\left(\mathbb{R}^N\right)$, endowed with the standard norm:

$$\|u\|_X = \|u\|_{H^1(\mathbb{R}^N)} + \|u\|_{D^{1,q}(\mathbb{R}^N)},$$

where $D^{1,q}\left(\mathbb{R}^N\right) := \{u \in L^{q^*}(\mathbb{R}^N) : \nabla u \in L^q(\mathbb{R}^N)\}$ which is equipped with the following semi-norm:

$$\|u\|_{D^{1,q}(\mathbb{R}^N)} = \|\nabla u\|_q$$

and $|\cdot|_\tau$ is the usual norm on $L^\nu(\mathbb{R}^N)$ for $\nu \in [1, +\infty)$. To research for the normalized solutions of equation (1.1), we analyze the critical points of the following functional

$$\mathcal{I}(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{q}\int\limits_{\mathbb{R}^N} |\nabla u|^q dx - \int\limits_{\mathbb{R}^N} H(u)dx$$

on the constraint manifold

$$S(a) = \left\{u \in X_{rad} := H^1_{rad}(\mathbb{R}^N) \cap D^{1,q}_{rad}\left(\mathbb{R}^N\right) : |u|_2 = a\right\},$$

where

$$H(t) = \frac{\mu}{\gamma}|t|^\gamma + \frac{1}{2^*}|t|^{2^*}, \quad t \in \mathbb{R}.$$

It is well known that $\mathcal{I} \in C^1(X, \mathbb{R})$ and the Fréchet derivative of $\mathcal{I}$ is given by

$$\langle \mathcal{I}'(u), u\rangle = \int\limits_{\mathbb{R}^N} |\nabla u|^2 dx + \int\limits_{\mathbb{R}^N} |\nabla u|^q dx - \mu\int\limits_{\mathbb{R}^N} |u|^\gamma dx - \int\limits_{\mathbb{R}^N} |u|^{2^*} dx$$

on $S(a)$. Then there exists the continuous embedding $X \hookrightarrow L^\nu(\mathbb{R}^N)$ for all $\nu \in [2, 2^*]$ and the compact embedding $X \hookrightarrow\hookrightarrow L^\nu(\mathbb{R}^N)$ for all $\nu \in (2, 2^*)$ and S denotes the best Sobolev constant by

$$S := \inf_{\substack{u\in X\\ u\neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}}. \tag{2.1}$$

Lemma 2.1 ([9, Lemma 2.1])**.** *Let $l \in (q, 2^*)$, a constant $C > 0$ exists such that for all $u \in X$*

$$|v|_l \leq C|\nabla v|_q^{\theta_{l,q}} |v|_2^{(1-\theta_{l,q})},$$

where $\theta_{l,q} = \frac{Nq(l-2)}{l[Nq-2(N-q)]}$.

Lemma 2.2 ([25, Lemma 2.1]). *Let $\tau \in (q, 2^*)$, there exists a constant*

$$C_\tau := (\tau\ /2\|W_\tau\|_2^{\tau-2})^{1/\tau} > 0$$

such that, for each $u \in X_{rad}(\mathbb{R}^N)$

$$\|v\|_{\ \tau} \leq C_\tau \|\nabla v\|_2^{\bar{a}} \|v\|_2^{(1-\bar{a})},$$

where $\bar{a} = N\left(\frac{1}{2} - \frac{1}{\tau}\right)$ and W_τ is the unique positive solution of

$$-\Delta W + \left(\frac{1}{\bar{a}} - 1\right) W = \frac{2}{\tau \bar{a}} |W|^{\tau-2} W.$$

Lemma 2.3 ([34, Lemma I.1.]). *Let $\{u_n\}$ be a weakly convergent sequence to u in X such that $|u_n|^{2^*} \rightharpoonup \nu$ and $|\nabla u_n|^2 + |\nabla u_n|^q \rightharpoonup \kappa$ in the sense of measures. Then, for some at most countable index set I,*

(i) $\nu = |u|^{2^*} + \sum_{i \in I} \delta_{x_i} \nu_i,\ \nu_i > 0$;
(ii) $\kappa \geq |\nabla u|^2 + |\nabla u|^q + \sum_{i \in I} \delta_{x_i} \mu_i,\ \mu_i > 0$,
(iii) $\kappa_i \geq S \nu_i^{2/2^*}$,

where S is the best Sobolev constant is given by (2.1), δ_{x_i} are Dirac measures at x_i, and κ_i, ν_i are positive constants.

Lemma 2.4 ([5, Lemma 8]). *Let the sequence $\{u_n\}$ weakly converge to u in X, and define*

$$\nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int\limits_{|x|>R} |u_n|^{2^*} dx,$$

$$\kappa_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int\limits_{|x|>R} (|\nabla u_n|^2 + |\nabla u_n|^q) dx.$$

The quantities ν_∞ and κ_∞ exist and satisfy

(i) $\limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty$,
(ii) $\limsup_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla u_n|^q) dx = \int_{\mathbb{R}^N} d\kappa + \kappa_\infty$,
(iii) $\kappa_\infty \geq S \nu_\infty^{2/2^*}$.

3. IN L^p-SUBCRITICAL PERTURBATION

We first review the genus theory to obtain the existence and multiplicity of normalized solutions for equation (1.1) in L^p-subcritical case.

Let X be a Banach space and M be a subset of X, if $u \in M$ and $-u \in M$, we call M is symmetric. Then we can define the following set:

$$\Xi := \{M \subset X \backslash \{0\} : M \text{ is closed and symmetric with respect to the origin}\}.$$

Set

$$\gamma(M)=\begin{cases}+\infty & \text{if is no such odd map exists,}\\ 0 & \text{if } M=\emptyset,\\ \inf\left\{k\in\mathbb{N}: \text{ there exists an odd } \varphi\in C\left(M,\mathbb{R}^k\backslash\{0\}\right)\right\},\end{cases}$$

where $\Xi_k=\{M\in\Xi:\gamma(M)\geq k\}$ and $M\in\Xi$.

In what follows, we need to make accurate analysis of the action functional $\mathcal{I}(u)$. If $u\in S(a)$, by Lemma 2.2, we can get

$$\begin{aligned}\mathcal{I}(u)&=\frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla u|^2dx+\frac{1}{q}\int\limits_{\mathbb{R}^N}|\nabla u|^qdx-\frac{\mu}{\gamma}\int\limits_{\mathbb{R}^N}|u|^\gamma dx-\frac{1}{2^*}\int\limits_{\mathbb{R}^N}|u|^{2^*}dx\\ &\geq\frac{1}{2}|\nabla u|_2^2-\frac{\mu}{\gamma}a^{\gamma-\frac{N(\gamma-2)}{2}}K|\nabla u|_2^{\frac{N(\gamma-2)}{2}}-\frac{1}{2^*}S^{-\frac{2^*}{2}}|\nabla u|_2^{2^*}\\ &:=h\left(|\nabla u|_2\right),\end{aligned}\tag{3.1}$$

where

$$h(t)=\frac{1}{2}t^2-\frac{\mu}{\gamma}a^{\gamma-\frac{N(\gamma-2)}{2}}Kt^{\frac{N(\gamma-2)}{2}}-\frac{1}{2^*}S^{-\frac{2^*}{2}}t^{2^*}.$$

Since $2<\gamma<2+\frac{4}{N}$, we have $\frac{N(\gamma-2)}{2}<2$, the following function

$$g(t)=\frac{1}{2}t^{2-\frac{N(\gamma-2)}{2}}-\frac{1}{2^*}S^{-\frac{2^*}{2}}t^{2^*-\frac{N(\gamma-2)}{2}},\quad\forall t\in(0,\infty)$$

attains a uniquely maximum point $(t_0,g(t_0))$, where $t_0>0, g(t_0)>0$. Hence, if

$$\frac{\mu}{\gamma}a^{\gamma-\frac{N(\gamma-2)}{2}}K<g(t_0):=\alpha,$$

then at t_0, the function $h(t)$ also achieve a unique positive local maximum. In particular, there exist two constants $0<\theta_1<t_0<\theta_2<+\infty$ such that

$$\begin{cases}h(t)>0 & \text{if } t\in(\theta_1,\theta_2),\\ h(t)<0 & \text{if } t\in(0,\theta_1)\cup(\theta_2,+\infty).\end{cases}$$

Based on the above facts, we conclude that the energy functional $\mathcal{I}$ is unbounded from below on $S(a)$. In order to guarantee that the energy functional $\mathcal{I}$ is bounded and coercive on $S(a)$, we will introduce a truncated function. Let $\tau:\mathbb{R}^+\to[0,1]$ be a C^∞ and non-increasing function that satisfies

$$\tau(t)=\begin{cases}0 & \text{if } t\geq\theta_2,\\ 1 & \text{if } t\leq\theta_1.\end{cases}$$

Given $u\in S(a)$, the corresponding truncated functional as follows:

$$\mathcal{I}_\tau(u)=\frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla u|^2dx+\frac{1}{q}\int\limits_{\mathbb{R}^N}|\nabla u|^qdx-\frac{\mu}{\gamma}\int\limits_{\mathbb{R}^N}|u|^\gamma dx-\frac{\tau\left(|\nabla u|\right)}{2^*}\int\limits_{\mathbb{R}^N}|u|^{2^*}dx.$$

From Lemma 2.2, we can get

$$\begin{aligned}\mathcal{I}_\tau(u) &\geq \frac{1}{2}|\nabla u|_2^2 - \frac{\mu}{\gamma}a^{\gamma-\frac{N(\gamma-2)}{2}}K|\nabla u|_2^{\frac{N(\gamma-2)}{2}} - \frac{\tau\left(|\nabla u|\right)}{2^*}S^{-\frac{2^*}{2}}|\nabla u|^{2^*}\\ &:= \tilde{h}\left(|\nabla u|_2\right) \geq h\left(|\nabla u|_2\right),\end{aligned}$$

where

$$\tilde{h}(t) = \frac{1}{2}t^2 - \frac{\mu}{\gamma}a^{\gamma-\frac{N(\gamma-2)}{2}}Kt^{\frac{N(\gamma-2)}{2}} - \frac{\tau\left(|\nabla u|\right)}{2^*}S^{-\frac{2^*}{2}}t^{2^*}.$$

Then, we derive $\tilde{h}(t) \geq h(t) > -\infty$ for $a \in \left(0, \left(\frac{\alpha}{\mu K}\right)^{\frac{2}{2N-\gamma(N-2)}}\right)$ and each $t \in (0, \theta_1)$. We assume $a \in \left(0, \left(\frac{\alpha}{\mu K}\right)^{\frac{2}{2N-\gamma(N-2)}}\right)$, and for all $t \in [\theta_1, +\infty)$, we have

$$\tilde{h}(t) = \frac{1}{2}t^2 - \frac{\mu}{\gamma}a^{\gamma-\frac{N(\gamma-2)}{2}}Kt^{\frac{N(\gamma-2)}{2}}.$$

Additionally, if we further assume the condition that

$$a \in \left(0, \left(\frac{\gamma}{2\mu K}\right)^{\frac{2}{2\gamma-N\gamma+2N}}\theta_1^{\frac{4-N\gamma+2N}{2\gamma-N\gamma+2N}}\right).$$

Then we can conclude that

$$0 < a < \min\left\{\left(\frac{\alpha}{\mu K}\right)^{\frac{2}{2N-\gamma(N-2)}}, \left(\frac{\gamma}{2\mu K}\right)^{\frac{2}{2\gamma-N\gamma+2N}}\theta_1^{\frac{4-N\gamma+2N}{2\gamma-N\gamma+2N}}\right\}.$$

Under this assumption, for any $t \geq \theta_1$, we have $\tilde{h}(t) > 0$. Therefore, we can choose $\theta_1 > 0$ small enough to ensure that

$$\frac{1}{2}t_1^2 - \frac{1}{2^*}S^{-\frac{2^*}{2}}t_1^{2^*} \geq 0 \text{ for all } t_1 \in [0, \theta_1] \text{ and } \theta_1 < S^{\frac{N}{4}}. \tag{3.2}$$

Lemma 3.1. *The energy functional $\mathcal{I}_\tau$ satisfies the following properties:*

(a) $\mathcal{I}_\tau \in C^1\left(X_{rad}, \mathbb{R}\right)$,
(b) *$\mathcal{I}_\tau$ is coercive and bounded from below on $S(a)$; furthermore, if $\mathcal{I}_\tau \leq 0$, we have $|\nabla u|_q + |\nabla u|_2 \leq \theta_1$ and $\mathcal{I}(u) = \mathcal{I}_\tau(u)$.*

Proof. Taking the same arguments as Willem [45, Lemma 2.16], we can deduce that $\mathcal{I}_\tau \in C^1\left(X_{rad}, \mathbb{R}\right)$, so we omit the proof here. We only prove (b). Fix $u \in S(a)$, the definition of τ gives that $\tau(|\nabla u|) \to 0$ as $|\nabla u| \to \infty$. Thus,

$$\begin{aligned}\mathcal{I}_\tau(u) &\geq \frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q dx + \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 dx - \frac{\mu}{\gamma}\int_{\mathbb{R}^N}|u|^\gamma dx\\ &\geq \frac{1}{q}\|\nabla u\|^q + \frac{1}{2}\|\nabla u\|^2 - \frac{\mu}{\gamma}Ca^{(1-\overline{a})\gamma}\|\nabla u\|^{\overline{a}\gamma} \to +\infty\end{aligned}$$

since $\overline{a}\gamma < 2$, where $\overline{a}$ is the constant introduced in Lemma 2.2, it follows that the energy functional $\mathcal{I}_\tau$ is coercive. Since $\widetilde{h}$ has a maximum value from the definition of $\widetilde{h}$, we deduce that $\mathcal{I}_\tau(u)$ is bounded from below on $S(a)$. In addition, $\widetilde{h}(t) < 0$ if $\mathcal{I}_\tau(u) < 0$. According to the definition of $\widetilde{h}(t)$, this further leads to $|\nabla u|_q + |\nabla u|_2 \leq \theta_1$. Hence, we obtain $\tau = 1$ from the definition of τ, i.e. $\mathcal{I}_\tau(u) = \mathcal{I}(u)$. We finish the proof of Lemma 3.1. □

Lemma 3.2. *Let $\{u_n\}$ be a $(PS)_c$ sequence at level $c < 0$ for $\mathcal{I}_\tau$ restricted to $S(a)$, then $u \not\equiv 0$.*

Proof. Assume for contradiction that $u \equiv 0$. Let $\{u_n\} \subset S(a)$ be a $(PS)_c$ sequence at level $c < 0$ for $\mathcal{I}_\tau$. According to Lemma 3.1(b), we have $|\nabla u_n|_2 + |\nabla u_n|_q \leq \theta_1$ for large n and $\{u_n\}$ is also a $(PS)_c$ sequence of $\mathcal{I}$ constrained to $S(a)$ with $c < 0$, i.e. $\mathcal{I}(u_n) \to c < 0$ and

$$\left\| \mathcal{I}|'_{S(a)}(u_n) \right\| \to 0 \text{ as } n \to \infty.$$

We now observe that the sequence $\{u_n\}$ is bounded in X_{rad}, then there exists some subsequence $u_n \rightharpoonup u$ in X_{rad} and for every $\nu \in (2, 2^*)$, $u_n \to u$ in $L^\nu(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^N$. Then we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\gamma \, dx = \int_{\mathbb{R}^N} |u|^\gamma dx, \tag{3.3}$$

since $2 < \gamma < 2 + \frac{4}{N}$. And

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\gamma \, dx = 0.$$

By the definition of $\mathcal{I}_\tau$ and (3.2), it follows that

$$\begin{aligned}
0 > c &= \lim_{n\to\infty} \mathcal{I}_\tau(u_n) = \lim_{n\to\infty} \mathcal{I}(u_n) \\
&= \lim_{n\to\infty} \left(\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u_n|^q dx - \frac{\mu}{\gamma} \int_{\mathbb{R}^N} |u_n|^\gamma dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u_n|^{2^*} dx \right) \\
&\geq \lim_{n\to\infty} \left(\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u_n|^q dx - \frac{\mu}{\gamma} \int_{\mathbb{R}^N} |u_n|^\gamma dx - \frac{1}{2^*} S^{-\frac{2^*}{2}} |\nabla u_n|^{2^*} \right) \\
&\geq -\frac{\mu}{\gamma} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\gamma dx = 0
\end{aligned}$$

which is absurd. This completes the proof of Lemma 3.2. □

We define the functional $\Psi(v) : X_{rad} \to \mathbb{R}$ by

$$\Psi(v) = \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 dx.$$

It follows that $S(a) = \Psi^{-1}\left(\left\{a^2/2\right\}\right)$. Thus, we can apply [45, Proposition 5.12], which guarantees the existence of $\lambda_n \in \mathbb{R}$ fulfilling

$$\left\|\mathcal{I}'\left(u_n\right) - \lambda_n \Psi'\left(u_n\right)\right\| \to 0 \quad \text{as } n \to \infty.$$

Then, we obtain

$$-\Delta u_n - \Delta_q u_n - \mu \left|u_n\right|^{\gamma-2} u_n - \left|u_n\right|^{2^*-2} u_n = \lambda_n u_n + o_n(1) \quad \text{in } X_{rad}^*, \tag{3.4}$$

where X_{rad}^* represents the dual space of X_{rad}.

The appearance of a critical term leads to a loss of compactness for the minimizing sequence $\{u_n\}$ in the entire space. To address this challenge, we apply the concentration-compactness principle developed in [34].

Lemma 3.3. *There holds $u_n \to u$ in $L^{2^*}\left(\mathbb{R}^N\right)$.*

Proof. We divide the proof into three steps.

Step 1. We prove that $\kappa\left(\{x_i\}\right) = \nu_i$, where $\kappa\left(\{x_i\}\right)$ is given in Lemma 2.3.

We first introduce a cut-off function $\varphi_\rho(x) := \varphi(\frac{x-x_i}{\rho})$ which satisfies

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in B_1, \\ 0 & \text{if } x \in B_2^c, \end{cases}$$

and $|\nabla \varphi| \leq 2$.

Next, we note that $\{u_n \varphi_\rho\}$ is bounded in X_{rad} and φ_ρ take values in $\mathbb{R}$, then $\langle \mathcal{I}'(\varphi_\rho u_n), \varphi_\rho u_n\rangle \to 0$ as $n \to \infty$. Based on these facts, we can conclude that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left|\nabla u_n\right|^2 \varphi_\rho(x) dx + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \varphi_\rho(x) dx \\ &+ \int_{\mathbb{R}^N} \left|\nabla u_n\right|^q \varphi_\rho(x) dx + \int_{\mathbb{R}^N} u_n \left|\nabla u_n\right|^{q-2} \nabla u_n \nabla \varphi_\rho(x) dx \\ &= \mu \int_{\mathbb{R}^N} |u_n|^\gamma \varphi_\rho(x) dx + \int_{\mathbb{R}^N} \left|u_n\right|^{2^*} \varphi_\rho(x) dx + o_n(1). \end{aligned} \tag{3.5}$$

In fact, since $\left|\nabla u_n\right|_2 + \left|\nabla u_n\right|_q \leq \theta_1$ as $n \to \infty$, then applying Lemmas 2.3 and 2.4 we deduce the existence of two positive measures ν, κ such that

$$\left|\nabla u_n\right|^2 + \left|\nabla u_n\right|^q \rightharpoonup \kappa \quad \text{and} \quad \left|u_n\right|^{2^*} \rightharpoonup \nu \tag{3.6}$$

as $n \to \infty$.

Due to the boundedness of $\{u_n\}$ in X_{rad}, together with (3.3) and Hölder's inequality, it follows that

$$\lim_{\rho \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} u_n |\nabla u_n|^{q-2} \nabla u_n \nabla \varphi_\rho(x) dx = 0$$

and

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} u_n\nabla u_n\nabla\varphi_\rho(x)dx = 0.$$

According to the definition of φ_ρ and $\gamma\in(2,2+\frac{4}{N})$, we obtain

$$\lim_{\rho\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^\gamma\varphi_\rho dx = 0.$$

From Lemma 2.3 and (3.6)

$$\begin{aligned}
&\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi_\rho(|\nabla u_n|^2+|\nabla u_n|^q)dx = \lim_{\rho\to 0}\int_{\mathbb{R}^N}\varphi_\rho d\kappa = \kappa\left(\{x_i\}\right),\\
&\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi_\rho\left|u_n\right|^{2^*}dx = \lim_{\rho\to 0}\int_{\mathbb{R}^N}\varphi_\rho d\nu = \nu\left(\{x_i\}\right) = \nu_i.
\end{aligned}$$

In summary, by letting $n\to\infty$ in (3.5), and then taking the limit as $\rho\to 0$, we arrive at the conclusion that $\kappa\left(\{x_i\}\right)=\nu_i$.

Step 2. We demonstrate that $\kappa_\infty=\nu_\infty$, where κ_∞ and ν_∞ are defined by Lemma 2.4.

We introduce a cut-off function, $\eta_R(x)=\eta(x/R)\in C^\infty\left(\mathbb{R}^N\right)$, which satisfies

$$\eta_R(x)=\begin{cases}0 & \text{if } x\in\mathbb{R}^N\backslash B_2,\\ 1 & \text{if } x\in B_1,\end{cases}$$

we observe that $\{u_n\eta_R\}$ is bounded in $X_{rad}(\mathbb{R}^N)$ and η_R take values in $\mathbb{R}$, $\langle\mathcal{I}'(\eta_R u_n),\eta_R u_n\rangle\to 0$ as $n\to\infty$. Based on the above, we can conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left|\nabla u_n\right|^2\eta_R(x)dx+\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\eta_R(x)dx\\
&+\int_{\mathbb{R}^N}\left|\nabla u_n\right|^q\eta_R(x)dx+\int_{\mathbb{R}^N}u_n\left|\nabla u_n\right|^{q-2}\nabla u_n\nabla\eta_R(x)dx\\
&=\mu\int_{\mathbb{R}^N}|u_n|^\gamma\eta_R(x)dx+\int_{\mathbb{R}^N}|u_n|^{2^*}\eta_R(x)dx+o_n(1).
\end{aligned}\tag{3.7}$$

This implies that

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}u_n|\nabla u_n|^{q-2}\nabla u_n\nabla\eta_R(x)dx=0$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\eta_R(x)dx=0.$$

From the definition of η_R, it follows that

$$\int\limits_{\{x\in\mathbb{R}^N:|x|>R\}} (|\nabla u_n|^2 + |\nabla u_n|^q)dx \le \int\limits_{\mathbb{R}^N} \eta_R(|\nabla u_n|^2 + |\nabla u_n|^q)dx$$
$$\le \int\limits_{\{x\in\mathbb{R}^N:|x|>R/2\}} (|\nabla u_n|^2 + |\nabla u_n|^q)dx.$$

Thus, by applying Lemma 2.4, we derive

$$\lim_{R\to\infty}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} \eta_R(|\nabla u_n|^2 + |\nabla u_n|^q)dx = \kappa_\infty \tag{3.8}$$

and

$$\lim_{R\to\infty}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} \eta_R\,|u_n|^{2^*}\,dx = \nu_\infty,$$

$$\lim_{R\to\infty}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} \eta_R\,|u_n|^{\gamma}\,dx = \lim_{R\to\infty}\int\limits_{\mathbb{R}^N} \eta_R|u|^\gamma dx = \lim_{R\to\infty}\int\limits_{|x|>R/2} \eta_R|u|^\gamma dx = 0.$$

As $R\to\infty$, (3.7) leads to $\kappa_\infty = \nu_\infty$.

Step 3. We demonstrate that for every $i_0\in\mathcal{I}$, $\nu_i = 0$ and $\nu_\infty = 0$.

To prove this, we proceed by contradiction. Assume that there exists $i_0\in\mathcal{I}$, such that $\nu_{i_0} > 0$ or $\nu_\infty > 0$. Steps 1 and 2 imply that $\kappa_i \ge S\kappa_i^{\frac{2}{2^*}}$ or $\kappa_\infty \ge S\kappa_\infty^{\frac{2}{2^*}}$. It yields that $\kappa_i \ge S^{\frac{N}{2}}$ or $\kappa_\infty \ge S^{\frac{N}{2}}$. If the former case is valid, then

$$\theta_1^2 \ge \lim_{\rho\to0}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}(|\nabla u_n|_2^2 + |\nabla u_n|_q^q)dx \ge \lim_{\rho\to0}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\widehat{\varphi}_\rho(|\nabla u_n|^2 + |\nabla u_n|^q)dx$$
$$= \lim_{\rho\to0}\int\limits_{\mathbb{R}^N}\widehat{\varphi}_\rho d\kappa = \kappa_{i_0} \ge S^{\frac{N}{2}}$$

which contradicts (3.2). The last case is the same as the first case, which also contradicts (3.2). Consequently, we have

$$u_n \to u \quad \text{in } L^{2^*}(\mathbb{R}^N/B_R(0)).$$

Thus, we conclude that

$$u_n \to u \quad \text{in } L^{2^*}(\mathbb{R}^N).$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *For all $c<0$, the functional $\mathcal{I}_\tau$ satisfies the $(PS)_c$ condition on $S(a)$.*

Proof. According to (3.4), we infer that

$$-\Delta u-\Delta_q|u|^{q-2}u-\mu\,|u|^{\gamma-2}\,u-|u|^{2^*-2}\,u=\lambda_a u.$$

Therefore, we obtain that

$$|\nabla u|_2^2+|\nabla u|_q^q-\mu\int\limits_{\mathbb{R}^N}|u|^\gamma dx-\int\limits_{\mathbb{R}^N}|u|^{2^*}dx=\lambda_a\int\limits_{\mathbb{R}^N}|u|^2dx. \tag{3.9}$$

Next, we show that $\lambda_a<0$. Indeed, if u is a weak solution of equation (3.9), then the Pohozaev identity [24] yields

$$\begin{aligned}&\frac{N-2}{2}\int\limits_{\mathbb{R}^N}|\nabla u|^2dx+\frac{N-q}{q}\int\limits_{\mathbb{R}^N}|\nabla u|^qdx\\&=\frac{N\mu}{\gamma}\int\limits_{\mathbb{R}^N}|u|^\gamma dx+\frac{N-2}{2}\int\limits_{\mathbb{R}^N}|u|^{2^*}dx+\frac{N\lambda_a}{2}\int\limits_{\mathbb{R}^N}|u|^2dx.\end{aligned} \tag{3.10}$$

Next, by $u\not\equiv 0, 2<\gamma<2+\frac{4}{N}$, we infer from (3.9) and (3.10) that

$$\begin{aligned}\lambda_a\int\limits_{\mathbb{R}^N}|u|^2dx&=-\frac{N-2}{2}\left(-\frac{2}{N-2}\frac{N-q}{q}+1\right)\int\limits_{\mathbb{R}^N}|u|^q\,dx\\&\quad+\left(-\frac{N-2}{2}\right)\left(\frac{2\mu N}{(N-2)\gamma}-\mu\right)\int\limits_{\mathbb{R}^N}|u|^\gamma dx\\&=\left(\frac{N-q}{q}-\frac{N-2}{2}\right)\int\limits_{\mathbb{R}^N}|u|^q\,dx-\mu\left(\frac{N}{\gamma}-\frac{N-2}{2}\right)\int\limits_{\mathbb{R}^N}|u|^\gamma dx\\&<0\end{aligned}$$

then $\lambda_a<0$. Thus, we conclude that

$$\begin{aligned}\lim_{n\to\infty}\left[|\nabla u_n|_2^2+|\nabla u_n|_q^q-\lambda_a\,|u_n|_2^2\right]&=\lim_{n\to\infty}\left[\mu\,|u_n|_\gamma^\gamma+|u_n|_{2^*}^{2^*}+o_n(1)\right]\\&=\mu|u|_\gamma^\gamma+|u|_{2^*}^{2^*}\\&=|\nabla u|_2^2+|\nabla u|_q^q-\lambda_a|u|_2^2.\end{aligned} \tag{3.11}$$

Since $\lambda_a<0$, we get $u_n\to u$ in X_{rad} and $|u|_2=a$. We finish the proof of Lemma 3.4. □

We consider the set

$$\mathcal{I}_\tau^{-\varepsilon}=\{u\in\omega_a:\mathcal{I}_\tau(u)\le-\varepsilon\}\subset X_{rad}\quad\text{for }\varepsilon>0.$$

The set $\mathcal{I}_\tau^{-\varepsilon}$ is symmetric and closed since $\mathcal{I}_\tau(u)$ is even and continuous on X_{rad}. The following lemma then holds; it is the same as Lemma 3.2 in [1].

Lemma 3.5. *Give $k \in \mathbb{N}$, for any $\mu \geq \mu_k$ and $0 < \varepsilon \leq \varepsilon_k$, there exist $\mu_k := \mu(k) > 0$ and $\varepsilon_k := \varepsilon(k) > 0$ such that $\gamma\left(\mathcal{I}_\tau^{-\varepsilon}\right) \geq k$.*

We define the set

$$\Xi_k := \{M_1 \subset S(a) : M_1 \text{ is symmetric and closed}, \gamma(M_1) \geq k\}$$

and

$$c_k := \inf_{M_1 \in \Xi_k} \sup_{u \in M_1} \mathcal{I}_\tau(u) > -\infty \quad \text{for any } k \in \mathbb{N}$$

according to Lemma 3.1(b). We proceed to establish Theorem 1.1 by setting

$$K_c := \left\{u \in \omega_a : \mathcal{I}_\tau(u) = c, \mathcal{I}_\tau'(u) = 0\right\}.$$

Lemma 3.6. *If $c = c_k = c_{k+1} = \ldots = c_{k+r}$, then it follows that $\gamma\left(K_c\right) \geq r+1$. Specially, there are at least $r+1$ nontrivial critical points of $\mathcal{I}_\tau(u)$.*

Proof. It is clear that $\mathcal{I}_\tau^{-\varepsilon} \in \Xi$ for $\varepsilon > 0$. Then, Lemma 3.5 assures the existence of $\mu_k := \mu(k) > 0$ and $\varepsilon_k := \varepsilon(k) > 0$ where $k \in \mathbb{N}$, such that if $\mu \geq \mu_k$ and $0 < \varepsilon \leq \varepsilon_k$, then $\gamma\left(\mathcal{I}_\tau^{-\varepsilon_k}\right) \geq k$, and so $\mathcal{I}_\tau^{-\varepsilon_k} \in \Sigma_k$. Furthermore, we get the following inequality:

$$c_k \leq \sup_{u \in \mathcal{I}_\tau^{-\varepsilon_k}} \mathcal{I}_\tau(u) = -\varepsilon_k < 0.$$

If $c = c_k = c_{k+1} = \ldots = c_{k+r} < 0$, we obtain that $\mathcal{I}_\tau(u)$ satisfies the $(PS)_c$ condition at the level $c < 0$ by using Lemma 3.4. As a result, the set K_c is a compact. Then, we can conclude that $\mathcal{I}_\tau(u)$ which is restricted on $S(a)$ has at least $r+1$ nontrivial critical points by applying Theorem 2.1 from [20]. □

Proof of Theorem 1.1. From Lemma 3.1(b), we obtain that $\mathcal{I}_\tau(u)$ in Lemma 3.6 and $\mathcal{I}$ have the same critical points, thus we complete the proof of Theorem 1.1. □

4. IN L^2-SUPERCRITICAL PERTURBATION

For L^2-supercritical case, $q + \frac{2q}{N} < \gamma < \frac{2N}{N-2}$. Given that $N(\gamma - q) > 2q$, we cannot use truncation technique directly to consider equation (1.1) cause the truncated functional $\mathcal{I}_\tau$ is still unbounded from below on $S(a)$. Inspired by the research of Jeanjean [18], for any $u \in X_{rad}$ and $\vartheta \in \mathbb{R}$, we define

$$\tau(u, \vartheta)(x) = e^{\frac{N\vartheta}{2}} u\left(e^\vartheta x\right).$$

Then we obtain the auxiliary functional $\tilde{\mathcal{I}} : X_{rad} \to \mathbb{R}$ as follows:

$$\begin{aligned}\tilde{\mathcal{I}}(u, \vartheta) = &\frac{1}{2} e^{2\vartheta} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &+ \frac{1}{q} e^{\frac{\vartheta(qN + 2q - 2N)}{2}} \int_{\mathbb{R}^N} |\nabla u|^q dx - \frac{1}{e^{N\vartheta}} \int_{\mathbb{R}^N} H\left(e^{\frac{N\vartheta}{2}} u(x)\right) dx,\end{aligned} \tag{4.1}$$

equivalent to

$$\tilde{\mathcal{I}}(u,\vartheta)=\frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{q}\int\limits_{\mathbb{R}^N}|\nabla v|^qdx-\int\limits_{\mathbb{R}^N}H(v(x))dx=\mathcal{I}(v)$$

for $v=\tau(u,\vartheta)(x)$.

4.1. THE MINIMAX APPROACH

We will show that $\tilde{\mathcal{I}}$ exhibits the mountain pass geometric structure on $S(a)\times\mathbb{R}$.

Lemma 4.1. *For any fixed $u\in S(a)$, we have:*

(a) $\tilde{\mathcal{I}}(u,\vartheta)\to 0^+$ *when* $\vartheta\to-\infty$,
(b) $\tilde{\mathcal{I}}(u,\vartheta)\to-\infty$ *when* $\vartheta\to+\infty$.

Proof. (a) Using $\tau(u,\vartheta)(x)=e^{\frac{N\vartheta}{2}}u\left(e^{\vartheta}x\right)$, for every $\varrho\geq 2$, we have

$$\int\limits_{\mathbb{R}^N}|\tau(u,\vartheta)(x)|^2\ \mathrm{d}x=a^2,\quad\int\limits_{\mathbb{R}^N}|\tau(u,\vartheta)(x)|^\varrho\mathrm{d}x=e^{N\vartheta\left(\frac{\varrho-2}{2}\right)}\int\limits_{\mathbb{R}^N}|u|^\varrho dx,\tag{4.2}$$

$$\int\limits_{\mathbb{R}^N}|\nabla\tau(u,\vartheta)(x)|^2dx=e^{2\vartheta}\int\limits_{\mathbb{R}^N}|\nabla u|^2dx\tag{4.3}$$

and

$$\int\limits_{\mathbb{R}^N}|\nabla\tau(u,\vartheta)(x)|^qdx=e^{\frac{\vartheta(qN+2q-2N)}{2}}\int\limits_{\mathbb{R}^N}|\nabla u|^qdx.\tag{4.4}$$

From the above equation, by setting $\delta>2$, we derive that

$$\int\limits_{\mathbb{R}^N}|\nabla\tau(u,\vartheta)(x)|^2dx\to 0,\quad\int\limits_{\mathbb{R}^N}|\nabla\tau(u,\vartheta)(x)|^qdx\to 0\quad\text{as }\vartheta\to-\infty$$

and

$$|\tau(u,\vartheta)|_\delta\to 0\quad\text{as }\vartheta\to-\infty.\tag{4.5}$$

From this, as $\vartheta\to-\infty$, it follows that

$$\int\limits_{\mathbb{R}^N}|H(\tau(u,\vartheta))|dx=\frac{\mu}{\gamma}\int\limits_{\mathbb{R}^N}|\tau(u,\vartheta)|^\gamma dx+\frac{1}{2^*}\int\limits_{\mathbb{R}^N}|\tau(u,\vartheta)|^{2^*}dx\to 0,$$

then we can get

$$\mathcal{I}(\tau(u,\vartheta))\to 0\quad\text{as }\vartheta\to-\infty,$$

proving (a).

(b) From (4.3) and (4.4), passing the limit as $\vartheta \to +\infty$, we obtain

$$\int_{\mathbb{R}^N} |\nabla \tau(u,\vartheta)(x)|^q dx \to +\infty \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla \tau(u,\vartheta)(x)|^2 dx \to +\infty.$$

Therefore, we have

$$\begin{aligned}
\mathcal{I}(\tau(u,\vartheta)) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla \tau(u,\vartheta)(x)|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla \tau(u,\vartheta)(x)|^q dx \\
&\quad - \frac{\mu}{\gamma}\int_{\mathbb{R}^N} |\tau(u,\vartheta)|^\gamma dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |\tau(u,\vartheta)|^{2^*} dx \\
&= \frac{e^{2\vartheta}}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{e^{\frac{\vartheta(qN+2q-2N)}{2}}}{q}\int_{\mathbb{R}^N} |\nabla u|^q dx \\
&\quad - \frac{\mu e^{N\vartheta\left(\frac{\gamma-2}{2}\right)}}{\gamma}\int_{\mathbb{R}^N} |u(x)|^\gamma dx - \frac{e^{\frac{2N\vartheta}{N-2}}}{2^*}\int_{\mathbb{R}^N} |u(x)|^{2^*} dx \\
&\to -\infty \quad \text{as } \vartheta \to +\infty,
\end{aligned}$$

due to $\gamma > q + \frac{2q}{N}$, $q \in (\frac{2N}{N+2}, 2)$. Therefore, the proof of Lemma 4.1 is complete. □

Lemma 4.2. *There exists $G(a) > 0$ such that*

$$0 < \sup_{u\in\Omega_1} \mathcal{I}(u) < \inf_{u\in\Omega_2} \mathcal{I}(u),$$

where

$$\begin{aligned}
\Omega_1 &= \left\{ u \in S(a) : \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^q dx \le G(a) \right\}, \\
\Omega_2 &= \left\{ u \in S(a) : \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^q dx = NG(a) \right\}.
\end{aligned}$$

Proof. Suppose $u, v \in S(a)$ are such that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^q dx < G(a)$$

and

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^q dx = NG(a),$$

where $G(a)$ is arbitrary but fixed. Then due to $\|\mathcal{I}'(u_n) - \lambda_n \psi'(u_n)\| \to 0$ as $n \to \infty$, we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} |\nabla u_n|^q dx - \mu \int_{\mathbb{R}^N} |u_n|^\gamma dx - \int_{\mathbb{R}^N} |u_n|^{2^*} dx = \lambda_n \int_{\mathbb{R}^N} |u_n|^2 dx. \tag{4.6}$$

Since $\gamma \in (q + \frac{2q}{N}, 2^*)$ and $q \in (\frac{2N}{N+2}, 2)$, we obtain

$$\begin{aligned}
\mathcal{I}(v) - \mathcal{I}(u) &\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla v|^q dx - \frac{\mu}{\gamma} \int_{\mathbb{R}^N} |v|^\gamma dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |v|^{2^*} dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla v|^q dx - \frac{1}{2^*} \Big(\mu \int_{\mathbb{R}^N} |v|^\gamma dx + \int |v|^{2^*} dx \Big) \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla v|^q dx - \frac{1}{2^*} \Big(\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^q dx \\
&\quad - \lambda_n \int_{\mathbb{R}^N} |v|^2 dx \Big) - \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q dx \\
&\geq \frac{1}{2} \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^q dx \right) - \frac{1}{2^*} \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^q dx \right) \\
&\quad - \frac{1}{q} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^q dx \right) \\
&\geq \Big(\frac{N}{2} - \frac{N}{2^*} - \frac{1}{q} \Big) G(a) \\
&\geq 0.
\end{aligned}$$

For any $u \in S(a)$, we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^q dx \leq G(a).$$

Combining this with (4.6), we obtain

$$\begin{aligned}\mathcal{I}(u) &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{q}\int |\nabla u|^q dx - \frac{1}{2^*}\left(\mu \int_{\mathbb{R}^N} |u|^r dx + \int_{\mathbb{R}^N} |u|^{2^*} dx\right) \\ &\geq \frac{1}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int |\nabla u|^q dx\right) - \frac{1}{2^*}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int |\nabla u|^q dx\right) \\ &\geq \frac{1}{2}G(a) - \frac{1}{2^*}G(a) \geq 0.\end{aligned}$$

Thus, the proof of Lemma 4.2 is complete. □

Following [45], we define the tangent space $S(a)$ at u as

$$T_u := \left\{ v \in X_{rad} : \int_{\mathbb{R}^N} uv dx = 0 \right\},$$

and the tangent space of $S(a) \times \mathbb{R}$ at (u,t) as

$$\widetilde{T}_{u,t} := \left\{ (v,k) \in X_{rad} \times \mathbb{R} : \int_{\mathbb{R}^N} uv dx = 0 \right\},$$

where $a > 0$.

Lemma 4.3. *Assume $G(a) > 0$ given in Lemma 4.2, then there exist $\hat{u}, \tilde{u} \in S(a)$ such that:*

(a) $|\nabla \hat{u}|_2^2 + |\nabla \hat{u}|_q^q \leq \frac{G(a)}{2}$,
(b) $|\nabla \tilde{u}|_2^2 + |\nabla \tilde{u}|_q^q > 2G(a)$,
(c) $\mathcal{I}(\tilde{u}) < 0 < \mathcal{I}(\hat{u})$.

Furthermore, we set

$$c_\mu(a) = \inf_{h \in \Gamma_a} \max_{t \in [0,1]} \mathcal{I}(h(t))$$

with

$$\Gamma_a = \{ h \in C([0,1], S(a)) : h(0) = \hat{u}, h(1) = \tilde{u} \},$$

and

$$\tilde{c}_\mu(a) = \inf_{\hat{h} \in \tilde{\Gamma}_a} \max_{t \in [0,1]} \tilde{\mathcal{I}}(\hat{h}(t))$$

with

$$\tilde{\Gamma}_a = \left\{ \hat{h} \in C([0,1], S(a) \times \mathbb{R}) : \hat{h}(0) = (\hat{u}, 0), \hat{h}(1) = (\tilde{u}, 0) \right\}$$

then we have

$$0 < \max\{\mathcal{I}(\hat{u}), \mathcal{I}(\tilde{u})\} \leq c_\mu(a) = \tilde{c}_\mu(a).$$

Proof. For every $u_0 \in S(a)$, Lemmas 4.1 and 4.2 ensure that there exist two numbers $\vartheta_1 << -1$ and $\vartheta_2 >> 1$ such that $\hat{u} = \varphi(u_0, \vartheta_1)$ and $\tilde{u} = \varphi(u_0, \vartheta_2)$ satisfy (a)–(c). We write each $\hat{h} \in \tilde{\Gamma}_a$ as

$$\hat{h}(t) = \left(\hat{h}_1(t), \hat{h}_2(t)\right) \in S(a) \times \mathbb{R}.$$

If we define $h(t) = \varphi\left(\hat{h}_1(t), \hat{h}_2(t)\right)$, then $h(t) \in \Gamma_a$ and

$$c_\mu(a) \leq \max_{t\in[0,1]} \mathcal{I}(h(t)) = \max_{t\in[0,1]} \tilde{\mathcal{I}}(\hat{h}(t)).$$

Due to the arbitrariness of $\hat{h} \in \tilde{\Gamma}_a$, it follows that $\tilde{c}_\mu(a) \geq c_\mu(a)$.

Moreover, for every $h \in \Gamma_a$, defining $\hat{h}(t) = (h(t), 0)$, then $\hat{h}(t) \in \tilde{\Gamma}_a$ and

$$\max_{t\in[0,1]} \mathcal{I}(h(t)) = \max_{t\in[0,1]} \tilde{\mathcal{I}}(\hat{h}(t)) \geq \tilde{c}_\mu(a).$$

Since $h \in \Gamma_a$ is arbitrary, we can conclude that $\tilde{c}_\mu(a) \leq c_\mu(a)$. Therefore, $\tilde{c}_\mu(a) = c_\mu(a)$, and $\max\{\mathcal{I}(\hat{u}), \mathcal{I}(\tilde{u})\} \leq c_\mu(a)$. $\square$

By applying Proposition 2.2 in Jeanjean [18], pseudo-gradient flow and the standard Ekeland variational principle, we can assert there exists $(PS)_{\tilde{c}_\mu(a)}$ sequence for $\tilde{\mathcal{I}}(u, \beta)$.

Proposition 4.4. *Let $\hat{h}_n \subset \tilde{\Gamma}_a$ satisfy*

$$\max_{t\in[0,1]} \tilde{\mathcal{I}}\left(\hat{h}_n(t)\right) \leq \tilde{c}_\mu(a) + \frac{1}{n}.$$

Then when $n \to \infty$ there exists a sequence $\{(v_n, \vartheta_n)\} \subset S_r(a) \times \mathbb{R}$ such that:

(a) $\tilde{\mathcal{I}}(v_n, \beta_n) \to \tilde{c}_\mu(a)$,

(b) $\tilde{\mathcal{I}}'\big|_{S_r(a)\times\mathbb{R}}(v_n, \vartheta_n) \to 0$, *i.e.,*

$$\partial_\vartheta \tilde{\mathcal{I}}(v_n, \vartheta_n) \to 0 \quad \text{and} \quad \left\langle \partial_v \tilde{\mathcal{I}}(v_n, \vartheta_n), \tilde{\psi} \right\rangle \to 0$$

for each

$$\tilde{\psi} \in T_{v_n,\vartheta_n} = \left\{ \tilde{\psi} = \left(\tilde{\psi}_1, \tilde{\psi}_2\right) \in X_{rad}\left(\mathbb{R}^N\right) \times \mathbb{R} : \int_{\mathbb{R}^N} v_n \tilde{\psi}_1 \, \mathrm{d}x = 0 \right\}.$$

Lemma 4.5. *Let $\{(v_n, \vartheta_n)\} \subset S_r(a) \times \mathbb{R}$, where $\{(v_n, \vartheta_n)\} \subset S_r(a) \times \mathbb{R}$ is from Proposition 4.4, setting $u_n = \varphi(v_n, \vartheta_n)$, then as $n \to \infty$, we have:*

(a) $\mathcal{I}(u_n) \to c_\mu(a)$,

(b) $P(u_n) \to 0$, *where*

$$P(u_n) = \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{qN + 2q - 2N}{2q} \int_{\mathbb{R}^N} |\nabla u_n|^q dx + N \int_{\mathbb{R}^N} H(u_n) dx - \frac{N}{2} \int_{\mathbb{R}^N} h(u_n) u_n dx.$$

Proof. For (a), since $\mathcal{I}(u_n) = \tilde{\mathcal{I}}\,(v_n, \vartheta_n)$ and $c_\mu(a) = \tilde{c}_\mu(a)$, we obtain the conclusion.
For (b), by (4.1) we have

$$\begin{aligned}
\lim_{\vartheta\to 0}\partial_\vartheta \tilde{\mathcal{I}}(v_n,\vartheta_n) &= \lim_{\vartheta\to 0}\partial_\vartheta\Bigg[\frac{e^{2\vartheta_n}}{2}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2dx \\
&\quad + \frac{e^{\frac{\vartheta_n(qN+2q-2N)}{2}}}{q}\int\limits_{\mathbb{R}^N}|\nabla v_n|^q dx \\
&\quad - \frac{1}{e^{N\vartheta_n}}\int\limits_{\mathbb{R}^N}H(e^{\frac{N\vartheta_n}{2}}v_n)dx\Bigg] \\
&= \lim_{\vartheta\to 0}\Bigg[e^{2\vartheta_n}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2dx + \frac{N}{e^{N\vartheta_n}}\int\limits_{\mathbb{R}^N}H(e^{\frac{N\vartheta_n}{2}}v_n)dx \\
&\quad + \frac{(qN+2q-2N)}{2q}e^{\frac{\vartheta_n(qN+2q-2N)}{2}}\int\limits_{\mathbb{R}^N}|\nabla v_n|^q dx \\
&\quad - \frac{N}{2e^{N\vartheta_n}}\int\limits_{\mathbb{R}^N}h(e^{\frac{N\vartheta_n}{2}}v_n)e^{\frac{N\vartheta_n}{2}}v_n dx\Bigg] \\
&= \int\limits_{\mathbb{R}^N}|\nabla u_n|^2dx + \frac{qN+2q-2N}{2q}\int\limits_{\mathbb{R}^N}|\nabla u_n|^q dx \\
&\quad + N\int\limits_{\mathbb{R}^N}H(u_n)dx \\
&\quad - \frac{N}{2}\int\limits_{\mathbb{R}^N}h(u_n)u_n dx \\
&= P(u_n).
\end{aligned}$$

Thus, $P(u_n) \to 0$, as $n \to \infty$. □

Lemma 4.6. *There holds* $\lim_{\mu\to+\infty} c_\mu(a) = 0$.

Proof. For every $u_0 \in X_{rad}$ and $t \in [0,1]$, $h_0(t) = \varphi(u_0, (1-t)\vartheta_1 + t\vartheta_2)$ is a path in Γ_a. Then, we obtain that

$$\begin{aligned}
c_\mu(a) &\leq \max_{t\in[0,1]} \mathcal{I}(h_0(t)) \\
&= \max_{t\in[0,1]} \Bigg\{ \frac{1}{2} e^{2[(1-t)\vartheta_1+t\vartheta_2]} \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \frac{1}{q} e^{\frac{[(1-t)\vartheta_1+t\vartheta_2](qN+2q-2N)}{2}} \int_{\mathbb{R}^N} |\nabla u_0|^q dx \\
&\qquad - \frac{\mu}{\gamma} e^{\frac{N(\gamma-2)}{2}[(1-t)\vartheta_1+t\vartheta_2]} \int_{\mathbb{R}^N} |u_0|^\gamma dx \\
&\qquad - \frac{1}{2^*} \mathrm{e}^{\frac{N(2^*-2)}{2}[(1-t)\vartheta_1+t\vartheta_2]} \int_{\mathbb{R}^N} |u_0|^{2^*} dx \Bigg\} \\
&\leq \max_{\sigma>0} \Bigg\{ \frac{\sigma^2}{2} \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \frac{1}{q} \sigma^{\frac{qN+2q-2N}{2}} \int_{\mathbb{R}^N} |\nabla u_0|^q dx - \frac{\mu \sigma^{N(\frac{\gamma-2}{2})}}{\gamma} |u_0|_\gamma^\gamma \Bigg\} \\
&= C \left(\frac{1}{\mu}\right)^{\frac{2}{N(\gamma-2)-4}},
\end{aligned}$$

where $\sigma := e^{(1-t)\vartheta_1+t\vartheta_2}$. Together with $\gamma \in \left(q + \frac{2q}{N}, 2^*\right)$, we can conclude that $\lim_{\mu\to\infty} c_\mu(a) = 0$. $\square$

By utilizing the Lagrange multipliers rule for u_n given in Lemma 4.5, we can identify a sequence $\{\lambda_n\} \subset \mathbb{R}$ satisfies

$$-\Delta u_n - \Delta_q u_n = h(u_n) + \lambda_n u_n + o_n(1). \tag{4.7}$$

We can demonstrate that $\{u_n\}$ is a bounded sequence on $S(a)$ by using Lemmas 2.3 and 2.4 of Jeanjean *et al.* [18]. Consequently, we deduce that λ_n must fulfill equality as follows:

$$\lambda_n = \frac{1}{|u_n|_2^2} \left\{ \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} |\nabla u_n|^q dx - \int_{\mathbb{R}^N} h(u_n) u_n dx \right\} + o_n(1)$$

or equivalently,

$$\lambda_n = \frac{1}{a^2} \left\{ \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} |\nabla u_n|^q dx - \int_{\mathbb{R}^N} h(u_n) u_n dx \right\} + o_n(1). \tag{4.8}$$

Lemma 4.7. *There exists a constant $D > 0$ such that*

$$\limsup_{n\to+\infty} \int\limits_{\mathbb{R}^N} H(u_n)dx \le Dc_\mu(a),$$
$$\limsup_{n\to+\infty} \int\limits_{\mathbb{R}^N} h(u_n)u_n dx \le Dc_\mu(a),$$
$$\limsup_{n\to+\infty} \int\limits_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla u_n|^q)dx \le Dc_\mu(a).$$

Proof. From Lemma 4.5(a) and (b), we get

$$\begin{aligned} Nc_\mu(a) + o_n(1) &= N\mathcal{I}(u_n) + P(u_n) \\ &= \frac{N+2}{2} \int\limits_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla u_n|^q)dx - \frac{N}{2} \int\limits_{\mathbb{R}^N} h(u_n)u_n dx \\ &= (N+2)\left[c_\mu(a) + \int\limits_{\mathbb{R}^N} H(u_n)dx + o_n(1)\right] \\ &\quad - \frac{N}{2} \int\limits_{\mathbb{R}^N} h(u_n)u_n dx + \frac{(N+2)(q-2)}{2q} \int\limits_{\mathbb{R}^N} |\nabla u_n|^q dx. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} &-(N+2)\int\limits_{\mathbb{R}^N} H(u_n)dx + \frac{N}{2}\int\limits_{\mathbb{R}^N} h(u_n)u_n dx \\ &= 2c_\mu(a) + o_n(1) + \frac{(N+2)(q-2)}{2q} \int\limits_{\mathbb{R}^N} |\nabla u_n|^q dx. \end{aligned} \tag{4.9}$$

Since $H(t) := \frac{\mu}{\gamma}|t|^\gamma + \frac{1}{2^*}|t|^{2^*}$, for all $t \in \mathbb{R}^N$ and $\gamma \in \left(q + \frac{2q}{N}, 2^*\right)$, we derive that for every $t \in \mathbb{R}$

$$\gamma H(t) \le h(t)t. \tag{4.10}$$

Combining with (4.9), we obtain that

$$\left(\frac{N\gamma}{2} - (N+2)\right) \int\limits_{\mathbb{R}^N} H(u_n)dx \le 2c_\mu(a) + o_n(1).$$

Thus,

$$\limsup_{n\to+\infty} \int\limits_{\mathbb{R}^N} H(u_n)dx \le Dc_\mu(a).$$

By (4.9), we also have

$$\limsup_{n\to+\infty}\int\limits_{\mathbb{R}^N} h(u_n)\,u_n dx \le Dc_\mu(a)$$

and

$$\limsup_{n\to+\infty}\int\limits_{\mathbb{R}^N} (|\nabla u_n|^2+|\nabla u_n|^q)dx \le Dc_\mu(a).$$

Therefore, we finish the proof of Lemma 4.7. □

Equation (4.9) reveals that the sequence $\left\{\int_{\mathbb{R}^N} H(u_n)\,dx\right\}_n$ is not closed to 0. In fact, if

$$\int\limits_{\mathbb{R}N} H(u_n)\,dx \to 0 \quad \text{as} \quad n\to+\infty,$$

then by $H(t)\ge \frac{h(t)t}{2^*}\ge 0$, for every $t\in\mathbb{R}$, we have

$$\int\limits_{\mathbb{R}^N} h(u_n)\,u_n dx \to 0 \quad \text{as} \quad n\to+\infty.$$

Together with (4.9), we can get $c_\mu(a)=0$, which is impossible. Then, we can choose a subsequence and assume that

$$\int\limits_{\mathbb{R}^N} H(u_n)\,dx \to D_1>0 \quad \text{as} \quad n\to\infty. \tag{4.11}$$

Lemma 4.8. *The sequence $\{\lambda_n\}$ is bounded in $\mathbb{R}^N$ with*

$$\lambda_n=\frac{-\mu}{a^2}\left(\frac{N}{\gamma}-\frac{N-2}{2}\right)\int\limits_{\mathbb{R}^N}|u_n|^\gamma\,dx-\frac{N(q-2)}{2qa^2}\int\limits_{\mathbb{R}^N}|\nabla u_n|^q\,dx+o_n(1)$$

and

$$\limsup_{n\to+\infty}\lambda_n\le\frac{D}{a^2}c_\mu(a)$$

for a suitable constant $D>0$.

Proof. Since $\{u_n\}$ is bounded, it follows that $\{\lambda_n\}$ is bounded. Indeed,

$$\lambda_n=\frac{1}{a^2}\left\{\int\limits_{\mathbb{R}^N}|\nabla u_n|^2dx+\int\limits_{\mathbb{R}^N}|\nabla u_n|^qdx-\int\limits_{\mathbb{R}^N}h(u_n)\,u_ndx\right\}+o_n(1). \tag{4.12}$$

By Lemma 4.7, we obtain

$$\begin{aligned}\lambda_n &\leq \frac{1}{a^2}\left\{\int\limits_{\mathbb{R}^N}|\nabla u_n|^2dx+\int\limits_{\mathbb{R}^N}|\nabla u_n|^q dx+\int\limits_{\mathbb{R}^N}h\left(u_n\right)u_n dx\right\}+o_n(1)\\ &\leq \frac{D}{a^2}c_\mu(a)+o_n(1).\end{aligned}$$

This guarantees the boundedness of $\{\lambda_n\}$. Moreover, we obtain that the second inequality holds. For the first equality, it follows from Lemma 4.5(b) that

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}|\nabla u_n|^2dx+\frac{qN+2q-2N}{2q}\int\limits_{\mathbb{R}^N}|\nabla u_n|^q dx\\ &=\frac{N}{2}\int\limits_{\mathbb{R}^N}h\left(u_n\right)u_n dx-N\int\limits_{\mathbb{R}^N}H\left(u_n\right)dx+o_n(1).\end{aligned}$$

Substituting this equality into (4.11), we get

$$\lambda_n a^2=-\mu\left(\frac{N}{\gamma}-\frac{N-2}{2}\right)\int\limits_{\mathbb{R}^N}|u_n|^\gamma\,dx-\frac{N(q-2)}{2q}\int\limits_{\mathbb{R}^N}|\nabla u_n|^q dx+o_n(1)$$

which completes the proof of Lemma 4.8. □

In this section, we address the space X_{rad}. Since $u_n \rightharpoonup u$ in X_{rad}, and $u_n \to u$ in $L^\gamma\left(\mathbb{R}^N\right)$ for $\gamma \in \left(q+\frac{2q}{N}, 2^*\right)$, then we can get

$$\lim_{n\to+\infty}\int\limits_{\mathbb{R}^N}|u_n|^\gamma\,dx=\int\limits_{\mathbb{R}^N}|u|^\gamma dx. \tag{4.13}$$

Lemma 4.9. *There exists $\mu^*>0$ for each $\mu\geq\mu^*>0$, such that $u\neq 0$.*

Proof. Assume by contradiction that $u=0$. Therefore, we observe that

$$\lim_{n\to+\infty}\int\limits_{\mathbb{R}^N}|u_n|^\gamma\,dx=0 \tag{4.14}$$

and by Lemma 4.8, we obtain

$$\limsup_{n\to+\infty}\lambda_n=0. \tag{4.15}$$

By (4.11), (4.14), (4.15) and the following equality

$$a^2\lambda_n=\int\limits_{\mathbb{R}^N}|\nabla u_n|^2dx+\int\limits_{\mathbb{R}^N}|\nabla u_n|^q dx-\int\limits_{\mathbb{R}^N}h\left(u_n\right)u_n dx+o_n(1),$$

we have

$$\int\limits_{\mathbb{R}^N}|\nabla u_n|^2dx+\int\limits_{\mathbb{R}^N}|\nabla u_n|^q dx-|u_n|_{2^*}^{2^*}=o_n(1). \tag{4.16}$$

Then we can assume that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} |\nabla u_n|^q dx = L + o_n(1) \text{ and } |u_n|_{2^*}^{2^*} = L + o_n(1), \tag{4.17}$$

where $L \geq 0$. If $L = 0$, by the definition of $\mathcal{I}(u_n)$, we can obtain that $c_\mu(a) = 0$, which is impossible. If $L > 0$, we can observe that

$$S \leq \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{\left(\int_{\mathbb{R}^N} |u_n|^{2^*} dx\right)^{\frac{2}{2^*}}} \leq \frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} |\nabla u_n|^q dx}{\left(\int_{\mathbb{R}^N} |u_n|^{2^*} dx\right)^{\frac{2}{2^*}}}. \tag{4.18}$$

Together with (4.17) and passing the limit as $n \to \infty$ in (4.18), we get that

$$L \geq S^{\frac{N}{2}}.$$

On the other hand, we obtain

$$\begin{aligned} o_n(1) + c_\mu(a) &= \frac{1}{2}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} |\nabla u_n|^q dx\right) - \frac{\mu}{\gamma}|u_n|_\gamma^\gamma - \frac{1}{2^*}|u_n|_{2^*}^{2^*} \\ &= \frac{1}{N} L \geq \frac{1}{N} S^{\frac{N}{2}} \end{aligned}$$

which is contradiction by Lemma 4.6. Therefore, we prove that $u \neq 0$ as $\mu > 0$ large enough. Hence, the proof of Lemma 4.9 is complete. □

Lemma 4.10. *Increasing if necessary μ^*, for any $\mu \geq \mu^*$, we have $u_n \to u$ in $L^{2^*}\left(\mathbb{R}^N\right)$.*

Proof. Using the same arguments as in Lemma 3.3, we obtain the desired result. Therefore, we omit the proof. □

4.2. PROOF OF THEOREM 1.2

From the above analysis, we obtain $u_n \rightharpoonup u$, where u is nontrivial. Since μ is large enough and by Lemma 4.8, we have

$$\begin{aligned} \lim_{n\to+\infty} \lambda_n &= \lim_{n\to+\infty} \frac{-\mu}{a^2}\left(\frac{N}{\gamma} - \frac{N-2}{2}\right)\int_{\mathbb{R}^N} |u_n|^\gamma dx - \frac{N(q-2)}{2qa^2}\int_{\mathbb{R}^N} |\nabla u_n|^q dx + o_n(1) \\ &= -\frac{-\mu}{a^2}\left(\frac{N}{\gamma} - \frac{N-2}{2}\right)\int_{\mathbb{R}^N} |u|^\gamma dx - \frac{N(q-2)}{2qa^2}\int_{\mathbb{R}^N} |\nabla u|^q dx < 0. \end{aligned}$$

Therefore, we can assume that

$$\lambda_n \to \lambda_a < 0 \quad \text{as} \quad n \to +\infty.$$

Using (4.7), it follows that

$$-\Delta u - \Delta_q u - h(u) = \lambda_a u \quad \text{in } \mathbb{R}^N. \tag{4.19}$$

Thus,

$$|\nabla u|_2^2 + |\nabla u|_q^q - \lambda_a |u|_2^2 = \int\limits_{\mathbb{R}^N} h(u) u dx.$$

On the other hand, we have that

$$|\nabla u_n|_2^2 + |\nabla u_n|_q^q - \lambda_n |u_n|_2^2 = \int\limits_{\mathbb{R}^N} h(u_n) u_n dx + o_n(1).$$

By Lemma 4.10, it follows that

$$u_n \to u \text{ in } L^{2^*}\left(\mathbb{R}^N\right).$$

Furthermore, we deduce that

$$\lim_{n\to+\infty} \int\limits_{\mathbb{R}^N} h\left(u_n\right) u_n dx = \int\limits_{\mathbb{R}^N} h(u) u dx.$$

Therefore, one has

$$\lim_{n\to+\infty} \left(|\nabla u_n|_2^2 + |\nabla u_n|_q^q - \lambda_n \left|u_n\right|_2^2\right) = |\nabla u|_2^2 + |\nabla u|_q^q - \lambda_a |u|_2^2.$$

By $\lambda_a < 0$, we obtain that

$$u_n \to u \text{ in } X_{rad}.$$

Thus, we conclude that $|u|_2^2 = a$, which confirms the required result.

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