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NONTRIVIAL SOLUTIONS OF DISCRETE KIRCHHOFF-TYPE PROBLEM VIA BIFURCATION THEORY

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Abstract. In this paper, we show that the bifurcation points for a discrete Kirchhoff-type problem with only local conditions, and we investigate the existence of positive and negative solutions for the problem when the nonlinear term f is asymptotically linear at zero and is asymptotically 3-linear at infinity. By using bifurcation techniques and the idea of taking limits of connected branches, under the assumption that f has some non-zero zeros, some results are also obtained.

Keywords: discrete Kirchhoff-type problem, nontrivial solution, bifurcation, superior limit.

Mathematics Subject Classification: 39A05, 39A28, 34B15.

1. INTRODUCTION

In the current paper, we will establish the existence of at least two constant-sign solutions for the discrete Kirchhoff-type problem with Dirichlet boundary condition

$$\begin{cases} -\left(a + b\sum_{t=1}^{T} |\Delta u(t)|^2\right) \Delta^2 u(t-1) = \lambda f(t, u(t)), & t \in [1, T]_Z, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where $a, b > 0, \lambda \neq 0$ is a parameter, $T \geq 3$ is a fixed positive integer, Z is the set of integers, for $i, j \in Z$ with $i < j$, $[i, j]_Z$ denotes the discrete interval $\{i, i+1, \ldots, j\}$, $\Delta u(t-1) := u(t) - u(t-1)$ is the forward difference operator, $f \in C([1, T]_Z \times \mathbb{R}, \mathbb{R})$ and there exist $f_0, f_\infty \in (0, +\infty)$ such that

$$f_0 = \lim_{|s| \to 0^+} \frac{f(t, s)}{s}, \quad f_\infty = \lim_{|s| \to +\infty} \frac{f(t, s)}{s^3}$$

uniformly with respect to $t \in [1, T]_Z$.

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Problem (1.1) can be regarded as the discrete counterpart of the following Kirchhoff-type problem

$$\begin{cases} -\left(a + b\int_0^1 |u'|^2 dx\right) u'' = \lambda f(x,u), \ x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

which is the stationary case of a nonlinear wave equation

$$u_{xx} - \left(a + b\int_0^1 |u'|^2 dx\right) u'' = \lambda f(x,u) \tag{1.3}$$

proposed by Kirchhoff [14] in 1883. (1.3) is an extension of the classical D'Alembert's wave equation, which takes into account the effects of string length variation during vibration. Due to its strong physical background, (1.2) has aroused great interest and has been extensively studied by various methods. For example, [19, 24] and [2, 5, 25] obtained nontrivial solutions via the Yang index, local linking theory and bifurcation theory, respectively. By variational methods, [6–8] gave existence of nontrivial solutions involving the $p(x)$-Laplacian. Recently, there have been very meaningful articles studying the existence, asymptotic behavior, uniqueness and other properties of non-local problems [4, 10, 21, 26].

We know that mathematical modeling involved in many fields, such as computer science, mechanical engineering, biological neural networks, control systems, etc., leads to the consideration of nonlinear discrete problems. When $b = 0$, there are many articles about the properties of solutions to the problem (1.1). In contrast, the research on nonlocal discrete problems is relatively few. In [23], thanks to variational methods and the computations of critical groups, Yang and Liu obtained the existence of nontrivial solutions for problem (1.1) depends on the local properties of f near zero and near infinity. Superlinear discrete Kirchhoff type equations involving functions with two discrete variables are investigated in [16, 17] by minimax methods and invariant sets of descending flow. Based on variational methods, Chakrone *et al.* [3] and Heidarkhani *et al.* [12] studied the existence and multiplicity solutions for a discrete boundary value problem of p-Kirchhoff type. We refer the reader to [11, 15, 27] for more information.

The results mentioned above do not take into account the effects on the existence of nontrivial solutions when the nonlinear terms have nonzero zeros. In addition, as far as we know, there are no studies on nonlocal discrete problems using bifurcation theory. Since the bifurcation theory can support us to obtain the results of global structure from local conditions, so it is very necessary to establish a bifurcation theory for discrete Kirchhoff problems. Due to the large differences between differential equation and difference equation, the results that are true in differential case may not be true in the difference case. Therefore, we will face some new challenges when establishing the bifurcation theory of discrete nonlocal problems.

We know from [13] that the linear eigenvalue problem

$$\begin{cases} -\Delta^2 u(t-1) = \lambda u(t), \quad t \in [1,T]_Z, \\ u(0) = u(T+1) = 0 \end{cases}$$

has T distinct eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_T$, where the eigenfunction ϕ_1 corresponding to λ_1 does not change its sign in $[1,T]_Z$. It can be seen from [17, p. 5] that the following problem

$$\begin{cases} -\left(\sum\limits_{t=1}^{T} |\Delta u(t)|^2\right) \Delta^2 u(t-1) = \mu u^3(t), & t \in [1,T]_Z, \\ u(0) = u(T+1) = 0 \end{cases}$$

has eigenvalues $0 < \mu_1 < \mu_2 \leq \ldots \leq \mu_T$, it is obvious that μ_1 is simple.

Let $\|\cdot\|$ denote the norm

$$\|u\| = \left(\sum_{t=1}^{T} |\Delta u(t)|^2\right)^{\frac{1}{2}}$$

of the Banach space

$$E := \{u : [0, T+1]_Z \to \mathbb{R} \mid u(0) = u(T+1) = 0\}.$$

In order to obtain the existence of constant-sign solutions by bifurcation technique, we must need to show the Dancer-type bifurcation theorem for the following discrete Kirchhoff-type problem

$$\begin{cases} -\left(a + b\|u\|^2\right) \Delta^2 u(t-1) = \lambda u(t) + g(t, u(t), \lambda), & t \in [1,T]_Z, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.4}$$

where $g \in C([1,T]_Z \times \mathbb{R}^2, \mathbb{R})$ satisfies

$$\lim_{|s| \to 0^+} \frac{g(t,s,\lambda)}{s} = 0$$

uniformly for all $t \in [1,T]_Z$ and in every bounded interval of λ. Since the equation contains a nonlocal norm, then classical bifurcation theorem cannot be used directly, and even some bifurcation theories are only applicable to difference equations. We benefit from the inspiration of [5], by means of a transformation, we can transform the problem (1.4) into the desired form. For the original Dancer-type bifurcation theorem (for linear operator), refer to [9, Theorem 2].

The following Dancer-type unilateral global bifurcation theorem of problem (1.4) is established as follows.

Theorem 1.1. $(a\lambda_1, 0)$ *is a bifurcation point of the positive solutions sets of problem* (1.4)*, and there exists two unbounded continuums in* $\mathbb{R} \times E$*,* $\mathcal{C}^+$ *and* $\mathcal{C}^-$*, bifurcating from* $(a\lambda_1, 0)$ *such that* $\mathcal{C}^\sigma \subseteq (\mathbb{R} \times \mathcal{P}^\sigma \cup \{(a\lambda_1, 0)\})$*, where* $\sigma = +$ *or* $-$.

Remark 1.2. We note that when $\lambda = 0$, (1.4) has only trivial solution. Therefore, $\mathcal{C}^+$ and $\mathcal{C}^-$ are separated by the hyperplane $\lambda = 0$.

As an application of Theorem 1.1, we give the existence of constant-sign solutions of the problem (1.1) for $sf(t,s) > 0$.

Theorem 1.3. *If $f_0, f_\infty \in (0,\infty)$, then problem* (1.1) *has at least one positive solution u^+ and one negative solution u^- for $\lambda \in \big(\min\{\frac{a\lambda_1}{f_0}, \frac{b\mu_1}{f_\infty}\}, \max\{\frac{a\lambda_1}{f_0}, \frac{b\mu_1}{f_\infty}\}\big)$.*

From the proof of Theorem 1.3, it is easy to get the following conclusion:

Corollary 1.4. *Suppose that there exists a positive constant M such that*

$$\frac{f(t,s)}{s} \geq M$$

for all $t \in [1,T]_Z$ and any $s \neq 0$. Then there exists $\lambda_* > 0$ such that problem (1.1) *has no positive or negative solution for any $\lambda \in (\lambda_*, +\infty)$.*

Further, we can provide more details about the connected components of constant-sign solutions under the assumptions that f has some nonzero zeros.

(G_1) There exist two functions $s_1, s_2 : [1,T]_Z \to (0,+\infty)$ with $s_1(t) \leq s_2(t)$ such that $f(t,0) = 0 = f(t,s_1(t)) = f(t,s_2(t))$, and $f(t,s) > 0$ for $s \in (0,s_1) \cup (s_2,+\infty)$, $f(t,s) < 0$ for $s \in (s_1,s_2)$.

(G_2) There exist two functions $s_3, s_4 : [1,T]_Z \to (-\infty,0)$ with $s_3(t) \leq s_4(t)$ such that $f(t,0) = 0 = f(t,s_3(t)) = f(t,s_4(t))$, and $f(t,s) < 0$ for $s \in (-\infty,s_3) \cup (s_4,0)$, $f(t,s) > 0$ for $s \in (s_3,s_4)$.

(G_3) There exist two constants $\alpha > 0$ and $\beta < 0$ such that

$$\lim_{s\to s_1^-} \frac{f(t,s)}{s_1 - s} = \alpha \quad \text{and} \quad \lim_{s\to s_4^+} \frac{f(t,s)}{s - s_4} = \beta$$

uniformly for $t \in [1,T]_Z$.

Now we are ready to give the existence conclusions of the signed solutions.

Theorem 1.5. *Assume that* (G_1)–(G_3) *hold.*

(1) *If $f_0 \in (0,\infty)$, then problem* (1.1) *has at least one positive solution u^+ and one negative solution u^- for $\lambda \in \big(\frac{a\lambda_1}{f_0}, +\infty\big)$ such that $\max_{t\in[1,T]_Z} u^+ < s_1$ and $\min_{t\in[1,T]_Z} u^- > s_4$. Further, there is $u^+ \to s_1^-$ and $u^- \to s_4^+$ as $\lambda \to +\infty$.*

(2) *If $f_0 = +\infty$, then problem* (1.1) *has at least one positive solution u^+ and one negative solution u^- for $\lambda \in \big(0,+\infty\big)$ such that $\max_{t\in[1,T]_Z} u^+ < s_1$ and $\min_{t\in[1,T]_Z} u^- > s_4$. Further, there is $u^+ \to s_1^-$ and $u^- \to s_4^+$ as $\lambda \to +\infty$.*

The present paper is built up as follows: In Section 2, we state some notations and preliminary results. The Dancer-type bifurcation theorem is developed in Section 3. We establish the existence of positive and negative solutions for problem (1.1) with $sf(t,s) > 0$ in Section 4. Finally in Section 5, let us consider that f has nonzero zeros, we will give detailed results of the global structure of the constant-sign solutions for problem (1.1).

2. SOME PRELIMINARY

Firstly, we give the relevant contents of the limit of connected branches.

Definition 2.1 ([22])**.** Let X be a Banach space, $\{C_n : n = 1, 2, 3, \ldots\}$ be a family of subsets of X. Then the limit superior D of C_n is defined by

$$D := \limsup_{n\to\infty} C_n = \{x \in X : \exists n_i \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \to x\}.$$

Definition 2.2 ([22])**.** Assume that X is a compact metric space, A and B are non-intersecting closed subsets of X, and no component of X interests both A and B. Then there exist two disjoint compact subsets X_A and X_B, such that $X = X_A \cup X_B$, $A \subset X_A$, $B \subset X_B$.

Lemma 2.3 ([18])**.** *Suppose that:*

(i) *there exists* $z_n \in C_n (n = 1, 2, \ldots)$ *and* $z^* \in X$*, such that* $z_n \to z^*$*,*
(ii) $\lim\limits_{n\to\infty} r_n = \infty$*, where* $r_n = \sup\{\|x\| : x \in C_n\}$*,*
(iii) *for every* $R > 0$*,* $(\bigcup\limits_{n=1}^{\infty} C_n) \cap \Omega_R$ *is a relative compact set of* X*, where* $\Omega_R = \{x \in X : \|x\| \le R\}$.

Then $D := \limsup\limits_{n\to\infty} C_n$ *contains an unbounded component* C *such that* $z^* \in C$*.*

Let us denote

$$\mathcal{P}^+ = \{u \in E : u > 0 \text{ in } t \in [1, T]_Z\}$$

and let $\mathcal{P}^- = -\mathcal{P}^+$ and $\mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}^-$.

We give the definitions of simple zeros and simple generalized zeros in the sense of difference.

Definition 2.4 ([13])**.** Let $u : [0, T+1]_Z \to \mathbb{R}$. If $u(t_0) = 0$, then t_0 is a zero of u. If $u(t_0) = 0$ and $u(t_0 - 1)u(t_0 + 1) < 0$ for some $t_0 \in [0, T+1]_Z$, then t_0 is a simple zero of u. If $u(t_0)u(t_0 + 1) < 0$ for some $t_0 \in [0, T]_Z$, then we say that u has a node at the points $t_* = \frac{t_0 u_{t_0+1} - (t_0+1) u_{t_0}}{u_{t_0} u_{t_0+1}} \in (t_0, t_0 + 1)$. The simple zeros and nodes of u are called the simple generalized zeros of u.

Lemma 2.5. *Let* (λ, u) *be a solution of* (1.4)*. If there exists* $t_0 \in [1, T]_Z$ *such that either*

$$u(t_0) = 0, \Delta u(t_0) = 0, \quad \text{or} \quad u(t_0) = 0, u(t_0 - 1)u(t_0 + 1) \ge 0,$$

then $u \equiv 0$ *in* $[0, T+1]_Z$*.*

Proof. Suppose $u(t_0) = 0, \Delta u(t_0) = 0$. In view of (1.4), we obtain that

$$(a + b\|u\|^2)\,[\Delta u(t_0 - 1) - \Delta u(t_0)] = \lambda u(t_0) + g(t, u(t_0), \lambda).$$

Since $\lim\limits_{|s|\to 0^+} \frac{g(t,s,\lambda)}{s} = 0$, combining this fact with assumptions, a direct computation shows that $(a + b\|u\|^2)(u(t_0) - u(t_0 - 1)) = 0$. Hence, $u(t_0 - 1) = 0$.

On the other hand, consider the equation

$$-(a+b\|u\|^2)\left[\Delta u(t_0+1)-\Delta u(t_0)\right]=\lambda u(t_0+1)+g(t,u(t_0+1),\lambda).$$

It is easy to deduce that $u(t_0+2)=0$. Therefore,

$$\Delta u(t_0-1)=\Delta u(t_0+1)=0.$$

Step by step, it follows that $u\equiv 0$.

Suppose $u(t_0)=0, u(t_0-1)u(t_0+1)\geq 0$. The following conclusion is found by calculation

$$-(a+b\|u\|^2)[u(t_0+1)+u(t_0-1)]=0.$$

Since $u(t_0-1)u(t_0+1)\geq 0$, this yields $u(t_0-1)=u(t_0+1)=0$. According to the above analysis, we conclude that $u\equiv 0$. □

Remark 2.6. If u satisfies the assumptions in Lemma 2.5, we also say that such u as $u\in\partial\mathcal{P}$.

3. UNILATERAL GLOBAL BIFURCATION

We introduce the auxiliary problem

$$\begin{cases}-\Delta^2 u(t-1)=h(t), & t\in[1,T]_Z,\\ u(0)=u(T+1)=0.\end{cases}\tag{3.1}$$

Let $S(h)$ denote the unique solution of problem (3.1), we know that $S:E\to E$ is a linear continuous operator. Further, the restriction of S to E is a completely continuous operator.

Consider problem (1.4), we first establish the following bifurcation theorem:

Theorem 3.1. $(a\lambda_1,0)$ *is a bifurcation point of the positive solutions sets of problem* (1.4)*, and there exists two continuums* $\mathcal{C}^+$ *and* $\mathcal{C}^-$*, bifurcating from* $(a\lambda_1,0)$ *such that either* $\mathcal{C}^+$ *and* $\mathcal{C}^-$ *are both unbounded or* $\mathcal{C}^+\cap\mathcal{C}^-\neq\{(a\lambda_1,0)\}$.

Proof. It is obvious that (λ,u) is a solution of (1.4) if and only if (λ,u) satisfies

$$u=S\left((a+b\|u\|^2)^{-1}(\lambda u+g(t,u,\lambda))\right).\tag{3.2}$$

Denote $Lu=a^{-1}S(u)$ and

$$G(\lambda,u)=(a+b\|u\|^2)^{-1}S\left(g(t,u,\lambda)\right)-\lambda b\|u\|^2[a(a+b\|u\|^2)]^{-1}S(u).$$

It is straightforward to see that $L\colon E\to E$ is linear completely continuous, $G:\mathbb{R}\times E\to E$ is compact. Since $a\lambda_1$ is the simple principle eigenvalue of L. Hence, (3.2) is equivalent to

$$u=\lambda Lu+G(\lambda,u).$$

Let

$$\overline{g}(t,u,\lambda) = \max_{0\le|s|\le u} \left|g(t,s,\lambda)\right| \quad \text{for any } t \in [1,T]_Z$$

and in every bounded interval of λ. It is clear that $\overline{g}$ is non-decreasing with respect to u and

$$\lim_{u\to 0^+} \frac{\overline{g}(t,u,\lambda)}{u} = 0 \tag{3.3}$$

uniformly for $t \in [1,T]_Z$ and in every bounded interval of λ. Moreover, (3.3) yields

$$\frac{|g(t,u,\lambda)|}{\|u\|} \le \frac{\overline{g}(t,|u|,\lambda)}{\|u\|} \le \frac{\overline{g}(t,\|u\|,\lambda)}{\|u\|} \to 0 \quad \text{as } u \to 0 \tag{3.4}$$

uniformly for $t \in [1,T]_Z$ and in every bounded interval of λ. This implies that $G = o(\|u\|)$ uniformly for $u \to 0$ and every λ in bounded intervals. According to the Theorem 2 of [9], we have the desired bifurcation results. □

Theorem 3.2. *If $\mathcal{C}^\sigma \subseteq (\mathbb{R}\times\mathcal{P}^\sigma \cup \{(a\lambda_1,0)\})$, then $\mathcal{C}^\sigma \cap (\mathbb{R}\times\{0\}) = (a\lambda_1,0)$, where $\sigma = +$ or $-$. In other words, the second case in Theorem 3.1 does not occur.*

Proof. Let $(\rho_n,u_n) \in \mathcal{C}^\sigma$ and $u_n \in \mathcal{P}^\sigma$ with $u_n \not\equiv 0$, We argue by contradiction and suppose $(\rho_n,u_n) \to (a\lambda_k,0)$ as $n\to+\infty$ with $k \ne 1$. Set $w_n = \frac{u_n}{\|u_n\|}$, then w_n satisfies

$$w_n = S\left((a+b\|u_n\|^2)^{-1}(\rho_n w_n + \frac{g(t,u_n,\rho_n)}{\|u_n\|})\right). \tag{3.5}$$

Combining (3.4), (3.5) and the compactness of S, it is easy to see that $w_n \to \overline{w}$ as $n\to+\infty$. Further, $\overline{w}$ satisfies the equation

$$-\Delta^2 w(t-1) = \lambda_k w(t)$$

and $\|\overline{w}\| = 1$. Therefore, $\overline{w}$ must change-sign in $[1,T]_Z$. Further, as a consequence for some n large enough, w_n must change-sign in $[1,T]_Z$, which provides a contradiction. □

Finally, we establish the global bifurcation phenomenon of constant-sign for problem (1.4).

Proof of Theorem 1.1. Following [1], we can find a neighborhood $\mathfrak{O}$ of $(a\lambda_1, 0)$ such that

$$(\mathfrak{O} \cap \mathcal{C}^\sigma) \subseteq (\mathbb{R} \times \mathcal{P}^\sigma \cup \{(a\lambda_1, 0)) \quad \text{or} \quad (\mathfrak{O} \cap \mathcal{C}^\sigma) \subseteq \left(\mathbb{R} \times \mathcal{P}^{-\sigma} \cup \{(a\lambda_1, 0)\right).$$

We only prove the case of $(\mathfrak{O} \cap \mathcal{C}^\sigma) \subseteq (\mathbb{R} \times \mathcal{P}^\sigma \cup \{(a\lambda_1, 0))$.

We claim that $\mathcal{C}^\sigma \subseteq (\mathbb{R} \times \mathcal{P}^\sigma \cup \{(a\lambda_1, 0))$. Suppose on the contrary that $\mathcal{C}^\sigma \nsubseteq (\mathbb{R} \times \mathcal{P}^\sigma \cup \{(a\lambda_1, 0))$. Taking a sequence $(\rho_i, u_i) \in \mathcal{C}^\sigma \cap (\mathbb{R} \times \mathcal{P}^\sigma)$, thus there exists $(\lambda, u) \in \mathcal{C}^\sigma \cap (\mathbb{R} \times \partial\mathcal{P}^\sigma)$ such that $(\lambda, u) \neq (a\lambda_1, 0)$ with $(\rho_i, u_i) \to (\rho, u)$. Since $u \in \partial\mathcal{P}^\sigma$, we note that $u \equiv 0$ follows directly from Lemma 2.5 and Remark 2.6. Set $\chi_i = \frac{u_i}{\|u_i\|}$, then χ_i satisfies

$$\chi_i = S\left((a + b\|u_i\|^2)^{-1}(\rho_i \chi_i + \frac{g(t, u_i, \rho_i)}{\|u_i\|})\right). \tag{3.6}$$

Combining (3.4), (3.6) and the compactness of S, it is easy to see that $\chi_i \to \overline{\chi}$ as $i \to +\infty$. Further, $\overline{\chi}$ satisfies the equation

$$-\Delta^2 \chi(t-1) = \frac{\rho}{a}\chi(t)$$

and $\|\overline{\chi}\| = 1$. Therefore, $\rho = a\lambda_k$ for $k \neq 1$. So, $(\rho_i, u_i) \to (a\lambda_k, 0)$ with $(\rho_i, u_i) \in \mathcal{C}^\sigma \cap (\mathbb{R} \times \mathcal{P}^\sigma)$. This provides a contradiction to Theorem 3.2. Hence, we conclude that $\mathcal{C}^\sigma \subseteq (\mathbb{R} \times \mathcal{P}^\sigma \cup \{(a\lambda_1, 0))$.

Next we prove that $\mathcal{C}^\sigma$ is unbounded, where $\sigma = +$ or $-$. We also argue by contradiction and suppose there exists $(\lambda', u') \in \mathcal{C}^+ \cap \mathcal{C}^-$ such that $(\lambda', u') \neq (a\lambda_1, 0)$ and $u' \in \mathcal{P}^+ \cap \mathcal{P}^-$. This provides a contradiction to the definition of $\mathcal{P}^\sigma$. □

4. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. Let $\xi \in C([1, T]_Z \times \mathbb{R}^+, \mathbb{R}^+)$ be such that

$$f(t, s) = f_0 s + \xi(t, s).$$

Obviously,

$$\lim_{|s| \to 0^+} \frac{\xi(t, s)}{s} = 0 \quad \text{and} \quad \lim_{|s| \to +\infty} \frac{\xi(t, s)}{s^3} = f_\infty$$

uniformly for $t \in [1, T]_Z$.

Applying Theorem 1.1, we obtain that there exist two unbounded continuums $\mathcal{C}^+$ and $\mathcal{C}^-$, bifurcating from $(\frac{a\lambda_1}{f_0}, 0)$ such that $\mathcal{C}^\sigma \subseteq \left(\mathbb{R} \times \mathcal{P}^\sigma \cup \{(\frac{a\lambda_1}{f_0}, 0)\}\right)$, where $\sigma = +$ or $-$.

Taking a sequence $(\rho_m, u_m) \in \mathcal{C}^\sigma$ such that

$$\rho_m + \|u_m\| \to +\infty$$

with $u_m \neq 0$. We know that $(0,0)$ is the only solution of (1.1) for $\lambda = 0$, thus $\mathcal{C}^\sigma \cap (\{0\} \times E) = \emptyset$. This implies that $\rho_m > 0$ for any $m \in \mathbb{N}$. We will show that there exists a positive constant K such that $\rho_m \in (0, K]$ for $m \in \mathbb{N}$ large enough. If $\mathcal{C}^\sigma$ is unbounded in λ axis, then $\lim\limits_{m\to+\infty} \rho_m = +\infty$. $(\rho_m, u_m) \in \mathcal{C}^\sigma$ implies that

$$-\Delta^2 u_m(t-1) = \frac{\rho_m}{a + b\|u_m\|^2} h(t, u_m) u_m(t),$$

where

$$h(t, u_m) = \begin{cases} \frac{f(t,u_m)}{u_m}, & u_m(t) \neq 0, \\ f_0, & u_m(t) = 0. \end{cases}$$

Obviously, $\frac{\rho_m}{a+b\|u_m\|^2} > 0$. Combining the assumptions $sf(t,s) > 0$ with $f_0, f_\infty \in (0, \infty)$, we deduce that there exists a positive constant M such that $h(t, u_m) \geq M$ for all $t \in [1, T]_Z$ and all $s \neq 0$ and all $m \in \mathbb{N}$. Hence, we observe that $\frac{\rho_m}{a+b\|u_m\|^2} h(t, u_m) > \lambda_1$ for m large enough and all $t \in [1, T]_Z$. By virtue of the Theorem 2 of [20], we conclude that u_m must change its sign in $[1, T]_Z$. This contradicts $(\rho_m, u_m) \in \mathcal{C}^\sigma$.

Therefore, $\mathcal{C}^\sigma$ is bounded in λ axis, we will prove that $\mathcal{C}^\sigma$ meets $(\frac{a\lambda_1}{f_0}, 0)$ to $(\frac{b\mu_1}{f_\infty}, \infty)$. Hence, one has

$$\|u_m\| \to +\infty \quad \text{as } n \to +\infty.$$

Let $\eta \in C([1, T]_Z \times \mathbb{R}^+, \mathbb{R}^+)$ be such that

$$f(t, s) = f_\infty s^3 + \eta(t, s).$$

Obviously,

$$\lim_{|s|\to 0} \frac{\eta(t,s)}{s} = f_0 \quad \text{and} \quad \lim_{|s|\to\infty} \frac{\eta(t,s)}{s^3} = 0$$

uniformly for $t \in [1, T]_Z$. Hence, u_m is a solution of the problem

$$u_m = S\left(\rho_m (a + b\|u_m\|^2)^{-1} (f_\infty u_m^3 + \eta(t, u_m))\right). \tag{4.1}$$

Set $v_m = \frac{u_m}{\|u_m\|}$, then v_m satisfies

$$v_m = S\left(\frac{\rho_m \|u_m\|^2}{a + b\|u_m\|^2} (f_\infty v_m^3 + \frac{\eta(t, u_m)}{\|u_m\|^3})\right). \tag{4.2}$$

Let $\widehat{\eta}(t, u) = \max\limits_{0 \leq s \leq u} |\eta(t, s)|$ for $t \in [1, T]_Z$. It is clear that $\widehat{\eta}$ is non-decreasing with respect to u. Let us define

$$\widetilde{\eta}(t, u) = \max_{\frac{u}{2} \leq s \leq u} |\eta(t, s)| \quad \text{for } t \in [1, T]_Z.$$

We observe that

$$\lim_{|u|\to+\infty} \frac{\widetilde{\eta}(t,u)}{u^3} = 0, \quad \widehat{\eta}(t, u) \leq \widehat{\eta}\left(t, \frac{u}{2}\right) + \widetilde{\eta}(t, u).$$

Further,

$$\limsup_{|u|\to+\infty}\frac{\widehat{\eta}(t,u)}{u^3}\le\limsup_{|u|\to+\infty}\frac{\widehat{\eta}(t,\frac{u}{2})}{u^3}\le\limsup_{\frac{|u|}{2}\to+\infty}\frac{\widehat{\eta}(t,\frac{u}{2})}{u^3}.$$

Hence, we deduce that

$$\lim_{|u|\to+\infty}\frac{\widehat{\eta}(t,u)}{u^3}=0$$

uniformly for $t\in[1,T]_Z$. Thanks to the above result, we conclude that

$$\frac{\eta(t,u_m)}{\|u_m\|^3}\le\frac{\widehat{\eta}(t,|u_m|)}{\|u_m\|^3}\le\frac{\widehat{\eta}(t,\|u_m\|)}{\|u_m\|^3}\to 0\quad\text{as } m\to+\infty$$

uniformly for $t\in[1,T]_Z$.

The compactness of S ensures that $v_m\to v$ and $v\in E$, and v verifies the equation

$$-\|v\|^2\Delta^2 v(t-1)=\frac{\rho}{b}f_\infty v(t)^3,$$

where $\rho=\lim_{m\to+\infty}\rho_m$. Again choosing a subsequence and relabeling it if necessary, we deduce that $\|v\|=1$ and $v\in\mathcal{C}^\sigma$. A direct computation show that $\rho=\frac{b\mu_1}{f_\infty}$.

Consequently, $\mathcal{C}^\sigma$ meets $(\frac{a\lambda_1}{f_0},0)$ to $(\frac{b\mu_1}{f_\infty},\infty)$. □

5. EXISTENCE OF CONSTANT-SIGN SOLUTIONS FOR f HAS SOME NON-ZERO ZEROS

To complete the proof of the main results, the next lemma is crucial.

Lemma 5.1. *If there exists $t_0\in[1,T]_Z$ such that $u(t_0)>0$ and*

$$-\left(a+b\|u\|^2\right)\Delta^2 u(t-1)+\iota u(t)\ge 0$$

for any $t\in[1,T]_Z$ and $\iota>0$, then $u(t)>0$ in $[1,T]_Z$.

Proof. Let $j\in[1,T]_Z$ be such that

$$u(j)=\min\{u(t):t\in[1,T]_Z\}.$$

If $u(j)>0$, thus it is clear that $u(t)>0$ for all $t\in[1,T]_Z$, and the proof is complete.

If $u(j)\le 0$, thus

$$u(j)=\min\{u(t):t\in[0,T+1]_Z\}.$$

It is obvious that $\Delta u(j-1)=u(j)-u(j-1)\le 0$ and $\Delta u(j)=u(j+1)-u(j)\ge 0$. Since $\iota>0$, by simply calculating, we have

$$-\left(a+b\|u\|^2\right)\Delta^2 u(j-1)+\iota u(j)=-\left(a+b\|u\|^2\right)[\Delta u(j)-\Delta u(j-1)]+\iota u(j)\le 0.$$

If the strict inequality holds, we immediately get a contradiction. If

$$-\left(a+b\|u\|^2\right)[\Delta u(j)-\Delta u(j-1)]+\iota u(j)=0,$$

we deduce that $u(j)=u(j-1)=u(j+1)=0$. Replacing j by $j-1$, we have $u(j-2)=u(j-1)$. Replacing $j+1$ by j, one has $u(j+2)=u(j+1)$. Repeating this process, we conclude that $u(j)=u(j+1)=\ldots=u(T+1)=u(0)=0$, this provides a contradiction to $u(t_0)>0$ for some $t_0\in[1,T]_Z$.

To summarize, we obtain $u(t)>0$ in $[1,T]_Z$. □

Similar to the above proof, we have the following corollary:

Corollary 5.2. *If there exists $t_0\in[1,T]_Z$ such that $u(t_0)<0$ and*

$$-\left(a+b\|u\|^2\right)\Delta^2u(t-1)+\kappa u(t)\le 0$$

for any $t\in[1,T]_Z$ and $\kappa>0$, then $u(t)<0$ in $[1,T]_Z$.

Here again, if $f_0\in(0,\infty)$, consider the following problem

$$\begin{cases}-\left(a+b\|u\|^2\right)\Delta^2u(t-1)=\lambda f_0u(t)+\lambda\xi(t,u), & t\in[1,T]_Z,\\ u(0)=u(T+1)=0,\end{cases}\tag{5.1}$$

as a bifurcation problem from the trivial solution $u\equiv 0$, where $\lim_{|s|\to 0^+}\frac{\xi(t,s)}{s}=0$. Clearly, (5.1) is equivalent to the problem (1.1).

We know that there exists a continuum $\mathcal{C}^\sigma$ of solutions of (5.1) meeting $(\frac{a\lambda_1}{f_0},0)$ to infinity. Moreover, $\mathcal{C}^\sigma\setminus\{(\frac{a\lambda_1}{f_0},0)\}\subset\mathbb{R}\times\mathcal{P}^\sigma$.

The following lemma shows the range of the signed solutions under the assumptions (G_1)–(G_3).

Lemma 5.3. *Let (G_1)–(G_3) hold and $f_0\in(0,+\infty)$. Then for $(\lambda,u)\in\mathcal{C}^+\cup\mathcal{C}^-$, we have*

$$s_4(t)<u(t)<s_1(t),\quad t\in[1,T]_Z.$$

Proof. We argue by contradiction and suppose there exist $(\lambda,u)\in\mathcal{C}^+\cup\mathcal{C}^-$ and $t_1,t_2\in[1,T]_Z$ such that

$$\max_{t\in[1,T]_Z}u(t)=s_1(t_1)\quad\text{or}\quad\min_{t\in[1,T]_Z}u(t)=s_4(t_2).$$

If $\max_{t\in[1,T]_Z}u(t)=s_1(t_1)$, next we show that there exists a constant $c_1>0$ such that $f(t,u)\le c_1(s_1(t)-u)$ for any $0\le u\le s_1(t)$. We see from the assumption (G_1) that this is correct for $u=0$ or $u=s_1(t)$. For any $\epsilon_1\in(0,\alpha)$, thanks to (G_3), and we deduce that there is $\tau_1>0$ such that

$$f(t,u)<(\alpha+\epsilon_1)(s_1(t)-u)$$

holds for all $s\in(s_1-\tau_1,s_1)$. It follows from (G_1) that

$$\max_{u\in[0,s_1-\tau_1]}\frac{f(t,u)}{s_1(t)-u}:=\gamma_1>0.$$

Taking $c_1=\max\{\gamma_1,\alpha+\epsilon_1\}$, this is a desired conclusion.

Now, we focus the nonlinear problem

$$\begin{cases} -\left(a+b\|s_1-u\|^2\right)\Delta^2(s_1-u)(t-1)+\lambda c_1(s_1(t)-u(t)) \\ \geq \lambda c_1(s_1(t)-u(t))-\lambda f(t,u(t)), \quad t\in[1,T]_Z, \\ s_1(0)-u(0)>0, \quad s_1(T+1)-u(T+1)>0, \end{cases}$$

Therefore, we have that

$$\begin{cases} -\left(a+b\|s_1-u\|^2\right)\Delta^2(s_1-u)(t-1)+\lambda c_1(s_1(t)-u(t))\geq 0, \quad t\in[1,T]_Z, \\ s_1(0)-u(0)>0, \quad s_1(T+1)-u(T+1)>0. \end{cases}$$

We note that the Lemma 5.1 ensures that $s_1(t)>u(t)$ for $t\in[1,T]_Z$. This is absurd.

If $\min\limits_{t\in[1,T]_Z} u(t)=s_4(t_2)$, we prove that there exists a constant $c_2>0$ such that $f(t,u)\geq c_2(s_4(t)-u)$ for any $s_4(t)\leq u\leq 0$. We see from the assumption (G_1) that this is correct for $u=0$ or $u=s_4(t)$. For any $\epsilon_2\in(0,-\beta)$, thanks to (G_3), and we deduce that there is $\tau_2>0$ such that

$$f(t,u)>(\epsilon_2-\beta)(s_4(t)-u)$$

holds for all $s\in(s_4,s_4+\tau_2)$. It follows from (G_1) that

$$\min_{u\in[s_4+\tau_2,0]}\frac{f(t,u)}{s_4(t)-u}:=\gamma_2>0.$$

Taking $c_2=\max\{\gamma_2,\epsilon_2-\beta\}$, this is a desired conclusion.

Now, we focus the nonlinear problem

$$\begin{cases} -\left(a+b\|s_4-u\|^2\right)\Delta^2(s_4-u)(t-1)+\lambda c_2(s_4(t)-u(t)) \\ \leq \lambda c_1(s_4(t)-u(t))-\lambda f(t,u(t)), \quad t\in[1,T]_Z, \\ s_4(0)-u(0)<0, \quad s_4(T+1)-u(T+1)<0, \end{cases}$$

Therefore, we have that

$$\begin{cases} -\left(a+b\|s_4-u\|^2\right)\Delta^2(s_4-u)(t-1)+\lambda c_2(s_4(t)-u(t))\leq 0, \quad t\in[1,T]_Z, \\ s_4(0)-u(0)<0, \quad s_4(T+1)-u(T+1)<0. \end{cases}$$

We note that the Corollary 5.2 ensures that $s_4(t)<u(t)$ for $t\in[1,T]_Z$. This is impossible. □

Remark 5.4. Although we give the range of signed solutions. In fact, if there exist sign-changing solutions, we can obtain the same range of sign-changing solutions under the conditions (G_1)–(G_3).

Proof of Theorem 1.5. (1) From the arguments of Theorem 1.3, the existence of u^+ and u^- are obtained immediately from the global structures of $\mathcal{C}^+$ and $\mathcal{C}^-$.

Now, we prove that $u^+ \to s_1^-$ as $\lambda \to +\infty$. We also argue by contradiction and suppose that there exists $\psi \in (0, s_1)$ such that $u^+ \leq \psi$ for λ large enough. In view of the condition (G_1), one concludes that there exists a constant $\varrho > 0$ such that

$$\frac{f(t, u^+)}{u^+} \geq \varrho$$

uniformly for $t \in [1, T]_Z$. It is straightforward to see from Corollary 1.4 that $u^+ \equiv 0$ for $\lambda \to +\infty$, this is absurd. The proof of $u^- \to s_4^+$ as $\lambda \to +\infty$ is similar.

(2) Define

$$f_n(t,s) = \begin{cases} ns, & s \in [-\frac{1}{n}, \frac{1}{n}], \\ \left(f(t, \frac{2}{n}) - 1\right)(ns - 1) + 1, & s \in (\frac{1}{n}, \frac{2}{n}), \\ -\left(f(t, -\frac{2}{n}) + 1\right)(ns + 1) - 1, & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ f(t,s), & s \in (-\infty, -\frac{2}{n}] \cup [\frac{2}{n}, +\infty). \end{cases}$$

Let us focus the problem

$$\begin{cases} -\left(a + b\|u\|^2\right)\Delta^2 u(t-1) = \lambda f_n(t, u(t)), & t \in [1, T]_Z, \\ u(0) = u(T+1) = 0. \end{cases} \tag{5.2}$$

We observe that $\lim\limits_{n \to +\infty} f_n(t,s) = f(t,s)$, $(f_n)_0 = n$. Then there exist two unbounded continuums $\mathcal{C}_n^+$ and $\mathcal{C}_n^-$ of constant-sign solutions of the problem (5.2) emanating from $(\frac{a\lambda_1}{n}, 0)$, such that they are disjoint, unbounded in the direction of λ, and

$$\mathcal{C}_n^+ \subseteq (\mathbb{R} \times \mathcal{P}^+) \cup \{(\frac{a\lambda_1}{n}, 0)\} \text{ and } \mathcal{C}_n^- \subseteq (\mathbb{R} \times \mathcal{P}^-) \cup \{(\frac{a\lambda_1}{n}, 0)\}.$$

Let us take $z_n = (\frac{a\lambda_1}{n}, 0)$ and $z^* = (0,0)$, it follows that $z_n \to z^*$. Thus, assumption (i) of the Lemma 2.3 is satisfied with $z^* = (0,0)$.

Denote $r_n = \sup\{\lambda + \|u\| : (\lambda, u) \in \mathcal{C}_n^\sigma\} \to \infty$, then (ii) of the Lemma 2.3 holds. (iii) can be concluded immediately from the Ascoli–Arzéla Theorem and the definition of f_n. Thanks to Lemma 2.3, we observe that $\limsup_{n \to +\infty} \mathcal{C}_n^\sigma$ contains unbounded connected components $\mathcal{C}^\sigma$ with

$$(0,0) \in \mathcal{C}^\sigma \subseteq \limsup_{n \to +\infty} \mathcal{C}_n^\sigma.$$

Naturally, we can easily get the existence of u^+ and u^-. The rest of proofs are the same as that of (1). □

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