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A NOTE ON NONLOCAL DISCRETE PROBLEMS INVOLVING SIGN-CHANGING KIRCHHOFF FUNCTIONS

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Abstract. In this note, we establish a multiplicity theorem for a nonlocal discrete problem of the type

$$\begin{cases} -\left(a\sum_{m=1}^{n+1}|x_m - x_{m-1}|^2 + b\right)(x_{k+1} - 2x_k + x_{k-1}) = h_k(x_k), \quad k = 1, \dots, n, \\ x_0 = x_{n+1} = 0 \end{cases}$$

assuming $a > 0$ and (for the first time) $b < 0$.

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1. INTRODUCTION

If Ω is a bounded domain of $\mathbb{R}^n$ and $K : [0, +\infty[\to \mathbb{R}$, $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ are two given functions, the nonlocal problem

$$\begin{cases} -K(\int_\Omega |\nabla u(x)|^2)\Delta u = \varphi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is certainly among the most studied ones in today's nonlinear analysis (we refer to [4] for an introduction to the subject).

In checking the relevant literature, one can realize that, in the majority of the papers, one assumes $K(t) = at + b$ with $a > 0$ and $b \geq 0$ and, in any case, that the Kirchhoff function K is assumed to be, in particular, continuous and non-negative in $[0, +\infty[$.

However, it is natural to ask what happens when at least one of these properties fails.

The case where K can be discontinuous in $[0, +\infty[$ has been considered for the first time in [5], for $n = 1$, and then in [6] for the general case (see also [1–3]). In these papers, however, K is non-negative.

The papers dealing with a sign-changing function K are more numerous, but in each of them it is assumed that $K(t) = at + b$ with $a < 0$ and $b \geq 0$. The first of these papers was [8].

In the present very short note, we are interested in the discrete counterpart of the above problem. That is to say, given n continuous functions $f_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \dots, n)$, we deal with the problem

$$\begin{cases} -K\left(\sum_{h=1}^{n+1} |x_h - x_{h-1}|^2\right)(x_{k+1} - 2x_k + x_{k-1}) = f_k(x_k), \quad k = 1, \dots, n, \\ x_0 = x_{n+1} = 0. \end{cases}$$

Also for this discrete problem, we can repeat what we said before, even in a stronger way: it seems that in each paper on the subject, the function K is continuous and non-negative in $[0, +\infty[$.

Our aim is to establish a multiplicity result for this problem where (for the first time) the Kirchhoff function K changes sign.

2. RESULTS

Before stating our result, we recall the following two theorems which will be key tools used in our proof.

Theorem 2.1 ([7])**.** *Let X be a topological space, let Y be a convex set in a topological vector space and let $h : X \times Y \to \mathbb{R}$ be lower semicontinuous and inf-compact in X, and continuous and quasi-concave in Y. Also, assume that*

$$\sup_Y \inf_X h < \inf_X \sup_Y h.$$

Moreover, let $\varphi : X \to \mathbb{R}$ be a lower semicontinuous function such that

$$\sup_X \varphi - \inf_X \varphi < \inf_X \sup_Y h - \sup_Y \inf_X h.$$

Then, for each convex set $S \subseteq Y$, dense in Y, there exists $\tilde{y} \in S$ such that the function $h(\cdot, \tilde{y}) + \varphi(\cdot)$ has at least two global minima.

Theorem 2.2 ([7])**.** *Let X be a topological space, let H be a real Hilbert, let Y be a closed ball in H centered at 0, and let $Q : X \to \mathbb{R}$, $\psi : X \to H$. Assume that the functional $x \mapsto Q(x) - \langle \psi(x), y \rangle$ is lower semicontinuous for each $y \in Y$, while the functional $x \mapsto Q(x) - \langle \psi(x), y_0 \rangle$ is inf-compact for some $y_0 \in Y$. Moreover, assume that, for each $x \in X$, there exists $u \in X$ such that*

$$Q(x) = Q(u)$$

and

$$\psi(x) = -\psi(u).$$

Finally, assume that there is no global minimum of Q at which ψ vanishes. Then, we have

$$\sup_{y \in Y} \inf_{x \in X} (Q(x) - \langle \psi(x), y \rangle) < \inf_{x \in X} \sup_{y \in Y} (Q(x) - \langle \psi(x), y \rangle).$$

Our main result is as follows:

Theorem 2.3. *Let $K : [0, +\infty[\to \mathbb{R}$, $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ be $n+1$ continuous functions satisfying the following conditions:*

(a) $\inf_{t>0} \int_0^t K(s)ds < 0$ *and* $\liminf_{t\to+\infty} \frac{\int_0^t K(s)ds}{t} > 0$,
(b) $\limsup_{|t|\to+\infty} \frac{\left|\int_0^t f_k(s)ds\right|}{t^2} < +\infty$ *for each* $k = 1, \dots, n$,
(c) *for each* $k = 1, \dots, n$, *the function* $t \to \int_0^t f_k(s)ds$ *is odd and vanishes only at* 0.

Then, for each $r > 0$, there exists a number $\delta > 0$ with the following property: for every n-tuple of continuous functions $g_1, \dots, g_n : \mathbb{R} \to \mathbb{R}$ satisfying

$$\max_{1\le k\le n} \left(\sup_{t\in\mathbb{R}} \int_0^t g_k(s)ds - \inf_{t\in\mathbb{R}} \int_0^t g_k(s)ds \right) < \delta,$$

there exists $(\tilde{\mu}_1, \dots \tilde{\mu}_n) \in \mathbb{R}^n$, with $\sum_{k=1}^n |\tilde{\mu}_k|^2 < r^2$, such that the problem

$$\begin{cases} -K\big(\sum_{h=1}^{n+1} |x_h - x_{h-1}|^2\big)(x_{k+1} - 2x_k + x_{k-1}) = g_k(x_k) + \tilde{\mu}_k f_k(x_k), & k=1, \dots, n, \\ x_0 = x_{n+1} = 0 \end{cases}$$

has at least three solutions.

Proof. Fix $r > 0$. First, we are going to apply Theorem 2.2. In this connection, take

$$X = \{(x_0, x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+2} : x_0 = x_{n+1} = 0\},$$

with the scalar product

$$\langle x, y\rangle_1 = \sum_{k=1}^{n+1} (x_k - x_{k-1})(y_k - y_{k-1}).$$

We denote by $\langle\cdot,\cdot\rangle_2$ the usual scalar product on $\mathbb{R}^n$, that is,

$$\langle x, y\rangle_2 = \sum_{k=1}^{n} x_k y_k.$$

Fix $\gamma > 0$ so that

$$\|x\|_2 \le \gamma \|x\|_1 \tag{2.1}$$

for all $x \in X$. Consider the functions $Q : X \to \mathbb{R}$ and $\psi : X \to \mathbb{R}^n$ defined by

$$Q(x) = \frac{1}{2} \int_0^{\|x\|_1^2} K(s)ds$$

and

$$\psi(x) = \left(\int_0^{x_1} f_1(s)ds, \dots, \int_0^{x_n} f_n(s)ds\right)$$

for all $x \in X$. Fix $\mu \in \mathbb{R}^n$. In view of (a) and (b), there exist $\eta_1, \eta_2, \eta_3 > 0$ such that

$$\int_0^t K(s)ds \geq \eta_1 t - \eta_2 \tag{2.2}$$

for all $t \geq 0$ and

$$\left|\int_0^t f_k(s)ds\right| \leq \eta_3(t^2+1) \tag{2.3}$$

for all $t \in \mathbb{R}$, $k = 1, \dots, n$. Fix $x \in X$. Using (2.2) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} Q(x) - \langle \psi(x), \mu\rangle_2 &\geq \frac{1}{2}\eta_1\|x\|_1^2 - \frac{1}{2}\eta_2 - |\langle \psi(x), \mu\rangle_2| \\ &\geq \frac{1}{2}\eta_1\|x\|_1^2 - \frac{1}{2}\eta_2 - \|\mu\|_2\|\psi(x)\|_2. \end{aligned} \tag{2.4}$$

On the other hand, in view of (2.3), for each $k = 1, \dots, n$, we have

$$\left|\int_0^{x_k} f_k(s)ds\right| \leq \eta_3(|x_k|^2+1)$$

and hence

$$\|\psi(x)\|_2 \leq \eta_3\sqrt{\sum_{k=1}^n (|x_k|^2+1)^2} \leq \eta_3\left(\sum_{k=1}^n |x_k|^2 + n\right). \tag{2.5}$$

Putting (2.1), (2.4) and (2.5) together, we get

$$Q(x) - \langle \psi(x), \mu\rangle_2 \geq \frac{1}{2}\eta_1\|x\|_1^2 - \|\mu\|_2\eta_3(\gamma^2\|x\|_1^2 + n) - \frac{1}{2}\eta_2. \tag{2.6}$$

Now, fix $\sigma > 0$ so that

$$\sigma < \min\left\{\frac{\eta_1}{2\eta_3\gamma^2}, r\right\}.$$

Let Y be the closed ball in $\mathbb{R}^n$ centered at 0, of radius σ. If $\mu \in Y$, in view of (2.6), we have

$$\lim_{\|x\|_1 \to +\infty}(Q(x) - \langle \psi(x), \mu\rangle_2) = +\infty$$

and so the function $x \to Q(x) - \langle \psi(x), \mu\rangle_2$ is inf-compact. Further, observe that, by (c), the function ψ vanishes only at 0, while, by (a), 0 is not a global minimum of Q.

Clearly, Q is even and ψ is odd, in view of (c). In other words, each assumption of Theorem 2.2 is satisfied. Consequently, the number

$$\delta := \frac{1}{n}\left(\inf_{x\in X}\sup_{\mu\in Y}(Q(x)-\langle\psi(x),\mu\rangle_2)-\sup_{\mu\in Y}\inf_{x\in X}(Q(x)-\langle\psi(x),\mu\rangle_2)\right) \tag{2.7}$$

is positive. At this point, we apply Theorem 2.1 taking

$$h(x,\mu)=Q(x)-\langle\psi(x),\mu\rangle_2$$

for all $(x,\mu)\in X\times Y$. Fix n continuous functions $g_1,\dots,g_n:\mathbb{R}\to\mathbb{R}$ satisfying

$$\max_{1\le k\le n}\left(\sup_{t\in\mathbb{R}}\int_0^t g_k(s)ds-\inf_{t\in\mathbb{R}}\int_0^t g_k(s)ds\right)<\delta \tag{2.8}$$

and consider the function $\varphi: X\to\mathbb{R}$ defined by

$$\varphi(x)=-\sum_{k=1}^{n}\int_0^{x_k} g_k(s)ds$$

for all $x\in X$. Clearly, in view of (2.7) and (2.8), we have

$$\begin{aligned}
\sup_X\varphi-\inf_X\varphi &\le \sum_{k=1}^{n}\left(\sup_{t\in\mathbb{R}}\int_0^t g_k(s)ds-\inf_{t\in\mathbb{R}}\int_0^t g_k(s)ds\right)\\
&\le n\max_{1\le k\le n}\left(\sup_{t\in\mathbb{R}}\int_0^t g_k(s)ds-\inf_{t\in\mathbb{R}}\int_0^t g_k(s)ds\right)\\
&<\inf_X\sup_Y h-\sup_Y\inf_X h.
\end{aligned}$$

So, each assumption of Theorem 2.1 is satisfied. As a consequence, there exists $\tilde{\mu}\in Y$ such that the function

$$J_{\tilde{\mu}}(\cdot):=h(\cdot,\tilde{\mu})+\varphi(\cdot)$$

has at least two global minima in X. It is clear that this function $J_{\tilde{\mu}}$ is C^1, with derivative given by

$$J'_{\tilde{\mu}}(x)(y)=K\left(\sum_{h=1}^{n+1}|x_h-x_{h-1}|^2\right)\langle x,y\rangle_1-\sum_{k=1}^{n}g_k(x_k)y_k-\sum_{k=1}^{n}\tilde{\mu}_k f_k(x_k)y_k$$

for all $x,y\in X$. So, taking into account that

$$\langle x,y\rangle_1=-\sum_{k=1}^{n}(x_{k+1}-2x_k+x_{k-1})y_k,$$

we have

$$
\begin{aligned}
J'_{\tilde{\mu}}(x)(y) = -K\left(\sum_{h=1}^{n+1}|x_h - x_{h-1}|^2\right)\sum_{k=1}^{n}(x_{k+1} - 2x_k + x_{k-1})y_k \\
- \sum_{k=1}^{n} g_k(x_k)y_k - \sum_{k=1}^{n}\tilde{\mu}_k f_k(x_k)y_k
\end{aligned}
\tag{2.9}
$$

for all $x, y \in X$. Since $J_{\tilde{\mu}}$ is coercive and has at least two global minima, by a classical theorem of Courant, it possesses at least three critical points which, by (2.9), are three solutions of the problem. □

Here is a remarkable corollary of Theorem 2.3.

Corollary 2.4. *Let $f_1, \dots f_n : \mathbb{R} \to \mathbb{R}$ be n continuous functions satisfying conditions (b) and (c) of Theorem 2.3. Then, for each $a, r > 0$ and $b < 0$, there exists a number $\delta > 0$ with the following property: for every n-tuple of continuous functions $g_1, \dots, g_n : \mathbb{R} \to \mathbb{R}$ satisfying*

$$
\max_{1\leq k\leq n}\left(\sup_{t\in\mathbb{R}}\int_0^t g_k(s)ds - \inf_{t\in\mathbb{R}}\int_0^t g_k(s)ds\right) < \delta,
$$

there exists $(\tilde{\mu}_1, \dots \tilde{\mu}_n) \in \mathbb{R}^n$, with $\sum_{k=1}^{n}|\tilde{\mu}_k|^2 < r^2$, such that the problem

$$
\begin{cases}
-\left(a\sum_{h=1}^{n+1}|x_h - x_{h-1}|^2 + b\right)(x_{k+1} - 2x_k + x_{k-1}) = g_k(x_k) + \tilde{\mu}_k f_k(x_k), \quad k = 1, \dots, n, \\
x_0 = x_{n+1} = 0
\end{cases}
$$

has at least three solutions.

Proof. It is enough to observe that the function $K(t) = at + b$ satisfies condition (a) of Theorem 2.3. □

Remark 2.5. It is important to remark that the technique adopted in the proof Theorem 2.3 cannot be used to treat the non-discrete problem, keeping condition (a). This is due to the fact that, under condition (a), the functional

$$
u \mapsto \int_0^{\int_\Omega |\nabla u(x)|^2 dx} K(s)ds
$$

is not weakly lower semicontinuous in $H_0^1(\Omega)$.

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