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NONTRIVIAL SOLUTIONS FOR NEUMANN FRACTIONAL p-LAPLACIAN PROBLEMS

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Abstract. In this paper, we investigate some classes of Neumann fractional p-Laplacian problems. We prove the existence and multiplicity of nontrivial solutions for several different nonlinearities, by using variational methods and critical point theory based on cohomological linking.

Keywords: fractional p-Laplacian, Neumann boundary condition, linking over cones.

Mathematics Subject Classification: 35A15, 47J30, 35S15, 47G10, 45G05.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the following problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{1.1}$$

where $p > 1$, $0 < s < 1$, $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and

$$(-\Delta)_p^s u(x) := P.V. \int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy,$$

while

$$\mathscr{N}_{s,p} u(x) := \int\limits_{\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy$$

for all $x \in \mathbb{R}^N \setminus \overline{\Omega}$. This kind of condition is called nonlocal Neumann boundary condition, see [12] for the case $p = 2$ and [4, 25, 26] for the general case. In order to find solutions for problem (1.1), we will work with the function space

$$X := \left\{ u : \mathbb{R}^N \to \mathbb{R} : u \text{ is measurable and such that } \|u\| < \infty \right\},$$

where

$$\|u\| := \left(\frac{1}{2}\iint\limits_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy + \int\limits_{\Omega} |u|^p dx\right)^{\frac{1}{p}},$$

and $\mathcal{Q} = \mathbb{R}^{2N} \setminus (C\Omega)^2, C\Omega = \mathbb{R}^N \setminus \Omega$.[1)] For the convenience of use, we also denote the fractional Gagliardo seminorm for a measurable function u by

$$[u] = \left(\iint\limits_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}},$$

see [13]. Finally, we denote by $\|\cdot\|_\nu$ the standard L^ν-norm, $\nu \in [1,\infty)$, that is

$$\|u\|_\nu = \left(\int\limits_{\Omega} |u(t)|^\nu dt\right)^{\frac{1}{\nu}}.$$

By the Sobolev embedding theorem, it is well known that the embedding mapping $X \hookrightarrow L^\nu(\Omega)$ is continuous for all $1 \le \nu \le p_s^*$ and compact for all $1 \le \nu < p_s^*$, see [11, Theorems 6.5, 6.7, 7.1], where p_s^* is the fractional Sobolev critical exponent of order s, defined as

$$p_s^* = \begin{cases} \frac{Np}{N-ps} & ps < N, \\ \infty & ps \ge N. \end{cases}$$

Hence, for $1 \le \nu \le p_s^*$, there is a positive constant M_0 such that

$$\|u\|_\nu \le M_0\|u\| \quad \text{for all } u \in X. \tag{1.2}$$

From [25] we take the following definition.

Definition 1.1. Let $u \in X$. If

$$\frac{1}{2}\iint\limits_{\mathcal{Q}} \frac{J_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy = \lambda\int\limits_{\Omega} |u|^{p-2}uvdx + \int\limits_{\Omega} f(x,u)vdx$$

for any $v \in X$, we say that u is a weak solution of problem (1.1), where

$$J_p(u(x)-u(y)) = |u(x)-u(y)|^{p-2}(u(x)-u(y)).$$

The corresponding functional Φ on X is defined by

$$\Phi(u) = \frac{1}{2p}\iint\limits_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy - \frac{\lambda}{p}\int\limits_{\Omega} |u|^p dx - \int\limits_{\Omega} F(x,u)dx,$$

[1)] The constant 1/2 multiplying the double integral is used just for useful normalization in the proofs of the main results.

where $F(x,t) = \int_0^t f(x,s)ds$. At least formally, finding weak solutions of problem (1.1) is equivalent to looking for critical points of Φ, and the equivalence depends on different regularity and growth assumptions on f, which we introduce below.

Before giving our results, we need to recall some concepts about the eigenvalues of the fractional p-Laplacian, see [25]: consider the nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{1.3}$$

where $\lambda \in \mathbb{R}$. If (1.3) has a nontrivial weak solution $u \in X$, that is

$$\frac{1}{2} \iint\limits_{\mathcal{Q}} \frac{J_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} dxdy = \lambda \int\limits_{\Omega} |u|^{p-2} uv dx$$

for all $v \in X$, we say that λ is an eigenvalue of $(-\Delta)_p^s$ with p-Neumann boundary condition and u is an associated eigenfunction. We denote the set of all the eigenvalues of $(-\Delta)_p^s$ in X by $\sigma(s,p)$. As in [25] (the definitions therein were slightly different), there is a sequence of eigenvalues defined by

$$\begin{aligned} \lambda_m := \inf \Bigg\{ \sup_{u \in A} \frac{1}{2} \iint\limits_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy : A \subseteq \Sigma, A \text{ is symmetric}, \\ \text{nonempty, closed and } i(A) \geq m \Bigg\}, \end{aligned} \tag{1.4}$$

where i is the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz, see [14], and

$$\Sigma := \Bigg\{ u \in X : \int\limits_{\Omega} |u|^p dx = 1 \Bigg\}.$$

It should be pointed out that $\lambda_1 = 0$ is the first (simple) eigenvalue, see [25], with associated eigenspace made of constant functions. Finally, as in [24], for every $m \in \mathbb{N}$, we introduce the following two cones

$$\mathcal{C}_m^- := \Bigg\{ u \in X : \frac{1}{2} \iint\limits_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \leq \lambda_m \int\limits_{\Omega} |u|^p dx \Bigg\}, \tag{1.5}$$

$$\mathcal{C}_m^+ := \Bigg\{ u \in X : \frac{1}{2} \iint\limits_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \geq \lambda_{m+1} \int\limits_{\Omega} |u|^p dx \Bigg\}. \tag{1.6}$$

In recent years, fractional problems have been widely investigated, mainly under Dirichlet, but also with Neumann or Robin boundary conditions, see, for instance, [1–4, 6–8, 10, 11, 13, 16–18, 22–25, 27–31], in which the authors studied regularity issues, as well as existence and multiplicity results. In particular, if $\lambda < 0$, by using the

Mountain Pass Theorem, in [25] the existence of nontrivial solutions for problem (1.1) is found. On the other hand, if $\lambda = 0$, problem (1.1) is simplified as

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{1.7}$$

and by applying the Weierstrass Theorem, in [24, Theorem 4.1] the following result was proved:

Theorem 1.2. *Suppose that $f(x,t)$ satisfies the following conditions.*

(H_0) *There exist $a_1 \in L^{\frac{p}{p-1}}(\Omega)$ and $b_1 \in \mathbb{R}$ such that*

$$|f(x,t)| \leq a_1(x) + b_1|t|^{p-1}$$

for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

(H_1)

$$\gamma(x) := \limsup_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} < \lambda_1 = 0.$$

Then problem (1.7) *admits a weak solution.*

In addition, in [24, Theorem 3.4] the existence of a nontrivial weak solution for problem (1.1) is given under the Ambrosetti–Rabinowitz condition (AR) generalized with the one introduced in [20]:

(H_2) there exist $\mu > p$ and $R_0 \geq 0$ such that

$$0 < \mu F(x,t) \leq f(x,t)t \tag{1.8}$$

for every $|t| > R_0$ and a.e. $x \in \Omega$, and there exist $\tilde{\mu} > p, b_2 > 0$ and $a_2 \in L^1(\Omega)$ such that

$$F(x,t) \geq b_2|t|^{\tilde{\mu}} - a_2(x) \tag{1.9}$$

for every $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

When the (AR) condition is not satisfied, by applying linking over cones introduced in [9], in [24, Theorem 3.7] the existence of nontrivial solutions for problem (1.1) is proved under (H_5)–(H_7) below and the following quasi-monotonocity condition introduced in [25] as a slight improvement of the one given in [21]:

(H_3) there exist $\vartheta \geq 1$ and $\beta^* \in L^1(\Omega), \beta^* \geq 0$ such that

$$\mathcal{F}\big(x,t_1\big) \leq \vartheta \mathcal{F}\big(x,t_2\big) + \beta^*(x)$$

for a.e. $x \in \Omega$ and all $0 \leq t_1 \leq t_2$ or $t_2 \leq t_1 \leq 0$, where

$$\mathcal{F}(x,t) := f(x,t)t - pF(x,t). \tag{1.10}$$

Here, motivated by [24, 25], we are interested in the existence of multiple nontrivial solutions for problem (1.1) by more general conditions. Now, we state our main results.

Theorem 1.3. *If hypotheses* (H_0), (H_1) *and*

(H_4) *there exists* $\rho > 0$ *such that*

$$F(x,t) \geq 0 \quad \text{for every } |t| \leq \rho \text{ and a.e. } x \in \Omega,$$

are satisfied, then problem (1.7) *admits at least two nontrivial solutions in* $\mathbb{R}^N$, *one being nonnegative and the other being nonpositive.*

Remark 1.4. Without further assumptions, we do not know whether the weak solution obtained by Theorem 1.2 is nontrivial. However, by adding condition (H_4), we get at least two nontrivial solutions in $\mathbb{R}^N$ for problem (1.7). So, Theorem 1.3 is a remarkable improvement of Theorem 1.2. Moreover, we have the following result.

Theorem 1.5. *Suppose that the following conditions hold:*

(H_5) $f(x,0)=0$ *and there exist constants* $b_3, b_4 > 0$ *and* $q \in (p, p_s^*)$ *such that*

$$|f(x,t)| \leq b_3 + b_4|t|^{q-1}$$

for every $t \in \mathbb{R}$ *and a.e.* $x \in \Omega$;

(H_6)

$$\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^p} = +\infty \quad \text{uniformly for a.e. } x \in \Omega;$$

(H_7)

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0 \quad \text{uniformly for a.e. } x \in \Omega;$$

(H_8)

$$F(x,t) \geq \frac{1}{p}|t|^p \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega;$$

(H_9) *there exist positive constants* R_1 *and* $\theta > 0$, $\kappa > \max\left\{1, \frac{N}{ps}\right\}$ *and a nonnegative function* $W(x) \in L^1(\Omega)$ *such that*

$$\left(\frac{F(x,t)}{|t|^p}\right)^{\kappa} \leq \theta \mathcal{F}(x,t) + W(x)$$

for all $|t| \geq R_1$ *and a.e.* $x \in \Omega$, *where* $\mathcal{F}$ *is the function defined in* (1.10).

Then problem (1.1) *admits one nontrivial solution for every* $\lambda \in \mathbb{R}$.

Remark 1.6. We anticipate that in Section 2 we prove that under assumption (H_5), condition (H_2) implies (H_6) and (H_9), while under assumptions (H_5) and (H_6), condition (H_3) implies (H_9). Hence, Theorem 1.5 extends the setting of both Theorem 3.4 and Theorem 3.7 in [24].

There are functions that satisfy our conditions but they do not satisfy (H_2) and (H_3). For example, let $N = p^2$,

$$f(x,t) = \begin{cases} \frac{p|t|^{p-2}t\int_1^{|t|} g(\tau)d\tau + |t|^{p-1}tg(|t|)}{\left(\ln|t|+1\right)^{\frac{1}{p}}} - \frac{|t|^{p-2}t\int_1^{|t|} g(\tau)d\tau}{p(\ln|t|+1)^{1+\frac{1}{p}}} + |t|^{p-2}t, & |t| \geq 1, x \in \Omega, \\ |t|^{p-2}t, & |t| \leq 1, x \in \Omega, \end{cases}$$

where $g : (-\infty, -1] \cup [1, +\infty) \to \mathbb{R}$ is defined by

$$g(t) = \begin{cases} n^3\Big(\frac{1}{n^2} - \big||t| - n\big|\Big) + \frac{1}{t}, & n - \frac{1}{n^2} \le |t| \le n + \frac{1}{n^2}, n = 2, 3, 4, \ldots, \\ \frac{1}{t}, & |t| < n - \frac{1}{n^2} \text{ or } |t| > n + \frac{1}{n^2}, n = 2, 3, 4, \ldots. \end{cases}$$

Then, $f(x, t)$ satisfies conditions of Theorem 1.5. As a matter of fact, by straightforward calculation, we have

$$g(n) = n + \frac{1}{n}, \quad g\left(n + \frac{1}{n^2}\right) = \frac{1}{n + \frac{1}{n^2}}, \quad n = 2, 3, 4, \ldots,$$

$$F(x,t) = \begin{cases} \frac{|t|^p \int_1^{|t|} g(\tau) d\tau}{\left(\ln|t|+1\right)^{\frac{1}{p}}} + \frac{1}{p}|t|^p, & |t| \ge 1, x \in \Omega, \\ \frac{1}{p}|t|^p, & |t| \le 1, x \in \Omega, \end{cases}$$

and

$$\mathcal{F}(x,t) := f(x,t)t - pF(x,t) = \begin{cases} \frac{|t|^{p+1} g(|t|)}{\left(\ln|t|+1\right)^{\frac{1}{p}}} - \frac{|t|^p \int_1^{|t|} g(\tau) d\tau}{p\left(\ln|t|+1\right)^{1+\frac{1}{p}}}, & |t| \ge 1, x \in \Omega, \\ 0, & |t| \le 1, x \in \Omega. \end{cases}$$

It is obvious that $f(x, t)$ satisfies conditions (H_5)–(H_8). Now, we only verify that (H_9) holds for any $\kappa \in (p, p+1)$. On the one hand, for $|t| \ge R_1 (\ge n + \frac{1}{n^2})$, we have

$$\begin{aligned} \left(\frac{F(x,t)}{|t|^p}\right)^\kappa &= \left(\frac{\ln|t|}{(\ln|t|+1)^{\frac{1}{p}}} + \frac{1}{p}\right)^\kappa \\ &< \left(\frac{\ln|t|}{(\ln|t|+1)^{\frac{1}{p}}} + 1\right)^\kappa \\ &< \left(\frac{\ln|t|+1}{(\ln|t|+1)^{\frac{1}{p}}} + \frac{\ln|t|+1}{(\ln|t|+1)^{\frac{1}{p}}}\right)^\kappa \\ &< 2^{p+1}\Big(\ln|t|+1\Big)^{p-\frac{1}{p}} \end{aligned}$$

for $\kappa \in (p, p+1)$. On the other hand, we obtain

$$\begin{aligned} \theta\mathcal{F}(x,t) + W(x) &= \theta\left(\frac{|t|^{p+1}\frac{1}{|t|}}{(\ln|t|+1)^{\frac{1}{p}}} - \frac{|t|^p \ln|t|}{p(\ln|t|+1)^{1+\frac{1}{p}}}\right) + W(x) \\ &\ge \theta\left(\frac{|t|^p}{(\ln|t|+1)^{\frac{1}{p}}} - \frac{|t|^p}{p(\ln|t|+1)^{\frac{1}{p}}}\right) + W(x) \\ &\ge \theta\left(\frac{p-1}{p}(\ln|t|+1)^{p-\frac{1}{p}}\right) + W(x). \end{aligned}$$

Hence, we can easily get that

$$2^{p+1}(\ln|t|+1)^{p-\frac{1}{p}} \leq \theta\left(\frac{p-1}{p}(\ln|t|+1)^{p-\frac{1}{p}}\right)+W(x)$$

for $\theta=\frac{2^{p+1}p}{p-1}>0$ and a nonnegative function $W(x)=x^2\in L^1(\Omega)$.

However, $f(x,t)$ does not satisfy condition (H_3). Actually, for $t_1:=n$, $t_2:=n+\frac{1}{n^2}$, we have

$$\begin{aligned}\mathcal{F}(x,t_1)&=\mathcal{F}(x,n)\\&=\frac{n^{p+1}g(n)}{\Big(\ln n+1\Big)^{\frac{1}{p}}}-\frac{n^p\int_1^n g(\tau)d\tau}{p\Big(\ln n+1\Big)^{1+\frac{1}{p}}}\\&=\frac{n^{p+2}+n^p}{\Big(\ln n+1\Big)^{\frac{1}{p}}}-\frac{n^p\int_1^n g(\tau)d\tau}{p\Big(\ln n+1\Big)^{1+\frac{1}{p}}}\end{aligned}$$

and

$$\begin{aligned}\vartheta\mathcal{F}(x,t_2)&=\vartheta\mathcal{F}\left(x,n+\frac{1}{n^2}\right)\\&=\frac{\vartheta\Big(n+\frac{1}{n^2}\Big)^{p+1}g\Big(n+\frac{1}{n^2}\Big)}{\Big(\ln\Big(n+\frac{1}{n^2}\Big)+1\Big)^{\frac{1}{p}}}-\frac{\vartheta\Big(n+\frac{1}{n^2}\Big)^{p}\int_1^{n+\frac{1}{n^2}}g(\tau)d\tau}{p\Big(\ln\Big(n+\frac{1}{n^2}\Big)+1\Big)^{1+\frac{1}{p}}}\\&=\frac{\vartheta\Big(n+\frac{1}{n^2}\Big)^{p}}{\Big(\ln\Big(n+\frac{1}{n^2}\Big)+1\Big)^{\frac{1}{p}}}-\frac{\vartheta\Big(n+\frac{1}{n^2}\Big)^{p}\int_1^{n+\frac{1}{n^2}}g(\tau)d\tau}{p\Big(\ln\Big(n+\frac{1}{n^2}\Big)+1\Big)^{1+\frac{1}{p}}}.\end{aligned}$$

Then, it is easy to get that

$$\mathcal{F}(x,t_1)-\vartheta\mathcal{F}(x,t_2)\to+\infty \quad \text{as } n\to\infty.$$

Hence, we can not find constants $\vartheta\geq 1,\beta^*(x)>0$ such that (H_3) holds. $f(x,t)$ does not satisfy condition (H_2) as well.

2. PRELIMINARY RESULTS

we recall an abstract critical point theorem which is based on the deformation lemma and a general linking structure. The deformation lemma is guaranteed by a compactness condition, the Palais–Smale condition or the Cerami condition – (PS) or (C) condition for short, while the geometrical structure is obtained by the notion of linking sets through the Alexander–Spanier cohomology, see [9, 15].

Definition 2.1. Let D, S, A, B be four subsets of a metric space X with $S \subseteq D$ and $B \subseteq A$. We say that (D, S) links (A, B) if $S \cap A = B \cap D = \emptyset$ and, for every deformation $\eta : D \times [0,1] \to X \setminus B$ with $\eta(S \times [0,1]) \cap A = \emptyset$, we have that $\eta(D \times 1) \cap A \neq \emptyset$. If $B = \emptyset$, we simply say that (D, S) links A.

Definition 2.2. Let $\Phi : X \to \mathbb{R}$ be a C^1 functional defined on a Banach space X. We say that Φ satisfies:

– the Palais–Smale condition at level $c \in \mathbb{R}$ $(PS)_c$, if for every $\{u_n\}_n$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in X', then, up to a subsequence, u_n converges strongly in X,
– the Cerami condition at level $c \in \mathbb{R}$ $(C)_c$, if for every $\{u_n\}_n$ such that $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ in X', then, up to a subsequence, u_n converges strongly in X.

Theorem 2.3. *Let X be a complete Finsler manifold of class C^1 and let $\Phi : X \to \mathbb{R}$ be a function of class C^1. Let D, S, A, B be four subsets of X, with $S \subseteq D$ and $B \subseteq A$, such that (D, S) links (A, B) and*

$$\sup_S \Phi < \inf_A \Phi, \quad \sup_D \Phi < \inf_B \Phi$$

(with $\sup \emptyset = -\infty$ *and* $\inf \emptyset = +\infty$*). Define*

$$c = \inf_{\eta \in \mathcal{N}} \sup \Phi(\eta(D \times \{1\})),$$

where $\mathcal{N}$ is the set of deformations $\eta : D \times [0,1] \to X \setminus B$ with $\eta(S \times [0,1]) \cap A = \emptyset$. Then we have

$$\inf_A \Phi \leq c \leq \sup_D \Phi.$$

Moreover, if Φ satisfies $(PS)_c$ (or $(C)_c$), then c is a critical value of Φ.

Theorem 2.4 (Mountain Pass Lemma)**.** *Let $\big(X, \|\cdot\|\big)$ be a Banach space, and let $\Phi \in C^1(X, \mathbb{R})$ satisfy the (PS) condition. Suppose that $\Phi(0) = 0$ and:*

(P_1) *there exist positive constants ϱ and α such that $\Phi(u) \geq \alpha > 0$ for all $u \in X$ with $\|u\| = \varrho$,*
(P_2) *there exists $e \in X$ with $\|e\| > \varrho$ such that $\Phi(e) < 0$.*

Then Φ possesses a critical value $c \geq \alpha$ given by

$$c := \inf_{\zeta \in \Gamma} \sup_{s \in [0,1]} \Phi(\zeta(s)),$$

where

$$\Gamma := \{\zeta \in C([0,1], X) \,:\, \zeta(0) = 0, \zeta(1) = e\}.$$

As shown in [5], the deformation lemma holds also replacing the usual (PS) condition with the weaker (C) condition. So, Theorem 2.3 holds with the $(PS)_c$ condition (as in the original [9, Theorem 2.2]), but also with the $(C)_c$ condition, for instance see [19, Theorem 5.40].

Definition 2.5. Let D, S, A, B be four subsets of a metric space X with $S \subseteq D$ and $B \subseteq A$; let m be a nonnegative integer and $\mathbb{K}$ be a field. We say that (D, S) links (A, B) cohomologically in dimension m over $\mathbb{K}$ if $S \cap A = B \cap D = \emptyset$ and the restriction homomorphism $H^m(X \setminus B, X \setminus A; \mathbb{K}) \to H^m(D, S; \mathbb{K})$ is not identically zero. If $B = \emptyset$, we simply say that (D, S) links A cohomologically in dimension m over $\mathbb{K}$.

Theorem 2.6 (Theorem 2.8, [9]). *Let X be a real normed space and let $\mathcal{C}^-, \mathcal{C}^+$ be two cones such that $\mathcal{C}^+$ is closed in X, $\mathcal{C}^- \cap \mathcal{C}^+ = \{0\}$ and $\big(X, \mathcal{C}^- \setminus \{0\}\big)$ links $\mathcal{C}^+$ cohomologically in dimension m over $\mathbb{K}$. Let $r_-, r_+ > 0$ and let*

$$\begin{aligned} D_- &= \big\{u \in \mathcal{C}^- : \|u\| \le r_-\big\}, \quad S_- = \big\{u \in \mathcal{C}^- : \|u\| = r_-\big\}, \\ D_+ &= \big\{u \in \mathcal{C}^+ : \|u\| \le r_+\big\}, \quad S_+ = \big\{u \in \mathcal{C}^+ : \|u\| = r_+\big\}. \end{aligned}$$

Then the following facts hold:

(d_1) *(D_-, S_-) links $\mathcal{C}^+$ cohomologically in dimension m over $\mathbb{K}$.*
(d_2) *(D_-, S_-) links $\big(D_+, S_+\big)$ cohomologically in dimension m over $\mathbb{K}$.*
Moreover, let $e \in X$ with $-e \notin \mathcal{C}^-$, $r_- > r_+$ and

$$\begin{aligned} Q &= \big\{u + te : u \in \mathcal{C}^-, t \ge 0, \|u + te\| \le r_-\big\}, \\ H &= \big\{u + te : u \in \mathcal{C}^-, t \ge 0, \|u + te\| = r_-\big\}, \end{aligned}$$

then the following facts hold:
(d_3) *$\big(Q, D_- \cup H\big)$ links S_+ cohomologically in dimension $m+1$ over $\mathbb{K}$.*
(d_4) *$D_- \cup H$ links $\big(D_+, S_+\big)$ cohomologically in dimension m over $\mathbb{K}$.*

Corollary 2.7 ([9, Corollary 2.9]). *Let X be a real normed space and $\mathcal{C}^-, \mathcal{C}^+$ be two symmetric cones in X such that $\mathcal{C}^+$ is closed in $X, \mathcal{C}^- \cap \mathcal{C}^+ = \{0\}$ and such that*

$$i\big(\mathcal{C}^- \setminus \{0\}\big) = i\big(X \setminus \mathcal{C}^+\big) < \infty.$$

Then the facts (d_1)–(d_4) of Theorem 2.6 hold for $m = i\big(\mathcal{C}^- \setminus \{0\}\big)$ and $\mathbb{K} = \mathbb{Z}_2$.

Proposition 2.8 ([9, Proposition 2.4]). *If (D, S) links (A, B) cohomologically (in some dimension), then (D, S) links (A, B).*

According to (1.5) and (1.6), we know that $\mathcal{C}_m^-$, $\mathcal{C}_m^+$ are two cones and satisfy the following identity.

Lemma 2.9 ([24, Theorem 2.6]). *Let $m \ge 1$ be such that $\lambda_m < \lambda_{m+1}$, then we have*

$$i(\mathcal{C}_m^- \setminus \{0\}) = i(X \setminus \mathcal{C}_m^+) = m.$$

We are now ready to prove that our assumptions are more general than the ones in [24].

Lemma 2.10. *Under condition (H_5), condition (H_2) implies conditions (H_6) and (H_9).*

Proof. It is obvious that (H_6) holds due to (1.9) in (H_2). Besides, since $q \in (p, p_s^*)$, one can easily get that $\frac{q}{q-p} > \frac{N}{ps}$. Then for any $\kappa \in \left(\frac{N}{ps}, \frac{q}{q-p}\right)$, by straightforward calculation, we obtain

$$q < \frac{p\kappa}{\kappa - 1}. \tag{2.1}$$

By integrating (H_5), we get that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\frac{p\kappa}{\kappa-1}}} = 0 \quad \text{uniformly for } a.e.\ x \in \Omega. \tag{2.2}$$

From (1.8) and (2.2), there exists a constant $R_2 \geq R_0$ such that

$$0 < \frac{F(x,t)}{|t|^{\frac{p\kappa}{\kappa-1}}} \leq (\mu - p)^{\frac{1}{\kappa-1}} \tag{2.3}$$

for $|t| \geq R_2$ and a.e. $x \in \Omega$. By taking the power $\kappa - 1$ in (2.3) and using (1.8), we immediately find that

$$\left(\frac{F(x,t)}{|t|^p}\right)^{\kappa} \leq (\mu - p)F(x,t) \leq f(x,t)t - pF(x,t) = \mathcal{F}(x,t).$$

for $|t| \geq R_2$ and a.e. $x \in \Omega$. So, condition (H_9) holds with $\theta = 1$ and $W = 0$. □

Lemma 2.11. *Under conditions* (H_5)*–*(H_6)*, condition* (H_3) *implies condition* (H_9)*.*

Proof. From assumptions (H_5) and (H_6), it follows that there exists a positive constant $R_3 > 1$ such that

$$\frac{F(x,t)}{|t|^q} \leq \frac{b_4}{q} + 1 \tag{2.4}$$

and

$$\frac{F(x,t)}{|t|^p} > 0 \tag{2.5}$$

for all $|t| \geq R_3$ and a.e. $x \in \Omega$. By (2.1), we obtain $p > q(\kappa - 1)/\kappa$, and since $\kappa < q/(q-p)$, we finally get $p > (\kappa - 1)\big(q - p\big)$. Setting $\xi = p - (\kappa - 1)(q - p)$, then one has $\xi > 0$. Now, let us consider the case of $t \geq R_3$, the case of $t \leq -R_3$ being analogous. In view of (2.4) and (H_3), it turns out that

$$\begin{aligned}\left(\frac{F(x,t)}{t^p}\right)^{\kappa} - \left(\frac{F(x,R_3)}{R_3{}^p}\right)^{\kappa} &= \int_{R_3}^{t} \frac{d}{ds}\left[\left(\frac{F(x,s)}{|s|^p}\right)^{\kappa}\right] ds \\ &= \int_{R_3}^{t} \kappa \left(\frac{F(x,s)}{|s|^p}\right)^{\kappa-1} \frac{f(x,s)s - pF(x,s)}{|s|^p s} ds\end{aligned}$$

$$
\begin{aligned}
&= \int_{R_3}^{t} \kappa\left(\frac{F(x,s)}{|s|^q}\right)^{\kappa-1} \frac{\mathcal{F}(x,s)}{s^{\xi+1}} ds \\
&\leq \kappa\left(\frac{b_4}{q}+1\right)^{\kappa-1} \int_{R_3}^{t} \frac{\mathcal{F}(x,s)}{s^{\xi+1}} ds \\
&\leq \kappa\left(\frac{b_4}{q}+1\right)^{\kappa-1} (\vartheta\mathcal{F}(x,t)+\beta^*) \int_{R_3}^{t} \frac{1}{s^{\xi+1}} ds \\
&\leq \kappa\left(\frac{b_4}{q}+1\right)^{\kappa-1} (\vartheta\mathcal{F}(x,t)+\beta^*) \frac{1}{\xi {R_3}^{\xi}},
\end{aligned}
$$

that is

$$
\left(\frac{F(x,t)}{t^p}\right)^{\kappa} \leq \left(\frac{b_4}{q}+1\right)^{\kappa-1} \frac{\kappa\vartheta}{\xi {R_3}^{\xi}} \mathcal{F}(x,t) + \left(\frac{b_4}{q}+1\right)^{\kappa-1} \frac{\kappa\beta^*}{\xi {R_3}^{\xi}} + \left(\frac{F(x,R_3)}{{R_3}^p}\right)^{\kappa}
$$

for all $t \geq R_3$ and a.e. $x \in \Omega$. Then, we have

$$
\left(\frac{F(x,t)}{t^p}\right)^{\kappa} \leq \theta\mathcal{F}(x,t) + W(x) \tag{2.6}
$$

for all $t \geq R_3$ and a.e. $x \in \Omega$, where

$$
\theta = \left(\frac{b_4}{q}+1\right)^{\kappa-1} \frac{\kappa\vartheta}{\xi {R_3}^{\xi}},
$$

and

$$
W(x) = \left(\frac{b_4}{q}+1\right)^{\kappa-1} \frac{\kappa\beta^*}{\xi {R_3}^{\xi}} + \left(\frac{F(x,R_3)}{{R_3}^p}\right)^{\kappa}
$$

is a nonnegative function according to (2.5). Analogously, it is easy to verify that inequality (2.6) holds for $t \leq -R_3$ and a.e. $x \in \Omega$. Hence, condition (H_9) holds. $\square$

In order to obtain multiple solutions for problem (1.7), we consider a truncated problem, that is

$$
\begin{cases} (-\Delta)_p^s u = f(x,u^{\pm}) & \text{in } \Omega, \\ \mathcal{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{2.7}
$$

Signed solutions of (2.7) are the critical points of the C^1 functional $I_{\pm}$ on X defined by

$$
\begin{aligned}
I_{\pm}(u) = \frac{1}{2p} \iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \frac{1}{p}\int_{\Omega} |u|^p dx \\
- \frac{1}{p}\int_{\Omega} \left|u^{\pm}\right|^p dx - \int_{\Omega} F(x,u^{\pm}) dx,
\end{aligned} \tag{2.8}
$$

where $F(x,t^{\pm}) = \int_0^t f(x,s^{\pm})ds$, $u^+ := \max\{u,0\}$, $u^- := \max\{-u,0\}$, $u = u^+ - u^-$ and $|u| = u^+ + u^-$. From (2.8) we get that

$$\begin{aligned}\langle I'_{\pm}(u), v\rangle = \frac{1}{2}\iint\limits_{\mathcal{Q}} \frac{J_p(u_n(x)-u_n(y))(v(x)-v(y))}{|x-y|^{N+ps}}dxdy + \int\limits_{\Omega} |u|^{p-2}uvdx \\ - \int\limits_{\Omega} |u^{\pm}|^{p-2}u^{\pm}vdx - \int\limits_{\Omega} f(x,u^{\pm})vdx\end{aligned}$$

for all $u, v \in X$. We will prove that I_+ admits a nonnegative critical point, which is a nonnegative solution of (2.7), and so of (1.7), as well. In the same way, I_- admits a nonpositive solution, which provides the second nontrivial solution to (1.7). In the following, we only consider I_+, the approach for I_- being similar. We also recall the following fact, to be used later on:

Lemma 2.12 ([24, Equation (15)]). *For any $x, y \in \mathbb{R}$, the following inequality holds:*

$$\big|x^- - y^-\big|^p \le |x-y|^{p-2}(x-y)\big(y^- - x^-\big). \tag{2.9}$$

Now, we are ready to prove our results.

3. PROOF OF THEOREM 1.3

We will divide the proof into three steps.

Step 1. Under conditions (H_0) and (H_1), we prove that I_+ is coercive, i.e., $I_+(u) \to \infty$ as $\|u\| \to \infty$.

By (H_1), for every $\varepsilon > 0$, there exists $M_1 > 0$ such that

$$\frac{f(x,t^+)}{|t|^{p-2}t} < \gamma(x) + \varepsilon$$

for all $|t| \ge M_1$ and a.e. $x \in \Omega$. By simple calculations, one has

$$F(x,t^+) \le \frac{\gamma(x)+\varepsilon}{p}(|t|^p - M_1{}^p) + \max\{F(x,M_1), F(x,-M_1)\}$$

for all $|t| \ge M_1$ and a.e. $x \in \Omega$. Hence, it holds that

$$\limsup_{|t|\to\infty} \frac{F(x,t^+)}{|t|^p} \le \frac{\gamma(x)}{p}. \tag{3.1}$$

Next, we show that

$$\liminf_{\|u\|\to\infty} \frac{I_+(u)}{\|u\|^p} > 0. \tag{3.2}$$

For this, let us choose any unbounded sequence $\{u_n\}$; by setting $z_n := \frac{u_n}{\|u_n\|}$, then $\{z_n\}_n$ is bounded and there exists a $z \in X$ such that, up to a subsequence,

$$\begin{cases} z_n \rightharpoonup z & \text{in } X, \\ z_n \to z & \text{in } L^\nu(\Omega), \quad \nu \in [1, p_s^*), \\ z_n \to z & \text{a.e. in } \Omega. \end{cases} \tag{3.3}$$

By (H_0), we immediately find that

$$\frac{F(x, u_n^+)}{\|u_n\|^p} \le \frac{a_1(x)|u_n| + \frac{b_1}{p}|u_n|^p}{\|u_n\|^p} \to \frac{b_1}{p}|z|^p \text{ a.e. in } \Omega,$$

as $n \to \infty$. It turns out from the Generalized Fatou Lemma that

$$\limsup_{n\to\infty} \int_\Omega \frac{F(x, u_n^+)}{\|u_n\|^p} dx \le \int_\Omega \limsup_{n\to\infty} \frac{F(x, u_n^+)}{\|u_n\|^p} dx.$$

If x is such that $\{|u_n(x)|\}$ is bounded, so that $z(x) = 0$, one has

$$\limsup_{n\to\infty} \frac{F(x, u_n^+)}{\|u_n\|^p} = 0.$$

If $\{|u_n(x)|\}$ is unbounded, one deduces from (3.1) that

$$\limsup_{n\to\infty} \frac{F(x, u_n^+)}{\|u_n\|^p} = \limsup_{n\to\infty} \frac{F(x, u_n^+)}{|u_n|^p} \frac{|u_n|^p}{\|u_n\|^p} \le \frac{\gamma(x)}{p}|z|^p.$$

In conclusion, considering the points where $\{|u_n(x)|\}$ is bounded or unbounded, we have

$$\limsup_{n\to\infty} \int_\Omega \frac{F(x, u_n^+)}{\|u_n\|^p} dx \le \int_\Omega \frac{\gamma(x)}{p}|z|^p \le 0. \tag{3.4}$$

Notice that

$$\frac{I_+(u_n)}{\|u_n\|^p} = \frac{1}{p} - \frac{1}{p}\int_\Omega \frac{|u_n^+|^p}{\|u_n\|^p} dx - \int_\Omega \frac{F(x, u_n^+)}{\|u_n\|^p} dx.$$

So, we find

$$\liminf_{n\to\infty} \frac{I_+(u_n)}{\|u_n\|^p} \ge \liminf_{n\to\infty} \left(\frac{1}{p} - \frac{1}{p}\int_\Omega \frac{|u_n|^p}{\|u_n\|^p} dx - \int_\Omega \frac{F(x, u_n^+)}{\|u_n\|^p} dx \right).$$

By (3.4), we get

$$\liminf_{n\to\infty} \frac{I_+(u_n)}{\|u_n\|^p} \ge \frac{1 - \int_\Omega \big(\gamma(x) + 1\big)|z|^p dx}{p}.$$

If $z = 0$, the $\liminf$ is at least $\frac{1}{p}$. If $z \neq 0$, then the measure of the set where $z \neq 0$ has positive measure. Thus, since $\gamma(x) < 0$ for a.e. $x \in \Omega$, by the weak semicontinuity of the norm in X, we find that

$$\liminf_{n\to\infty} \frac{I_+(u_n)}{\|u_n\|^p} > \frac{1 - \int\limits_\Omega |z|^p dx}{p} \geq \frac{\|z\|^p - \int\limits_\Omega |z|^p dx}{p} = \frac{[z]^p}{p} \geq 0.$$

This fact being true for any diverging sequence $\{u_n\}$, we get that (3.2) is satisfied, and so I_+ is coercive.

Step 2. I_+ has a minimum point $\bar{u}$.

Indeed, I_+ is sequentially lower semicontinuous with respect to the weak convergence, since the norm is sequentially lower semicontinuous with respect to the weak convergence, while $\int_\Omega |u^+|^p dx$ and $\int_\Omega F(x,u)dx$ are continuous.

Thus, by the Weierstrass Theorem I_+ has a minimum point $\bar{u}$.

Step 3. $\bar{u}$ is nonnegative and nontrivial.

Let us start showing that, under conditions (H_0) and (H_4), 0 is not an isolated minimizer of I_+. Indeed, from assumption (H_4), for $t \in (0,\rho)$, one has

$$F(x,t^+) = F(x,t) \geq 0.$$

Choose $\phi_1 > 0$ in Ω be a λ_1-eigenfunction, that is ϕ_1 is a constant (see [25]), for instance let us fix $\phi_1 = 1$. Hence, taking $\tau \in (0,\rho)$, it holds that

$$\begin{aligned} I_+(\tau) = I_+(\tau\phi_1) &= \frac{\tau^p}{2p} \iint\limits_{\mathcal{Q}} \frac{|\phi_1(x) - \phi_1(y)|^p}{|x-y|^{N+ps}} dxdy + \frac{1}{p} \int\limits_\Omega |\tau\phi_1|^p dx \\ &\quad - \frac{1}{p} \int\limits_\Omega |\tau\phi_1^+|^p dx - \int\limits_\Omega F(x,\tau\phi_1^+)dx \\ &= - \int\limits_\Omega F(x,\tau\phi_1^+)dx \leq 0 = I_+(0). \end{aligned}$$

Therefore, 0 is not an isolated minimizer of I_+.

Conclusion. Finally, assume that u is a critical point of I_+, so that, in particular, $\langle I'_+(u), -u^- \rangle = 0$, that is

$$\begin{aligned} 0 = &-\frac{1}{2} \iint\limits_{\mathcal{Q}} \frac{J_p(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}} dxdy - \int\limits_\Omega |u|^{p-2}uu^- dx \\ &+ \int\limits_\Omega |u^+|^{p-2}u^+u^- dx + \int\limits_\Omega f(x,u^+)u^- dx \\ = &\frac{1}{2} \iint\limits_{\mathcal{Q}} \frac{J_p(u(x)-u(y))(u^-(y)-u^-(x))}{|x-y|^{N+ps}} dxdy + \int\limits_\Omega (u^-)^p dx, \end{aligned}$$

since $f(x,u^+)u^- = 0$ in Ω. By (2.9), we find

$$\frac{1}{2}\iint\limits_{Q} \frac{|u^-(x)-u^-(y)|^p}{|x-y|^{N+ps}}dxdy + \int\limits_{\Omega} (u^-)^p dx \le 0.$$

So, $u^- = 0$ in X. Being $u \ge 0$ a critical point of I_+, then it is also a critical point of I. By Step 2, I_+ has a nonnegative minimizer $\bar{u} \in X$, that is

$$I_+(\bar{u}) = \inf_{u\in X} I_+(u).$$

By Step 3, 0 is not an isolated minimizer of I_+. So, if $\bar{u}$ is an isolated critical point of I_+, we have $\bar{u} \neq 0$. Therefore, $\bar{u}$ is a nonzero critical point of I, and thus a nontrivial nonnegative solution of (1.7). On the other hand, if $\bar{u}$ is not an isolated critical point of I_+, then I already has infinitely many nontrivial critical points; in any case we find a nontrivial solution. By applying the same reasoning to I_-, one can find a nonpositive critical point of I, say $\underline{u} \neq 0$. Hence, $\bar{u}$ and $\underline{u}$ are two nontrivial signed solutions of problem (1.7).

4. PROOF OF THEOREM 1.5

In order to get a nontrivial solution to (1.1), we introduce the related functional

$$\begin{aligned}\Phi(u) &= \frac{1}{2p}\iint\limits_{Q} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy + \frac{1}{p}\int\limits_{\Omega} |u|^p dx \\ &\quad - \frac{\lambda+1}{p}\int\limits_{\Omega} |u|^p dx - \int\limits_{\Omega} F(x,u)dx \\ &= \frac{1}{p}\|u\|^p - \frac{\lambda+1}{p}\int\limits_{\Omega} |u|^p dx - \int\limits_{\Omega} F(x,u)dx.\end{aligned}$$

We will divide the proof into two steps.

Step 1. We show that Φ satisfies the $(C)_c$ condition for every $c \in \mathbb{R}$. Let $\{u_n\} \subset X$ be a $(C)_c$ sequence for Φ, that is

$$(1 + \|u_n\|)\Phi'(u_n) \to 0, \tag{4.1}$$

and

$$\Phi(u_n) \to c \in \mathbb{R} \quad \text{as} \quad n \to \infty. \tag{4.2}$$

We want to prove that $\{u_n\}$ admits a strongly convergent subsequence. By standard argument due to the reflexivity of X and the compact embedding of X into Lebesgue spaces of order less than p^*, in order to prove that $\{u_n\}$ admits a strongly convergent subsequence, it is enough to prove that $\{u_n\}$ is bounded.

Now, assume by contradiction that $\{u_n\}$ is unbounded. Up to a subsequence, we can assume that $\|u_n\| \to +\infty$ as $n \to \infty$ and that there exists $w \in X$ such that, set $w_n = \frac{u_n}{\|u_n\|}$, we have

$$\begin{cases} w_n \rightharpoonup w & \text{in } X, \\ w_n \to w & \text{in } L^\nu(\Omega), \quad \nu \in [1, p_s^*), \\ w_n \to w & \text{a.e. in } \Omega. \end{cases}$$

Define the set

$$\Omega_{\neq} := \{x \in \Omega : w(x) \neq 0\}.$$

If $|\Omega_{\neq}| > 0$, then one has

$$|u_n(x)| \to +\infty \quad \text{for a.e. } x \in \Omega_{\neq} \quad \text{as} \quad n \to \infty.$$

Therefore, by (H_6), we have

$$\lim_{n\to\infty} \frac{F(x, u_n)}{\|u_n\|^p} = \lim_{n\to\infty} \frac{F(x, u_n)}{|u_n|^p} |w_n|^p = +\infty \quad \text{for a.e. } x \in \Omega_{\neq}.$$

Again by (H_6) we can invoke Fatou's Lemma, obtaining

$$\int_\Omega \liminf_{n\to\infty} \frac{F(x, u_n)}{\|u_n\|^p} dx \le \liminf_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^p} dx,$$

which leads to

$$\lim_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^p} dx = +\infty. \tag{4.3}$$

From (4.2), one knows that there exists $M_2 \in \mathbb{R}$ such that

$$-\frac{1}{2p}[u_n]^p - \frac{1}{p}\int_\Omega |u_n|^p dx + \frac{\lambda+1}{p}\int_\Omega |u_n|^p dx + \int_\Omega F(x, u_n) dx \le M_2 \text{ for all } n \in \mathbb{N}.$$

So there exists some $M_3 = M_3(\lambda) > 0$ such that

$$\int_\Omega F(x, u_n) dx \le M_2 + M_3 \|u_n\|^p.$$

The above inequality implies that

$$\limsup_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^p} dx \le M_3,$$

which contradicts with (4.3). Hence, $|\Omega_{\neq}| = 0$, namely $w = 0$ a.e. in Ω. Thus, we have that

$$w_n \to 0 \text{ in } L^\nu(\Omega) \quad \text{for all } \nu \in [1, p_s^*). \tag{4.4}$$

From (H_5) we know that if $M_4 > R_1$, then

$$|f(x,t)| \le b_3 + b_4 M_4^{q-1}$$

for all $(x,t) \in \Omega \times [-M_4, M_4]$. So it is easy to get that

$$|F(x,t)| \le M_5 \tag{4.5}$$

for all $(x,t) \in \Omega \times [-M_4, M_4]$ and $M_5 = b_3 M_4 + \frac{b_4}{q} M_4^q$, which implies that there exists $M_6 > 0$ such that

$$|\mathcal{F}(x,t)| = |f(x,t)t - pF(x,t)| \le M_6 \tag{4.6}$$

for all $(x,t) \in \Omega \times [-M_4, M_4]$. Set

$$\Omega_n := \{x \in \Omega : |u_n(x)| \ge M_4\}.$$

Then, from (4.2), (4.5), the Hölder inequality and (H_9), we find that

$$\begin{aligned}
\frac{1}{p} - \frac{\Phi(u_n)}{\|u_n\|^p} &= \frac{1}{p} - \frac{c + o(1)}{\|u_n\|^p} \\
&= \int\limits_{\Omega} \frac{F(x,u_n)}{\|u_n\|^p} dx + \frac{\lambda+1}{p} \int\limits_{\Omega} \frac{|u_n|^p}{\|u_n\|^p} dx \\
&= \int\limits_{\Omega_n} \frac{F(x,u_n)}{\|u_n\|^p} dx + \int\limits_{\Omega \setminus \Omega_n} \frac{F(x,u_n)}{\|u_n\|^p} dx + \frac{\lambda+1}{p} \int\limits_{\Omega} |w_n|^p dx \\
&\le \int\limits_{\Omega_n} \frac{F(x,u_n)}{|u_n|^p} |w_n|^p dx + \frac{M_5 |\Omega|}{\|u_n\|^p} + \frac{\lambda+1}{p} \int\limits_{\Omega} |w_n|^p dx \\
&\le \left[\int\limits_{\Omega_n} \left(\frac{F(x,u_n)}{|u_n|^p} \right)^{\kappa} dx \right]^{\frac{1}{\kappa}} \left[\int\limits_{\Omega_n} |w_n|^{\frac{p\kappa}{\kappa-1}} dx \right]^{\frac{\kappa-1}{\kappa}} + \frac{M_5|\Omega|}{\|u_n\|^p} \\
&\quad + \frac{\lambda+1}{p} \|w_n\|_p^p \\
&\le \left[\int\limits_{\Omega_n} (\theta \mathcal{F}(x,u_n) + W(x)) dx \right]^{\frac{1}{\kappa}} \|w_n\|^p_{\frac{p\kappa}{\kappa-1}} + \frac{M_5|\Omega|}{\|u_n\|^p} + \frac{\lambda+1}{p} \|w_n\|_p^p.
\end{aligned}$$

Writing $\int_{\Omega_n} = \int_{\Omega} - \int_{\Omega \setminus \Omega_n}$, by (4.6) we can estimate the previous quantity with

$$\begin{aligned}
&\le \Big[\theta(p\Phi(u_n) - \langle \Phi'(u_n), u_n \rangle) + \theta M_6 |\Omega| + \|W\|_1 \Big]^{\frac{1}{\kappa}} \|w_n\|^p_{\frac{p\kappa}{\kappa-1}} \\
&\quad + \frac{M_5|\Omega|}{\|u_n\|^p} + \frac{\lambda+1}{p} \|w_n\|_p^p.
\end{aligned}$$

Since $\{u_n\}$ is a Cerami sequence, we have that

$$\{\Phi(u_n)\} \text{ is bounded and } \langle \Phi'(u_n), u_n \rangle \to 0 \text{ as } n \to \infty.$$

Hence, from the previous inequalities we are finally led to

$$\frac{1}{p} - \frac{c + o(1)}{\|u_n\|^p} \le c_n \|w_n\|^p_{\frac{p\kappa}{\kappa-1}} + o(1),$$

where c_n is a bounded sequence of real numbers and $o(1) \to 0$ as $n \to \infty$. Since $\kappa > \max\{1, \frac{N}{ps}\}$, one has $\frac{p\kappa}{\kappa-1} \in (1, p_s^*)$ and so from (4.4), by letting $n \to \infty$ in the inequality above, we finally have

$$\frac{1}{p} \le 0.$$

This is an obvious contradiction. So, $\{u_n\}$ is bounded and Φ satisfies the $(C)_c$ condition.
Step 2. Let $\{\lambda_m\}$ be the sequence defined in (1.4), then either $\lambda + 1 < \lambda_1 = 0$ or there exists $m \ge 1$ such that

$$\lambda_m \le \lambda + 1 < \lambda_{m+1}. \tag{4.7}$$

First case: $\lambda + 1 < \lambda_1$. By assumptions (H_5) and (H_7), for a fixed $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$|F(x,t)| \le \frac{\varepsilon}{p}|t|^p + M_\varepsilon |t|^q$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. Let us choose ε such that $\lambda + 1 + \varepsilon < 0$. Then, for any $u \in X$, we get by (1.2) that

$$\begin{aligned}
\Phi(u) &\ge \frac{1}{p}\|u\|^p - \frac{\lambda+1}{p}\int\limits_\Omega |u|^p dx - \frac{\varepsilon}{p}\int\limits_\Omega |u|^p dx - -M_\varepsilon \int\limits_\Omega |u|^q dx \\
&\ge \frac{1}{p}\|u\|^p - \frac{\lambda+1+\varepsilon}{p}\int\limits_\Omega |u|^p dx - -M_0^q M_\varepsilon \|u\|^q \\
&\ge \frac{1}{p}\|u\|^p - M_0^q M_\varepsilon \|u\|^q = \|u\|^p\left(\frac{1}{p} - M_0^q M_\varepsilon \|u\|^{q-p}\right).
\end{aligned}$$

Since $q > p$, we set

$$\varrho := \left(\frac{1}{2pM_0^q M_\varepsilon}\right)^{\frac{1}{q-p}} > 0,$$

and

$$\alpha := \frac{\varrho^p}{2p} > 0,$$

so that $\Phi(u) \ge \alpha$ for $\|u\| = \varrho$.

Now, take $u \ne 0$ and $t > 0$; by (H_6) we have

$$\begin{aligned}
\Phi(tu) &= \frac{t^p}{p}\|u\|^p - \frac{\lambda+1}{p}t^p \int\limits_\Omega |u|^p dx - \int\limits_\Omega F(x,tu)dx \\
&\ge t^p \|u\|^p \left(\frac{1}{p} - \int\limits_\Omega \frac{F(x,tu)}{t^p|u|^p}\frac{|u|^p}{\|u\|^p}dx\right) \to -\infty,
\end{aligned}$$

as $t \to +\infty$.

Hence, there exists $\bar{t} > \varrho$ such that

$$\Phi(\bar{t}u) < 0.$$

Hence, by Theorem 2.4, there exists a critical point $\bar{u}$ of Φ such that $\Phi(\bar{u}) > 0$, so that $\bar{u} \neq 0$.

Second case: (4.7) *holds.*

By assumptions (H_5) and (H_7), fixed $\varepsilon > 0$ with $\lambda + 1 + \varepsilon < \lambda_{m+1}$, there exists $M_\varepsilon > 0$ such that

$$|F(x,t)| \leq \frac{\varepsilon}{p}|t|^p + M_\varepsilon |t|^q$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. Now, let $\mathcal{C}_m^-$ and $\mathcal{C}_m^+$ be as in (1.5) and (1.6). So, for each $u \in \mathcal{C}_m^+$, we get by (1.2) that

$$\begin{aligned}\Phi(u) &\geq \frac{1}{p}\|u\|^p - \frac{\lambda+1}{p}\int\limits_\Omega |u|^p dx - \frac{\varepsilon}{p}\int\limits_\Omega |u|^p dx - -M_\varepsilon \int\limits_\Omega |u|^q dx \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{p\lambda_{m+1}}(\lambda+1+\varepsilon)[u]^p - -M_\varepsilon \int\limits_\Omega |u|^q dx \\ &\geq \frac{1}{p}\left(1 - \frac{\lambda+1+\varepsilon}{\lambda_{m+1}}\right)\|u\|^p - -M_0^q M_\varepsilon \|u\|^q.\end{aligned}$$

Since $q > p$, we set

$$r_+ := \left[\frac{1}{2pM_0^q M_\varepsilon}\left(1 - \frac{\lambda+1+\varepsilon}{\lambda_{m+1}}\right)\right]^{\frac{1}{q-p}} > 0,$$

and

$$\alpha := \left[\frac{1}{2p}\left(1 - \frac{\lambda+1+\varepsilon}{\lambda_{m+1}}\right)\right]^{\frac{q}{q-p}} \left(\frac{1}{M_0^q M_\varepsilon}\right)^{\frac{p}{q-p}} > 0,$$

so that $\Phi(u) \geq \alpha$ for $\|u\| = r_+$.

From (H_8) and (4.7), for all $u \in \mathcal{C}_m^-$, we have

$$\begin{aligned}\Phi(u) \leq \frac{1}{2p}\iint\limits_{\mathcal{Q}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \frac{1}{p}\int\limits_\Omega |u|^p dx \\ - \frac{\lambda_m}{p}\int\limits_\Omega |u|^p dx - \frac{1}{p}\int\limits_\Omega |u|^p dx \leq 0.\end{aligned} \tag{4.8}$$

Now, take $e \in X \setminus \mathcal{C}_m^-$, so that for each $u \in \mathcal{C}_m^-$ and $t > 0$, by (H_6) we have

$$\begin{aligned}\Phi(u+te) &= \frac{1}{2p}\|u+te\|^p - \frac{\lambda+1}{p}\int\limits_\Omega |u+te|^p dx - \int\limits_\Omega F(x, u+te) dx \\ &\leq \frac{1}{2p}\|u+te\|^p\left(1 - p\int\limits_\Omega \frac{F(x,u+te)}{|u+te|^p}\frac{|u+te|^p}{\|u+te\|^p}dx\right) \to -\infty,\end{aligned}$$

as $t \to +\infty$. Thus, for every $u \in \mathcal{C}_m^-$ with $\|u\| = 1$, there is r_u such that $\Phi(tu) \le 0$ for all $t > r_u$. On the other hand, being $\mathcal{C}_m^- \cap S_1$[2)] a compact set, in which all norms are equivalent, it is easy to prove that r_u depends continuously on u, so that there exists $r_- > r_+$ such that

$$\Phi(u + te) < 0 \tag{4.9}$$

for all $u \in \mathcal{C}_m^- \cap S_1$ and all $t \ge r_-$.

Now, choose

$$\begin{aligned}
D_- &= \{u \in \mathcal{C}_m^- : \|u\| \le r_-\},\\
S_+ &= \{u \in \mathcal{C}_m^+ : \|u\| = r_+\},\\
Q &= \{u + te : u \in \mathcal{C}_m^-, t > 0, \|u + te\| \le r_-\},\\
H &= \{u + te : u \in \mathcal{C}_m^-, t > 0, \|u + te\| = r_-\}.
\end{aligned}$$

By the definitions of $\mathcal{C}_m^-$ and $\mathcal{C}_m^+$, it follows from Lemma 2.9 that

$$i(\mathcal{C}_m^- \setminus \{0\}) = i(X \setminus \mathcal{C}_m^+) = m.$$

By Corollary 2.7, we know that point (d_3) of Theorem 2.6 holds, namely, $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $m+1$ over $\mathbb{Z}_2$. In particular, $(Q, D_- \cup H)$ links S_+ thanks to Proposition 2.8. Moreover, by (4.8) and (4.9), together with the fact that Q is compact, one has

$$\sup_{D_- \cup H} \Phi < \inf_{S_+} \Phi, \quad \sup_Q \Phi < +\infty.$$

Furthermore, by Step 1, the $(C)_c$ condition holds. Setting $D = Q$, $S = D_- \cup H$, $B = \emptyset$, $A = S_+$, by Theorem 2.3, we have that Φ has a critical value $c \ge \alpha$. Hence, Φ has a nontrivial critical point u_* such that $\Phi(u_*) > 0$.

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[2)] As usual, we have set $S_1 = \{u \in X : \|u\| = 1\}$.

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