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NOVEL OSCILLATION CRITERIA FOR THIRD-ORDER SEMI-CANONICAL DIFFERENTIAL EQUATIONS WITH AN ADVANCED NEUTRAL TERM

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Abstract. The main purpose of this paper is to present new oscillation results for nonlinear semi-canonical third-order differential equations with an advanced neutral term. The main idea is first by reducing the studied semi-canonical equation into standard canonical type equation without assuming any extra conditions. Then, by using the comparison method and integral averaging technique, sufficient conditions are established to ensure the oscillation of the reduced canonical equation, which in turn leads to the oscillation of the original equation. Therefore, the technique used here is very useful since the results already known for the canonical equations can be applied to obtain the oscillation of the semi-canonical equations. Two examples are provided to illustrate the importance of the main results.

Keywords: oscillation, third-order, neutral differential equations, semi-canonical.

Mathematics Subject Classification: 34C10, 34K11, 34K40.

1. INTRODUCTION

The aim of this paper is to investigate the oscillation property of the semi-canonical nonlinear third-order differential equation with an advanced neutral term

$$\Big(a_2(t)\,(a_1(t)z'(t))'\Big)' + f(t)g(x(\sigma(t))) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$ and the following conditions are assumed to hold without further mention:

(H1) $a_i \in C^{(3-i)}([t_0,\infty),\mathbb{R})$, $i = 1,2$, such that $a_1(t) > 0$, $a_2(t) > 0$, and

$$A_1(t_0) := \int_{t_0}^{\infty} \frac{1}{a_1(t)} dt = \infty \text{ and } A_2(t_0) := \int_{t_0}^{\infty} \frac{1}{a_2(t)} dt < \infty, \tag{1.2}$$

where condition (1.2) means that equation (1.1) is in semi-canonical form;

(H2) $\tau, \sigma \in C^1([t_0,\infty),\mathbb{R})$ such that $\tau(t) > t$, $\sigma(t) < t$, $\tau'(t) > 0$, $\sigma'(t) > 0$, and $\lim_{t\to\infty}\sigma(t) = \infty$;
(H3) $f, p \in C([t_0,\infty),[0,\infty))$ such that $p(t) \le p_0 < 1$, and $f(t) > 0$ for large t;
(H4) $g \in C([t_0,\infty),\mathbb{R})$ such that $xg(x) > 0$ for $x \neq 0$ and $g(x)/x^\alpha \ge k > 0$, where α is a ratio of positive odd integers.

Through the operators below

$$L_0z = z, \quad L_1z = a_1z', \quad L_2z = a_2\left(a_1z'\right)', \quad L_3z = \Big(a_2\left(a_1z'\right)'\Big)', \tag{1.3}$$

for $t \in [t_0,\infty)$ and $L_3z \in C([t_0,\infty),\mathbb{R})$; equation (1.1) can be rewritten as

$$L_3z(t) + f(t)g(x(\sigma(t))) = 0, \quad t \ge t_0 > 0. \tag{1.4}$$

By a solution of (1.1) (or (1.4)), we mean a function $x \in C([t_x,\infty),\mathbb{R})$ for some $t_x \ge t_0$ such that $L_0z \in C^1([t_x,\infty),\mathbb{R})$, $L_1z \in C^1([t_x,\infty),\mathbb{R})$, $L_2z \in C^1([t_x,\infty),\mathbb{R})$ and x satisfies (1.1) (or (1.4)) on $[t_x,\infty)$. We exclude from our consideration those solutions of (1.1) (or (1.4)) which vanish identically in some neighborhood of infinity; and we tacitly assume that (1.1) (or (1.4)) possesses such solutions. Such a solution $x(t)$ of (1.1) (or (1.4)) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x,\infty)$, i.e., for any $t_1 \in [t_x,\infty)$ there exists $t_2 \ge t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1.1) (or (1.4)) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been increasing interest in studying the oscillation properties of third-order neutral functional differential equations of type (1.1) or its particular cases or its generalizations. From the review of literature, depending on various ranges $p(t)$, there are many results reported on the oscillatory behavior of equations of type (1.1) when it is in canonical form and semi-canonical form, that is,

$$A_2(t_0) = \infty \quad \text{and} \quad A_1(t_0) = \infty, \tag{1.5}$$

or

$$A_2(t_0) < \infty \quad \text{and} \quad A_1(t_0) = \infty, \tag{1.6}$$

or

$$A_2(t_0) = \infty \quad \text{and} \quad A_1(t_0) < \infty, \tag{1.7}$$

see, for example, [2–14, 18–22, 24–33, 35, 36] and the references cited therein. On the other hand in [6, 20, 24, 25], the authors studied the oscillation properties of (1.1) for the noncanonical case, that is,

$$A_2(t_0) < \infty \quad \text{and} \quad A_1(t_0) < \infty. \tag{1.8}$$

Recently in [31], the authors studied the existence of oscillatory solutions of (1.1) when $p(t) > 1$ for $t \ge t_0$; and in [10] the authors obtained oscillation criteria for (1.1) using the following conditions:

$$\sigma \circ \tau = \tau \circ \sigma, \tag{1.9}$$

and

$$p(t) \leq p_0 < 1, \tag{1.10}$$

and condition (1.6). The following theorem was provided in [10].

Theorem 1.1 ([10, Theorem 2.1])**.** *If there exists a positive real-valued differentiable function $m(t)$ such that, for any real number d,*

$$\frac{d}{a_1(t)} + \frac{m'(t)}{m(t)} < 0, \quad \inf_{t \geq t_0} \left\{ 1 - \frac{p(\tau(t))m(\tau(t))}{m(\sigma(\tau(t)))} \right\} > 0,$$

and

$$\int_{t_0}^{\infty} \frac{1}{a_2(t)} \int_{t_0}^{t} f(s) ds dt = \infty, \tag{1.11}$$

then every solution of (1.1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Note that, if we closely look at the proof of [10, Theorem 2.1], there is a mistake in the proof for eliminating the case (2) of [10, Lemma 1.1]. In the proof of [10, Theorem 2.1], they used $x(t) \geq z(t)$ to eliminate the case (2), which is not correct.

The objective of this paper is to obtain sufficient conditions for the oscillation of (1.1) without assumption (1.9). Moreover, the oscillation results are obtained by transforming the semi-canonical equation (1.1) into canonical form, which reduces the number of classes of positive solutions from 3 to 2. This is done without assuming extra conditions, and greatly simplifies the process of obtaining conditions for the oscillation of solutions. Therefore, the results presented here are new and improve some results appearing in the literature. Examples are given to show the importance and novelty over the known results.

2. MAIN RESULTS

We start with a theorem that transforms the semi-canonical equation (1.1) (or (1.4)) into a canonical equation. In order to make our notation more concise, we define the following expressions:

$$A_2(t) = \int_t^{\infty} \frac{1}{a_2(s)} ds \quad \text{and} \quad A_{12}(t) = \int_t^{\infty} \frac{A_2(s)}{a_1(s)} ds.$$

Theorem 2.1. *The semi-canonical operator $L_3 z$ can be written in the canonical form*

$$\Delta_3 z = \begin{cases} \frac{1}{A_2}\left(a_2 A_2^2 \left(\frac{a_1}{A_2} z'\right)'\right)', & \text{if } A_{12}(t_0) = \infty, \\ \frac{1}{A_2}\left(\frac{a_2 A_2^2}{A_{12}}\left(\frac{a_1}{A_2} A_{12}^2 \left(\frac{z}{A_{12}}\right)'\right)'\right)', & \text{if } A_{12}(t_0) < \infty. \end{cases} \tag{2.1}$$

Proof. The proof of the theorem can be found in [13, Theorem 2] and also in [28, Theorem 2.1], hence the details are omitted. □

Based on Theorem 2.1, we can rewrite equation (1.1) (or (1.4)) in the equivalent canonical form as follows:

$$\Delta_3 y(t) + F(t)g(x(\sigma(t))) = 0 \tag{2.2}$$

with

$$\int_{t_0}^{\infty} \frac{1}{b_1(t)} dt = \int_{t_0}^{\infty} \frac{1}{b_2(t)} dt = \infty, \tag{2.3}$$

where

$$y(t) = \begin{cases} z(t), & \text{if } A_{12}(t_0) = \infty, \\ \frac{z(t)}{A_{12}(t)}, & \text{if } A_{12}(t_0) < \infty, \end{cases} \tag{2.4}$$

$$\Delta_0 y = y, \quad \Delta_1 y = b_1 y', \quad \Delta_2 y = b_2(b_1 y')', \quad \Delta_3 y = (\Delta_2 y)',$$

$$b_1(t) = \begin{cases} \frac{a_1(t)}{A_2(t)}, & \text{if } A_{12}(t_0) = \infty, \\ a_1(t)\frac{A_{12}^2(t)}{A_2(t)}, & \text{if } A_{12}(t_0) < \infty, \end{cases}$$

$$b_2(t) = \begin{cases} a_2(t)A_2^2(t), & \text{if } A_{12}(t_0) = \infty, \\ a_2(t)\frac{A_2^2(t)}{A_{12}(t)}, & \text{if } A_{12}(t_0) < \infty, \end{cases}$$

and

$$F(t) = f(t)A_2(t), \quad \text{if } A_{12}(t_0) = \infty \quad \text{or} \quad A_{12}(t_0) < \infty.$$

Corollary 2.2. *The semi-canonical nonlinear neutral differential equation* (1.1) (*or* (1.4)) *possesses a solution $x(t)$ if and only if the canonical equation* (2.2) *has the same solution.*

Corollary 2.3. *The semi-canonical nonlinear neutral differential equation* (1.1) (*or* (1.4)) *has an eventually positive solution if and only if the canonical equation* (2.2) *has an eventually positive solution.*

It is well-known from a generalization of lemma of Kiguradze [16, Lemma 1.1] (see also [10, 15]) that the set of positive solutions of (1.1) (or (1.4)) has the following structure.

If $x(t)$ is an eventually positive solution of (1.1) (or (1.4)), then there exists $t_1 \geq t_0$ such that for all $t \geq t_1$, the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ belongs to one of the following three classes:

$$\mathcal{S}_0 = \{z : z > 0,\ L_1 z < 0,\ L_2 z > 0,\ L_3 z < 0\},$$
$$\mathcal{S}_1 = \{z : z > 0,\ L_1 z > 0,\ L_2 z < 0,\ L_3 z < 0\},$$
$$\mathcal{S}_2 = \{z : z > 0,\ L_1 z > 0,\ L_2 z > 0,\ L_3 z < 0\},$$

where $L_i z$ $(i = 1, 2, 3)$ are defined as in (1.3). Hence, the set $\mathcal{S}$ of all positive solutions of (1.1) (or (1.4)) has the decomposition $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$.

From the above classification, we see that (1.1) (or (1.4)) has two types of monotonically increasing solutions and one type of monotonically decreasing solution. Corollary 2.3 simplifies the study of (1.1) (or (1.4)), since the canonical equation (2.2) has only of two types of solutions: one eventually decreasing and the other eventually increasing, as stated in the following lemma. This lemma follows from a generalization of the well-known Kiguradze lemma [16, Lemma 1.1] (also see [15]) applied to (2.2) under condition (2.3).

Lemma 2.4. *If $x(t)$ is an eventually positive solution of* (2.2)*, then there exists $t_1 \geq t_0$ such that for all $t \geq t_1$, the corresponding function $y(t)$* (*see* (2.4)) *belongs to one of the following two classes:*

$$\mathcal{N}_0 = \{y : y > 0,\ \Delta_1 y < 0,\ \Delta_2 y > 0,\ \Delta_3 y < 0\},$$
$$\mathcal{N}_2 = \{y : y > 0,\ \Delta_1 y > 0,\ \Delta_2 y > 0,\ \Delta_3 y < 0\}.$$

Therefore, the set $\mathcal{N}$ of positive nonoscillatory solutions of (2.2) can be expressed in the form:

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2.$$

First, we find the relation between $x(t)$ and $y(t)$ in the case when $y \in \mathcal{N}_0$.

Lemma 2.5. *Let x be an eventually positive solution of* (2.2) *and the corresponding function $y \in \mathcal{N}_0$. Then*

$$x(t) \geq E(t)y(t), \tag{2.5}$$

for all $t \geq t_1 \geq t_0$, where

$$E(t) = \begin{cases} 1 - p_0, & \text{if } A_{12}(t_0) = \infty, \\ (1 - p_0)\, A_{12}(t), & \text{if } A_{12}(t_0) < \infty. \end{cases}$$

Proof. Let $x(t)$ be an eventually positive solution of (2.2), say $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ and for some $t_1 \geq t_0$. Then, the corresponding function $y(t)$ satisfies (2.4). At first, we consider the case when $A_{12}(t_0) = \infty$. Since $y \in \mathcal{N}_0$, we observe from the definition of $y(t)$ (see (2.4)) that

$$x(t) = y(t) - p(t)x(\tau(t)) \geq y(t) - p_0 y(\tau(t)) \geq (1 - p_0)y(t). \tag{2.6}$$

Next, we consider the case when $A_{12}(t_0) < \infty$. Using again the definition of $y(t)$ and taking into account that $A_{12}(t)$ is decreasing and $\tau(t) > t$, we obtain

$$\begin{aligned} x(t) = A_{12}(t)y(t) - p(t)x(\tau(t)) &\geq A_{12}(t)y(t) - p_0 A_{12}(\tau(t))y(\tau(t)) \\ &\geq A_{12}(t)\left(1 - \frac{p_0 A_{12}(\tau(t))}{A_{12}(t)}\right) y(t) \\ &\geq A_{12}(t)\,(1 - p_0)\, y(t). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) yields (2.5) and completes the proof. □

Lemma 2.6. *Let x be an eventually positive solution of* (2.2) *and the corresponding function $y \in \mathcal{N}_0$. Then $y(t)$ satisfies the inequality*

$$\Delta_3 y(t) + kE^{\alpha}(\sigma(t))F(t)y^{\alpha}(\sigma(t)) \leq 0, \quad t \geq t_1 \geq t_0. \tag{2.8}$$

Proof. The proof follows from (H4), (2.5), and (2.2); so we omit the details. □

Before starting the next results, let us define the following notation:

$$B_i(t) = \int_{t_*}^{t} \frac{1}{b_i(s)} ds \quad \text{for} \quad i = 1, 2, \quad \text{and} \quad B_{12}(t) = \int_{t_*}^{t} \frac{1}{b_1(s)} \int_{t_*}^{s} \frac{1}{b_2(s_1)} ds_1 ds$$

for $t \geq t_* \geq t_0$.

Lemma 2.7. *Let $y \in \mathcal{N}_2$. Then*

(i) $\Delta_1 y(t) \geq B_2(t)\Delta_2 y(t)$ *and* $\frac{\Delta_1 y(t)}{B_2(t)}$ *is decreasing,*
(ii) $\frac{y(t)}{B_{12}(t)}$ *is decreasing,*
(iii) $y(t) \geq B_{12}(t)\Delta_2 y(t)$.

Proof. Since $y \in \mathcal{N}_2$, $\Delta_2 y(t)$ is decreasing for $t \geq t_1$. Therefore,

$$\Delta_1 y(t) \geq \Delta_2 y(t) \int_{t_1}^{t} \frac{1}{b_2(s)} ds = B_2(t)\Delta_2 y(t), \tag{2.9}$$

and so

$$\left(\frac{\Delta_1 y(t)}{B_2(t)}\right)' = \frac{B_2(t)\Delta_2 y(t) - \Delta_1 y(t)}{B_2^2(t) b_2(t)} \leq 0,$$

that is, $\Delta_1 y(t)/B_2(t)$ is decreasing. This proves (i).

Next, from (i), we have

$$y(t) = y(t_1) + \int_{t_1}^{t} \frac{B_2(s)\Delta_1 y(s)}{B_2(s) b_1(s)} ds \geq B_{12}(t) \frac{\Delta_1 y(t)}{B_2(t)}, \tag{2.10}$$

and so

$$\left(\frac{y(t)}{B_{12}(t)}\right)' = \frac{B_{12}(t)\Delta_1 y(t) - B_2(t) y(t)}{b_1(t) B_{12}^2(t)} \leq 0,$$

that is, $y(t)/B_{12}(t)$ is decreasing. This proves (ii).

On the other hand, inequalities (2.9) and (2.10) leads to

$$y(t) \geq B_{12}(t)\Delta_2 y(t),$$

which proves (iii). This ends the proof. □

Next, we find the relation between $x(t)$ and $y(t)$ in the case where $y \in \mathcal{N}_2$. To prove our next results, we use the additional hypothesis:

$$\frac{B_{12}(\tau(t))}{B_{12}(t)} < \frac{1}{p_0} \quad \text{for } t \geq t_0. \tag{2.11}$$

Lemma 2.8. *Let* (2.11) *hold. Suppose that x is an eventually positive solution of* (2.2) *and the corresponding function $y \in \mathcal{N}_2$. Then $y(t)$ satisfies the inequality*

$$\Delta_3 y(t) + kE_1^{\alpha}(\sigma(t))F(t)y^{\alpha}(\sigma(t)) \leq 0, \quad t \geq t_1, \tag{2.12}$$

where

$$E_1(t) = \begin{cases} 1 - \frac{p_0 B_{12}(\tau(t))}{B_{12}(t)}, & \text{if } A_{12}(t_0) = \infty, \\ A_{12}(t)\left(1 - \frac{p_0 B_{12}(\tau(t))}{B_{12}(t)}\right), & \text{if } A_{12}(t_0) < \infty. \end{cases}$$

Proof. Since $y \in \mathcal{N}_2$, we see that Lemma 2.7(ii) holds. Now, in the case when $A_{12}(t_0) = \infty$, we observe from the definition of $y(t)$ (see (2.4)) and Lemma 2.7(ii) that

$$x(t) \geq y(t) - \frac{p_0 B_{12}(\tau(t))}{B_{12}(\tau(t))} y(\tau(t)) \geq \left(1 - \frac{p_0 B_{12}(\tau(t))}{B_{12}(t)}\right) y(t), \tag{2.13}$$

and also, by (2.11),

$$E_1(t) = \left(1 - \frac{p_0 B_{12}(\tau(t))}{B_{12}(t)}\right) > 0, \quad \text{if } A_{12}(t_0) = \infty.$$

Next, we consider the case when $A_{12}(t_0) < \infty$. From the definition of $y(t)$ and Lemma 2.7(ii), we obtain

$$\begin{aligned} x(t) &\geq A_{12}(t)y(t) - p_0 A_{12}(\tau(t)) \frac{B_{12}(\tau(t))}{B_{12}(\tau(t))} y(\tau(t)) \\ &\geq A_{12}(t)\left(1 - \frac{p_0 A_{12}(\tau(t)) B_{12}(\tau(t))}{A_{12}(t) B_{12}(t)}\right) y(t) \\ &\geq A_{12}(t)\left(1 - \frac{p_0 B_{12}(\tau(t))}{B_{12}(t)}\right) y(t), \end{aligned} \tag{2.14}$$

and also, by (2.11),

$$E_1(t) = A_{12}(t)\left(1 - \frac{p_0 B_{12}(\tau(t))}{B_{12}(t)}\right) > 0, \quad \text{if } A_{12}(t_0) < \infty.$$

Combining (2.13) and (2.14) and taking (2.2) into account, we see that (2.12) holds. This completes the proof of the lemma. □

Theorem 2.9. *In addition to condition* (2.11)*, assume that there exists a function* $\rho \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$\rho'(t) \geq 0, \quad \rho(t) > t, \quad \eta(t) = \sigma(\rho(\rho(t))) < t. \tag{2.15}$$

If the first-order delay differential equations

$$H'(t) + kQ_1(t)H^\alpha(\eta(t)) = 0, \tag{2.16}$$

and

$$V'(t) + kQ_2(t)V^\alpha(\sigma(t)) = 0, \tag{2.17}$$

where

$$Q_1(t) = \frac{1}{b_1(t)} \int_t^{\rho(t)} \frac{1}{b_2(s)} \int_s^{\rho(s)} E^\alpha(\sigma(s_1))F(s_1)ds_1 ds,$$
$$Q_2(t) = E_1^\alpha(\sigma(t))F(t)B_{12}^\alpha(\sigma(t)),$$

are oscillatory, then every solution of (1.1) *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ and for some $t_1 \geq t_0$. Then, by Corollary 2.3, the function $x(t)$ is a positive nonoscillatory solution of (2.2). Hence, Lemma 2.4 leads to the following: either

$$y(t) \in \mathcal{N}_0 \quad \text{or} \quad y(t) \in \mathcal{N}_2 \quad \text{for } t \geq t_1.$$

At first, we consider the case when $y \in \mathcal{N}_0$. Then, $y(t)$ satisfies inequality (2.8). Integrating (2.8) from t to $\rho(t)$ gives

$$\begin{aligned}\Delta_2 y(t) &\geq k \int_t^{\rho(t)} E^\alpha(\sigma(s))F(s)y^\alpha(\sigma(s))ds \\ &\geq ky^\alpha(\sigma(\rho(t))) \int_t^{\rho(t)} E^\alpha(\sigma(s))F(s)ds.\end{aligned}$$

Then,

$$(\Delta_1 y(t))' \geq \frac{ky^\alpha(\sigma(\rho(t)))}{b_2(t)} \int_t^{\rho(t)} E^\alpha(\sigma(s))F(s)ds.$$

Again integrating the last inequality from t to $\rho(t)$, we get

$$\begin{aligned}-y'(t) &\geq \frac{1}{b_1(t)} \int_t^{\rho(t)} \frac{ky^\alpha(\sigma(\rho(s)))}{b_2(s)} \int_s^{\rho(s)} E^\alpha(\sigma(s_1))F(s_1)ds_1 ds \\ &\geq \frac{ky^\alpha(\eta(t))}{b_1(t)} \int_t^{\rho(t)} \frac{1}{b_2(s)} \int_s^{\rho(s)} E^\alpha(\sigma(s_1))F(s_1)ds_1 ds.\end{aligned}$$

Thus,

$$y(t) \geq k \int_{t}^{\infty} \frac{y^{\alpha}(\eta(s))}{b_1(s)} \int_{s}^{\rho(s)} \frac{1}{b_2(s_1)} \int_{s_1}^{\rho(s_1)} E^{\alpha}(\sigma(s_2))F(s_2)ds_2ds_1ds := H(t).$$

Since $y(t) \geq H(t) > 0$, we conclude that H is a positive solution of the inequality

$$H'(t) + kQ_1(t)H^{\alpha}(\eta(t)) \leq 0. \tag{2.18}$$

Hence, by [23, Theorem 1], we conclude that (2.16) has a positive solution, which contradicts the fact that (2.16) oscillates.

Next, we consider the case $y(t) \in \mathcal{N}_2$. Then, we again see that (2.12) and Lemma 2.7(iii) hold. By Lemma 2.7(iii), we arrive at the following inequality

$$y(\sigma(t)) \geq B_{12}(\sigma(t))\Delta_2 y(\sigma(t)). \tag{2.19}$$

Letting $V(t) = \Delta_2 y(t)$, we observe from (2.12) and (2.19) that V is a positive solution of the inequality

$$V'(t) + kE_1^{\alpha}(\sigma(t))F(t)B_{12}^{\alpha}(\sigma(t))V^{\alpha}(\sigma(t)) \leq 0. \tag{2.20}$$

The rest of the proof is similar to that of the case $y(t) \in \mathcal{N}_0$ and hence is omitted. This ends the proof. □

Corollary 2.10. *Let* $\alpha = 1$ *and let* (2.11) *and* (2.15) *hold. If*

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} Q_1(s)ds > \frac{1}{ke}, \tag{2.21}$$

and

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q_2(s)ds > \frac{1}{ke}, \tag{2.22}$$

then equation (1.1) *is oscillatory.*

Proof. Application of [17, Theorem 1] (also see [1, Lemma 2.2.9]) with (2.21) and (2.22) implies that equations (2.16) and (2.17) are oscillatory. Now the conclusion follows from Theorem 2.9. This ends the proof. □

Corollary 2.11. *Let* $\alpha < 1$, (2.11) *and* (2.15) *hold. If*

$$\int_{t_0}^{\infty} Q_1(s)ds = \int_{t_0}^{\infty} Q_2(s)ds = \infty, \tag{2.23}$$

then equation (1.1) *is oscillatory.*

Proof. It is not difficult to see that the proof follows from $\eta(t) < t$ and (2.18), and $\sigma(t) < t$ and (2.20), respectively. □

Corollary 2.12. *Let $\alpha > 1$ and (2.11) hold. Suppose that $\sigma(t) = \theta_1 t$, $\eta(t) = \theta_2 t$ with $\theta_1, \theta_2 \in (0,1)$ and suppose also that there exists $\rho(t) = \theta_3 t$ with $\theta_3 > 1$ such that $\theta_2 = \theta_1\theta_3^2$. If there exists $\lambda_1 > -\frac{\ln\alpha}{\ln\theta_2}$ such that*

$$\liminf_{t\to\infty} \left[Q_1(t)\exp\left(-t^{\lambda_1}\right)\right] > 0, \tag{2.24}$$

and there exists $\lambda_2 > -\frac{\ln\alpha}{\ln\theta_1}$ such that

$$\liminf_{t\to\infty} \left[Q_2(t)\exp\left(-t^{\lambda_2}\right)\right] > 0, \tag{2.25}$$

then every solution of (1.1) oscillates.

Proof. Application of [34, Theorem 4] with (2.24) and (2.25) implies that equations (2.16) and (2.17) are oscillatory. Now the conclusion follows from Theorem 2.9. This ends the proof. □

Corollary 2.13. *Let $\alpha > 1$ and (2.11) hold. Assume $\sigma(t) = t - \theta_1$, $\eta(t) = t - \theta_2$ with $\theta_1, \theta_2 \in (0,\infty)$ and there exists $\rho(t) = t + \theta_3$ with $\theta_3 \in (0,\infty)$ such that $2\theta_3 < \theta_1$. If there exists $\lambda_1 > \frac{1}{\theta_2}\ln\alpha$ such that*

$$\liminf_{t\to\infty} \left[Q_1(t)\exp\left(-e^{\lambda_1 t}\right)\right] > 0, \tag{2.26}$$

and there exists $\lambda_2 > \frac{1}{\theta_1}\ln\alpha$ such that

$$\liminf_{t\to\infty} \left[Q_2(t)\exp\left(-e^{\lambda_2 t}\right)\right] > 0, \tag{2.27}$$

then every solution of (1.1) oscillates.

Proof. Application of [34, Theorem 3(i)] with (2.26) and (2.27) yields that equations (2.16) and (2.17) are oscillatory. Now the conclusion follows from Theorem 2.9. This ends the proof. □

Theorem 2.14. *Let (2.11) holds. If*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} F(s)E^\alpha(\sigma(s))N^\alpha(\sigma(t),\sigma(s))ds \begin{cases} > \frac{1}{k}, & \text{if } \alpha = 1, \\ = \infty, & \text{if } \alpha < 1, \end{cases} \tag{2.28}$$

where

$$N(\sigma(t),\sigma(s)) = \int_{\sigma(s)}^{\sigma(t)} \frac{1}{b_1(s_1)} \int_{s_1}^{\sigma(t)} \frac{1}{b_2(s_2)} ds_2 ds_1,$$

and

$$\limsup_{t\to\infty} B_{12}^\alpha(\sigma(t)) \int_t^\infty F(s)E_1^\alpha(\sigma(s))ds \begin{cases} > \frac{1}{k}, & \text{if } \alpha = 1, \\ = \infty, & \text{if } \alpha < 1, \end{cases} \tag{2.29}$$

then equation (1.1) is oscillatory.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ and for some $t_1 \geq t_0$. Then, by Corollary 2.3, the function $x(t)$ is a positive nonoscillatory solution of (2.2). Hence, Lemma 2.4 leads to the following: Either

$$y(t) \in \mathcal{N}_0 \quad \text{or} \quad y(t) \in \mathcal{N}_2 \quad \text{for} \quad t \geq t_1.$$

First, we assume that $y \in \mathcal{N}_0$. Then $y(t)$ satisfies (2.8). Using the monotonicity of $\Delta_2 y(t)$ together with $j \geq u$ leads to

$$-\Delta_1 y(u) \geq \int_u^j \frac{b_2(s)(\Delta_1 y(s))'}{b_2(s)} ds \geq \Delta_2 y(j) \int_u^j \frac{1}{b_2(s)} ds.$$

An integration of the latter inequality from u to $j \geq u$ in u yields

$$\begin{aligned} y(u) &\geq \Delta_2 y(j) \int_u^j \frac{1}{b_1(s)} \int_s^j \frac{1}{b_2(s_1)} ds_1 ds \\ &= N(j,u)\Delta_2 y(j). \end{aligned} \tag{2.30}$$

Putting $u = \sigma(s)$ and $j = \sigma(t)$ into (2.30), we get

$$y(\sigma(s)) \geq N(\sigma(t), \sigma(s))\Delta_2 y(\sigma(t)). \tag{2.31}$$

Integrating (2.8) from $\sigma(t)$ to t and using (2.31), we obtain

$$\Delta_2 y(\sigma(t)) \geq k \left(\int_{\sigma(t)}^t F(s)E^\alpha(\sigma(s))N^\alpha(\sigma(t), \sigma(s))ds \right) (\Delta_2 y(\sigma(t)))^\alpha ,$$

which can be written as

$$(\Delta_2 y(\sigma(t)))^{1-\alpha} \geq k \int_{\sigma(t)}^t F(s)E^\alpha(\sigma(s))N^\alpha(\sigma(t), \sigma(s))ds. \tag{2.32}$$

Taking $\limsup$ as $t \to \infty$ in (2.32), we obtain a contradiction with (2.28).

Next assume that $y \in \mathcal{N}_2$. Then y satisfies (2.12) for $t \geq t_1$. Integrating (2.12) from t to ∞ and using Lemma 2.7(iii), we obtain

$$\begin{aligned} \Delta_2 y(t) &\geq k \int_t^\infty F(s)E_1^\alpha(\sigma(s))y^\alpha(\sigma(s))ds \\ &\geq kB_{12}^\alpha(\sigma(t))(\Delta_2 y(\sigma(t)))^\alpha \int_t^\infty F(s)E_1^\alpha(\sigma(s))ds \\ &\geq kB_{12}^\alpha(\sigma(t))(\Delta_2 y(t))^\alpha \int_t^\infty F(s)E_1^\alpha(\sigma(s))ds, \end{aligned}$$

which can be written as

$$(\Delta_2 y(t))^{1-\alpha} \geq k B_{12}^{\alpha}(\sigma(t)) \int_t^{\infty} F(s) E_1^{\alpha}(\sigma(s)) ds. \tag{2.33}$$

Taking lim sup as $t \to \infty$ in (2.33), we obtain a contradiction to (2.29). This completes the proof. □

3. EXAMPLES

In this section, we provide two examples to illustrate the importance of the main results.

Example 3.1. Consider the linear functional differential equation with an advanced argument

$$\left(t^3 \left(\frac{1}{t^2} z'(t)\right)'\right)' + 4t^3 x\left(\frac{\sqrt{t}}{2}\right) = 0, \quad t \geq 1, \tag{3.1}$$

where $z(t) = x(t) + \frac{1}{16} x(2t)$. By a simple computation, we have

$$A_2(t) = \frac{1}{2t^2}, \quad A_{12}(1) = \infty,$$

and semi-canonical equation (3.1) reduces to

$$\left(\frac{1}{t} y''\right)' + 4tx\left(\frac{\sqrt{t}}{2}\right) = 0, \quad t \geq 1,$$

which is in canonical form. With a further calculation, we see that $B_1(t) \approx t$, $B_2(t) \approx t^2/2$, $B_{12}(t) \approx t^3/6$, $k = 1$, $E(t) = 15/16$, and $E_1(t) = 1/2$. Choose $\rho(t) = 2t$, we see that $\eta(t) = \sqrt{t}$ and so (2.15) holds. Also,

$$Q_1(t) = \frac{675}{32} t^4 \quad \text{and} \quad Q_2(t) = \frac{1}{24} t^{5/2}.$$

The conditions (2.21) and (2.22) are clearly satisfied, and therefore by Corollary 2.10, equation (3.1) is oscillatory.

Note that $\sigma \circ \tau = \sqrt{\frac{t}{2}}$ and $\tau \circ \sigma = \sqrt{t}$, hence the condition $\sigma \circ \tau = \tau \circ \sigma$ used in [10, 14, 27, 35] is not satisfied and the results in these papers cannot be used for this example. Hence, the results of this paper are more general then the above mentioned results.

Example 3.2. Consider the third-order sublinear neutral functional differential equation

$$\left(t^3 z''(t)\right)' + atx^{1/3}\left(\frac{t}{6}\right) = 0, \quad t \geq 1, \tag{3.2}$$

where $z(t) = x(t) + \frac{1}{8}x(2t)$ and $a > 0$. By a simple calculation, we have $A_2(t) = 1/2t^2$ and $A_{12}(t) = 1/2t$. The transformed equation is

$$y'''(t) + \frac{2a}{t}x^{1/3}\left(\frac{t}{6}\right) = 0, \quad t \geq 1,$$

which is in canonical form. With further calculation, we see that $B_1(t) \approx t$, $B_2(t) \approx t$, $B_{12}(t) = t^2/2$, $k = 1$, $E(t) = 7/16t$, and $E_1(t) = 1/4t$. Choose $\rho(t) = 2t$, we see that $\eta(t) = 2t/3$ and so (2.15) holds. Also,

$$Q_1(t) = dt^{2/3},$$

where

$$d = \frac{9a(21)^{1/3}(2^{1/3} - 1)(2^{2/3} - 1)}{2^{4/3}}t^{2/3},$$

and

$$Q_2(t) = \frac{a}{6^{1/3}}\frac{1}{t^{2/3}}.$$

The conditions (2.23) is clearly satisfied. Therefore, by Corollary 2.11, equation (3.2) is oscillatory.

4. CONCLUSION

In this paper, first we transform the semi-canonical equation (1.1) into a canonical form without assuming any extra conditions. Then we apply the comparison method and integral averaging technique to obtain oscillation criteria for the canonical equation, which gives the oscillation criteria for the considered semi-canonical equation. Therefore, one can get many oscillation criteria for equation (1.1) since there are several oscillation results available in the literature for canonical type equations. Also, the criteria obtained here rectified the mistake in the proof of [10, Theorem 2.1].

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