Opuscula Math. 45, no. 4 (2025), 523–542
https://doi.org/10.7494/OpMath.2025.45.4.523 Opuscula Mathematica

MULTIPLICITY RESULT FOR MIXED LOCAL AND NONLOCAL KIRCHHOFF PROBLEMS INVOLVING CRITICAL GROWTH

Vinayak Mani Tripathi

Communicated by Vicenţiu D. Rădulescu

Abstract. In this paper, we study the multiplicity of nonnegative solutions for the following nonlocal elliptic problem

$$\begin{cases} M\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \Big)\mathcal{L}(u) \\ = \lambda f(x)|u|^{p-2}u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary, $1 < p < 2 < 2^* = \frac{2N}{N-2}$, $N \geq 3$, $\lambda > 0$, M is a Kirchhoff coefficient and $\mathcal{L}$ denotes the mixed local and nonlocal operator. The weight function $f \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is allowed to change sign. By applying variational approach based on constrained minimization argument, we show the existence of at least two nonnegative solutions.

Keywords: mixed local and nonlocal operators, Kirchhoff type problem, critical nonlinearity, Nehari manifold.

Mathematics Subject Classification: 35A01, 35A15, 35B33, 35R11.

1. INTRODUCTION

We study the following mixed local and nonlocal Kirchhoff problem

$$\begin{cases} M\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)\mathcal{L}(u) \\ = \lambda f(x)|u|^{p-2}u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_λ}$$

where, $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary, $1 < p < 2 < 2^* = \frac{2N}{N-2}$, $N \geq 3$, $\lambda > 0$ and $\mathcal{L}$ denotes the mixed local and nonlocal operator

$$\mathcal{L}(u) := -\Delta u + (-\Delta)^s u, \quad s \in (0,1).$$

The weight function $f \in L^{\frac{2^*}{2^*-p}}(\Omega)$ is allowed to change sign. The fractional Laplacian operator $(-\Delta)^s$ is defined, up to a normalization constant, as

$$(-\Delta)^s \phi(x) = \int\limits_{\mathbb{R}^N} \frac{2\phi(x) - \phi(x+y) - \phi(x-y)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

for any $\phi \in C_0^\infty(\Omega)$ (see, [19] for more details). The continuous map $M : [0, \infty) \to [0, \infty)$ is defined by

$$M(t) = a + bt^{\theta - 1}, \tag{1.1}$$

where $\theta \in [1, \frac{2^*}{2})$, $a > 0$ and $b > 0$.

The above class of problems can be seen as stationary state of the following problem

$$\begin{cases} u_{tt} - M\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = F(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which was initially introduced by Kirchhoff to deal with free transversal oscillations of elastic strings (see [28]). The term M measures the stress in the string resulting from a change in its length during vibration and is directly proportional to the Sobolev norm of the string displacement. Nonlocal diffusion problems have received a lot of attention in the recent past, especially those that are driven by the fractional Laplace operator. A potential reason for this is that the fractional operator naturally arises in a number of physical phenomena including population dynamics, geophysical fluid dynamics, flame propagation and liquid chemical reactions. Furthermore, in probability theory, it provides a basic model to explain specific jump Lévy processes (see, [3, 29, 43]). The study of a fractional Kirchhoff model arising from the analysis of string vibrations has been discussed in [23]. For recent advances in nonlocal Kirchhoff problems, we suggest interested readers to the survey [35].

After the pioneering work of Ambrosetti *et al.* in [2], many researchers have shown their interest in the particular class of nonlinearity that is the subject of this study, known as concave-convex nonlinearity. The existence and multiplicity of solutions dealing with nonlinearity, as involved in the problem (P_λ), have been studied in extensive detail for both local and nonlocal elliptic problems. For p-Laplacian and fractional operator we refer interested readers to see the work in [8, 18, 26] and the references therein, where the authors have studied such class of problems. There are numerous papers dealing with Kirchhoff-type nonlocal problems involving fractional Laplacian. In [34, 45] authors have considered semilinear and fractional Kirchhoff problems with concave-convex critical nonlinearity involving sign changing weight. Using the Nehari manifold idea (see [32, 33]) authors proved the existence of at least two positive solutions for a suitable choice of parameter λ involved in the problem. In the same direction, the Kirchhoff-type equations have also been studied for the double phase problems when the nonlinearity combines the singular and the subcritical terms (see [4]). Recently, in [37], the author has discussed a new approach to obtain a unique solution for the nonlocal problems with discontinuous Kirchhoff functions. We refer the reader to [42] for the existence, multiplicity and the concentration of solutions to fractional p-Laplace Kirchhoff equations with exponential growth. With no attempt to provide the complete list of references, we cite [5, 7, 16, 17, 23, 36, 38, 39] and references therein for problems involving fractional Laplacian and fractional Kirchhoff operator.

The mixed operator has been the subject of the study in the recent past. This naturally arises from the superposition of two different-scale stochastic processes: a Lévy flight and a classical random walk. For instance, about the study of the best foraging techniques and the spread of biological species [22], the species survival problem [21], and so on. Inspired by the interesting application of mixed local and nonlocal problems the following class of problem is addressed in the literature for the analysis of several qualitative properties of solutions

$$-\Delta_q u + (-\Delta)_q^s u = f \quad \text{in } \Omega,$$

where $s \in (0,1), q \in (1,\infty), -\Delta_q$ and $(-\Delta)_q^s$ denotes q-Laplacian and fractional q-Laplacian, respectively. In case $q = 2$, we refer interested readers to see the works in [1, 6, 9] wherein the authors have shown the existence of weak solutions, the strong maximum principle, local boundedness, interior Sobolev and Lipschitz regularity. By using variational techniques, authors in [30] exhibited the existence of a weak solution when f has the most linear growth. Boundedness and strong maximum principle for the inhomogeneous case has been proved in [12]. In the case $q \in (1,\infty)$ and $f = 0$, the local boundedness of weak subsolutions, local Hölder continuity of weak solutions, Harnack inequality for weak solutions and weak Harnack inequality for weak supersolutions has been established in [24]. Recently, in [44] authors proved the existence result for Kirchhoff type mixed local and nonlocal elliptic problem with the concave-convex and Choquard nonlinearities in a modified supercritical range. To obtain the result authors have used the nonsmooth variational principle. Additionally, we suggest interested readers view the results in [10, 13, 20, 25, 40] and the references therein for problems related to mixed local and nonlocal operators.

In this paper, we aim to show multiplicity results for the problem (P_λ). The major challenge here is to show the existence of a second solution for (P_λ) as the optimal constant in mixed type Sobolev inequality is not achieved (see, [11]). By using Talenti functions, Brezis–Lieb result, and estimates from [11], we have ensured the existence of a second solution under some restriction on dimension and on the fractional parameter. To the best of our knowledge, this is the first attempt to study the concave-convex class of problems for mixed local and nonlocal operators using the Nehari manifold idea.

This work is organized into the following sections. In Section 2 we recall all the relevant notation, definitions and preliminary results used throughout this work. In Section 3 we have provided the framework of the Nehari setup and some technical results. We have shown the existence of minimizers for the energy functional in Sections 4 and 5. We conclude this section by stating the main results of our work.

Theorem 1.1. *There exists $\Lambda_0 > 0$ such that the problem (P_λ) has a nontrivial nonnegative solution for all $\lambda \in (0, \Lambda_0)$.*

Theorem 1.2. *Let $N+4s<6$, then there exists $0<\Lambda_{00}\leq\Lambda_0$, such that the problem (P_λ) has at least two nontrivial nonnegative solutions for all $\lambda\in(0,\Lambda_{00})$ and for sufficiently small values of b in* (1.1).

Remark 1.3. As $N+4s<6$ in Theorem 1.2, we emphasize that the results hold only in dimension 3, 4 and 5, under certain restriction on the fractional exponent.

2. PRELIMINARIES

Let $s\in(0,1)$. If $u:\mathbb{R}^N\to\mathbb{R}$ is a measurable function, then we define

$$[u]_s := \left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)^{\frac{1}{2}},$$

the so called the Gagliardo semi norm of u of order s. Let $\Omega\subset\mathbb{R}^N$ is an open set (with Lipschitz boundary), not necessarily bounded. We denote $\mathcal{X}^{1,2}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the following global norm

$$\rho(u) := \left(\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + [u]_s^2\right)^{\frac{1}{2}}, \quad u\in C_0^\infty(\Omega).$$

Remark 2.1. A few remarks are in order concerning the space $\mathcal{X}^{1,2}(\Omega)$.

(1) The norm $\rho(u)$ is induced by the scalar product

$$\langle u,v\rangle_\rho := \left(\int\limits_{\mathbb{R}^N}\nabla u\cdot\nabla v dx + \iint\limits_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy\right),$$

where $\cdot$ denotes the usual scalar product on the Euclidean space $\mathbb{R}^N$ and $\mathcal{X}^{1,2}(\Omega)$ is a Hilbert space.

(2) Despite of $u\in C_0^\infty(\Omega)$ the L^2-norm of ∇u is considered on the whole space. This is to emphasize that elements in $\mathcal{X}^{1,2}(\Omega)$ are functions defined on the entire space $\mathbb{R}^N$ and not only in Ω. The benefit of having this global functional setting is that these functions can be globally approximated on $\mathbb{R}^N$ (with respect to the norm $\rho(\cdot)$) by smooth functions with compact support in Ω.

In particular when $\Omega\neq\mathbb{R}^N$, we can see that this global definition of $\rho(\cdot)$, implies that the functions in $\mathcal{X}^{1,2}(\Omega)$ naturally satisfy the nonlocal Dirichlet condition specified in problem (P_λ), that is,

$$u\equiv 0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega \text{ for every } u\in\mathcal{X}^{1,2}(\Omega). \tag{2.1}$$

For detail understanding of nature of the space $\mathcal{X}^{1,2}(\Omega)$ and the validity of (2.1), we refer interested readers to see Remark 2.1 in [11, 13].

(3) The embedding $\mathcal{X}^{1,2}(\Omega)\hookrightarrow L^r(\Omega)$ is compact for every $r\in[1,2^*)$.

The meaning of solution of the underline problem is stated as follows.

Definition 2.2. A function $u \in \mathcal{X}^{1,2}(\Omega)$ is said to be a (weak) solution of the problem (P_λ) if

$$M(\rho(u)^2)\langle u, \phi\rangle_\rho = \lambda \int\limits_\Omega f|u|^{p-2}u\phi\,dx + \int\limits_\Omega |u|^{2^*-2}u\phi\,dx,$$

for all $\phi \in \mathcal{X}^{1,2}(\Omega)$.

To study the problem (P_λ) via variational methods, we define the associated energy functional $J_\lambda : \mathcal{X}^{1,2}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\hat{M}(\rho(u)^2) - \frac{\lambda}{p}\int\limits_\Omega f|u|^p dx - \frac{1}{2^*}\int\limits_\Omega |u|^{2^*}dx \\ &= \frac{a}{2}\rho(u)^2 + \frac{b}{2\theta}\rho(u)^{2\theta} - \frac{\lambda}{p}\int\limits_\Omega f|u|^p dx - \frac{1}{2^*}\int\limits_\Omega |u|^{2^*}dx, \end{aligned}$$

where

$$\hat{M}(t) = \int\limits_0^t M(s)ds = at + \frac{b}{\theta}t^\theta.$$

One can see that J_λ is of class C^1 and in light of Definition 2.2 the critical points of J_λ corresponds to the solutions of (P_λ). Let us recall that for any arbitrary open set $\Omega \subset \mathbb{R}^N$, the sharp constant for the embedding $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ is given by

$$S_N = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*}dx\right)^{\frac{2}{2^*}}}.$$

Also, the sharp constant for a mixed type Sobolev inequality is given by

$$S_{N,s}(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\rho(u)^2}{\left(\int_\Omega |u|^{2^*}dx\right)^{\frac{2}{2^*}}}. \tag{2.2}$$

In [11], the authors proved that the best constant in the natural mixed Sobolev inequality is never attained, and it coincides with the one coming from the purely local one, i.e., $S_N = S_{N,s}(\Omega)$.

3. NEHARI MANIFOLD AND FIBERING MAPS

To study the minimizers of the energy functional J_λ, it is necessary that J_λ be bounded from below on the space $\mathcal{X}^{1,2}(\Omega)$, which is not the case in our setting. A natural constraint for studying the minimization problem is the Nehari manifold. The Nehari set associated with problem (P_λ) is given by

$$\mathcal{N}_\lambda = \{u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\} : D_u J_\lambda(u)u = 0\}.$$

Observe that the set $\mathcal{N}_\lambda$ contains all the critical points of J_λ. For a fixed $u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\}$, we introduce the fibering map $m_{\lambda,u} : (0,\infty) \to \mathbb{R}$ given as $m_{\lambda,u}(t) = J_\lambda(tu)$. It is easy to see that $u \in \mathcal{N}_\lambda$ if and only if $m'_{\lambda,u}(1) = 0$. More generally, $tu \in \mathcal{N}_\lambda$ if and only if $m'_{\lambda,u}(t) = 0$. It is evident that we can split the set $\mathcal{N}_\lambda$ the following disjoint subsets:

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : m''_{\lambda,u}(1) > 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : m''_{\lambda,u}(1) < 0\}, \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : m''_{\lambda,u}(1) = 0\}, \end{aligned}$$

which correspond to $t = 1$ as local minimum, maximum and the inflection point of the fibering map. In order to study minimization problem over the Nehari decompositions, we aim to show these sets are nonempty. Observe that for any $u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\}$, with $\int_\Omega f|u|^q dx > 0$, $t(u)u \in \mathcal{N}_\lambda^0$ if the pair $(t = t(u), \lambda = \lambda(u))$ is unique solution of the system

$$\begin{aligned} &at^2\rho(u)^2 + bt^{2\theta}\rho(u)^{2\theta} - \lambda t^p \int_\Omega f|u|^p dx - t^{2^*} \int_\Omega |u|^{2^*} = 0, \\ &2at^2\rho(u)^2 + 2\theta bt^{2\theta}\rho(u)^{2\theta} - \lambda pt^p \int_\Omega f|u|^p dx - 2^* t^{2^*} \int_\Omega |u|^{2^*} = 0. \end{aligned} \tag{3.1}$$

The uniqueness of $\lambda(u)$ follows from uniqueness of $t(u)$ as nontrivial zero of the following scalar equation:

$$h(t) = a(2-p)t^2\rho(u)^2 + b(2\theta - p)t^{2\theta}\rho(u)^{2\theta} - (2^* - p)t^{2^*} \int_\Omega |u|^{2^*}.$$

To show the uniqueness of $t(u)$, we define

$$m_u(t) = a(2-p)\rho(u)^2 + b(2\theta - p)t^{2\theta-2}\rho(u)^{2\theta} - (2^* - p)t^{2^*-2} \int_\Omega |u|^{2^*} dx,$$

where $h_u(t) = t^2 m_u(t)$. Since

$$m'_u(t) = b(2\theta - 2)(2\theta - p)t^{2\theta-1}\rho(u)^{2\theta} - (2^* - p)(2^* - 2)t^{2^*-1} \int_\Omega |u|^{2^*} dx,$$

$m_u(t)$ has a unique critical point

$$t^* = \left(\frac{b(2\theta - 2)(2\theta - p)\rho(u)^{2\theta}}{(2^* - p)(2^* - 2)\int_\Omega |u|^{2^*}} \right)^{\frac{1}{2^*-2\theta}}.$$

Moreover, as $m_u'(t) > 0$ as $t \to 0^+$ and $m_u'(t) < 0$ as $t \to +\infty$, and we can conclude that $m_u(t)$ has a unique zero $t(u) > t^* > 0$. Consequently, $h_u(t)$ has a unique nontrivial zero.

Solving the above system (3.1), we have following implicit form of the nonlinear generalized Rayleigh quotient $\lambda(u)$:

$$\lambda(u) = \frac{a(2^* - 2)t(u)^{2-p}\rho(u)^2 + b(2^* - 2\theta)t(u)^{2\theta-p}\rho(u)^{2\theta}}{(2^* - p)\int_\Omega f|u|^p dx}.$$

From expression of $\lambda(u)$ it is clear that $\lambda(u)$ is 0-homogeneous and $t(u)$ is (-1)-homogeneous. Note that while dealing with degenerate local or nonlocal Kirchhoff problems with such class of nonlinearity, we have explicit representations for $\lambda(u)$ (see, for instance, [31]).

We define the extremal value for the Nehari manifold method (see [27]) by

$$\lambda^* = \inf_{\mathcal{X}^{1,2}(\Omega)\setminus\{0\}} \left\{ \lambda(u) : \int_\Omega f|u|^p dx > 0 \right\}.$$

In order to study the minimization problem, we show in the following proposition that the decompositions of the Nehari manifold are nonempty.

Proposition 3.1. *Given $u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\}$. There are two possibilities:*

(a) *if $\int_\Omega f|u|^p dx > 0$, then there exists two critical points $t^+(u), t^-(u)$ of the fibering map $m_{\lambda,u}$ such that $t^+_\lambda(u)u \in \mathcal{N}^+_\lambda$ and $t^-(u)u \in \mathcal{N}^-_\lambda$. Moreover, $\phi_{\lambda,u}$ is decreasing in $(0, t^+(u)]$ and $[t^-(u), \infty)$ and increasing in $[t^+(u), t^-(u)]$ for any $\lambda \in (0, \lambda(u))$,*

(b) *if $\int_\Omega f|u|^p dx \leq 0$, then there exists unique t^* such that $t^*u \in \mathcal{N}^+_\lambda$ for any $\lambda > 0$.*

Proof. For a fixed $u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\}$, define a map $\Phi_u : (0, \infty) \to \mathbb{R}$ by

$$\Phi_u(t) = at^{2-p}\rho(u)^2 + bt^{2\theta-p}\rho(u)^{2\theta} - t^{2^*-p} \int_\Omega |u|^{2^*} dx.$$

Then it is clear that $tu \in \mathcal{N}_\lambda$ if and only if t is root of the scalar equation

$$\Phi_u(t) = \lambda \int_\Omega f|u|^p dx.$$

We have $\Phi_u(t) > 0^+$ as $t \to 0^+$ and $\Phi(t) \to -\infty$ as $t \to \infty$. Using

$$\Phi_u'(t) = a(2 - p)t^{1-p}\rho(u)^2 + b(2\theta - p)t^{2\theta-p-1}\rho(u)^{2\theta} - (2^* - p)t^{2^*-p-1} \int_\Omega |u|^{2^*} dx,$$

one can observe that $m''_{\lambda,tu}(1) = t^{p-1}\Phi_u'(t)$, where $\Phi_u'(t) = t^{-1-p}h_u(t)$. Thus, it is enough to analyze the nature of $h_u(t)$. We already know that $h_u(t)$ has unique zero at $t(u)$, therefore, $\Phi_u(t)$ has a global maximum at $t(u)$ in light of $\Phi_u'(t) > 0$ when $t \to 0^+$ and $\Phi_u'(t) < 0$ for large t. Consequently, when $\lambda < \lambda(u)$, using the relation $m''_{\lambda,tu}(1) = t^{p-1}\Phi_u'(t)$, there exists $t^\pm(u)$ satisfying (a). In case when $\int_\Omega f|u|^p dx \leq 0$ from the above analysis of the map Φ_u, we can conclude the case (b). This completes the proof. □

The following remark is a direct consequence of Proposition 3.1. Moreover, as a consequence of implicit function theorem, the set $\mathcal{N}_\lambda$ is a manifold.

Remark 3.2. For all $\lambda > 0$, $\mathcal{N}_\lambda^\pm \neq \emptyset$. Moreover, $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda < \lambda^*$.

3.1. AN ESTIMATE OF EXTREMAL VALUE λ^*:

Let $u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\}$ be fixed and for $t > 0$, define the map

$$\tilde{\Phi}_u(t) = bt^{2\theta - p}\rho(u)^{2\theta} - t^{2^*-p}\int\limits_\Omega |u|^{2^*}dx.$$

Then, we have $\max_{t>0} \tilde{\Phi}_u(t) = \tilde{\Phi}_u(\tilde{t}_{\max})$, where

$$\tilde{t}_{\max} = \left(\frac{b(2\theta - p)\rho(u)^{2\theta}}{(2^* - p)\int_\Omega |u|^{2^*}dx}\right)^{\frac{1}{2^*-2\theta}} \geq \left(\frac{b(2\theta - p)}{(2^* - p)}S_{N,s}(\Omega)^{\frac{2^*}{2}}\right)^{\frac{1}{2^*-2\theta}} \frac{1}{\rho(u)} := \tilde{t}^0_{\max}.$$

Therefore, as $\tilde{\Phi}_u$ is increasing from $[0, \tilde{t}_{\max}]$, we get

$$\max_{t>0} \Phi_u(t) \geq \tilde{\Phi}_u(\tilde{t}^0_{\max}) \geq \left(\frac{2^* - 2\theta}{2\theta - p}\right)\left(\frac{b(2\theta - p)}{2^* - p}\right)^{\frac{2^*-p}{2^*-2\theta}} (S_{N,s}(\Omega)^{\frac{2^*}{2}})^{\frac{2\theta-p}{2^*-2\theta}}\rho(u)^p.$$

Thus, if

$$\lambda < \Lambda_1 := \left(\frac{2^* - 2\theta}{2\theta - p}\right)\left(\frac{b(2\theta - p)}{2^* - p}\right)^{\frac{2^*-p}{2^*-2\theta}} (S_{N,s}(\Omega)^{\frac{2^*}{2}})^{\frac{2\theta-p}{2^*-2\theta}} \frac{S_{N,s}(\Omega)^{\frac{p}{2}}}{\|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}},$$

it holds

$$\Phi_u(\tilde{t}^0_{\max}) > \lambda \int\limits_\Omega f|u|^p, \quad \forall\, \lambda \in (0, \Lambda_1).$$

Consequently, for every $u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\}$, $t^\pm(u)u \in \mathcal{N}_\lambda^\pm$ and $\mathcal{N}_\lambda^0 = \emptyset$. Thus, $\lambda^* \geq \Lambda_1$.

We have following observation on J_λ.

Lemma 3.3. *The energy functional J_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Take $0 \neq u \in \mathcal{N}_\lambda$. We have

$$\begin{aligned} J_\lambda(u) &= \frac{a}{2}\rho(u)^2 + \frac{b}{2\theta}\rho(u)^{2\theta} - \frac{\lambda}{p}\int\limits_\Omega f|u|^p dx - \frac{1}{2^*}\int\limits_\Omega |u|^{2^*}dx \\ &\geq \left(\frac{1}{2\theta} - \frac{1}{2^*}\right) b\rho(u)^{2\theta} - \lambda\left(\frac{1}{p} - \frac{1}{2^*}\right)\int\limits_\Omega f|u|^p dx \\ &\geq \left(\frac{1}{2\theta} - \frac{1}{2^*}\right) b\rho(u)^{2\theta} - \lambda\left(\frac{1}{p} - \frac{1}{2^*}\right) S_{N,s}(\Omega)^{-p/2}\rho(u)^p\|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}, \end{aligned}$$

which implies that J_λ is coercive. Define the map

$$g(t) = \left(\frac{1}{2\theta} - \frac{1}{2^*}\right) t^{2\theta} - \lambda \left(\frac{1}{p} - \frac{1}{2^*}\right) S_{N,s}(\Omega)^{-p/2} t^p \|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}.$$

One can observe that $g(t)$ is bounded from below. In fact, it follows that there exists $C > 0$ such that $J_\lambda(u) > -C$. This completes the proof. □

The following lemma ensures that the local minimizers of the energy functional J_λ on $\mathcal{N}_\lambda$ are critical points for J_λ (see [15, Theorem 2.3]).

Lemma 3.4. *If u is a local minimizer of J_λ in $\mathcal{N}_\lambda$ and $u \notin \mathcal{N}_\lambda^0$. Then u is a critical point for J_λ.*

Now we are ready to introduce the minimization problem. Define

$$J_\lambda = \inf\{J_\lambda(u) : u \in \mathcal{N}_\lambda\},$$
$$J_\lambda^+ = \inf\{J_\lambda(u) : u \in \mathcal{N}_\lambda^+\},$$
$$J_\lambda^- = \inf\{J_\lambda(u) : u \in \mathcal{N}_\lambda^-\}.$$

In the upcoming lemmas and proposition, we prove several technical results necessary for the analysis of the above minimization problems.

Lemma 3.5. *There exists $\delta > 0$ such that $\rho(u) \geq \delta$ for $u \in \mathcal{N}_\lambda^-$. Moreover, $\mathcal{N}_\lambda^-$ is closed in the topology of $\mathcal{X}^{1,2}(\Omega)$ for $\lambda \in (0, \lambda^*)$.*

Proof. If $u \in \mathcal{N}_\lambda^-$, then

$$(2\theta - p) b \rho(u)^{2\theta} < (2^* - p) \int_\Omega |u|^{2^*} dx \leq (2^* - p) S_{N,s}(\Omega)^{-2^*/2} \rho(u)^{2^*},$$

which implies that

$$\rho(u) > \left(\frac{1}{S_{N,s}(\Omega)^{-2^*/2}} \frac{2\theta - p}{2^* - p}\right)^{\frac{1}{2^*-2\theta}}.$$

Thus, $\rho(u) \geq \delta$ for some $\delta > 0$. To prove the remaining part, consider a sequence $\{u_k\} \subset \mathcal{N}_\lambda^-$ such that $u_k \to u$ in $\mathcal{X}^{1,2}(\Omega)$. Then $u \in \mathcal{N}_\lambda^- \cup \{0\}$ as $\mathcal{N}_\lambda^0 = \emptyset$, for $\lambda \in (0, \lambda^*)$. Since $\rho(u) = \lim_{k\to\infty} \rho(u_k) \geq \delta$, we get $u \neq 0$ and $u \in \mathcal{N}_\lambda^-$. □

Lemma 3.6. *For each $u \in \mathcal{N}_\lambda$ and $\lambda \in (0, \Lambda_1)$, there exists $\epsilon > 0$ and a differentiable map $\xi : B(0, \epsilon) \subset \mathcal{X}^{1,2}(\Omega) \to \mathbb{R}$ such that $\xi(v)(u - v) \in \mathcal{N}_\lambda$ and*

$$\langle \xi'(0), v\rangle = \frac{2\langle u, v\rangle_\rho - \lambda \int_\Omega f|u|^{p-2} uv dx - 2^* \int_\Omega |u|^{2^*-2} uv dx}{(2 - p)\rho(u)^2 - (2^* - p) \int_\Omega |u|^{2^*} dx}. \tag{3.2}$$

Proof. Fix $u \in \mathcal{N}_\lambda$ and define the map $F_u : \mathbb{R}^+ \times \mathcal{X}^{1,2}(\Omega) \to \mathbb{R}$ by

$$F_u(t, v) = t^2 \rho(u - v)^2 - \lambda t^p \int_\Omega f|u - v|^p dx - t^{2^*} \int_\Omega |u - v|^{2^*} dx.$$

Then $F_u(1,0) = 0$ and $\frac{\partial F_u}{\partial t}(1,0) \neq 0$ for $\lambda \in (0, \Lambda_1)$. Using implicit function theorem, there exists $\epsilon > 0$ and a differentiable functional $\xi : B(0,\xi) \subset \mathcal{X}^{1,2}(\Omega) \to \mathbb{R}$ such that $\xi(0) = 1$, (3.2) holds and $F_u(\xi(v), v) = 0$ for all $v \in B(0, \epsilon)$. Hence, $\xi(v)(u - v) \in \mathcal{N}_\lambda$. □

From Lemma 3.3, we know that J_λ is bounded below in $\mathcal{N}_\lambda$. Therefore, for any $\lambda \in (0, \Lambda_1)$ by the Ekeland variational principle, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda$ such that

$$J_\lambda(u_k) \leq J_\lambda + \frac{1}{k} \quad \text{and} \quad J_\lambda(u_k) \leq J_\lambda(v) + \frac{1}{k}\rho(u_k - v) \text{ for all } v \in \mathcal{N}_\lambda.$$

The following result is a consequence of the Ekeland variational principle and the Lemma 3.6. The idea of the proof is similar to [34, Proposition 3.8], and for this reason it has been omitted here.

Lemma 3.7. *Let $\lambda \in (0, \Lambda_1)$. Then there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda$ such that*

$$J_\lambda(u_k) = J_\lambda + o_k(1) \quad \textit{and} \quad J_\lambda^{'}(u_k) = o_k(1).$$

The next result ensures that the compactness of J_λ can be recovered below a suitable value.

Proposition 3.8. *Let $\{u_k\}$ be a sequence in $\mathcal{X}^{1,2}(\Omega)$ such that*

$$J_\lambda(u_k) \to c, \;\; J_\lambda'(u_k) \to 0 \;\textit{in}\; (\mathcal{X}^{1,2}(\Omega))^* \;\textit{as}\; k \to \infty, \tag{3.3}$$

where

$$c < c_\lambda := \frac{1}{N}(aS_{N,s}(\Omega))^{\frac{N}{2}} - \lambda^{\frac{2\theta}{2\theta-p}} \left(\frac{2\theta - p}{2^* p 2\theta}\right) \frac{\left((2^* - p)S_{N,s}(\Omega)^{\frac{-p}{2}} \|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}\right)^{\frac{2\theta}{2\theta-p}}}{((2^* - 2\theta)b)^{\frac{p}{2\theta-p}}},$$

then $\{u_k\}$ possesses a strongly convergent subsequence.

Proof. Let $\{u_k\}$ be a sequence of bounded functions in $\mathcal{X}^{1,2}(\Omega)$ (as J_λ is bounded below and coercive in $\mathcal{N}_\lambda$). Then, by using compact embedding of $\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^r(\Omega)$ for $r \in [1, 2^*)$, there exists $u_0 \in \mathcal{X}^{1,2}(\Omega)$ such that, up to a subsequence again denoted by $\{u_k\}$, $u_k \rightharpoonup u_0$ in $\mathcal{X}^{1,2}(\Omega)$, $\rho(u_k) \to \mu$, $u_k \to u_0$ in $L^r(\Omega)$ for $r \in [1, 2^*)$, $u_k(x) \to u_0(x)$ a.e. in Ω. If $\mu = 0$, then it follows that $u_k \to 0$ in $\mathcal{X}^{1,2}(\Omega)$. Thus assume $\mu > 0$. By the Brezis–Lieb lemma [14], we have

$$\begin{aligned} \rho(u_k)^2 &= \rho(u_k - u_0)^2 + \rho(u_0)^2 + o_k(1), \\ \int_\Omega |u_k|^{2^*} dx &= \int_\Omega |u_k - u_0|^{2^*} dx + \int_\Omega |u_0|^{2^*} dx + o_k(1). \end{aligned} \tag{3.4}$$

Testing (2.2) with $(u_k - u_0)$ and using (3.4), we have

$$
\begin{aligned}
o_k(1) &= (a + b(\rho(u_k)^{2\theta-2}))\Bigg(\int\limits_{\mathbb{R}^n} \nabla u_k \cdot \nabla(u_k - u_0)dx \\
&\quad + \iint\limits_{\mathbb{R}^{2N}} \frac{(u_k(x)-u_k(y))((u_k-u_0)(x)-(u_k-u_0)(y))}{|x-y|^{N+2s}} dxdy\Bigg) \\
&\quad - \lambda \int\limits_{\Omega} f|u_k|^{p-2}u_k(u_k-u_0)dx - \int\limits_{\Omega} |u_k|^{2^*-2}u_k(u_k-u_0)dx \\
&= (a + b\mu^{2\theta-2})(\mu^2 - \rho(u_0)^2) - \lambda\int\limits_{\Omega} f|u_k|^{p-2}u_k(u_k-u_0) - \int\limits_{\Omega}|u_k|^{2^*}dx + \int\limits_{\Omega} u_0^{2^*}dx \\
&= (a + b\mu^{2\theta-2}) \lim_{k\to\infty} \rho(u_k-u_0)^2 - \lambda\int\limits_{\Omega} f|u_k|^{p-2}u_k(u_k-u_0)dx - \int\limits_{\Omega}|u_k-u_0|^{2^*}dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(a + b\mu^{2\theta-2}) \lim_{k\to\infty} \rho(u_k-u_0)^2 = \lambda \lim_{k\to\infty} \int\limits_{\Omega} f|u_k|^{p-2}u_k(u_k-u_0)dx \\
+ \lim_{k\to\infty} \int\limits_{\Omega} |u_k-u_0|^{2^*}dx.
\end{aligned}
\tag{3.5}
$$

Applying the Lebesgue dominated convergence theorem, we obtain

$$
(a + b\mu^{2\theta-2}) \lim_{k\to\infty} \rho(u_k-u_0)^2 = l^{2^*}, \tag{3.6}
$$

where we denote $\lim_{k\to\infty}\int_\Omega |u_k-u_0|^{2^*}dx = l^{2^*}$. From (3.5) we can conclude that $l \geq 0$. If $l = 0$, we have $u_k \to u_0$, and we are done in this case. Suppose the case when $l > 0$. From (2.2) we get

$$
\rho(u_k-u_0)^2 \geq S_{N,s}(\Omega)l^2. \tag{3.7}
$$

Using (3.6) and (3.7), we obtain

$$
l^{2^*-2} \geq S_{N,s}(\Omega)(a + b\mu^{2\theta-2}). \tag{3.8}
$$

Also, note that from (3.6), we have

$$
(a + b\mu^{2\theta-2})(\mu^2 - \rho(u_0)^2) \leq l^{2^*}. \tag{3.9}
$$

Using (3.8) and (3.9), we obtain

$$
\mu^2 \geq S_{N,s}(\Omega)^{\frac{N}{2}} a^{\frac{2}{2^*-2}}. \tag{3.10}
$$

For any $\phi \in \mathcal{X}^{1,2}(\Omega)$, denote

$$\begin{aligned} H(u_k,\phi) = (a+b\rho(u_k)^{2\theta-2})\Big(&\int\limits_{\mathbb{R}^N} \nabla u_k\cdot\nabla\phi dx \\ &+\iint\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dxdy\Big) \\ &-\lambda\int\limits_{\Omega} f|u_k|^{p-2}u_k\phi dx - \int\limits_{\Omega}|u_k|^{2^*-2}u_k\phi dx. \end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned} J_\lambda(u_k) - \frac{1}{2^*}H(u_k,u_k) &= \left(\frac{1}{2}-\frac{1}{2^*}\right)a\rho(u_k)^2 + \left(\frac{1}{2\theta}-\frac{1}{2^*}\right)b\rho(u_k)^{2\theta} \\ &\quad -\lambda\left(\frac{1}{p}-\frac{1}{2^*}\right)\int\limits_{\Omega} f|u_k|^p dx \\ &\geq \left(\frac{1}{2}-\frac{1}{2^*}\right)a\rho(u_k)^2 + \left(\frac{1}{2\theta}-\frac{1}{2^*}\right)b\rho(u_k)^{2\theta} \\ &\quad -\lambda\left(\frac{1}{p}-\frac{1}{2^*}\right)S_{N,s}(\Omega)^{\frac{-p}{2}}\|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}\rho(u_k)^p. \end{aligned} \tag{3.11}$$

Now define

$$F_b(t) = \left(\frac{1}{2\theta}-\frac{1}{2^*}\right)bt^{2\theta} - \lambda\left(\frac{1}{p}-\frac{1}{2^*}\right)S_{N,s}(\Omega)^{\frac{-p}{2}}\|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}t^p.$$

Then, a straightforward computation implies that

$$F_b(t) \geq -\lambda^{\frac{2\theta}{2\theta-p}}\left(\frac{2\theta-p}{2^*p2\theta}\right)\frac{\left((2^*-p)S_{N,s}(\Omega)^{\frac{-p}{2}}\|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}\right)^{\frac{2\theta}{2\theta-p}}}{((2^*-2\theta)b)^{\frac{p}{2\theta-p}}}.$$

Taking limit $k\to\infty$ in (3.11), together with (3.3) and (3.10), we get

$$c \geq \frac{1}{N}(aS_{N,s}(\Omega))^{\frac{N}{2}} - \lambda^{\frac{2\theta}{2\theta-p}}\left(\frac{2\theta-p}{2^*p2\theta}\right)\frac{\left((2^*-p)S_{N,s}(\Omega)^{\frac{-p}{2}}\|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}\right)^{\frac{2\theta}{2\theta-p}}}{((2^*-2\theta)b)^{\frac{p}{2\theta-p}}}.$$

This leads to a contradiction with the assumption, completing the proof. □

4. EXISTENCE OF FIRST SOLUTION IN $\mathcal{N}_\lambda^+$

In this section, using the standard minimization argument, we prove the existence of first solution of the problem (P_λ) in $\mathcal{N}_\lambda^+$. We start by proving that the energy level is negative in $\mathcal{N}_\lambda^+$.

Lemma 4.1. *For $\lambda > 0$, we have that $J_\lambda^+ = \inf\{J_\lambda(u) : u \in \mathcal{N}_\lambda^+\} < 0$.*

Proof. Let $0 \neq u \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we have

$$\begin{aligned}
J_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{2^*}\right) a\rho(u)^2 + \left(\frac{1}{2\theta} - \frac{1}{2^*}\right) b\rho(u)^{2\theta} - \lambda \left(\frac{1}{p} - \frac{1}{2^*}\right) \int\limits_\Omega f|u|^p dx \\
&< \left(\frac{1}{2} - \frac{1}{2^*}\right) a\rho(u)^2 + \left(\frac{1}{2\theta} - \frac{1}{2^*}\right) b\rho(u)^{2\theta} \\
&\quad - \left(\frac{1}{p} - \frac{1}{2^*}\right) \left(\frac{2^* - 2}{2^* - p} a\rho(u)^2 + \frac{2^* - 2\theta}{2^* - p} b\rho(u)^{2\theta}\right) \\
&= \frac{2^* - 2}{2^*} \left(\frac{1}{2} - \frac{1}{p}\right) \rho(u)^2 + \frac{2^* - 2\theta}{2^*} \left(\frac{1}{2\theta} - \frac{1}{p}\right) b\rho(u)^{2\theta} < 0,
\end{aligned}$$

and as $J_\lambda(0) = 0$, we get $m_\lambda^+ < 0$. $\square$

Choose $\Lambda_2 > 0$ such that

$$\Lambda_2^{\frac{2\theta}{2\theta-p}} \left(\frac{2\theta - p}{2^* p 2\theta}\right) \frac{\left((2^* - p) S_{N,s}(\Omega)^{\frac{-p}{2}} \|f\|_{L^{\frac{2^*}{2^*-p}}(\Omega)}\right)^{\frac{2\theta}{2\theta-p}}}{((2^* - 2\theta)b)^{\frac{p}{2\theta-p}}} \leq \frac{1}{N} (aS_{N,s}(\Omega))^{\frac{N}{2}}.$$

Proof of Theorem 1.1. Considering $\Lambda_0 = \min\{\Lambda_1, \Lambda_2\}$ and using Proposition 3.7, we get a minimizing Palais–Smale sequence $\{u_k\} \subset \mathcal{N}_\lambda^+$ such that $\{u_k\}$ is bounded in $\mathcal{X}^{1,2}(\Omega)$ and $J_\lambda(u_k) \to m_\lambda^+$. Also, from Lemma 4.1, we have $J_\lambda^+ < 0$. In view of the Proposition 3.8, there exists $u_0 \in \mathcal{X}^{1,2}(\Omega)$ such that $u_k \to u_0$ in $\mathcal{X}^{1,2}(\Omega)$. Thus, u_0 is a minimizer of J_λ in $\mathcal{N}_\lambda$ for all $\lambda \in (0, \Lambda_1)$ since $J_\lambda(u_0) < 0$, we have that $u_0 \not\equiv 0$. Next, we claim that $u_0 \in \mathcal{N}_\lambda^+$. If not then $u_0 \in \mathcal{N}_\lambda^0$ which is not possible as $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \Lambda_0)$ (see Remark 3.2). Therefore, $u_0 \in \mathcal{N}_\lambda^+$. Using Lemma 3.4, we can conclude that the obtained minimizer $u_0 \in \mathcal{N}_\lambda^+$ is a critical point and equivalently the solution of the problem (P_λ).

Next, it remains to show that obtained solution is nonnegative. It is required to be verified due to presence of nonlocal fractional operator, we have $\rho(u) \neq \rho(|u|)$ in $\mathcal{X}^{1,2}(\Omega)$ and thus $J_\lambda(u) \neq J_\lambda(|u|)$. To overcome this difficulty, we consider positive part of the problem and the corresponding energy functional as follows

$$J_\lambda^+(u) = \frac{1}{2} \hat{M}(\rho(u)^2) - \lambda \frac{1}{p} \int\limits_\Omega f(u^+)^p dx - \frac{1}{2^*} \int\limits_\Omega (u^+)^{2^*} dx,$$

where $u^+ := \max\{u, 0\}$ is the positive part of u. Then it is easy to see that the critical points of J_λ are also critical points of J_λ^+. Thus,

$$\begin{aligned}
&M(\rho(u_0)^2) \left(\int\limits_{\mathbb{R}^N} \nabla u_0 \cdot \nabla \phi dx + \iint\limits_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy \right) \\
&= \lambda \int\limits_\Omega f(u_0^+)^{p-1} \phi dx + \int\limits_\Omega (u_0^+)^{2^*-1} \phi dx
\end{aligned} \tag{4.1}$$

for all $\phi \in \mathcal{X}^{1,2}(\Omega)$. Testing (4.1) by $\phi = u^-$ and using the inequality

$$\begin{aligned}(u_0(x) - u_0(y))(u_0^-(x) - u_0^-(y)) &= -u_0^+(x)u_0^-(y) - u_0^-(x)u_0^+(y) - (u_0^-(x) - u_0^-(y))^2 \\ &\leq -|u_0^-(x) - u_0^-(y)|^2,\end{aligned}$$

we have $\rho(u_0^-) = 0$, thus u_0 is nonnegative solution of the problem (P_λ). This completes the proof. □

5. EXISTENCE OF SECOND SOLUTION IN $\mathcal{N}_\lambda^-$

To show the existence of the second solution to the problem (P_λ), we have followed the idea and the estimates from [11, 45]. Consider the test function $\eta \in C_0^\infty(\Omega)$ such that

$$0 \leq \eta(x) \leq 1 \text{ in } \Omega, \quad \eta(x) = 1 \text{ in } B_{\frac{\rho}{2}}(0), \quad \text{and } \eta(x) = 0 \text{ in } (B_\rho(0))^c,$$

for ρ sufficiently small. For $\epsilon > 0$, let

$$u_\epsilon(x) := \frac{\epsilon^{\frac{(N-2)}{2}}}{(|x|^2 + \epsilon^2)^{\frac{N-2}{2}}} \quad \text{and} \quad u_{\epsilon,\eta} := \frac{\eta(x)u_\epsilon(x)}{\|\eta u_\epsilon\|_{L^{2^*}(\Omega)}}.$$

Consider the following sets:

$$\begin{aligned} U_1 &= \left\{ u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\} : \frac{1}{\rho(u)} t^- \left(\frac{u}{\rho(u)} \right) > 1 \right\} \cup \{0\}, \\ U_2 &= \left\{ u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\} : \frac{1}{\rho(u)} t^- \left(\frac{u}{\rho(u)} \right) < 1 \right\}, \end{aligned}$$

where t^- is as in Proposition 3.1. Then

$$\mathcal{N}_\lambda^- = \left\{ u \in \mathcal{X}^{1,2}(\Omega) \setminus \{0\} : \frac{1}{\rho(u)} t^- \left(\frac{u}{\rho(u)} \right) = 1 \right\}$$

is a connected component of U_1 and U_2. Also, one can observe that $\mathcal{N}_\lambda^+ \subset U_1$, and thus $u_0 \in U_1$. We now provide the subsequent technical lemma. The idea of its proof is similar to [45].

Lemma 5.1. *Let $\lambda \in (0, \Lambda_0)$ and u_0 be a local minimizer for the functional J_λ in $\mathcal{X}^{1,2}(\Omega)$ obtained in Theorem 1.1. Then, for any $\epsilon > 0$ and η defined above, there exists $l_0 > 0$ such that $u_0 + l_0 u_{\epsilon,\eta} \in U_2$.*

Proposition 5.2. *Let $\lambda \in (0, \Lambda_0)$ and let u_0 be the local minimizer achieved in last case for the functional J_λ in $\mathcal{X}^{1,2}(\Omega)$. Then, for any $r > 0$ and η there exists $\epsilon_0 = \epsilon_0(r, N)$ and $\Lambda_3 > 0$ such that $J_\lambda(u_0 + r u_{\epsilon,\eta}) < c_\lambda$ for any $\epsilon \in (0, \epsilon_0)$ and $\lambda \in (0, \min\{\Lambda_0, \Lambda_3\})$.*

Proof. We have

$$\begin{aligned} J_\lambda(u_0 + ru_{\epsilon,\eta}) = \frac{a}{2}\rho(u_0)^2 + \frac{a}{2}r^2\rho(u_{\epsilon,\eta})^2 + ar\Bigg(\int\limits_{\mathbb{R}^N} \nabla u_0 \cdot \nabla u_{\epsilon,\eta} dx \\ + \iint\limits_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(u_{\epsilon,\eta}(x) - u_{\epsilon,\eta}(y))}{|x-y|^{N+2s}} dxdy \Bigg) \\ + \frac{b}{2\theta}\Bigg(\rho(u_0)^2 + r^2\rho(u_{\epsilon,\eta})^2 + 2r \int\limits_{\mathbb{R}^N} \nabla u_0 \cdot \nabla u_{\epsilon,\eta} \\ + 2r \iint\limits_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(u_{\epsilon,\eta}(x) - u_{\epsilon,\eta}(y))}{|x-y|^{N+2s}} dxdy \Bigg)^\theta \\ - \frac{\lambda}{p} \int\limits_\Omega f|u_0 + ru_{\epsilon,\eta}|^p dx - \frac{1}{2^*} \int\limits_\Omega |u_0 + ru_{\epsilon,\eta}|^{2^*} dx. \end{aligned} \tag{5.1}$$

In order to estimate terms in the above expression, we have used the following inequalities:

$$(\alpha+\beta)^\theta \leq 2^{\theta-1}(\alpha^\theta+\beta^\theta) \leq \alpha^\theta + C_\theta(\alpha^\theta+\beta^\theta) + \theta\alpha^{\theta-1}\beta, \quad \text{for any } \alpha, \beta \geq 0, \theta \geq 1, \tag{5.2}$$

$$(\alpha+\beta)^p - \alpha^p - \beta^p - p\alpha^{p-1}\beta \geq C_1\alpha\beta^{p-1}, \quad \text{for any } \alpha, \beta \geq 0, p > 2, \tag{5.3}$$

where $C_\theta > 0$ depending on θ and $C_1 > 0$. Using Young's inequality, for some $D_\theta > 0$, we get

$$\begin{aligned} \frac{b}{2\theta}\left(\rho(u_0 + ru_{\epsilon,\eta})^{2\theta}\right) \leq \frac{b}{2\theta}\rho(u_0)^{2\theta} + bC_\theta(\rho(u_0))^{2\theta} + bD_\theta r^{2\theta}(\rho(u_{\epsilon,\eta}))^{2\theta} \\ + b(\rho(u_0))^{2\theta-2} r\Bigg(\int\limits_{\mathbb{R}^N} \nabla u_0 \cdot \nabla u_{\epsilon,\eta} dx \\ + \iint\limits_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(u_{\epsilon,\eta}(x) - u_{\epsilon,\eta}(y))}{|x-y|^{N+2s}} dxdy \Bigg). \end{aligned}$$

Since u_0 is a solution, (5.1) reduces to

$$\begin{aligned} J_\lambda(u_0 + ru_{\epsilon,\eta}) \leq J_\lambda(u_0) + \frac{a}{2}r^2(\rho(u_{\epsilon,\eta}))^2 + bC_\theta(\rho(u_0))^{2\theta} + bD_\theta r^{2\theta}(\rho(u_{\epsilon,\eta}))^{2\theta} \\ - \frac{\lambda}{p}\int\limits_\Omega f\left((u_0 + ru^p_{\epsilon,\eta})dx - u_0^p - pru_0^{p-1}u_{\epsilon,\eta}\right) \\ - \frac{1}{2^*}\int\limits_\Omega \left((u_0 + ru^{2^*}_{\epsilon,\eta})dx - u_0^{2^*} - 2^* ru_0^{2^*-1}u_{\epsilon,\eta}\right). \end{aligned}$$

Denote $\rho(u_0) = R$. Using $f > 0$ in the support of $u_{\epsilon,\eta}$ and the inequalities (5.2), (5.3), we can conclude that

$$J_\lambda(u_0 + ru_{\epsilon,\eta}) \leq J_\lambda(u_0) + \frac{a}{2}r^2(\rho(u_{\epsilon,\eta}))^2 + bC_\theta R^{2\theta} + bD_\theta r^{2\theta}(\rho(u_{\epsilon,\eta}))^{2\theta} - \frac{1}{2^*}r^{2^*}\int_\Omega u_{\epsilon,\eta}^{2^*}dx - C_1 r^{2^*-1}\int_\Omega u_{\epsilon,\eta}^{2^*-1}dx. \tag{5.4}$$

Considering the estimates used in [11] and taking $b = \epsilon^q$ with $q > N-2$, we get

$$J_\lambda(u_0 + ru_{\epsilon,\eta}) \leq \frac{a}{2}r^2(S_{N,s}(\Omega) + O(\epsilon^{k_{s,N}})) + C_2\epsilon^q + C_3\epsilon^{N-2} - \frac{r^{2^*}}{2^*} - C_4 r^{2^*-1}\epsilon^{(N-2)/2},$$

where $k_{s,N} = \min\{N-2, 2-2s\}$. Next, we define

$$G(t) = \frac{a}{2}t^2(S_{N,s}(\Omega) + O(\epsilon^{k_{s,N}})) - \frac{t^{2^*}}{2^*} - C_4 t^{2^*-1}\epsilon^{(N-2)/2}.$$

By noting that $G(t) \to -\infty$ as $t \to \infty$ and $G(t) \to 0^+$ as $t \to 0^+$, one can ensure that there exists t_ϵ such that $\frac{d}{dt}G(t)|_{t=t_\epsilon} = 0$ and $G(t_\epsilon) = \sup_{t\geq 0} G(t)$. Also, $G'(t_\epsilon) = 0$, implies there exists $\nu > 0$ such that $t_\epsilon \geq \nu > 0$. Using the fact $\inf J_\lambda = m_\lambda^+ < 0$ in $\mathcal{N}_\lambda^+$, we get

$$\begin{aligned}J_\lambda(u_0 + ru_\epsilon) &\leq \frac{a}{2}t^2(S_{N,s}(\Omega) + O(\epsilon^{k_{s,N}})) - \frac{t^{2^*}}{2^*} - C_4 t^{2^*-1}\epsilon^{(N-2)/2} + C_5\epsilon^{N-2} \\ &\leq \frac{a}{2}t^2(S_{N,s}(\Omega) + O(\epsilon^{k_{s,N}})) - \frac{t^{2^*}}{2^*} + C_5\epsilon^{N-2} - C_6\epsilon^{(N-2)/2},\end{aligned}$$

where $C_5, C_6 > 0$ are positive constants independent of ϵ, λ. Since, the map

$$t \to \frac{a}{2}t^2(S_{N,s}(\Omega) + O(\epsilon^{k_{s,N}})) - \frac{t^{2^*}}{2^*}$$

is increasing in $[0, (a(S_{N,s}(\Omega) + C\epsilon^{k_{s,N}}))^{\frac{1}{2^*-2}})$, we have

$$\begin{aligned}J_\lambda(u_0 + ru_\epsilon) &\leq \left(\frac{1}{2} - \frac{1}{2^*}\right)(a(S_{N,s}(\Omega) + O(\epsilon^{k_{s,N}})))^{\frac{2^*}{2^*-2}} + C_5\epsilon^{N-2} - C_6\epsilon^{(N-2)/2} \\ &\leq \frac{1}{N}(aS_{N,s}(\Omega))^{\frac{N}{2}} + C_7\epsilon^{\min\{N-2,2-2s\}} - C_6\epsilon^{\frac{N-2}{2}},\end{aligned}$$

where $C_7 > 0$ and ϵ is sufficiently small. Choosing $0 < \epsilon < \epsilon_1$ sufficiently small such that

$$C_7\epsilon^{\min\{N-2,2-2s\}} - C_5\epsilon^{\frac{N-2}{2}} < 0.$$

Therefore, taking $\lambda \in (0, \Lambda_3)$, where Λ_3 satisfies the following inequality

$$\Lambda_3^{\frac{2\theta}{2\theta-p}}\left(\frac{2\theta-p}{2^*p2\theta}\right)\frac{\left((2^*-p)S_{N,s}(\Omega)^{\frac{-p}{2}}\|f\|_{L^{\frac{2^*}{2^*-2}}(\Omega)}\right)^{\frac{2\theta}{2\theta-p}}}{((2^*-2\theta)b)^{\frac{p}{2\theta-p}}} < C_5\epsilon^{\frac{N-2}{2}} - C_7\epsilon^{\min\{N-2,2-2s\}},$$

we have $J_\lambda(u_0 + ru_\epsilon) < c_\lambda$, where c_λ is defined in Proposition 3.8. This completes the proof. □

Proof of Theorem 1.2. Assume that Λ_0 and Λ_3 are as in Section 4, Proposition 5.2, and fix $\lambda < \Lambda_{00} := \min\{\Lambda_0, \Lambda_3\}$. Also, as we have $u_0 \in U_1$ and $u_0 + l_0 u_{\epsilon,\eta} \in U_2$, we can define a continuous path connecting U_1 and U_2 as $t \to \gamma(t) := u_0 + tl_0 u_{\epsilon,\eta}$. Therefore, there exists $t \in (0,1)$ such that $\gamma(t) \in \mathcal{N}_\lambda^-$ and consequently

$$m_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u) \leq J_\lambda(\gamma(t)).$$

Furthermore, from Proposition 5.2, we get $J_\lambda^- < c_\lambda$ for $\lambda < \Lambda_{00}$. Now using Lemma 3.7, we can ensure existence of a minimizing Palais–Smale sequence $\{u_k\} \subset \mathcal{N}_\lambda^-$ such that $J_\lambda(u_k) \to m_\lambda^-$. Since $J_\lambda^+ < c_\lambda$, consequently, from Proposition 3.8 there exists $u_1 \in \mathcal{X}^{1,2}(\Omega)$ such that $u_k \to u_1$ in $\mathcal{X}^{1,2}(\Omega)$. Therefore, u_1 is a minimizer of J_λ. As from Lemma 3.5, $\mathcal{N}_\lambda^-$ is closed we have that $u_1 \in \mathcal{N}_\lambda^-$ for all $\lambda \in (0, \Lambda_{00})$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, we can conclude that u_0, u_1 are distinct. This completes the proof of the Theorem 1.2. Following the arguments as in the proof of Theorem 1.1, we can conclude that u_1 is a nonnegative solution of the problem (P_λ). This ends the proof. □

Remark 5.3. We believe that in case the sublinear term in the problem (P_λ) get perturbed with subcritical growth then using compact embedding $\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^r(\Omega)$ for every $r \in [1, 2^*)$ we can establish the multiplicity result in $(0, \Lambda^*)$, where

$$\Lambda^* = \inf_{\mathcal{X}^{1,2}(\Omega) \cap \int_\Omega f|u|^p dx > 0} \frac{a(q-2)t(u)^{2-p}\rho(u)^2 + b(q-2\theta)t(u)^{2\theta-p}\rho(u)^{2\theta}}{(q-p)\int_\Omega f|u|^p dx},$$

for $1 < p < 2 < q < 2^*, \theta \in [1, \frac{q}{2})$ and $q \in (2\theta, 2^*)$. Precisely, in this case we do not required, the Proposition 3.8. Moreover, following the idea in [31, 41], one can ensure the multiplicity when $\lambda \in (0, \Lambda^* + \epsilon)$, for some $\epsilon > 0$. Note that, when $\lambda \geq \Lambda^*$, the Nehari set $\mathcal{N}_\lambda$ is no longer a manifold.

Acknowledgements

The author would like to thank Dr. Pawan Kumar Mishra for providing incisive feedback on the results and discussions. The author wish to thank the anonymous referee for his/her comments and remark on the work.

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Vinayak Mani Tripathi
vinayakm@iitbhilai.ac.in
https://orcid.org/0009-0007-1710-3443

Indian Institute of Technology Bhilai
Department of Mathematics
491002, Durg, Chhattisgarh, India

Received: March 25, 2025.
Revised: May 19, 2025.
Accepted: May 20, 2025.