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# SPREADING IN CLAW-FREE CUBIC GRAPHS

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**Abstract.** Let $p \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{\infty\}$. We study a dynamic coloring of the vertices of a graph $G$ that starts with an initial subset $S$ of blue vertices, with all remaining vertices colored white. If a white vertex $v$ has at least $p$ blue neighbors and at least one of these blue neighbors of $v$ has at most $q$ white neighbors, then by the spreading color change rule the vertex $v$ is recolored blue. The initial set $S$ of blue vertices is a $(p,q)$-spreading set for $G$ if by repeatedly applying the spreading color change rule all the vertices of $G$ are eventually colored blue. The $(p,q)$-spreading set is a generalization of the well-studied concepts of $k$-forcing and $r$-percolating sets in graphs. For $q \geq 2$, a $(1,q)$-spreading set is exactly a $q$-forcing set, and the $(1,1)$-spreading set is a 1-forcing set (also called a zero forcing set), while for $q = \infty$, a $(p,\infty)$-spreading set is exactly a $p$-percolating set. The $(p,q)$-spreading number, $\sigma_{(p,q)}(G)$, of $G$ is the minimum cardinality of a $(p,q)$-spreading set. In this paper, we study $(p,q)$-spreading in claw-free cubic graphs. While the zero-forcing number of claw-free cubic graphs was studied earlier, for each pair of values $p$ and $q$ that are not both 1 we either determine the $(p,q)$-spreading number of a claw-free cubic graph $G$ or show that $\sigma_{(p,q)}(G)$ attains one of two possible values.

**Keywords:** bootstrap percolation, zero forcing set, $k$-forcing set, spreading.

**Mathematics Subject Classification:** 05C35, 05C75.

## 1. INTRODUCTION

In this paper we continue the study of dynamic graph colorings and explore the concept of $(p,q)$-spreading in claw-free cubic graph. The concept of spreading in graphs is a generalization of the well-studied concepts of $k$-forcing and $r$-percolating sets in graphs.

Consider a dynamic coloring of the vertices of a graph $G$ that starts with an initial subset $S$ of blue vertices, with all remaining vertices colored white. For $k \in \mathbb{N}$, the color change rule in $k$-forcing is defined as follows: *if a blue vertex $u$ has at most $k$*

 

*white neighbors, then all white neighbors of $u$ are recolored blue.* In particular, when $k = 1$, this is the color change rule for zero forcing in graphs. The initial set $S$ of blue vertices is a $k$-forcing set for $G$ if by repeatedly applying the color change rule in $k$-forcing all the vertices of $G$ are eventually colored blue. The *$k$-forcing number* of $G$, denoted $F_k(G)$, is the minimum cardinality among all $k$-forcing set of $G$. When $k = 1$, the $k$-forcing number of $G$ is called the *zero forcing number* and is denoted by $Z(G)$, and so $Z(G) = F_1(G)$.

The concept of bootstrap percolation has a similar definition to that of zero forcing, yet a different motivation. It was introduced as a simplified model of a magnetic system in 1979 [10], and was later studied on random graphs and also on deterministic graphs. In particular, already some early papers considered bootstrap percolation in grids [7, 8].

As before, we consider a dynamic coloring of the vertices of a graph $G$ that starts with an initial subset $S$ of blue vertices, with all remaining vertices colored white. We refer to blue vertices as "infected" and white vertices as "uninfected". For $r \in \mathbb{N}$, the color change rule in $r$-percolation is defined as follows: *if a white (uninfected) vertex $u$ has at least $r$ blue (infected) neighbors, then the vertex $u$ is recolored blue.* The initial set $S$ of blue vertices is an *$r$-neighbor bootstrap percolating set*, or simply an *$r$-percolating set*, of $G$ if by repeatedly applying the color change rule in $r$-percolation all the vertices of $G$ are eventually colored blue. The *$r$-neighbor bootstrap percolation number*, or simply the *$r$-percolation number*, of $G$, denoted $m(G, r)$, is the minimum cardinality among all $r$-neighbor bootstrap percolating sets of $G$.

Motivated by the similarity of the definitions of the above two concepts, a common generalization of $k$-forcing and $r$-bootstrap percolation was introduced in [9]. Let $p \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{\infty\}$. As before, we consider a dynamic coloring of the vertices of a graph $G$ that starts with an initial subset $S$ of blue vertices, with all remaining vertices colored white. The color change rule in $(p, q)$-spreading is defined as follows: *if a white vertex $w$ has at least $p$ blue neighbors, and one of the blue neighbors of $w$ has at most $q$ white neighbors, then the vertex $w$ is recolored blue.* The initial set $S$ of blue vertices is a $(p, q)$-*spreading set* of $G$, if by repeatedly applying the $(p, q)$-spreading color change rule all the vertices of $G$ are eventually colored blue. The $(p, q)$-*spreading number*, denoted $\sigma_{(p,q)}(G)$, of a graph $G$ is the minimum cardinality among all $(p, q)$-spreading sets.

The $(p, q)$-spreading set is a generalization of the well-studied concepts of $k$-forcing and $r$-percolating sets in graphs. For $q \geq 2$, a $(1, q)$-spreading set is exactly a $q$-forcing set [3], and the $(1, 1)$-spreading set is a 1-forcing set (also called a zero forcing set [2, 21]), while if $q \geq \Delta(G)$ (including $q = \infty$), then a $(p, q)$-spreading set is exactly a $p$-percolating set. In the foundational paper [9], the complexity of the decision version of the $(p, q)$-spreading number was proved to be NP-complete for all $p$ and $q$, while efficient algorithms for determining these numbers were found in trees. In addition, for almost all values of $p$ and $q$, the $(p, q)$-spreading numbers of Cartesian grids $P_n \,\square\, P_m$ were established in [9].

The following trivial observation will be (at least implicitly) used several times in the paper.

**Observation 1.1.** *Let $G$ be a graph, $p \geq 2$ and $q \in \mathbb{N} \cup \{\infty\}$. If $P$ is a $(p,q)$-spreading set in $G$, then $P$ is also a $(p,q+1)$-spreading set in $G$ as well as a $(p-1,q)$-spreading set in $G$. In particular,*

$$\sigma_{(p,q)}(G) \geq \sigma_{(p,q+1)}(G) \text{ and } \sigma_{(p,q)}(G) \geq \sigma_{(p-1,q)}(G).$$

A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. A *cubic graph* (also called a 3-*regular graph*) is a graph in which every vertex has degree 3. In this paper we study spreading in claw-free cubic graphs.

### 1.1. GRAPH THEORY NOTATION AND TERMINOLOGY

For notation and graph theory terminology, we in general follow [17]. Specifically, let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. A *neighbor* of a vertex $v$ in $G$ is a vertex $u$ that is adjacent to $v$, that is, $uv \in E(G)$. The *open neighborhood* $N_G(v)$ of a vertex $v$ in $G$ is the set of neighbors of $v$, while the *closed neighborhood* of $v$ is the set $N_G[v] = \{v\} \cup N(v)$. We denote the *degree* of $v$ in $G$ by $\deg_G(v) = |N_G(v)|$, and $\Delta(G) = \max\{\deg_G(v) : v \in V(G)\}$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \cup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$.

For a set $S \subseteq V(G)$, the subgraph induced by $S$ is denoted by $G[S]$. Further, the subgraph of $G$ obtained from $G$ by deleting all vertices in $S$ and all edges incident with vertices in $S$ is denoted by $G - S$; that is, $G - S = G[V(G) \setminus S]$. If $S = \{v\}$, then we also denote $G - S$ simply by $G - v$. If $F$ is a graph, then an $F$-*component* of $G$ is a component isomorphic to $F$. We denote by $\alpha(G)$ the independence number of $G$, and so $\alpha(G)$ is the maximum cardinality among all independent sets of $G$. A *vertex cover* of $G$ is a set $S$ of vertices such that every edge of $G$ is incident with at least one vertex in $S$. The *vertex covering number* $\beta(G)$ (also denoted $\tau(G)$ in the literature), equals the minimum cardinality among all vertex covers of $G$. By the well-known Gallai theorem, $\alpha(G) + \beta(G) = n(G)$ holds in any graph $G$.

We denote the *path*, *cycle*, and *complete graph* on $n$ vertices by $P_n$, $C_n$, and $K_n$, respectively, and we denote the *complete bipartite graph* with partite sets of cardinality $n$ and $m$ by $K_{n,m}$. A *triangle* in $G$ is a subgraph isomorphic to $K_3$, whereas a *diamond* in $G$ is an induced subgraph of $G$ isomorphic to $K_4$ with one edge missing, denoted by $K_4 - e$. A graph is *diamond-free* if it does not contain $K_4 - e$ as an induced subgraph. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \ldots, k\}$.

### 1.2. MAIN RESULTS AND ORGANIZATION OF THE PAPER

Davila and Henning studied zero forcing in claw-free cubic graphs [11]. They established two upper bounds, which are summarized in the following result (for the definition of diamond-necklace $N_k$, see Section 2).

**Theorem 1.2** ([11])**.** *If $G \neq K_4$ is a connected, claw-free, cubic graph of order $n$, then the following properties hold:*

(a) $Z(G) \leq \alpha(G) + 1$,
(b) $Z(G) \leq \frac{n}{3} + 1$, *unless $G = N_2$ in which case $Z(G) = \frac{1}{3}(n + 4)$.*

It is proved in [11] that both upper bounds in Theorem 1.2 are sharp. He *et al.* [18] recently characterized the connected, claw-free cubic graphs $G$ for which $Z(G) = \alpha(G) + 1$. Notably, there are only three sporadic graphs that attain this value, namely $N_2$, $N_3$ and the Hamming graph $K_3 \Box K_2$. Since these graphs have $8, 12$ and 6 vertices, respectively, the following result immediately follows.

**Theorem 1.3** ([18]). *If $G \neq K_4$ is a connected, claw-free, cubic graph of order at least* 14*, then $Z(G) \leq \alpha(G)$.*

Note that $Z(G) = \sigma_{(1,1)}(G)$, and the mentioned result (Theorem 1.2(a)) is placed in the $(1,1)$-entry of Table 1.

From properties of connected, claw-free, cubic graphs $G$ which we discuss in Section 2, if $G \neq K_4$ then there is a unique partition of the vertex set $V(G)$ into subsets each of which induces either a triangle or a diamond. The number of these subsets, called units, in $G$ is denoted by $u(G)$. Table 1 summarizes the values of $(p,q)$-spreading numbers in claw-free cubic graphs $G \neq K_4$ for all possible $p$ and $q$ in terms of the parameters $\alpha(G)$, $\beta(G)$, $u(G)$ and $n(G)$. Besides the $(1,1)$-entry giving the mentioned upper bound on $\sigma_{(1,1)}(G)$ all other values in Table 1 are obtained in this paper. It is easy to see that $\sigma_{(1,2)}(G) = 2$ for any claw-free cubic graph $G$. Indeed, any adjacent pair of vertices in $G$ is a $(1,2)$-spreading set in $G$. The results $\sigma_{(1,q)}(G) = 1$ for any $q \geq 3$ and $\sigma_{(p,q)}(G) = n(G)$ for any $p \geq 4$ and any $q \in \mathbb{N} \cup \{\infty\}$ are trivial consequences of definitions.

**Table 1**

$(p,q)$-spreading numbers of claw-free cubic graphs $G$, $G \neq K_4$

| $p$ \ $q$ | 1 | 2 | $\geq 3$ |
|---|---|---|---|
| 1 | $\leq \alpha(G)(+1)$, $n \geq 14$ [Thm. 1.3] ([Thm. 1.2]) | 2 | 1 |
| 2 | $u(G)+1$ or $u(G)+2$ [Thm. 4.9] | $u(G)$ or $u(G)+1$ [Cor. 4.8] | $u(G)$ or $u(G)+1$ [Cor. 4.5] |
| 3 | $\beta(G)$ or $\beta(G)+1$ [Prop. 3.12] | $\beta(G)$ [Prop. 3.8] | $\beta(G)$ [Prop. 3.8] |
| $\geq 4$ | $n(G)$ | $n(G)$ | $n(G)$ |

The paper is organized as follows. In Section 3, we consider $(3,q)$-spreading numbers in claw-free cubic graphs; their values can be found in Line 3 of Table 1. (Note that 3-percolation coincides with $(3,q)$-spreading when $q \geq \Delta(G)$.) In Section 4, we deal with $(2,q)$-spreading numbers of claw-free cubic graphs; see Line 2 in Table 1. In the case of $(2,q)$-spreading, where $q \geq 3$, $\sigma_{(2,q)}(G) = u(G) + 1$ precisely when $G$ is a diamond necklace $N_k$, while in all other claw-free cubic graphs $\sigma_{(2,q)}(G) = u(G)$. Thus, the 2-percolation number of claw-free cubic graphs is fully determined. On the other hand, when $q \leq 2$ a claw-free cubic graph can attain two values: $\sigma_{(2,2)}(G) \in \{u(G), u(G)+1\}$, and $\sigma_{(2,1)}(G) \in \{u(G)+1, u(G)+2\}$, but we have not established exactly which claw-free cubic graphs attain which of the two possible values. We conclude the paper with some open problems that arise from this study.

## 2. CLAW-FREE CUBIC GRAPHS

The following property of connected, claw-free, cubic graphs is established in [20].

**Lemma 2.1** ([20]). *If $G \neq K_4$ is a connected, claw-free, cubic graph of order $n$, then the vertex set $V(G)$ can be uniquely partitioned into sets each of which induces a triangle or a diamond in $G$.*

By Lemma 2.1, the vertex set $V(G)$ of connected, claw-free, cubic graph $G \neq K_4$ can be uniquely partitioned into sets each of which induce a triangle or a diamond in $G$. Following the notation introduced in [20], we refer to such a partition as a *triangle-diamond partition* of $G$, abbreviated $\Delta$-D-partition. We call every triangle and diamond induced by a set in our $\Delta$-D-partition a *unit* of the partition. A unit that is a triangle is called a *triangle-unit* and a unit that is a diamond is called a *diamond-unit*. (We note that a triangle-unit is a triangle that does not belong to a diamond.) Two units in the $\Delta$-D-partition are *adjacent* if there is an edge joining a vertex in one unit to a vertex in the other unit. In what follows, we define several structures in claw-free, cubic graphs that we will need when proving our main results. Some of these definitions have already appeared in the literature, see, for example, [4, 5, 20].

For $k \geq 2$ an integer, let $N_k$ be the connected cubic graph constructed as follows. Take $k$ disjoint copies $D_1, D_2, \ldots, D_k$ of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where $a_ib_i$ is the missing edge in $D_i$. Let $N_k$ be obtained from the disjoint union of these $k$ diamonds by adding the edges $\{a_ib_{i+1} : i \in [k-1]\}$ and adding the edge $a_kb_1$. We call $N_k$ a *diamond-necklace* with $k$ diamonds. Let $\mathcal{N}_{\text{cubic}} = \{N_k \mid k \geq 2\}$. The diamond-necklace, $N_4$, with four diamonds is illustrated in Figure 1.

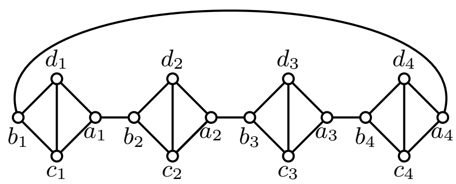


**Fig. 1.** The diamond-necklace $N_4$

For $k \geq 1$ an integer, let $F_{2k}$ be the connected cubic graph constructed as follows. Take $2k$ disjoint copies $T_1, T_2, \ldots, T_{2k}$ of a triangle, where $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2k]$. Let

$$E_a = \{x_{2i-1}x_{2i} : i \in [k]\},$$
$$E_b = \{y_{2i-1}y_{2i} : i \in [k]\},$$
$$E_c = \{z_{2i}z_{2i+1} : i \in [k]\},$$

where addition is taken modulo $2k$ (and so, $z_1 = z_{2k+1}$). Let $F_{2k}$ be obtained from the disjoint union of these $2k$ triangles by adding the edges $E_a \cup E_b \cup E_c$. The resulting graph $F_{2k}$ we call a *triangle-necklace* with $2k$ triangles. Let $\mathcal{T}_{\text{cubic}} = \{F_{2k} \colon k \geq 1\}$. The triangle-necklace $F_6$ is shown in Figure 2.

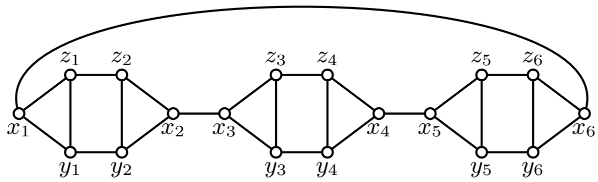


**Fig. 2.** The triangle-necklace $F_6$

For $k \geq 2$ an integer, take $2k$ disjoint copies $T_1, T_2, \ldots, T_{2k}$ of a triangle, where $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2k]$, and take $k$ disjoint copies $D_1, D_2, \ldots, D_k$ of a diamond, where $V(D_j) = \{a_j, b_j, c_j, d_j\}$ and where $a_j b_j$ is the missing edge in $D_j$ for $j \in [k]$. Let

$$\begin{aligned}
E_1 &= \{x_{2i-1}a_i \colon i \in [k]\},\\
E_2 &= \{x_{2i}b_i \colon i \in [k]\},\\
E_3 &= \{y_{2i-1}z_{2i+1} \colon i \in [k-1]\} \cup \{y_{2k-1}z_1\},\\
E_4 &= \{y_{2i}z_{2i+2} \colon i \in [k-1]\} \cup \{y_{2k}z_2\}.
\end{aligned}$$

Let $H_{2k}$ be obtained from the disjoint union of these $2k$ triangles and $k$ diamonds by adding the edges $E_1 \cup E_2 \cup E_3 \cup E_4$. The resulting graph $H_{2k}$ we call a *triangle-diamond-necklace* with $k$ diamonds. Let $\mathcal{H}_{\text{cubic}} = \{H_{2k} \colon k \geq 2\}$. The triangle-diamond-necklace $H_6$ is shown in Figure 3.

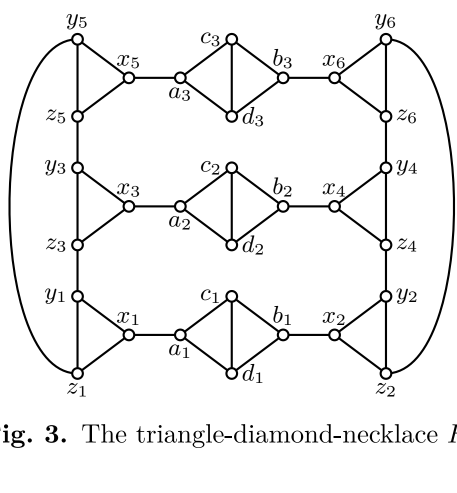


**Fig. 3.** The triangle-diamond-necklace $H_6$

## 3. 3-PERCOLATION IN CLAW-FREE, CUBIC GRAPHS

In this section, we study 3-percolation in claw-free, cubic graphs. The following lemma is already known in the literature and follows readily from the definition of $r$-bootstrap percolation.

**Lemma 3.1** ([19])**.** *For $r \geq 2$ if $H$ is a subgraph of a graph $G$ such that every vertex in $H$ has strictly less than $r$ neighbors in $G$ that belong to $V(G) \setminus V(H)$, then every $r$-percolating set of $G$ contains at least one vertex of $H$.*

As a consequence of Lemma 3.1, we establish a relationship between the $r$-percolation number and the independence number of an $r$-regular graph. Recall that $\beta(G)$ is the cardinality of a minimum vertex cover of $G$.

**Proposition 3.2.** *Let $G$ be an $r$-regular graph of order $n$, where $r \geq 2$. A set $S \subset V(G)$ is an $r$-percolating set in $G$ if and only if $S$ is a vertex cover in $G$. In particular, $m(G, r) = \beta(G) = n - \alpha(G)$.*

*Proof.* Let $S$ be an $r$-percolating set of an $r$-regular graph $G$. Suppose that $G - S$ contains an edge $uv$. Let $H = G[\{u, v\}]$, and so $H = K_2$ is a subgraph of $G$ such that every vertex in $H$ has strictly less than $r$ neighbors in $G$ that belong to $V(G) \setminus V(H)$. By Lemma 3.1, we infer that every $r$-percolating set of $G$ contains at least one vertex of $H$, which is a contradiction with our assumption. Thus, $S$ is a vertex cover of $G$. Conversely, suppose that $S$ is a vertex cover of $G$, and let $I = V(G) \setminus S$. Since the set $I$ is an independent set of $G$, every vertex in $I$ has all its $r$ neighbors in the set $V(G) \setminus I$. The set $S$ is therefore an $r$-percolating set of $G$. Since $m(G, r)$ is the cardinality of a minimum $r$-percolating set, we infer that $m(G, r) = \beta(G)$. Consequently, by the famous Gallai theorem ($\alpha(G) + \beta(G) = n(G)$ in any graph $G$) we also have $m(G, r) = n - \alpha(G)$. □

Let us recall the statement of the well-known Brooks' Coloring Theorem.

**Theorem 3.3** (Brooks' Coloring Theorem)**.** *If $G$ is a connected graph, which is not a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

As a consequence of Theorem 3.3, we have the following trivial lower bound on the independence number of a graph.

**Theorem 3.4.** *If $G \neq K_n$ is a connected graph of order $n$ with maximum degree $\Delta \geq 3$, then $\alpha(G) \geq \frac{n}{\Delta}$.*

*Proof.* As a consequence of Theorem 3.3, if $G \neq K_n$ is a connected graph of order $n$ with maximum degree $\Delta \geq 3$, then the chromatic number of $G$ is at most $\Delta$, that is, $\chi(G) \leq \Delta$. Alternatively, viewing a $\Delta$-coloring of $G$ as a partitioning of its vertices into $\Delta$ independent sets, called color classes, we infer by the Pigeonhole Principle that $G$ contains an independent set of cardinality at least $n/\Delta$, implying that $\alpha(G) \geq n/\Delta$. □

The following upper bound on the independence number of a claw-free graph is given by several authors.

**Theorem 3.5** ([14, 23]). *If $G$ is a claw-free graph of order $n$ with minimum degree $\delta$, then*

$$\alpha(G) \le \left(\frac{2}{\delta+2}\right) n.$$

As a consequence of Theorems 3.4 and 3.5, we have the following bounds on the independence number of a claw-free graph.

**Theorem 3.6.** *If $G \neq K_n$ is a connected, claw-free graph of order $n$ with minimum degree $\delta$ and maximum degree $\Delta \ge 3$, then*

$$\frac{n}{\Delta} \le \alpha(G) \le \left(\frac{2}{\delta+2}\right) n.$$

As a consequence of Proposition 3.2 and Theorem 3.6, we have the following result.

**Theorem 3.7.** *If $G \neq K_4$ is a connected, claw-free, cubic graph of order $n$, then the following properties hold.*

(a) $\frac{1}{3}n \le \alpha(G) \le \frac{2}{5}n$.

(b) $\frac{3}{5}n \le m(G,3) \le \frac{2}{3}n$.

We show next that the bounds of Theorem 3.7 are tight (in the sense that they hold for connected graphs of arbitrarily large orders). Suppose that $G \in \mathcal{T}_{\mathrm{cubic}}$. Thus, $G$ is a triangle-necklace $F_{2k}$ for some $k \ge 1$, and so $G$ has order $n = 6k$ and contains $2k$ vertex disjoint triangles. Moreover, every unit of the (unique) $\Delta$-D-partition of $G$ is a triangle-unit, implying that $\alpha(G) \le \frac{1}{3}n$. By Theorem 3.7(a), $\alpha(G) \ge \frac{1}{3}n$. Consequently, $\alpha(G) = \frac{1}{3}n$. For example, the white vertices in the triangle-necklace $F_6$ of order $n = 18$ shown in Figure 4 form an $\alpha$-set of $F_6$ (of cardinality $6 = \frac{1}{3}n$). By Proposition 3.2, we infer that $m(G,3) = \frac{2}{3}n$, and the corresponding set of shaded vertices in Figure 4 is a $\beta$-set of $G$. This shows that the lower bound of Theorem 3.7(a) is tight, as is the upper bound of Theorem 3.7(b).

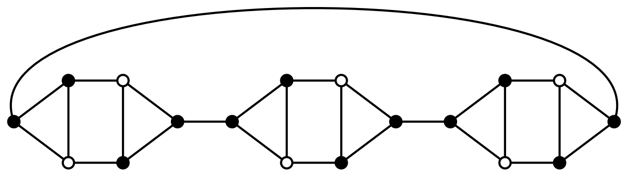

**Fig. 4.** A $\beta$-set in the triangle-necklace $F_6$

Suppose next that $G \in \mathcal{H}_{\mathrm{cubic}}$. Thus, $G$ is a triangle-diamond-necklace $H_{2k}$ for some $k \ge 2$, and so $G$ has order $n = 10k$. We can choose an independent set of $G$ to contain one vertex from every triangle-unit and two vertices from every diamond-unit, implying that $\alpha(G) \ge 4k = \frac{2}{5}n$. For example, the white vertices in the triangle-diamond-necklace $H_6$ of order $n = 30$ shown in Figure 5 form an $\alpha$-set of $H_6$ (of cardinality $12 = \frac{2}{5}n$). By Proposition 3.2, we infer that $m(G,3) = \frac{3}{5}n$, and the corresponding set of shaded vertices in Figure 5 is a $\beta$-set of $H_6$. This shows that the upper bound of Theorem 3.7(a) is tight, as is the lower bound of Theorem 3.7(b).

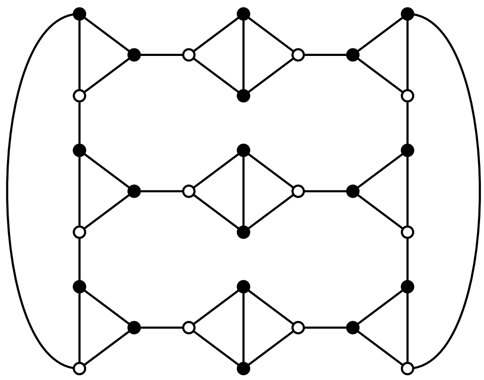

**Fig. 5.** A $\beta$-set in the triangle-diamond-necklace $H_6$

Note that $m(G,3)$ in (claw-free) cubic graphs coincides with $\sigma_{(3,q)}(G)$ for all $q \geq 3$, establishing the $(3,3)$-entry of Table 1. Hence, by Proposition 3.2, we have $\sigma_{(3,q)}(G) = m(G,3) = \beta(G) = n(G) - \alpha(G)$ for all $q \geq 3$. The result can be extended to $\sigma_{(3,2)}(G)$, which yields the following result.

**Proposition 3.8.** *If $G$ is a connected, claw-free cubic graph of order $n$, then $\sigma_{(3,q)}(G) = \beta(G)$ for every $q \geq 2$.*

*Proof.* We have already established that $\sigma_{(3,q)}(G) = \beta(G)$ for every $q \geq 3$, so it remains to resolve the case $q = 2$. Consider a 3-percolating set $P$ of a connected, claw-free cubic graph $G$, and color all its vertices blue. By Proposition 3.2, $P$ is a vertex cover and so $V(G) \setminus P$ is an independent set in $G$. Now, since an independent set in $G$ contains at most one vertex from each triangle, a vertex cover contains at least two vertices in each triangle. This implies that every vertex in $P$ has a neighbor in $P$. Therefore, since $\deg_G(x) = 3$ for all $x \in V(G)$, we infer that every (blue) vertex in $P$ has at most two (white) neighbors in $V(G) \setminus P$. Thus, $P$ is a $(3,2)$-spreading set. Since this is true for every 3-percolating set $P$ of $G$, we infer that $\sigma_{(3,2)}(G) \leq m(G,3) = \beta(G)$.

Conversely, by Observation 1.1, $\sigma_{(3,2)}(G) \geq \sigma_{(3,3)}(G)$. Combining the obtained inequalities, we infer

$$\beta(G) \geq \sigma_{(3,2)}(G) \geq \sigma_{(3,3)}(G) = m(G,3) = \beta(G). \qquad \square$$

We remark that $\sigma_{(3,1)}(G) > \beta(G)$ if $G \in \mathcal{H}_{\text{cubic}}$ is a triangle-diamond necklace. Indeed, in any vertex cover $P$ of $G$, every vertex in $P$ has a neighbor in $P$, and so it does not have at most one neighbor in $V(G) \setminus P$.

Recall that two units in the $\Delta$-D-partition are adjacent if there exists at least one edge joining a vertex in one unit to a vertex in the other unit. We say that a unit in the $\Delta$-D-partition of $G$ is *infected* if all vertices in the unit are infected.

**Lemma 3.9.** *If $G$ is a connected, claw-free cubic graph that contains only triangle-units, then there exists an independent set in $G$ that contains a vertex from every triangle of $G$.*

*Proof.* By supposition, every unit in the $\Delta$-D-partition of the connected, claw-free cubic $G$ is a triangle-unit. Since $G$ is diamond-free, we note that the triangle-units in $G$ correspond to the triangles in $G$. The graph $G$ has order $n = 3t$ where $t$ denotes the number of triangle-units in $G$. By Theorem 3.7(a), we have $t = \frac{1}{3}n \le \alpha(G)$. Since every independent set in $G$ contains at most one vertex from every triangle-unit in $G$, we infer that $\alpha(G) \le t = \frac{1}{3}n$. Consequently, $\alpha(G) = \frac{1}{3}n = t$ and every maximum independent set in $G$ contains a vertex from every triangle of $G$. □

**Lemma 3.10.** *If $G$ is a connected, claw-free cubic graph, then there exists an independent set in $G$ that contains a vertex from every triangle of $G$.*

*Proof.* If the connected, claw-free cubic $G$ that contains only triangle-units, then the result follows from Lemma 3.9. Hence, we may assume that $G$ contains at least one diamond-unit. We now construct an independent set $I$ of $G$ as follows. Initially, we set $I = \emptyset$. For each diamond-unit $D$ of $G$, we add to the set $I$ exactly one of the two vertices of degree 3 in the diamond-unit $D$. Thereafter, we delete all diamond-units from the graph $G$. If $G$ contains only diamond-units, then the resulting set $I$ has the desired property that it contains a vertex from every triangle of $G$. Hence, we may assume that $G$ contains at least one triangle-unit.

Let $G'$ be the graph obtained by deleting all diamond-units from $G$. We note that every component, $C$, of $G_1$ has the following properties: (1) the component $C$ contains no diamond, (2) every vertex in $C$ belongs to a triangle in the component $C$ and (3) the component $C$ has minimum degree 2 (and maximum degree at most 3). We now select an arbitrary vertex $u_1$ of degree 2 in $C$, add the vertex $u_1$ to the set $I$, and delete the triangle that contains the vertex $u_1$. In the resulting graph $G_2$, once again properties (1), (2) and (3) hold, and we select an arbitrary vertex $u_2$ of degree 2 in $G_2$, add the vertex $u_2$ to the set $I$, and delete from $G_2$ the triangle that contains the vertex $u_2$. Upon completion of this process, the resulting set $I$ is an independent set in $G$ that contains a vertex from every triangle of $G$. □

We note that the statement in Lemma 3.10 is equivalent to the following statement.

**Lemma 3.11.** *If $G$ is a connected, claw-free cubic graph, then there exists a vertex cover $P$ such that every triangle of $G$ contains exactly two vertices from $P$.*

**Proposition 3.12.** *If $G$ is a connected, claw-free cubic graph, then*

$$\sigma_{(3,1)}(G) \le \beta(G) + 1,$$

*and this bound is sharp.*

*Proof.* Let $G$ be a connected, claw-free cubic graph, and let $P$ be a 3-percolating set of $G$ satisfying $|P| = m(G,3) = \beta(G)$. We note that $P$ is a vertex cover of $G$ and $V(G) \setminus P$ is a maximum independent set of $G$. Clearly, every vertex, which is not in $P$, has three neighbors in $P$, and so the first condition of the $(3,1)$-spreading rule is always satisfied with respect to $P$. In addition, due to Lemma 3.11 we may assume that $P$ contains exactly two vertices from every triangle in $G$, or, equivalently, so that the independent set $V(G) \setminus P$ contains a vertex from every triangle in $G$. Now, consider an arbitrary triangle $T$ in $G$, and let $P^* = P \cup \{v\}$, where $v$ is the unique vertex in $V(T) \setminus P$. Let the vertices of $P^*$ be initially infected, and note that (with respect to $P^*$) the triangle $T$ is completely infected.

Let $u$ be a vertex that is not yet infected and is adjacent to an infected triangle $T'$ (with all three vertices of $T'$ infected). In particular, we note that $u \notin P^*$. Suppose firstly that $u$ is adjacent to two vertices of $T'$. Thus, the vertices in $V(T') \cup \{u\}$ induce a diamond-unit. In this case, the vertex $u$ immediately becomes infected, since both of its neighbors in $T'$ have no other uninfected neighbor. As a result, the (unique) triangle containing the vertex $u$ becomes an infected triangle, and so the diamond-unit containing $u$ is infected. Suppose next that $u$ is adjacent to exactly one vertex $w'$ of $T'$. Once again in this case, the vertex $u$ immediately becomes infected, since its neighbor $w'$ in $T'$ has no other uninfected neighbor. On the other hand, suppose that $u$ is infected and is adjacent to an infected triangle $T'$ (with all three vertices of $T'$ infected), and there exists an uninfected neighbor $v$ of $u$ that belongs to the same unit as $u$. Then $v$ becomes infected, because $u$ has only one uninfected neighbor, namely $v$.

In all of the above cases, the unit, which is adjacent to an infected unit, becomes infected. Since $G$ is connected, and initially there is a triangle infected (making the unit in which the triangle lies also infected), we infer that due to the above arguments all vertices in $G$ become infected. Thus, $\sigma_{(3,1)}(G) \le |P^*| = |P| + 1 = \beta(G) + 1$, thereby proving the desired upper bound. The case of a triangle-diamond necklace $G \in \mathcal{H}_{\mathrm{cubic}}$ shows that the bound is sharp. □

## 4. 2-PERCOLATION IN CLAW-FREE, CUBIC GRAPHS

In this section, we study 2-percolation in claw-free, cubic graphs. It is easy to see that $m(K_4, 2) = 2$. In what follows we therefore restrict our attention to 2-percolation in connected, claw-free, cubic graphs $G$ different from $K_4$. Thus, by Lemma 2.1, such a graph $G$ has a $\Delta$-D-partition (in which every unit is a triangle-unit or a diamond-unit). Recall that $u(G)$ is defined as the number of units in this (unique) $\Delta$-D-partition. In what follows, if $D$ is a diamond-unit in the $\Delta$-D-partition of $G$, then a *dominating vertex* in $D$ is a vertex that is adjacent to the three other vertices in the diamond-unit $D$.

**Proposition 4.1.** *If $G \in \mathcal{N}_{\mathrm{cubic}}$, then $m(G, 2) = u(G) + 1$.*

*Proof.* Suppose that $G \in \mathcal{N}_{\mathrm{cubic}}$, and so $G$ is a diamond-necklace $N_k$ for some $k \ge 2$. Thus, $G$ has $u(G) = k$ units, each of which is a diamond-unit. Let $S$ be a 2-percolating set of $G$ of minimum cardinality, and so $S$ is an 2-percolating set of $G$ satisfying $|S| = m(G, 2)$. By Lemma 3.1, we infer that the set $S$ contains at least one vertex from every diamond-unit of $G$. Moreover, if $D$ is a diamond-unit of $G$ and the set $S$ contains exactly one vertex from $D$, then by Lemma 3.1 we infer that such a vertex of $S$ is a dominating vertex of $D$. In particular, since $S$ contains at least one vertex from every unit in the $\Delta$-D-partition of $G$, we note that $m(G, 2) = |S| \ge u(G) = k$.

Suppose that $|S| = k$. By our earlier observations, the set $S$ contains exactly one vertex from every diamond-unit, namely a vertex from each diamond-unit that dominates that unit. The resulting set $S$ is a 2-packing in $G$; that is, $d_G(u, v) \ge 3$ for every two distinct vertices $u$ and $v$ that belong to $S$. Moreover in this case, letting $H = G - S$, we note that $H$ is a cycle $C_{3k}$ and every vertex in $H$ has exactly one neighbor that belongs to $V(G) \setminus V(H) = S$. Thus by Lemma 3.1, we infer that every 2-percolating set of $G$ contains at least one vertex of $H$. However, this contradicts our supposition that $S$ is a 2-percolating set of $G$ that contains no vertex of $H$. Hence, $m(G, 2) = |S| \geq k + 1$.

To establish an upper bound on $m(G, 2)$ in this case when $G = N_k$, if $S^*$ consists of a dominating vertex from $k - 1$ diamond-units and two dominating vertices from the remaining diamond-unit, then $S^*$ is a 2-percolating set of $G$ and $|S^*| = k + 1$. For example, if $G = N_4$ (here $k = 4$), then such a set $S^*$ illustrated in Figure 6 by the shaded vertices is a 2-percolating set of $G$ and $|S^*| = 5$.

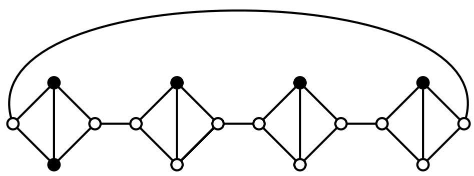

**Fig. 6.** The diamond-necklace $N_4$ and a 2-percolating set of size 5

This implies that $m(G, 2) \leq |S^*| = k + 1$. Consequently, $m(G, 2) = k + 1$. Thus in this case when $G \in \mathcal{N}_{\text{cubic}}$, we have shown that $m(G, 2) = u(G) + 1$. □

If two distinct units are joined by at least two edges, then we say that these two units are *double-bonded*.

**Lemma 4.2.** *If $G \neq K_4$ is a connected, claw-free, cubic graph and $G \notin \mathcal{N}_{\text{cubic}}$, then there exists a triangle-unit $T'$, such that the graph $G - T'$ has at most two components.*

*Proof.* Let $\mathcal{T}$ be the set of all triangle-units in $G$. Suppose to the contrary that for every $T \in \mathcal{T}$ the graph $G - T$ consists of three components. Let $T_1 \in \mathcal{T}$ be a triangle-unit such that at least one of the three components of the graph $G - T_1$ does not contain a triangle-unit. Denote this component by $G_1$ and let $D_1$ be the diamond-unit in $G_1$ adjacent to $T_1$. Then in $G_1$, $D_1$ is adjacent to exactly one diamond-unit $D_2$, which is in turn adjacent to another diamond-unit $D_3$ and so on, which yields a contradiction since $G$ is finite. Therefore, $G_1$ contains a triangle-unit $T' \notin \mathcal{T}$, which is the final contradiction. □

The following observation will be necessary for proving the subsequent theorem.

**Observation 4.3.** *Let $G \neq K_4$ be a connected, claw-free, cubic graph, and $U, V$ adjacent units in the $\Delta$-$D$-partition of $G$. Also, let $uv$ be an edge connecting units $U$ and $V$, where $u \in U$ and $v \in V$. If $u$ is infected and another vertex $v_1 \in V$, where $d(v_1, u) = 2$, is also infected, then the whole unit $V$ becomes infected.*

*Proof.* Vertex $v$ has two infected neighbors, namely $u$ and $v_1$. After that, if $V$ is a triangle unit, the remaining vertex of $V$ is also infected, and if $V$ is a diamond unit, the remaining two vertices of $V$ become infected after two steps. □

**Theorem 4.4.** *If $G \neq K_4$ is a connected, claw-free, cubic graph of order $n$ that contains $u(G)$ units, then*

$$m(G,2) = \begin{cases} u(G)+1, & \text{if } G \in \mathcal{N}_{\text{cubic}}, \\ u(G), & \text{otherwise.} \end{cases}$$

*Proof.* By Lemma 3.1, we infer that every 2-percolating set of $G$ contains at least one vertex from every triangle-unit and every diamond-unit of $G$. Moreover, if $D$ is a diamond-unit of $G$ and a 2-percolating set contains exactly one vertex from $D$, then by Lemma 3.1 we infer that such a vertex is a dominating vertex of $D$. In particular, since every 2-percolating set of $G$ contains at least one vertex from every unit in the $\Delta$-D-partition of $G$, we note that $m(G,2) \geq u(G)$. If $G \in \mathcal{N}_{\text{cubic}}$, then by Proposition 4.1, $m(G,2) = u(G)+1$. Hence, we may assume that $G \notin \mathcal{N}_{\text{cubic}}$, for otherwise the desired result follows.

Since every triangle-unit contributes 3 to the order of the graph and every diamond-unit contributes 4 to the order of the graph, we observe that if $G$ has order $n$ with $u_t$ triangle-units and $u_d$ diamond-units, then $u(G) = u_t + u_d$ and $n = 3u_t + 4u_d$. By assumption, $u_t \geq 1$. Thus, since $n$ is even, $u_t \geq 2$, that is, $G$ contains at least two triangle-units. According to Lemma 4.2, there exists a triangle-unit $T_1$ of $G$, where $V(T_1) = \{t_1, t_2, t_3\}$, such that $G - T_1$ consists of at most two components. This implies that at most one of the edges incident with exactly one vertex of $T_1$ is a bridge. Since $G$ is connected, the triangle-unit $T_1$ is adjacent to at least one other unit. Let $U_1$ be a unit that is adjacent to $T_1$ such that the edge between $U_1$ and $T_1$ is not a bridge, or $T_1$ and $U_1$ are double-bonded. Finally denote as $G_1$ the component of $G - T_1$ containing $U_1$.

As observed earlier, $m(G,2) \geq u(G)$. Hence, it suffices for us to show that $m(G,2) \leq u(G)$. For this purpose, we construct a 2-percolating set $S$ of $G$ that contains exactly one vertex from each unit of $G$. Initially, we let $S = \emptyset$. We consider three cases. First, suppose that the units $T_1$ and $U_1$ are not double-bonded.

*Case 1. The unit $U_1$ is a triangle-unit.* Let $V(U_1) = \{a_1, b_1, c_1\}$ and where $t_1c_1$ is an edge. In this case, we add to $S$ the vertices $t_1$ and $a_1$, and so $S = \{t_1, a_1\}$. The vertices in $S$ are indicated by the shaded vertices in Figure 7(a). Due to Observation 4.3, every vertex in the triangle-unit $U_1$ becomes infected.

*Case 2. The unit $U_1$ is a diamond-unit.* Let $V(U_1) = \{a_1, b_1, c_1, d_1\}$ and where $a_1b_1$ is the missing edge in $U_1$ and where $t_1b_1$ is an edge of $G$. In this case, we add to $S$ the vertices $t_1$ and $c_1$, and so $S = \{t_1, c_1\}$. The vertices in $S$ are indicated by the shaded vertices in Figure 7(b). Due to Observation 4.3, every vertex in the diamond-unit $U_1$ becomes infected.

*Case 3*. The units $T_1$ and $U_1$ are double-bonded. Thus, there is a vertex $u \in U_1$ that is joined to a vertex of $T_1$ different from $t_1$. Renaming the vertices $t_2$ and $t_3$ if necessary, we may assume that $ut_2$ is such an edge between the units $T_1$ and $U_1$. In particular, we note that $t_2 \notin S$ and let $S = \{t_1, u\}$. The vertices in $S$ are indicated by the shaded vertices in Figure 7(c). Again, due to Observation 4.3, every vertex in both units $T_1$ and $U_1$ becomes infected. In this case, the set $S$ consists of two vertices, one vertex from each of the units $U_1$ and $T_1$.

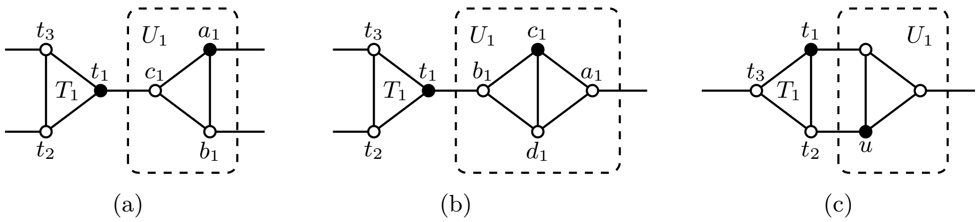


**Fig. 7.** Possible adjacent units in the proof of Theorem 4.4

Cases 1, 2 and 3 occur in Step 1 of our procedure to construct the set $S$. Thus after Step 1, the initial set $S$ consists of two vertices, namely a vertex $t_1$ from the triangle-unit $T_1$ and one vertex from the unit $U_1$ that is adjacent to $T_1$. Moreover in all three cases, every vertex in the unit $U_1$ becomes infected. We now proceed to Step 2 of our procedure to construct the set $S$.

Let $U_2$ be a unit different from $T_1$ that is adjacent to the infected unit $U_1$, and let $uv$ be an edge that joins a vertex $u \in U_1$ and a vertex $v \in U_2$. Note that if there is no such unit $U_2$, then $G$ consist only of units $T_1$ and $U_1$ and is now fully infected, yielding the desired result. Since $U_2 \neq T_1$, we note that $v \neq t_1$. In particular, we note that $v \notin S$ since at this stage of the construction the set $S$ only contains the vertex $t_1$ and one other vertex, namely a vertex from $U_1$.

Suppose that $U_2$ is a triangle-unit. Let $V(U_2) = \{v, v_1, v_2\}$. We now add to the set $S$ exactly one of the vertices $v_1$ and $v_2$. By symmetry, we may assume that $v_1$ is added to the set $S$. Once again, due to Observation 4.3, every vertex in the triangle-unit $U_2$ becomes infected.

Suppose next that $U_2$ is a diamond-unit. Let $V(U_2) = \{v, v_1, v_2, v_3\}$, where $v_1$ and $v_2$ are the dominating vertices of $U_2$ (and so, $vv_3$ is the missing edge in $D_v$). We now add to the set $S$ exactly one of the dominating vertices of $U_2$. By symmetry, we may assume that dominating vertex $v_1$ of $U_2$ is added to $S$. We again invoke Observation 4.3, thus every vertex in the diamond-unit $U_2$ becomes infected.

After Step 2 of our procedure to construct the set $S$, every vertex in the unit $U_2$ becomes infected, and the set $S$ contains exactly one vertex from each of the units $T_1, U_1$ and $U_2$.

Suppose that $i \geq 2$ and after Step $i$ of our procedure to construct the set $S$, the units $U_1, U_2, \ldots, U_i$ are all infected, and the set $S$ contains exactly one vertex from each of the units $U_1, U_2, \ldots, U_i$. Moreover, if $T_1 = U_j$ for some $j \in [i] \setminus \{1\}$, then after Step $i$ we have $|S| = i$ and the set $S$ contains exactly one vertex from each of the units $U_1, U_2, \ldots, U_i$, while if $T_1 \neq U_j$ for any $j \in [i]$, then $|S| = i + 1$ and $S$ contains one vertex from each of the units $U_1, U_2, \ldots, U_i, T_1$.

In Step $i+1$, let $U_{i+1}$ be a unit different the units $U_1, U_2, \ldots, U_i$ that is adjacent to an infected unit $U_j$ for some $j \in [i]$, and let $uv$ be an edge that joins a vertex $u \in U_j$ and a vertex $v \in U_{i+1}$.

Suppose that $U_{i+1}$ is a triangle-unit. Let $V(U_{i+1}) = \{v, v_1, v_2\}$. In this case, either $U_{i+1} = T_1$ and renaming vertices of $T_1$ if necessary we may assume that $v_1 = t_1$, or $U_{i+1} \neq T_1$, and we add to the set $S$ the vertex $v_1$. Suppose next that $U_{i+1}$ is a diamond-unit. Let $V(U_{i+1}) = \{v, v_1, v_2, v_3\}$, where $v_1$ and $v_2$ are the dominating vertices of $U_{i+1}$ (and so, $vv_3$ is the missing edge in $D_v$). We now add to the set $S$ the dominating vertex $v_1$ of $U_{i+1}$. In either of these cases the vertices $u$ and $v_1$ satisfy the conditions of Observation 4.3, therefore the whole unit $U_{i+1}$ becomes infected.

Continuing in the described way, the above process continues until every unit of $G_1$ and also the unit $T_1$ become infected. If $G - T_1$ is connected, then all vertices of $G$ have become infected and $S$ contains exactly one vertex from each unit of $G$, thus the proof is complete. Otherwise, if $G - T_1$ is not connected, then it has at most two components, and let $G_2$ be the component of $G - T_1$ different from $G_1$. Let $U_1'$ be the unit in $G_2$ that is adjacent to $T_1$, and let $v_2a$ be the bridge of $G$ connecting $T_1$ with $U_1'$. Now, adding a dominating vertex $b$ of $U_1'$, different from $a$, we infer that all vertices of $U_1'$ become infected (using the analogous arguments as in the previous cases). Now, we can continue with the same process, continuing by infecting the units $U_i'$ in $G_2$ that are adjacent to an already infected unit by adding to $S$ appropriately selected dominating vertex of $U_i'$. Since $G_2$ is connected, we infer that all vertices of $G_2$ become infected, where we put in $S$ exactly one vertex of each unit. Thus, $S$ is a 2-percolating set of $G$, implying that $m(G, 2) \leq u(G)$. As observed earlier, $m(G, 2) \geq u(G)$. Consequently, $m(G, 2) = u(G)$. This completes the proof of Theorem 4.4. □

By definition of a $(p, q)$-spreading set in a graph, if $G$ is cubic graph, then a set is a $(2, 3)$-spreading set if and only if it is 2-percolating set, implying that $\sigma_{(2,3)}(G) = m(G, 2)$. As an immediate consequence of Theorem 4.4, we infer the following result.

**Corollary 4.5.** *If $G \neq K_4$ is a connected, claw-free, cubic graph of order $n$ that contains $u(G)$ units, then*

$$\sigma_{(2,3)}(G) = \begin{cases} u(G) + 1, & \text{if } G \in \mathcal{N}_{\text{cubic}}, \\ u(G), & \text{otherwise.} \end{cases}$$

We are now in a position to establish the following relationships between the $(2, 2)$-spreading number and the $(2, 3)$-spreading number of a connected, claw-free, cubic graph.

**Proposition 4.6.** *If $G = K_4$ or if $G \in \mathcal{N}_{\text{cubic}}$, then $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G)$.*

*Proof.* If $G = K_4$, then $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G) = 2$. Hence, we may assume that $G \neq K_4$. Suppose that $G \in \mathcal{N}_{\text{cubic}}$. By Corollary 4.5, we have $\sigma_{(2,3)}(G) = u(G) + 1$. Moreover, as shown in the proof of Proposition 4.1, if $G = N_k$ for some $k \geq 2$, then we can choose the $(2, 3)$-spreading set $S$ of $G$ to consists of a dominating vertex from $k - 1$

diamond-units and two dominating vertices from the remaining diamond-unit in $G$ (as illustrated in Figure 6 in the case when $k = 4$). However, such a set $S$ is also a $(2,2)$-spreading set $S$ of $G$, and so $\sigma_{(2,2)}(G) \leq |S| = \sigma_{(2,3)}(G)$. By Observation 1.1, $\sigma_{(2,3)}(G) \leq \sigma_{(2,2)}(G)$. Consequently, if $G \in \mathcal{N}_{\mathrm{cubic}}$, then $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G)$. □

**Proposition 4.7.** *If $G$ is a connected, claw-free, cubic graph, then*

$$\sigma_{(2,2)}(G) \leq \sigma_{(2,3)}(G) + 1.$$

*Proof.* Let $G$ be a connected, claw-free, cubic graph. If $G = K_4$ or if $G \in \mathcal{N}_{\mathrm{cubic}}$, then by Proposition 4.6 we have $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G)$. Hence, we may assume that $G \neq K_4$ and $G \notin \mathcal{N}_{\mathrm{cubic}}$. To prove that $\sigma_{(2,2)}(G) \leq \sigma_{(2,3)}(G)+1$, let $S$ be a $(2,3)$-spreading set of $G$ as constructed in the proof of Theorem 4.4. Adopting the notation from the proof of Theorem 4.4, in Step 1 of our procedure to construct the set $S$ we add one additional vertex to the set $S$ as follows. If the unit $U_1$ is a triangle-unit (see Figure 7(a)), then we add the common neighbor of $t_1$ and $a_1$, namely the vertex $c_1$, to the set $S$. If the unit $U_1$ is a diamond-unit (see Figure 7(b)), then we add the common neighbor of $t_1$ and $c_1$, namely the vertex $b_1$, to the set $S$. Let $S^*$ denote the resulting set upon completion of the construction of the set $S$, and so $|S^*| = |S| + 1 = \sigma_{(2,3)}(G) + 1$. With the addition of the vertex $c_1$ in Case 1 or the vertex $b_1$ in Case 2, the set $S^*$ is a $(2,2)$-spreading set $S$ of $G$, implying that $\sigma_{(2,2)}(G) \leq |S^*| = |S| + 1 = \sigma_{(2,3)}(G) + 1$. □

Every $(2,3)$-spreading set of $G$ is by definition also a $(2,2)$-spreading set of $G$, and so $\sigma_{(2,3)}(G) \leq \sigma_{(2,2)}(G)$. Hence, as a consequence of Proposition 4.7 and applying also Corollary 4.5, we have the following result.

**Corollary 4.8.** *If $G$ is a connected, claw-free, cubic graph, then*

$$\sigma_{(2,3)}(G) \leq \sigma_{(2,2)}(G) \leq \sigma_{(2,3)}(G) + 1.$$

*In addition,*

$$u(G) \leq \sigma_{(2,2)}(G) \leq u(G) + 1.$$

If $G = F_{2k} \in \mathcal{T}_{\mathrm{cubic}}$ then it is not difficult to see that $S = \{z_i, : i \in [2k-1]\} \cup \{x_{2k}\}$ is a $(2,2)$-spreading set. Notably, the process starts with the infection of vertex $x_1$, and then continues with $y_1, y_2$ and so on. Therefore, $\sigma_{(2,2)}(G) = u(G)$. On the other hand, if $G \in \mathcal{N}_{\mathrm{cubic}}$, then $\sigma_{(2,2)}(G) = u(G) + 1$. In both of these cases it holds that $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G)$, however this does not hold in general. For instance, see the graph $G$ in Figure 8 for which we have $\sigma_{(2,3)} = u(G)$, where the shaded vertices indicated in the figure form a $(2,3)$-spreading set of $G$. Yet, one can verify that $\sigma_{(2,2)} = u(G) + 1$.

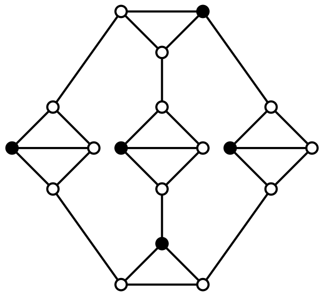

**Fig. 8.** A graph $G$ with $\sigma(2,3) = u(G)$ and $\sigma(2,2) = u(G) + 1$

Finally, we consider $(2,1)$-spreading, and bound the corresponding invariant in claw-free cubic graphs. Again we see that it can achieve only two possible values.

**Theorem 4.9.** *If $G \neq K_4$ is a connected, claw-free, cubic graph of order $n$ that contains $u(G)$ units, then*

$$u(G) + 1 \leq \sigma_{(2,1)}(G) \leq u(G) + 2.$$

*Proof.* First note that due to Lemma 3.1 a $(2,1)$-spreading set $S$ needs to contain at least one vertex in each unit of $G \neq K_4$. Now, if $S$ is a set of vertices that contains exactly one vertex from each unit of $G$, then every vertex $v \in S$ has at least two neighbors in $V(G) \setminus S$, and therefore such a set $S$ cannot be a $(2,1)$-spreading set. Hence, $u(G) + 1 \leq \sigma_{(2,1)}(G)$.

We show next that $\sigma_{(2,1)}(G) \leq u(G) + 2$ by constructing a $(2,1)$-spreading set $S$ of $G$ satisfying $|S| \leq u(G) + 2$. Let $T$ be an arbitrary triangle in $G$ (that may also be part of a diamond-unit). Initially, we let $S = V(T)$, and so the vertices in $T$ form the set of initially infected vertices. We now enlarge the set $S$ as follows. Suppose that $T$ belongs to a diamond-unit $D$. In this case, we let $v$ be the vertex in the diamond-unit $D$ that is not in $T$. The vertex $v$ immediately becomes infected in the $(2,1)$-spreading process, since both of its neighbors in $T$ have no other uninfected neighbor. As a result, all vertices in the diamond-unit $D$ become infected, that is, $D$ becomes an infected unit. Suppose next that $T$ is a triangle-unit of $G$. Then $T$ is already an infected unit.

Finally let $T'$ be the infected unit containing $T$ (that is, $T' = D$ or $T' = T$). We now extend the set $S$ in the same manner as in the proof of Theorem 4.4. Let $U$ be a unit adjacent to $T'$. If $U$ is a triangle-unit, then we add to $S$ a vertex in $U$ that is not adjacent to any vertex in the unit $T'$. If $U$ is a diamond-unit, then we add to $S$ a vertex in $U$ that is a dominating vertex of the unit $U$. Proceeding analogously as in the proof of Theorem 4.4, this results in all vertices of the unit $U$ becoming infected in the $(2,1)$-spreading process.

Continuing in this way by considering a vertex not yet infected that is adjacent to an infected triangle, we obtain a $(2,1)$-spreading set $S$ of $G$ starting with the set $V(T)$ and adding exactly one vertex from every unit of $G$ that does not contain the triangle $T$. Thus, $\sigma_{(2,1)}(G) \leq |S| = u(G) + 2$. $\square$

## 5. CONCLUDING REMARKS

We end this paper with some remarks concerning the computational complexity of determining $\sigma_{(p,q)}(G)$ in claw-free cubic graphs $G$. We have already mentioned that it is unclear whether one can determine the zero forcing number of a claw-free cubic graph efficiently. For some of the values of $p$ and $q$, where $p > 1$ or $q > 1$, there exists a polynomial-time algorithm to determine $\sigma_{(p,q)}(G)$. First, determining the independence number in claw-free graphs can be done in polynomial time (see [24], where a polynomial-time algorithm is presented even for the weighted version of the problem). Therefore, the problem of determining $\sigma_{(3,q)}(G)$ is polynomial in claw-free cubic graphs $G$ for all $q \geq 2$. Similarly, one can efficiently determine the number of units in a claw-free cubic graph. Thus, the problem of determining $\sigma_{(2,q)}(G)$ is also polynomial in claw-free cubic graphs $G$ for any $q \geq 3$.

The following problems remain open:

**Problem 5.1.** Provide a structural characterization of the connected claw-free cubic graphs $G$ for which $\sigma_{(2,2)} = u(G)$ (resp., $u(G)+1$). Is there a polynomial-time algorithm to recognize the corresponding classes of connected claw-free cubic graphs?

**Problem 5.2.** Provide a structural characterization of the connected claw-free cubic graphs $G$ for which $\sigma_{(2,1)} = u(G) + 1$ (resp., $u(G) + 2$). Is there a polynomial-time algorithm to recognize the corresponding classes of connected claw-free cubic graphs?

**Problem 5.3.** Provide a structural characterization of the connected claw-free cubic graphs $G$ for which $\sigma_{(3,1)} = \beta(G)$ (resp., $\beta(G) + 1$). Is there a polynomial-time algorithm to recognize the corresponding classes of connected claw-free cubic graphs?

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