Opuscula Math. 45, no. 5 (2025), 665–684
https://doi.org/10.7494/OpMath.2025.45.5.665 Opuscula Mathematica

SYSTEMS OF DIFFERENTIAL INCLUSIONS WITH COMPETING OPERATORS AND VARIABLE EXPONENTS

Francesca Vetro and Rakib Efendiev

Communicated by Vicenţiu D. Rădulescu

Abstract. In this paper, we study a system of differential inclusions with Dirichlet boundary condition, involving competing operators and variable exponents. More precisely, we investigate the existence of both generalized solutions and weak solutions to the problem under consideration. In order to archive our results, we make use of approximation through finite dimensional subspaces via a Galerkin basis along with minimization and nonsmooth analysis.

Keywords: systems of differential inclusions, hemivariational inequalities, competing operators, Galerkin basis.

Mathematics Subject Classification: 49Q20, 49N60, 46E35.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ be a bounded domain whose boundary $\partial\Omega$ is Lipschitz. For $m \in C(\overline{\Omega})$ with $m(x) > 1$ for all $x \in \overline{\Omega}$, we put

$$m^- = \min_{x\in\overline{\Omega}} m(x) \quad \text{and} \quad m^+ = \max_{x\in\overline{\Omega}} m(x).$$

Let now p_i and q_i with $i = 1, 2$ be functions satisfying the following assumptions:

$(H0)$ $p_i, q_i \in C(\overline{\Omega})$ are such that

$$1 < q_i^- \leq q_i(x) \leq q_i^+ < p_i^- \leq p_i(x) \leq p_i^+ < +\infty$$

for all $x \in \overline{\Omega}$ and $i = 1, 2$.

In the present paper, we focus on the following problem of differential inclusions with Dirichlet boundary condition

$$\begin{cases} (-\Delta_{p_1(\cdot)}u_1 + \mu_1\Delta_{q_1(\cdot)}u_1, -\Delta_{p_2(\cdot)}u_2 + \mu_2\Delta_{q_2(\cdot)}u_2) \in \partial F(u_1, u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mu_1, \mu_2 \in \mathbb{R}$ are parameters and $-\Delta_{p_i(\cdot)}$ and $\Delta_{q_i(\cdot)}$, for $i = 1, 2$, denote the negative $p_i(\cdot)$-Laplace operator and the positive $q_i(\cdot)$-Laplace operator, respectively. In the right-hand side of problem (1.1), we find the generalized gradient ∂F of a locally Lipschitz function $F : \mathbb{R}^2 \to \mathbb{R}$. We stress that pointwise $\partial F(u_1, u_2)$ is a subset of $\mathbb{R}^2$. Therefore, (1.1) is a system of two differential inclusions, which are said hemivariational inclusions because they involve generalized gradients. Further, we point out that, according to the Clarke's subdifferentiation theory (see [2]), corresponding to (1.1) we have the following hemivariational inequality

$$\begin{aligned} &\langle -\Delta_{p_1(\cdot)} u_1, h_1 \rangle + \mu_1 \langle \Delta_{q_1(\cdot)} u_1, h_1 \rangle + \langle -\Delta_{p_2(\cdot)} u_2, h_2 \rangle \\ &+ \mu_2 \langle \Delta_{q_2(\cdot)} u_2, h_2 \rangle \le \int_\Omega F^\circ(u_1, u_2; h_1, h_2)\, dx \end{aligned} \tag{1.2}$$

for all $(h_1, h_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$, with F° being the generalized directional derivative of F. We mention that hemivariational inequalities have relevant applications, for example, in the context of mechanics, as one can see by Chang [1] and Panagiotopoulos [12] where the study of such a type of inequalities was developed. We also cite the recent paper of Jleli *et al.* [7] (hyperbolic differential inequality).

The driven operators involved in (1.1) are sum of a negative $p_i(\cdot)$-Laplace operator and a positive $q_i(\cdot)$-Laplace operator weighted by the parameter $\mu_i \in \mathbb{R}$. When $\mu_i \ge 0$ such operators do not satisfy the ellipticity condition and they become competing operators. We recall that problems involving competing operators with constant exponents were first considered in Liu *et al.* [8] and then in Motreanu [9, 10], Galewski *et al.* [4] and Gambera *et al.* [5], see also their references. For the case of variable exponents, we instead mention the recent works of Ghasemi *et al.* [6] (competing operator), Moussaoui *et al.* [11] and Vetro [14] ($p(\cdot)$-Laplace operator).

Taking into account the possible loss of ellipticity for system (1.1) due to the presence of parameters μ_i for $i = 1, 2$, as explored in [9] we need to consider a new type of solution, called generalized solution.

Definition 1.1. We say that $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$ is a generalized solution to problem (1.1) if there exists a sequence

$$\{(u_{1n}, u_{2n})\}_{n\in\mathbb{N}} \subset W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$$

such that

(i) $u_{in} \xrightarrow{w} u_i$ in $W_0^{1,p_i(\cdot)}(\Omega)$, as $n \to +\infty$ for $i = 1, 2$,
(ii) $-\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in} - \zeta_{in} \xrightarrow{w} 0$ in $(W_0^{1,p_i(\cdot)}(\Omega))^*$, as $n \to +\infty$, with $\zeta_{in} \in (W_0^{1,p_i(\cdot)}(\Omega))^*$ and $\zeta_{in} \in \partial F(u_{1n}, u_{2n})$ a.e. in Ω for $i = 1, 2$,
(iii) $\langle -\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in}, u_{in} - u_i \rangle \to 0$, as $n \to +\infty$ for $i = 1, 2$.

In addition, using the hemivariational inequality given in (1.2), we can provide a notion of weak solution to problem (1.1) as follows.

Definition 1.2. We say that $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$ is a weak solution to problem (1.1) if it solves the inequality (1.2). This means that there exists $(\zeta_1, \zeta_2) \in (W_0^{1,p_1(\cdot)}(\Omega))^* \times (W_0^{1,p_2(\cdot)}(\Omega))^*$ satisfying $(\zeta_1, \zeta_2) \in \partial F(u_1, u_2)$ a.e. on Ω such that

$$\begin{aligned} &\langle -\Delta_{p_1(\cdot)} u_1, h_1\rangle + \mu_1 \langle \Delta_{q_1(\cdot)} u_1, h_1\rangle \\ &+ \langle -\Delta_{p_2(\cdot)} u_2, h_2\rangle + \mu_2 \langle \Delta_{q_2(\cdot)} u_2, h_2\rangle = \int\limits_\Omega (\zeta_1 h_1 + \zeta_2 h_2) dx. \end{aligned}$$

Remark 1.3. We point out that any weak solution to problem (1.1) is also a generalized solution. In fact, if $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$ is a weak solution to problem (1.1), then it is sufficient to take

$$u_{in} := u_i \quad \text{and} \quad \zeta_{in} := \zeta_i \quad \text{for all } n \in \mathbb{N} \text{ and } i = 1, 2$$

in Definition 1.1, in order to conclude that (u_1, u_2) is a generalized solution to problem (1.1).

Thus, our first aim is in exploring the link between weak solutions and generalized solutions to problem (1.1). As above mentioned, any weak solution to problem under consideration is in addition a generalized solution. We here show that the reverse implication holds, under very general assumptions on F (see hypothesis (H_F)), if driven operators in (1.1) are both elliptic, which means to claim that $\mu_1 \leq 0$ as well as $\mu_2 \leq 0$ (see Theorem 4.1). Then, we establish the existence, for any $\mu_1, \mu_2 \in \mathbb{R}$, of at least a generalized solution to problem (1.1), supposed again that hypothesis (H_F) holds (see Theorem 5.1). Finally, combining the results in Theorems 4.1 and 5.1, we achieve the existence of at least a weak solution to problem (1.1), supposed $\mu_1, \mu_2 \leq 0$ (see Theorem 5.2). We point out that in order to obtain our results, we make use of approximation through finite dimensional subspaces via a Galerkin basis along with minimization and nonsmooth analysis. In this way, we are able to derive some a priori estimates, which are of independent interest in the context of competing operators (see Propositions 3.3 and 3.4).

We emphasize that the results of the present paper extend ones in Motreanu [10] to the case of variable exponents and ones in Ghasemi *et al.* [6] to the case of systems.

2. PRELIMINARIES

In this section, we collect some notions and results which we need for our study. More precisely, we first give any remarks on variable exponent Lebesgue and Sobolev spaces, and then we recall some basic elements of nonsmooth analysis. We refer the reader to books of Diening *et al.* [3] and Rădulescu *et al.* [13] for more details on variable exponent Lebesgue and Sobolev spaces. A detailed treatment on nonsmooth analysis can be find in Clarke [2].

Let $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$ be a bounded domain whose boundary is Lipschitz. Also, let $m \in C(\overline{\Omega})$ such that $m(x) > 1$ for all $x \in \overline{\Omega}$. We here use m' in order to denote the conjugate variable exponent of m, that is, we have that

$$\frac{1}{m(x)} + \frac{1}{m'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

By $L^{m(\cdot)}(\Omega)$ we denote the variable exponent Lebesgue space, that is,

$$L^{m(\cdot)}(\Omega) = \left\{u \in M(\Omega) \,:\, \rho_{m(\cdot)}(u) < +\infty\right\}$$

where $M(\Omega)$ stands for the set of all measurable functions $u\colon \Omega \to \mathbb{R}$, while $\rho_{m(\cdot)}$ denotes the modular given by

$$\rho_{m(\cdot)}(u) := \int_\Omega |u|^{m(x)} \, \mathrm{d}x \quad \text{for all } u \in M(\Omega). \tag{2.1}$$

As usual, we consider on $L^{m(\cdot)}(\Omega)$ the Luxemburg norm, that is,

$$\|u\|_{m(\cdot)} := \inf\left\{\lambda > 0 \,:\, \rho_{m(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

for all $u \in L^{m(\cdot)}(\Omega)$. With such norm $L^{m(\cdot)}(\Omega)$ becomes a separable, uniformly convex and hence reflexive Banach space whose dual space is given by $L^{m'(\cdot)}(\Omega)$. We point out that the modular $\rho_{m(\cdot)}$ and the norm $\|\cdot\|_{m(\cdot)}$ are strictly related, like one can see from the following result.

Proposition 2.1. *Let $m \in C(\overline{\Omega})$ be such that $m(x) > 1$ for all $x \in \overline{\Omega}$. Then the following hold:*

(i) $\|u\|_{m(\cdot)} < 1$ *(resp.* $> 1, = 1$*) if and only if* $\rho_{m(\cdot)}(u) < 1$ *(resp.* $> 1, = 1$*),*
(ii) *if* $\|u\|_{m(\cdot)} < 1$ *then* $\|u\|_{m(\cdot)}^{m^+} \leq \rho_{m(\cdot)}(u) \leq \|u\|_{m(\cdot)}^{m^-}$,
(iii) *if* $\|u\|_{m(\cdot)} > 1$ *then* $\|u\|_{m(\cdot)}^{m^-} \leq \rho_{m(\cdot)}(u) \leq \|u\|_{m(\cdot)}^{m^+}$.

We note that Proposition 2.1 in particular assures that the following inequality

$$\|u\|_{m(\cdot)}^{m^-} - 1 \leq \rho_{m(\cdot)}(u) \leq \|u\|_{m(\cdot)}^{m^+} + 1 \tag{2.2}$$

is verified for all $u \in L^{m(\cdot)}(\Omega)$. Also, we recall that the Hölder-type inequality

$$\int_\Omega |u\,v| \, \mathrm{d}x \leq \left[\frac{1}{m^-} + \frac{1}{(m')^-}\right] \|u\|_{m(\cdot)} \, \|v\|_{m'(\cdot)}$$

holds for all $u \in L^{m(\cdot)}(\Omega)$ and all $v \in L^{m'(\cdot)}(\Omega)$. Moreover, for $m_1, m_2 \in C(\overline{\Omega})$ such that $1 < m_1(x) \leq m_2(x)$ for all $x \in \overline{\Omega}$, we have the continuous embedding

$$L^{m_2(\cdot)}(\Omega) \hookrightarrow L^{m_1(\cdot)}(\Omega).$$

Using $L^{m(\cdot)}(\Omega)$ we introduce the variable exponent Sobolev space $W^{1,m(\cdot)}(\Omega)$ by

$$W^{1,m(\cdot)}(\Omega) = \left\{u \in L^{m(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{m(\cdot)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{1,m(\cdot)} = \|u\|_{m(\cdot)} + \|\nabla u\|_{m(\cdot)},$$

where $\|\nabla u\|_{m(\cdot)} = \|\,|\nabla u|\,\|_{m(\cdot)}$ for all $u \in W^{1,m(\cdot)}(\Omega)$. We write $W_0^{1,m(\cdot)}(\Omega)$ by the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,m(\cdot)}$. We point out that $W^{1,m(\cdot)}(\Omega)$ and $W_0^{1,m(\cdot)}(\Omega)$ are uniformly convex, separable and reflexive Banach spaces. Further, we stress that the Poincaré inequality is valid for $W_0^{1,m(\cdot)}(\Omega)$. Therefore, there exists a constant $c_m > 0$ such that the following inequality

$$\|u\|_{m(\cdot)} \le c_m \|\nabla u\|_{m(\cdot)}$$

holds for all $u \in W_0^{1,m(\cdot)}(\Omega)$. Taking this into account, we can consider on $W_0^{1,m(\cdot)}(\Omega)$ the equivalent norm given by

$$\|u\| := \|\nabla u\|_{m(\cdot)} \quad \text{for all } u \in W_0^{1,m(\cdot)}(\Omega).$$

Finally, we have the following Sobolev embedding result.

Proposition 2.2. *Let $m \in C(\overline{\Omega})$ be such that $1 < m(x) \le p_i^*(x) := \frac{Np_i(x)}{N-p_i(x)}$ for all $x \in \overline{\Omega}$ and $i = 1, 2$. Then, we have the continuous embedding*

$$W_0^{1,p_i(\cdot)}(\Omega) \hookrightarrow L^{m(\cdot)}(\Omega).$$

If $1 < m(x) < p_i^*(x)$ for all $x \in \overline{\Omega}$, then the above embedding is compact.

We emphasize that, according to Proposition 2.2, if $m(x) < p_i^*(x)$ for all $x \in \overline{\Omega}$, we can find a constant $\bar{c}_i > 0$ such that

$$\|u\|_{m(\cdot)} \le \bar{c}_i \|\nabla u\|_{p_i(\cdot)} \quad \text{for all } u \in W_0^{1,p_i(\cdot)}(\Omega) \text{ and } i = 1, 2. \tag{2.3}$$

Next, we recall that the negative $m(\cdot)$-Laplace operator

$$-\Delta_{m(x)} : W_0^{1,m(x)}(\Omega) \to (W_0^{1,m(x)}(\Omega))^*$$

is defined by

$$\langle -\Delta_{m(\cdot)} u, h\rangle = \int\limits_\Omega |\nabla u|^{m(x)-2} \nabla u \cdot \nabla h dx$$

for all $u, h \in W_0^{1,m(\cdot)}(\Omega)$, while the positive $m(\cdot)$-Laplace operator $\Delta_{m(\cdot)}$ is given by

$$\langle -\Delta_{m(\cdot)} u, h\rangle = -\int\limits_\Omega |\nabla u|^{m(x)-2} \nabla u \cdot \nabla h dx$$

for all $u, h \in W_0^{1,m(\cdot)}(\Omega)$. Such operators have several notable properties. In particular, both $-\Delta_{m(\cdot)}$ and $\Delta_{m(\cdot)}$ are bounded (that is, they map bounded sets to bounded sets) and continuous. In addition, $-\Delta_{m(\cdot)}$ is strictly monotone and satisfies the $(S)_+$-properties, which means that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,m(\cdot)}(\Omega) \quad \text{and} \quad \limsup_{n\to+\infty} \langle -\Delta_{m(\cdot)} u_n, u_n - u\rangle \le 0$$

imply

$$u_n \to u \text{ in } W_0^{1,m(\cdot)}(\Omega).$$

We conclude this section with some basic fact from the theory of nonsmooth analysis. Let X be a Banach space with topological dual X^*. Also, let $\langle\cdot,\cdot\rangle$ be the duality pairing between X and X^*. A function $\psi : X \to \mathbb{R}$ is called locally Lipschitz if for every $u \in X$ there are an open neighborhood U of u and a constant $c_U > 0$ such that the following inequality

$$|\psi(x) - \psi(y)| \le c_U \|x - y\|$$

is verified for all $x, y \in U$. The generalized directional derivative of ψ at $u \in X$ in the direction $h \in X$ is defined by

$$\psi^\circ(u; h) = \limsup_{x\to u, t\downarrow 0} \frac{\psi(x + th) - \psi(x)}{t},$$

while the generalized gradient of ψ at $u \in X$ is given by

$$\partial\psi(u) = \{u^* \in X^* : \langle u^*, h\rangle \le \psi^\circ(u; h), \text{ for all } h \in X\}.$$

The multifunction $u \to \partial\psi(u)$ is called the Clarke subdifferential of ψ. We remark that such subdifferential has several interesting properties. In particular, we recall the following.

Proposition 2.3. *Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz function at $u \in X$.*

(P1) *If ψ has a local minimum or maximum at $u \in X$, then $0 \in \partial\psi(u)$.*
(P2) *The Clarke subdifferential $\partial\psi(u)$ is a nonempty, convex, weak*-compact subset of X^*.*
(P3) *If $\{u_n\}_{n\in\mathbb{N}} \subset X$ and $\{\zeta_n\}_{n\in\mathbb{N}} \subset X^*$ are two sequences such that*

$$\zeta_n \in \partial\psi(u_n), \quad u_n \to u \textit{ in } X \quad \textit{and} \quad \zeta_n \xrightarrow{w^*} \zeta,$$

then $\zeta \in \partial\psi(u)$.
(P4) (*Mean-value theorem*) *If ψ is locally Lipschitz on an open neighborhood containing the segment $[u, v]$, then there exists $w \in (u, v)$ and $\zeta \in \partial\psi(w)$ satisfying*

$$\psi(v) - \psi(u) = \langle\zeta, v - u\rangle.$$

We point out that if $\phi \in C^1(X)$, then ϕ is locally Lipschitz with $\partial\phi(u) = \{\phi'(u)\}$ for all $u \in X$.

Finally, as we will make use of the Galerkin basis of $W_0^{1,p_i(x)}(\Omega)$, for the sake of reader convenience, we here recall such notion.

Definition 2.4. A sequence $\{X_n\}_{n\in\mathbb{N}}$ of vector subspaces of $W_0^{1,p_i(x)}(\Omega)$ is a Galerkin basis of $W_0^{1,p_i(x)}(\Omega)$ if the following conditions hold:

(i) $\dim(X_n) < +\infty$ for all $n \in \mathbb{N}$,
(ii) $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$,
(iii) $\overline{\bigcup_{n=1}^{\infty} X_n} = W_0^{1,p_i(x)}(\Omega)$.

3. ASSOCIATED ENERGY FUNCTIONAL

In this section, we introduce the energy functional J associated to problem (1.1) and examine its properties. More precisely, we here show that such functional is locally Lipschitz and coercive. Then, we explore some usefull properties of the local minimizers of J. We stress that in order to archive such results we make use of embedding results along with the theory of nonsmooth analysis.

First, we are going to formulate the precise assumptions on F.

(H_F) $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function satisfying the following growth condition: there exist positive constants $a_0, a_1, a_2, b_0, b_1, b_2, \alpha_1, \alpha_2$ with $1 < \alpha_1 < p_1^-$, $1 < \alpha_2 < p_2^-$, $a_1\, p_1^+ A_1 < p_1^-$ and $b_2\, p_2^+ B_1 < p_2^-$, such that

$$|z_1| \le a_0 + a_1|t|^{p_1^- -1} + a_2|s|^{\frac{p_2^-}{\alpha_1'}}$$

and

$$|z_2| \le b_0 + b_1|t|^{\frac{p_1^-}{\alpha_2'}} + b_2|s|^{p_2^- -1}$$

for all $t, s \in \mathbb{R}$ and $(z_1, z_2) \in \partial F(t,s)$, where A_1 and B_1 are the best constants such that

$$\int\limits_\Omega |u_1|^{p_1^-}\, dx \ \le\ A_1 \|\nabla u_1\|_{p_1(\cdot)}^{p_1^-} \text{ for all } u_1 \in W_0^{1,p_1(\cdot)}(\Omega), \tag{3.1}$$

$$\int\limits_\Omega |u_2|^{p_2^-}\, dx \le\ B_1 \|\nabla u_2\|_{p_2(\cdot)}^{p_2^-} \text{ for all } u_2 \in W_0^{1,p_2(\cdot)}(\Omega), \tag{3.2}$$

respectively (see (2.3)).

We point out that, according to (2.3), as $1 < \alpha_i < p_i^-$ due to hypothesis (H_F), from now on we will denote by A_2, A_3, B_2, B_3 constants satisfying the following estimates

$$\int_\Omega |u_1|dx \leq A_2\|\nabla u_1\|_{p_1(\cdot)}, \quad \int_\Omega |u_1|^{\alpha_1}dx \leq A_3\|\nabla u_1\|^{\alpha_1}_{p_1(\cdot)} \text{ for all } u_1 \in W_0^{1,p_1(\cdot)}(\Omega), \tag{3.3}$$

$$\int_\Omega |u_2|dx \leq B_2\|\nabla u_2\|_{p_2(\cdot)}, \quad \int_\Omega |u_2|^{\alpha_2}\,dx \leq B_3\|\nabla u_2\|^{\alpha_2}_{p_2(\cdot)} \text{ for all } u_2 \in W_0^{1,p_2(\cdot)}(\Omega). \tag{3.4}$$

Let now $\Psi : W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega) \to \mathbb{R}$ be the functional given by

$$\Psi(u_1, u_2) = \int_\Omega F(u_1, u_2)dx \tag{3.5}$$

for all $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$. We stress that, as (H_F) holds, we have that Ψ is Lipschitz continuous on the bounded subsets of $W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$. This in particular guarantees that Ψ has everywhere a well-defined Clarke subdifferential

$$\partial\Psi : W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega) \to 2^{(W_0^{1,p_1(\cdot)}(\Omega))^* \times (W_0^{1,p_2(\cdot)}(\Omega))^*}.$$

Using Ψ, we define the functional $J : W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega) \to \mathbb{R}$ corresponding to problem (1.1) by

$$J(u_1, u_2) = \sum_{i=1}^{2} \left(\int_\Omega \frac{1}{p_i(x)}|\nabla u_i|^{p_i(x)}dx - \mu_i \int_\Omega \frac{1}{q_i(x)}|\nabla u_i|^{q_i(x)}dx \right) - \Psi(u_1, u_2) \tag{3.6}$$

for all $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$.

Now, our first aim is in showing that J is locally Lipschitz and coercive, which means that $J(u_1, u_2) \to +\infty$ as $\|(u_1, u_2)\| = \|u_1\| + \|u_2\| \to +\infty$.

Proposition 3.1. *Let hypotheses* $(H0)$ *and* (H_F) *be satisfied. Also, let* J *be the functional defined in* (3.6)*. Then,* J *is locally Lipschitz and*

$$\partial J(u_1,u_2)=\sum_{i=1}^{2}\left(\int_\Omega |\nabla u_i|^{p_i(x)-2}\nabla u_i\cdot\nabla h_i dx-\mu_i\int_\Omega |\nabla u_i|^{q_i(x)-2}\nabla u_i\cdot\nabla h_i dx\right)\\ -\partial\Psi(u_1,u_2) \tag{3.7}$$

for all $(u_1,u_2),(h_1,h_2)\in W_0^{1,p_1(\cdot)}(\Omega)\times W_0^{1,p_2(\cdot)}(\Omega)$*. Furthermore,* J *is coercive on* $W_0^{1,p_1(\cdot)}(\Omega)\times W_0^{1,p_2(\cdot)}(\Omega)$*.*

Proof. First, we note that as (H_F) holds, the functional Ψ given in (3.5) is locally Lipschitz. This, with a view to (3.6), permits us to we affirm that J is locally Lipschitz as well. Thus, in order to achieve (3.7) we have just to derive the Clarke subdifferential of Ψ and then of J.

Therefore, we are going to show that J is a coercive functional. To this end, we point out that using (2.2) and the continuous embedding $W_0^{1,p_i(\cdot)}(\Omega)\hookrightarrow W_0^{1,q_i^+}(\Omega)$ for $i=1,2$, we get that

$$\begin{aligned}\int_\Omega |\nabla u_i|^{q_i(x)}dx&\le \|\nabla u_i\|_{q_i(\cdot)}^{q_i^+}+1\\ &\le C_i\|\nabla u_i\|_{p_i(\cdot)}^{q_i^+}+1\end{aligned}\tag{3.8}$$

for some $C_i>0$, for all $u_i\in W_0^{1,p_i(\cdot)}(\Omega)$ and $i=1,2$.

Also, from hypothesis (H_F) and mean-value theorem (see (P4) of Proposition 2.3), we derive that

$$\begin{aligned}|F(t,s)|&\le |F(0,0)|+a_0|t|+b_0|s|+\frac{a_1}{p_1^-}|t|^{p_1^-}+\frac{b_2}{p_2^-}|s|^{p_2^-}\\ &\quad+\frac{a_2\alpha_1'}{p_2^-+\alpha_1'}|t||s|^{\frac{p_2^-}{\alpha_1'}}+\frac{b_1\alpha_2'}{p_1^-+\alpha_2'}|t|^{\frac{p_1^-}{\alpha_2'}}|s|\\ &\le |F(0,0)|+a_0|t|+b_0|s|+\frac{a_1}{p_1^-}|t|^{p_1^-}+\frac{b_2}{p_2^-}|s|^{p_2^-}\\ &\quad+a(\varepsilon)|t|^{\alpha_1}+\varepsilon|s|^{p_2^-}+\varepsilon|t|^{p_1^-}+b(\varepsilon)|s|^{\alpha_2}\ \text{(by Young's inequality with }\varepsilon)\\ &\le |F(0,0)|+a_0|t|+b_0|s|+\left(\frac{a_1}{p_1^-}+\varepsilon\right)|t|^{p_1^-}\\ &\quad+\left(\frac{b_2}{p_2^-}+\varepsilon\right)|s|^{p_2^-}+a(\varepsilon)|t|^{\alpha_1}+b(\varepsilon)|s|^{\alpha_2},\end{aligned}\tag{3.9}$$

for all $t, s \in \mathbb{R}$, where we have $\varepsilon, a(\varepsilon), b(\varepsilon) > 0$. Thus, using (2.2) along with the estimates (3.9), (3.1), (3.2), (3.3), (3.4) and (3.8), from (3.6) we see that

$$\begin{aligned}
J(u_1, u_2) \geq{}& \sum_{i=1}^{2} \left(\frac{1}{p_i^+} \int_\Omega |\nabla u_i|^{p_i(x)} dx - \frac{|\mu_i|}{q_i^-} \int_\Omega |\nabla u_i|^{q_i(x)} dx \right) - |F(0,0)||\Omega| \\
& - \int_\Omega \Big(a_0 |u_1| + b_0 |u_2| + \left(\frac{a_1}{p_1^-} + \varepsilon \right) |u_1|^{p_1^-} + \left(\frac{b_2}{p_2^-} + \varepsilon \right) |u_2|^{p_2^-} \\
& + a(\varepsilon)|u_1|^{\alpha_1} + b(\varepsilon)|u_2|^{\alpha_2} \Big) dx \\
\geq{}& \sum_{i=1}^{2} \left(\frac{1}{p_i^+} (\|\nabla u_i\|_{p_i(\cdot)}^{p_i^-} - 1) - \frac{|\mu_i|}{q_i^-} (C_i \|\nabla u_i\|_{p_i(\cdot)}^{q_i^+} + 1) \right) - |F(0,0)||\Omega| \\
& - a_0\, A_2 \|\nabla u_1\|_{p_1(\cdot)} - b_0 B_2 \|\nabla u_2\|_{p_2(\cdot)} - a(\varepsilon) A_3 \|\nabla u_1\|_{p_1(\cdot)}^{\alpha_1} \\
& - b(\varepsilon) B_3 \|\nabla u_2\|_{p_2(\cdot)}^{\alpha_2} - \left(\frac{a_1}{p_1^-} + \varepsilon \right) A_1 \|\nabla u_1\|_{p_1(\cdot)}^{p_1^-} - \left(\frac{b_2}{p_2^-} + \varepsilon \right) B_1 \|\nabla u_2\|_{p_2(\cdot)}^{p_2^-} \\
\geq{}& \frac{1}{p_1^+} \left(1 - \frac{a_1\, p_1^+}{p_1^-} A_1 - \varepsilon A_1 p_1^+ \right) \|\nabla u_1\|_{p_1(\cdot)}^{p_1^-} + \frac{1}{p_2^+} \left(1 - \frac{b_2\, p_2^+}{p_2^-} B_1 - \varepsilon B_1 p_2^+ \right) \\
& \times \|\nabla u_2\|_{p_2(\cdot)}^{p_2^-} - \frac{|\mu_1|}{q_1^-} C_1 \|\nabla u_1\|_{p_1(\cdot)}^{q_1^+} - \frac{|\mu_2|}{q_2^-} C_2 \|\nabla u_2\|_{p_2(\cdot)}^{q_2^+} \\
& - a_0\, A_2 \|\nabla u_1\|_{p_1(\cdot)} - b_0 B_2 \|\nabla u_2\|_{p_2(\cdot)} \\
& - a(\varepsilon) A_3 \|\nabla u_1\|_{p_1(\cdot)}^{\alpha_1} - b(\varepsilon) B_3 \|\nabla u_2\|_{p_2(\cdot)}^{\alpha_2},
\end{aligned}$$

for all $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$, with $|\Omega|$ being the Lebesgue measure of Ω in $\mathbb{R}^N$. Now, we recall that $a_1\, p_1^+ A_1 < p_1^-$ and $b_2\, p_2^+ B_1 < p_2^-$ from hypothesis (H_F). According to this, for $\varepsilon > 0$ small we have that

$$1 - \frac{a_1\, p_1^+}{p_1^-} A_1 - \varepsilon A_1 p_1^+ > 0 \quad \text{as well as} \quad 1 - \frac{b_2\, p_2^+}{p_2^-} B_1 - \varepsilon B_1 p_2^+ > 0.$$

Taking this into account, as from hypothesis $(H0)$ for $i = 1, 2$ we have that $p_i^- > q_i^+$, we conclude that the functional J is coercive. This proves the claim. □

Let now $\{X_n\}$ and $\{Y_n\}$ be a Galerkin basis of $W_0^{1,p_1(\cdot)}(\Omega)$ and $W_0^{1,p_2(\cdot)}(\Omega)$, respectively. We stress that $\{X_n \times Y_n\}$ is a Galerkin basis of the product space $W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$. Using such Galerkin basis, we are going to state and prove the following result for local minimizers of the functional J.

Proposition 3.2. *Let hypotheses* $(H0)$ *and* (H_F) *be satisfied. Also, let* J *be the functional given in* (3.6). *Then, for any* $n \in \mathbb{N}$ *we have that*

$$J(u_{1n}, u_{2n}) = \inf\{J(v_1, v_2) : (v_1, v_2) \in X_n \times Y_n\}$$

for some $(u_{1n}, u_{2n}) \in X_n \times Y_n$ *and* $(\zeta_{1n}, \zeta_{2n}) \in (W_0^{1,p_1(\cdot)}(\Omega))^* \times (W_0^{1,p_2(\cdot)}(\Omega))^*$ *with* $(\zeta_{1n}, \zeta_{2n}) \in \partial F(u_{1n}, u_{2n})$ *a.e. on* Ω *satisfying*

$$\int_\Omega |\nabla u_{1n}|^{p_1(x)-2}\nabla u_{1n} \cdot \nabla h_1 dx - \mu_1 \int_\Omega |\nabla u_{1n}|^{q_1(x)-2}\nabla u_{1n} \cdot \nabla h_1 dx - \int_\Omega \zeta_{1n} h_1\, dx = 0, \tag{3.10}$$

$$\int_\Omega |\nabla u_{2n}|^{p_2(x)-2}\nabla u_{2n} \cdot \nabla h_2 dx - \mu_2 \int_\Omega |\nabla u_{2n}|^{q_2(x)-2}\nabla u_{2n} \cdot \nabla h_2\, dx - \int_\Omega \zeta_{2n} h_2\, dx = 0, \tag{3.11}$$

for all $h_1 \in X_n$ *and all* $h_2 \in Y_n$.

Proof. First, we point out that as $\{X_n \times Y_n\}$ is a Galerkin basis of the product space $W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$, we have that

$$\dim(X_n \times Y_n) < +\infty \quad \text{for all } n \in \mathbb{N}.$$

Also, from Proposition 3.1 we know that J is locally Lipschitz and coercive. For way of this, we can find $(u_{1n}, u_{2n}) \in X_n \times Y_n$ such that

$$J(u_{1n}, u_{2n}) = \inf\{J(v_1, v_2) : (v_1, v_2) \in X_n \times Y_n\}.$$

Thus, from (P_1) of Proposition 2.3 we deduce that

$$(0,0) \in \partial J|_{X_n \times Y_n}(u_{1n}, u_{2n}). \tag{3.12}$$

Now, according to (3.12), we have that there exists $(z_{1n}, z_{2n}) \in \partial\Psi(u_{1n}, u_{2n})$ such that

$$\langle -\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in} - z_{in}, h_i\rangle = 0 \quad \text{for } i = 1, 2. \tag{3.13}$$

Then, using Theorem 2.7.5 and Remark 2.7.6 of Clarke [2], we can affirm that

$$\partial\Psi(u_{1n}, u_{2n}) \subset \int_\Omega \partial F(u_{1n}, u_{2n})\, dx.$$

This assures that, corresponding to $(z_{1n}, z_{2n}) \in \partial\Psi(u_{1n}, u_{2n})$, we can find $(\zeta_{1n}, \zeta_{2n}) \in \partial F(u_{1n}, u_{2n})$ a.e. on Ω such that

$$\langle z_{in}, h_i\rangle = \int_\Omega \zeta_{in} h_i\, dx \quad \text{for } i = 1, 2. \tag{3.14}$$

Using (3.14) in (3.13), we derive that both (3.10) and (3.11) hold. Therefore, the assertion of proposition is proved. □

Next, we focus on the minimizer sequence $\{(u_{1n}, u_{2n})\}_{n\in\mathbb{N}}$ obtained in Proposition 3.2. We are in the position to produce the following result.

Proposition 3.3. *Let hypotheses* $(H0)$ *and* (H_F) *be satisfied. Also, let*

$$\{(u_{1n}, u_{2n})\}_{n\in\mathbb{N}} \subset W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$$

be a sequence as given in Proposition 3.2. Then, the following a priori estimate

$$\|\nabla u_{1n}\|_{p_1(\cdot)} + \|\nabla u_{2n}\|_{p_2(\cdot)} \leq K \tag{3.15}$$

holds, for some $K > 0$ *and all* $n \in \mathbb{N}$

Proof. We note that choosing as test function $h_1 = u_{1n} \in X_n$ in (3.10) and then using (3.1), (3.3), (3.8), hypothesis (H_F) and Young's inequality with ε, we derive that

$$\begin{aligned}
\int_\Omega |\nabla u_{1n}|^{p_1(x)}dx &= \mu_1 \int_\Omega |\nabla u_{1n}|^{q_1(x)}dx + \int_\Omega \zeta_{1n}u_{1n}dx \\
&\leq |\mu_1|(C_1\|\nabla u_{1n}\|_{p_1(\cdot)}^{q_1^+} + 1) + a_0A_2\|\nabla u_{1n}\|_{p_1(\cdot)} + a_1A_1\|\nabla u_{1n}\|_{p_1(\cdot)}^{p_1^-} \\
&\quad + a_2a(\varepsilon)A_3\|\nabla u_{1,n}\|_{p_1(\cdot)}^{\alpha_1} + a_2\varepsilon B_1\|\nabla u_{2n}\|_{p_2(\cdot)}^{p_2^-}.
\end{aligned}$$

Analogously, choosing as test function $h_2 = u_{2n} \in Y_n$ in (3.11) and using (3.2), (3.4), (3.8), hypothesis (H_F) and the Young inequality with ε, we see that

$$\begin{aligned}
\int_\Omega |\nabla u_{2n}|^{p_2(x)}dx &= \mu_2 \int_\Omega |\nabla u_{2n}|^{q_2(x)}dx + \int_\Omega \zeta_{2n}u_{2n}dx \\
&\leq |\mu_2|(C_2\|\nabla u_{2n}\|_{p_2(\cdot)}^{q_2^+} + 1) + b_0B_2\|\nabla u_{2n}\|_{p_2(\cdot)} + b_2B_1\|\nabla u_{2n}\|_{p_2(\cdot)}^{p_2^-} \\
&\quad + b_1b(\varepsilon)B_3\|\nabla u_{2,n}\|_{p_2(\cdot)}^{\alpha_2} + b_1\varepsilon A_1\|\nabla u_{1n}\|_{p_1(\cdot)}^{p_1^-}.
\end{aligned}$$

The previous inequalities along with (2.2) yield that

$$\begin{aligned}
&(1 - A_1(a_1 + \varepsilon b_1))\|\nabla u_{1n}\|_{p_1(\cdot)}^{p_1^-} + (1 - B_1(b_2 + \varepsilon a_2))\|\nabla u_{2n}\|_{p_2(\cdot)}^{p_2^-} \\
&\leq |\mu_1|C_1\|\nabla u_{1n}\|_{p_1(\cdot)}^{q_1^+} + |\mu_2|C_2\|\nabla u_{2n}\|_{p_2(\cdot)}^{q_1^+} + a_0A_2\|\nabla u_{1n}\|_{p_1(\cdot)} \\
&\quad + b_0B_2\|\nabla u_{2n}\|_{p_2(\cdot)} + a_2a(\varepsilon)A_3\|\nabla u_{1,n}\|_{p_1(\cdot)}^{\alpha_1} + b_1b(\varepsilon)B_3\|\nabla u_{2,n}\|_{p_2(\cdot)}^{\alpha_2} + \hat{C}
\end{aligned}$$

for all $n \in \mathbb{N}$ and some $\hat{C} > 0$. Now, we recall that $1 - A_1a_1 > 0$ and $1 - B_1b_2 > 0$ from hypothesis (H_F). According to this, for $\varepsilon > 0$ small we have that

$$1 - A_1(a_1 + \varepsilon b_1) > 0 \quad \text{as well as} \quad 1 - B_1(b_2 + \varepsilon a_2) > 0.$$

This, along with the fact that $p_i^- > q_i^+ > 1$ for $i = 1, 2$ due to hypothesis $(H0)$, permits us to conclude that the sequence

$$\{\|\nabla u_{1n}\|_{p_1(\cdot)} + \|\nabla u_{2n}\|_{p_2(\cdot)}\}_{n\in\mathbb{N}}$$

is bounded by a constant in $W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$. This means that the a priori estimate (3.15) holds and hence the proof is concluded. □

Our next result gives indications about the boundedness of the operators involved in (3.10) and (3.11).

Proposition 3.4. *Let hypotheses* $(H0)$ *and* (H_F) *be satisfied. Also, let*

$$\{(u_{1n}, u_{2n})\}_{n\in\mathbb{N}} \subset W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$$

and

$$(\zeta_{1n}, \zeta_{2n}) \in (W_0^{1,p_1(\cdot)}(\Omega))^* \times (W_0^{1,p_2(\cdot)}(\Omega))^*$$

be as given in Proposition 3.2. Then, there exists a positive constant $\bar{K} > 0$ *such that the following inequality*

$$\| -\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in} - \zeta_{in}\|_{(W_0^{1,p_i(\cdot)}(\Omega))^*} \le \bar{K} \tag{3.16}$$

holds for all $n \in \mathbb{N}$ *and* $i = 1, 2$.

Proof. In order to prove the claim, we point out that the following inequality

$$\begin{aligned}
&|\langle -\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in} - \zeta_{in}, h_i\rangle| \\
&= \left| \int_\Omega |\nabla u_{in}|^{p_i(x)-2} \nabla u_{in} \cdot \nabla h_i dx - \mu_i \int_\Omega |\nabla u_{in}|^{q_i(x)-2} \nabla u_{in} \cdot \nabla h_i dx - \int_\Omega \zeta_{in} h_i dx \right| \\
&\le \int_\Omega |\nabla u_{in}|^{p_i(x)-1} |\nabla h_i| dx + |\mu_i| \int_\Omega |\nabla u_{in}|^{q_i(x)-1} |\nabla h_i| dx + \int_\Omega |\zeta_{in}||h_i| dx
\end{aligned} \tag{3.17}$$

holds for any $h_i \in W_0^{1,p_i(\cdot)}(\Omega)$ and $i = 1, 2$. Also, we note that according to (2.1) we have that

$$\rho_{p_i'(x)}(|\nabla u_{in}|^{p_i(x)-1}) = \rho_{p_i(x)}(|\nabla u_{in}|) \quad \text{for } i = 1, 2.$$

This in particular guarantees that for $i = 1, 2$ there exists $\gamma_i > 0$ satisfying

$$\| |\nabla u_{in}|^{p_i(x)-1} \|_{p_i'(\cdot)} \le \|\nabla u_{in}\|_{p_i(\cdot)}^{\gamma_i}.$$

Now, using the previous inequality along with (3.15) and the Hölder inequality, we can estimate the first two terms in the right-hand side of (3.17). Thus, we see that

$$\begin{aligned}
\int_\Omega |\nabla u_{in}|^{p_i(x)-1} |\nabla h_i| dx &\le 2 \| |\nabla u_{in}|^{p_i(x)-1} \|_{p_i'(\cdot)} \|\nabla h_i\|_{p_i(\cdot)} \\
&\le 2 \|\nabla u_{in}\|_{p_i(\cdot)}^{\gamma_i} \|\nabla h_i\|_{p_i(\cdot)} \\
&\le \widehat{M}_i \|\nabla h_i\|_{p_i(\cdot)}
\end{aligned} \tag{3.18}$$

for some $\widehat{M}_i > 0$, and

$$\begin{aligned}
\int_\Omega |\nabla u_{in}|^{q_i(x)-1} |\nabla h_i| dx &\le \int_\Omega (1 + |\nabla u_{in}|^{p_i(\cdot)-1}) |\nabla h_i| dx \\
&\le \widetilde{C} \|\nabla h_i\|_{p_i(\cdot)} + 2 \|\nabla u_{in}\|_{p_i(\cdot)}^{\gamma_i} \|\nabla h_i\|_{p_i(\cdot)} \\
&\le \widetilde{M}_i \|\nabla h_i\|_{p_i(\cdot)}
\end{aligned} \tag{3.19}$$

for some $\widetilde{C}, \widetilde{M}_i > 0$, with $\widetilde{C}$ being an embedding constant.

Next, in according to hypothesis (H_F) and (3.15), we have that

$$\begin{aligned}\int_\Omega |\zeta_{1n}||h_1|dx &\le \int_\Omega (a_0 + a_1|u_{1n}|^{p_1^- -1} + a_2|u_{2n}|^{\frac{p_2^-}{\alpha_1'}})|h_1|dx \\ &\le a_0 A_2 \|\nabla h_1\|_{p_1(\cdot)} + a_1 \bar{C}\|\nabla u_{1n}\|_{p_1(\cdot)}^{p_1^- -1}\|\nabla h_1\|_{p_1(\cdot)} \\ &\quad + a_2\,\bar{D}\,\|\nabla u_{2n}\|_{p_2(\cdot)}^{\frac{p_2^-}{\alpha_1'}}\|\nabla h_1\|_{p_1(\cdot)} \\ &\le \bar{M}_1\,\|\nabla h_1\|_{p_1(\cdot)}\end{aligned} \tag{3.20}$$

for some $\bar{M}_1 > 0$, with $\bar{C}, \bar{D}$ being embedding constants. We stress that, in a similar way, we can also see that

$$\int_\Omega |\zeta_{2n}||h_2|\,dx \;\le\; \bar{M}_2\,\|\nabla h_2\|_{p_2(\cdot)} \tag{3.21}$$

for some $\bar{M}_2 > 0$.

Now, using (3.17), (3.18), (3.19) and (3.20) we arrive to the following inequality

$$|\langle -\Delta_{p_1(\cdot)}u_{1n} + \mu_1\Delta_{q_1(\cdot)} - \zeta_{1n}, h_1\rangle| \le (\widehat{M}_1 + \widetilde{M}_1 + \bar{M}_1)\|\nabla h_1\|_{p_1(\cdot)},$$

while from (3.17), (3.18), (3.19) and (3.21) we derive that

$$|\langle -\Delta_{p_2(\cdot)}u_{2n} + \mu_2\Delta_{q_2(\cdot)} - \zeta_{2n}, h_2\rangle| \le (\widehat{M}_2 + \widetilde{M}_2 + \bar{M}_2)\|\nabla h_2\|_{p_2(\cdot)}.$$

On the base of the previous inequalities, we are able to conclude that (3.16) holds for some positive constant $\bar{K}$. $\square$

4. WEAK SOLUTIONS VIA GENERALIZED SOLUTIONS

In this section, we focus on link between weak solutions and generalized solutions to problem (1.1). As previous mentioned, any weak solution to problem (1.1) is in addition a generalized solution (see Remark 1.3). We here show that the reverse implication holds if driven operators in (1.1) are both elliptic, which means to claim that $\mu_1 \le 0$ as well as $\mu_2 \le 0$.

Theorem 4.1. *Let hypotheses* $(H0)$ *and* (H_F) *be satisfied. If* $\mu_1 \le 0$ *and* $\mu_2 \le 0$*, then any generalized solution to problem* (1.1) *is in addition a weak solution.*

Proof. In order to produce the claim, we suppose that $\mu_1 \le 0$ and $\mu_2 \le 0$ (that is, we assume that the driven operators in (1.1) are elliptic). Also, let $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$ be a generalized solution to problem (1.1). Now, our aim is in showing that (u_1, u_2) is in addition a weak solution to (1.1). To this purpose, we remark that, as (u_1, u_2) is a generalized solution to problem (1.1), we can find a sequence

$$\{(u_{1n}, u_{2n})\}_{n\in\mathbb{N}} \subset W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$$

and

$$\zeta_{in} \in (W_0^{1,p_i(\cdot)}(\Omega))^* \text{ with } \zeta_{in} \in \partial F(u_{1n}, u_{2n}) \text{ a.e. in } \Omega \text{ for } i = 1, 2,$$

satisfying all the conditions in Definition 1.1. Furthermore, we recall that the negative q_i-Laplacian operator, $-\Delta_{q_i(\cdot)}$, for $i = 1, 2$ is monotone. Hence, we know that

$$\langle -\Delta_{q_i(\cdot)} u + \Delta_{q_i(\cdot)} v, u - v\rangle \geq 0$$

for all $u, v \in W_0^{1,q_i(\cdot)}(\Omega)$ and $i = 1, 2$. This, along with the fact that $\mu_i \leq 0$, assures that the following inequality

$$\mu_i\langle -\Delta_{q_i(\cdot)} u_{in} + \Delta_{q_i(\cdot)} u_i, u_{in} - u_i\rangle \leq 0$$

holds for $n \in \mathbb{N}$ and $i = 1, 2$. Keeping this and Definition 1.1 (i), (iii) in mind, for $i = 1, 2$ we can write that

$$\begin{aligned}
&\limsup_{n\to+\infty}\langle -\Delta_{p_i(\cdot)} u_{in}, u_{in} - u_i\rangle \\
&= \limsup_{n\to+\infty}\big[\langle -\Delta_{p_i(\cdot)} u_{in} + \mu_i\Delta_{q_i(\cdot)} u_{in}, u_{in} - u_i\rangle + \mu_i\langle -\Delta_{q_i(\cdot)} u_{in} + \Delta_{q_i(\cdot)} u_i, u_{in} - u_i\rangle \\
&\quad + \mu_i\langle -\Delta_{q_i(\cdot)} u_i, u_{in} - u_i\rangle\big] \\
&\leq \limsup_{n\to+\infty}\langle -\Delta_{p_i(\cdot)} u_{in} + \mu_i\Delta_{q_i(\cdot)} u_{in}, u_{in} - u_i\rangle + \mu_i \lim_{n\to+\infty}\langle -\Delta_{q_i(\cdot)} u_i, u_{in} - u_i\rangle = 0.
\end{aligned}$$

From the previous inequality, as the operator $-\Delta_{p_i(\cdot)}$ is of $(S)_+$-type and continuous, we deduce for $i = 1, 2$ that

$$u_{in} \to u_i \quad \text{in } W_0^{1,p_i(\cdot)}(\Omega) \quad \text{as } n \to +\infty$$

and further

$$-\Delta_{p_i(\cdot)} u_{in} \to -\Delta_{p_i(\cdot)} u_i \ \text{ in } (W_0^{1,p_i(\cdot)}(\Omega))^* \quad \text{as } n \to +\infty.$$

Moreover, as $u_{in} \to u_i$ in $W_0^{1,p_i(\cdot)}(\Omega)$ and the operator $-\Delta_{q_i(\cdot)}$ is continuous, for $i = 1, 2$ we have also that

$$\lim_{n\to+\infty} \Delta_{q_i(\cdot)} u_{in} = \Delta_{q_i(\cdot)} u_i \quad \text{in } (W_0^{1,q_i(\cdot)}(\Omega))^*.$$

At this point, we stress that from Propositions 3.2 and 3.4 we know that

$$\{\zeta_{in}\}_{n\in\mathbb{N}} \subset (W_0^{1,p_i(\cdot)}(\Omega))^* \quad \text{is bounded for } i = 1, 2.$$

Therefore, for $i = 1, 2$ we can assume that

$$\zeta_{in} \xrightarrow{w} \zeta_i \quad \text{in } (W_0^{1,p_i(\cdot)}(\Omega))^* \quad \text{as } n \to +\infty.$$

Thus, according to (P3) of Proposition 2.3, we have that $(\zeta_1, \zeta_2) \in \partial F(u_1, u_2)$ a.e. on Ω. Then, using Definition 1.1 (ii) along with the fact that $u_{in} \to u_i$ in $W_0^{1,p_i(\cdot)}(\Omega)$ and $(\zeta_{1n}, \zeta_{2n}) \in \partial F(u_{1n}, u_{2n}) \subset (W_0^{1,p_1(\cdot)}(\Omega))^* \times (W_0^{1,p_2(\cdot)}(\Omega))^*$, we derive for $i = 1, 2$ that

$$-\Delta_{p_i(\cdot)} u_i + \mu_i \Delta_{q_i(\cdot)} u_i - \zeta_i = 0 \quad \text{in } (W_0^{1,p_i(\cdot)}(\Omega))^*,$$

where $(\zeta_1, \zeta_2) \in \partial F(u_1, u_2) \subset (W_0^{1,p(\cdot)}(\Omega))^* \times (W_0^{1,p_2(\cdot)}(\Omega))^*$ a.e. on Ω. With a view to Definition 1.2, this assures that the generalized solution (u_1, u_2) to problem (1.1) is in addition a weak solution. Therefore, the claim is now proved. $\square$

5. EXISTENCE RESULT

In this section, we are ready finally to state and prove the main result of the present paper. Precisely, we here show that problem (1.1) admits for any $\mu_1, \mu_2 \in \mathbb{R}$ at least a generalized solution in $W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$, supposed that hypotheses $(H0)$ and (H_F) hold. We stress that, according to Theorem 4.1, we have that such solution is in addition a weak solution to problem (1.1) whenever $\mu_1 \le 0$ and $\mu_2 \le 0$.

Now, our main result reads as follows.

Theorem 5.1. *Let hypotheses $(H0)$ and (H_F) be satisfied. Then, for any $\mu_1, \mu_2 \in \mathbb{R}$ we have that problem (1.1) admits at least a generalized solution (u_1, u_2) in $W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$.*

Proof. In order to prove the claim, we have to show that there exists $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$ such that all the assumptions in Definition 1.1 are satisfied. To this end, we consider a Galerkin basis $\{X_n\}$ of $W_0^{1,p_1(\cdot)}(\Omega)$ and a Galerkin basis $\{Y_n\}$ of $W_0^{1,p_2(\cdot)}(\Omega)$. Also, as hypotheses $(H0)$ and (H_F) hold, let

$$\{(u_{1n}, u_{2n})\}_{n\in\mathbb{N}} \subset W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$$

and

$$(\zeta_{1n}, \zeta_{2n}) \in (W_0^{1,p_1(\cdot)}(\Omega))^* \times (W_0^{1,p_2(\cdot)}(\Omega))^*$$

with $(\zeta_{1n}, \zeta_{2n}) \in \partial F(u_{1n}, u_{2n})$ a.e. in Ω, as given in Proposition 3.2. Now, according to Propositions 3.2 and 3.3, we have that

$$\{(u_{1n}, u_{2n})\}_{n\in\mathbb{N}} \subset W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega) \quad \text{is bounded.}$$

For way of this, we can suppose that

$$u_{in} \xrightarrow{w} u_i \text{ in } W_0^{1,p_i(\cdot)}(\Omega) \quad \text{and} \quad u_{in} \to u_i \text{ in } L^{p_i(\cdot)}(\Omega)$$

as $n \to +\infty$, for some $u_i \in W_0^{1,p_i(\cdot)}(\Omega)$ and $i = 1, 2$. Consequently, we have that condition (i) of Definition 1.1 is verified.

Next, we point out that Proposition 3.2 along with Proposition 3.4 assures that

$$\{-\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in} - \zeta_{in}\}_{n\in\mathbb{N}} \subset (W_0^{1,p_i(\cdot)}(\Omega))^*$$

is bounded for $i = 1, 2$. Therefore, we can suppose that

$$-\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in} - \zeta_{in} \xrightarrow{w} y_i \ \text{ in } \ (W_0^{1,p_i(\cdot)}(\Omega))^* \tag{5.1}$$

as $n \to +\infty$, for some $y_i \in (W_0^{1,p_i(\cdot)}(\Omega))^*$ and $i = 1, 2$.

Now, we take $h_1 \in \cup_{n=1}^{+\infty} X_n$ in (3.10) and $h_2 \in \cup_{n=1}^{+\infty} Y_n$ in (3.11). We remark that according to Definition 2.4 we have that $(h_1, h_2) \in X_n \times Y_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Then, as

$$\langle -\Delta_{p_i(\cdot)} u_{in}, h_i\rangle + \langle \mu_i \Delta_{q_i(\cdot)} u_{in}, h_i\rangle - \int_\Omega \zeta_{in} h_i dx = 0$$

for all $n \geq n_0$ and $i = 1, 2$ from (3.10) and (3.11), passing to the limit as $n \to +\infty$ and using (5.1), we deduce that the following equality

$$\lim_{n\to+\infty} \langle -\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in} - \zeta_{in}, h_i\rangle = 0$$

holds for $i = 1, 2$. This in particular guarantees that

$$\langle y_i, h_i\rangle = 0 \quad \text{for } i = 1, 2.$$

Consequently, on the base of Definition 2.4 (iii), we can affirm that

$$y_i = 0 \quad \text{for } i = 1, 2.$$

Thus, we have that

$$-\Delta_{p_i(\cdot)} u_{in} + \mu_i \Delta_{q_i(\cdot)} u_{in} - \zeta_{in} \xrightarrow{w} 0 \quad \text{in } (W_0^{1,p_i(\cdot)}(\Omega))^* \tag{5.2}$$

as $n \to +\infty$ and $i = 1, 2$. Therefore, Definition 1.1 (ii) is satisfied as well.

Next, we point out that if in (3.10) we choose $h_i = u_{in} - u_i \in W_0^{1,p_i(\cdot)}(\Omega)$ for $i = 1, 2$, pass to the limit as $n \to +\infty$ and use (5.2), then according to the properties of the sequences $\{u_{in}\}_{n\in\mathbb{N}}$ and $\{\zeta_{in}\}_{n\in\mathbb{N}}$, we get that

$$\lim_{n\to+\infty} \int_\Omega \zeta_{in}(u_{in} - u_i)\, dx = 0.$$

Taking this into account, from

$$\lim_{n\to+\infty} \left[\langle -\Delta_{p_i(\cdot)} u_{in}, u_{in} - u_i\rangle + \mu_i \Delta_{q_i(\cdot)} u_{in}, u_{in} - u_i\rangle - \int_\Omega \zeta_{in}(u_{in} - u_i) dx\right] = 0,$$

we derive that

$$\lim_{n\to+\infty} \left[\langle -\Delta_{p_i(\cdot)} u_{in}, u_{in} - u_i\rangle + \mu_i \langle \Delta_{q_i(\cdot)} u_{in}, u_{in} - u_i\rangle\right] = 0.$$

This means that Definition 1.1 (iii) is true too.

Finally, as all the assumptions in Definition 1.1 are satisfied, we conclude that $(u_1, u_2) \in W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$ is a generalized solution to problem (1.1). Taking into account that $\mu_1, \mu_2 \in \mathbb{R}$ are arbitrary, this produces the claim. $\square$

We emphasize that as immediate consequence of Theorems 4.1 and 5.1 we have the following result.

Theorem 5.2. *Let hypotheses* $(H0)$ *and* (H_F) *be satisfied. Also, let* $\mu_1 \le 0$ *and* $\mu_2 \le 0$. *Then, problem* (1.1) *admits at least a weak solution in* $W_0^{1,p_1(\cdot)}(\Omega) \times W_0^{1,p_2(\cdot)}(\Omega)$.

For the sake of reader convenience, an example illustrating how our results can be applied is provided next.

Example 5.3. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be the function so defined

$$f_1(t) = \begin{cases} t^2 \cos \frac{1}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

We stress that f_1 is a locally Lipschitz function with Clarke subdifferential given by

$$\partial f_1(t) = \begin{cases} 2t \cos \frac{1}{t} + \sin \frac{1}{t} & \text{if } t \neq 0, \\ [-1, 1] & \text{if } t = 0. \end{cases}$$

Next, we consider the function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(t) = |t| \quad \text{for all } t \in \mathbb{R}.$$

Such function has instead Clarke subdifferential given by

$$\partial f_2 = \begin{cases} -1 & \text{if } t < 0, \\ [-1, 1] & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Using f_1 and f_2, we introduce the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(t, s) = f_1(t) + t\, f_2(s) + s^2 \tag{5.3}$$

for all $t, s \in \mathbb{R}$. We point out that F is a locally Lipschitz function. Further, we can easily see that the Clarke subdifferential of F is given by

$$\partial F(t, s) = (\partial f_1(t) + f_2(s),\, t\, \partial f_2(s) + 2s)$$

for all $t, s \in \mathbb{R}$. Taking this into account, for any $(\zeta_1, \zeta_2) \in \partial F(t, s)$ we have that the following inequalities

$$|\zeta_1| \leq 1 + 2|t| + |s| \quad \text{and} \quad |\zeta_2| \leq |t| + 2|s|$$

are verified for all $(t, s) \in \mathbb{R}^2$.

Now, we assume $\alpha_i = 2$ and $p_i^- > 2$ for $i = 1, 2$. Such choice of α_i and p_i permits us to affirm that hypothesis (H_F) is satisfied by the function F introduced in (5.3). Thus, we are going to consider the following problem

$$\begin{cases} (-\Delta_{p_1(\cdot)}u_1 + \mu_1\Delta_{q_1(\cdot)}u_1, -\Delta_{p_2(\cdot)}u_2 + \mu_2\Delta_{q_2(\cdot)}u_2) \in \partial F(u_1, u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, F is from (5.3), and we suppose $p_i^- > 2$ for $i = 1, 2$, with p_i, q_i satisfying hypothesis $(H0)$. As hypothesis (H_F) holds as well, we are in the position to apply both Theorem 5.1 and Theorem 5.2. Consequently, according to Theorem 5.1, we have that for any $\mu_1, \mu_2 \in \mathbb{R}$ the problem under consideration admits at least a generalized solution. From Theorem 5.2 we instead derive the existence of a weak solution to the problem whenever $\mu_1 \leq 0$ and $\mu_2 \leq 0$.

REFERENCES

[1] K.C. Chang, *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102–129.

[2] F.H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons Inc., New York, 1983.

[3] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Math., vol. 2017, Springer-Verlag, Berlin Heidelberg, 2011.

[4] M. Galewski, D. Motreanu, *On variational competing (p, q)-Laplacian Dirichlet problem with gradient depending weight*, Appl. Math. Letters **148** (2024), 108881.

[5] L. Gambera, S.A. Marano, D. Motreanu, *Quasi-linear Dirichlet systems with competing operators and convection*, J. Math. Anal. Appl. **530** (2024), 127718.

[6] M. Ghasemi, C. Vetro, Z. Zhang, *Dirichlet μ-parametric differential problem with multivalued reaction term*, Mathematics **13** (2025), 1295.

[7] M. Jleli, M. Kirane, B. Samet, *A general blow-up result for a degenerate hyperbolic inequality in an exterior domain*, Bull. Math. Sci. **13** (2023), no. 3, 2150012.

[8] Z. Liu, R. Livrea, D. Motreanu, S. Zeng, *Variational differential inclusions without ellipticity condition*, Electron J. Qual. Theory Differ. Equ. **2020** (2020), no. 43, 17 pp.

[9] D. Motreanu, *Quasilinear Dirichlet problems with competing operators and convection*, Open Math. **18** (2020), no. 1, 1510–1517.

[10] D. Motreanu, *Systems of hemivariational inclusions with competing operators*, Mathematics **12** (2024), no. 1766, 11 pp.

[11] A. Moussaoui, D. Nabab, J. Vélin, *Singular quasilinear convective systems involving variable exponents*, Opuscula Math. **44** (2024), no. 1, 105–134.

[12] P.D. Panagiotopoulos, *Hemivariational Inequalities*, Berlin, Heidelberg: Springer, 1993.

[13] V.D. Rădulescu, D.D. Repovš, *Partial Differential Equations with Variable Exponents. Variational Methods and Qualitative Analysis*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015.

[14] C. Vetro, *Variable exponent $p(x)$-Kirchhoff type problem with convection*, J. Math. Anal. Appl. **506** (2022), no. 2, 125721.

Francesca Vetro (corresponding author)
francescavetro80@gmail.com
https://orcid.org/0000-0001-7448-5299

Baku Engineering University
Scientific Research Center
Khirdalan City, Baku, Absheron, Azerbaijan

Rakib Efendiev
refendiyev@beu.edu.az
https://orcid.org/0000-0001-9864-5410

Baku Engineering University
Department of Mathematics and Computer Science
Khirdalan City, Baku, Absheron, Azerbaijan

Received: May 16, 2025.
Revised: August 11, 2025.
Accepted: August 12, 2025.