Opuscula Math. 45, no. 4 (2025), 509–521
https://doi.org/10.7494/OpMath.2025.45.4.509 Opuscula Mathematica

ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF OPERATORS THAT ARE SUM OF PSEUDO p-LAPLACE TYPE

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Communicated by J.I. Díaz

Abstract. The article investigates a Poisson-type problem for operators that are finite sum of pseudo p-Laplace-type operators within long cylindrical domains. It establishes that the rate of convergence is exponential, which is considered optimal. In addition, the study analyzes the asymptotic behavior of the related energy functional. This research contributes to a deeper understanding of the mathematical properties and asymptotic analysis of solutions in this context.

Keywords: pseudo p-Laplace equation, cylindrical domains, asymptotic analysis.

Mathematics Subject Classification: 35B27, 46E35, 49J52.

1. INTRODUCTION

Let $\Omega_\ell := \ell\omega_1 \times \omega_2$ be a cylindrical domain of length $\ell > 0$, where $\omega_1 \subset \mathbb{R}^{n-r}$ is convex, bounded and $\omega_2 \subset \mathbb{R}^r$ is a bounded open set. It is also assumed that $0 \in \omega_1$. Let us denote a generic point in $\mathbb{R}^n$ by $x = (X_1, X_2)$ with $X_1 = (x_1, \ldots, x_{n-r}) \in \mathbb{R}^{n-r}$ and $X_2 = (x_{n-r+1}, \ldots, x_n) \in \mathbb{R}^m$ respectively. ∇, ∇_{X_1} and ∇_{X_2} will denote the gradient in $\mathbb{R}^n$, $\mathbb{R}^{n-r}$ and $\mathbb{R}^r$, respectively. u_{x_i} will denote the partial derivative of u along x_i-th direction. The set ω_2 will be referred to as the cross-section of the cylindrical domains Ω_ℓ. We consider now $q_j \in \mathbb{R}$, $j = 1, \ldots, m$, such that

$$2 \leq q_m \leq q_{m-1} \leq \ldots \leq q_1.$$

For Ω a bounded subset of $\mathbb{R}^d$, using Hölder's inequality it is easy to see

$$L^{q_1}(\Omega) \subseteq L^{q_2}(\Omega) \subseteq \ldots \subseteq L^{q_m}(\Omega) \subseteq L^2(\Omega),$$

$$W_0^{1,q_1}(\Omega) \subseteq W_0^{1,q_2}(\Omega) \subseteq \ldots \subseteq W_0^{1,q_m}(\Omega) \subseteq W_0^{1,2}(\Omega).$$

For a real number $p > 1$, we define p' as its conjugate, which satisfies

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Recall the definition of the pseudo p-Laplace operator:

$$L_p(u) := -\sum_{i=1}^{n} \left(|u_{x_i}|^{p-2} u_{x_i}\right)_{x_i}.$$

In this article, we are interested in studying the Poisson problem for the operator

$$L(u) := \sum_{i=1}^{m} L_{q_i}(u).$$

More precisely, for $f \in L^2(\omega_2)$, consider the following problem

$$\begin{cases} -\sum_{i=1}^{n} \left(\left(\sum_{j=1}^{m} |(u_\ell)_{x_i}|^{q_j-2} \right) (u_\ell)_{x_i} \right)_{x_i} = f(X_2) & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell. \end{cases} \tag{1.1}$$

The solution of the above equation is understood in the weak sense and will be defined explicitly in the next section. The typical functional space for considering solutions to (1.1) is the Sobolev space $W_0^{1,q_1}(\Omega_\ell)$. Existence and uniqueness of the weak solution to the previous equation are standard; however, for the sake of completeness, we provide a proof in the appendix. The existence of solutions for the problem above (and for more general cases) can be found in [19]. The operator under consideration can be seen as the "finite sums" of pseudo Laplace operators. Pseudo Laplace operators are special case of anisotropic Laplace operator. For a detailed study of anisotropic Laplacian operators, we refer to [15, 17].

This article focuses on studying the asymptotic behavior of u_ℓ as ℓ (the length of the cylinder) tends to infinity. Recently, a similar analysis was conducted in [5] for operators that are finite sums of p-Laplace operators, which serves as the primary motivation for this work. For studies involving a single anisotropic p-Laplace operator, we refer to [6] and [18].

Consider an analogous equation on the cross-section ω_2, which will be useful for our analysis: follows

$$\begin{cases} -\sum_{i=n-r+1}^{n} \left(\left(\sum_{j=1}^{m} |W_{x_i}|^{q_j-2} \right) W_{x_i} \right)_{x_i} = f(X_2), & \text{in } \omega_2, \\ W = 0 & \text{on } \partial\omega_2. \end{cases} \tag{1.2}$$

It is well known that u_ℓ uniquely satisfies

$$J_\ell(u_\ell) = \inf_{u \in W_0^{1,q_1}(\Omega_\ell)} J_\ell(u),$$

where J_ℓ is the energy functional associated to the problem (1.1) and is defined as

$$J_\ell(u) = \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{1}{q_j} \int_{\Omega_\ell} |u_{x_i}|^{q_j}\, dx - \int_{\Omega_\ell} f u dx.$$

Moreover, u_ℓ is also the unique solution of (1.1) (see Section 4 for proof of this fact). That is, u_ℓ satisfies

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\int_{\Omega_\ell} |(u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i}v_{x_i}dx = \int_{\Omega_\ell} fvdx, \quad \forall v \in W_0^{1,q_1}(\Omega_\ell) \tag{1.3}$$

Similarly, let W be the unique function that satisfies

$$J_{\omega_2}(W) = \inf_{u\in W_0^{1,q_1}(\omega_2)} J_{\omega_2}(u), \tag{1.4}$$

where

$$J_{\omega_2}(u) = \sum_{j=1}^{m}\sum_{i=n-r+1}^{n}\frac{1}{q_j}\int_{\omega_2}|u_{x_i}|^{q_j}dX_2 - \int_{\omega_2} fudX_2.$$

As before, W is the weak solution of(1.2).

Theorem 1.1. *For some $\alpha \in (0,1)$ and for some constant $C > 0$ independent of ℓ,*

$$\sum_{j=1}^{m}\int_{\Omega_{\alpha\ell}} |\nabla(u_\ell - W)|^{q_j}dx \le Ce^{-\alpha\ell},$$

where W is extended as a function of X_2 in the whole $\Omega_{\alpha\ell}$.

In [18], the proof of Theorem 1.1 is provided for the particular case where $q_1 = q_2 = \ldots = q_m = p$.

Our next theorem in this direction is the following.

Theorem 1.2 (Convergence of the energy)**.** *For some constant $C > 0$, independent of ℓ, we have*

$$J_{\omega_2}(W) \le \frac{J_\ell(u_\ell)}{\mu_{n-r}(\ell\omega_1)} \le J_{\omega_2}(W) + \frac{C}{\ell}.$$

For related work on semilinear equations, polynomial rate of convergence are established in [2, 12, 13]. We also refer to [1, 3, 4, 7, 9–11, 14, 20] and the references therein for a comprehensive survey in this direction and related areas. In particular, the paper [13] (see also [2]) examines semilinear elliptic problems on infinite cylindrical domains.

From an application perspective, the above two theorems have significant implications from a numerical standpoint. They help to reduce computational costs that arise from the curse of dimensionality, by enabling the study of lower-dimensional problems. For direct applications, interested readers may refer to [8].

The structure of this article is as follows. In the next section, we introduce the necessary function spaces, preliminaries, and some key estimates in the form of lemmas. In the third section, we provide detailed proofs of the main theorems. The final section is the Appendix, where we provide the proof of existence and uniqueness of u_ℓ.

2. PRELIMINARIES

Throughout this article, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ will denote a generic point. $|x| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$ will denote its Euclidean norm. $|x|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ will denote the ℓ_p norm of the point x. The following inequality (the equivalence of all finite dimensional norms) will be used in several places, without making any references:

$$c_1|x| \leq |x|_p \leq c_2|x|, \; x \in \mathbb{R}^n, \text{ for some constant } c_1, c_2 > 0.$$

For $p \geq 1$, $W^{1,p}(\Omega), W_0^{1,p}(\Omega)$ will denote usual Sobolev spaces (see [16]). The space

$$V_p(\Omega_\ell) := \left\{\phi \in W^{1,p}(\Omega_\ell) \mid \phi = 0 \text{ on } \ell\omega_1 \times \partial\omega_2\right\}$$

is a subspace of $W^{1,p}(\Omega)$. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^k$ will be denoted by $\mu_k(E)$. Throughout this article, the value of the constants will be denoted by a generic number $C > 0$ and may change from line to line. We say $u_\ell \in W_0^{1,q_1}(\Omega_\ell)$ is a weak solution of the problem (1.1) if (1.3) is satisfied.

Now we present some lemmas that will be used in the proofs of the main theorems.

Lemma 2.1 (Uniform Poincaré inequality)**.** *Let $p > 1$. Then there exist a constant $C > 0$ (independent of ℓ, ℓ' and $\ell^{''}$) such that for $\ell^{'} < \ell^{''} \leq \ell$,*

$$\int\limits_{\Omega_{\ell''} \setminus \Omega_{\ell'}} |\phi|^p dx \leq \int\limits_{\Omega_{\ell''} \setminus \Omega_{\ell'}} \sum_{i=n-r+1}^{n} |\phi_{x_i}|^p dx \leq C \int\limits_{\Omega_{\ell''} \setminus \Omega_{\ell'}} \sum_{i=1}^{n} |\phi_{x_i}|^p dx$$

for all $\phi \in V_p(\Omega_\ell)$.

Proof. The first inequality is trivial. Let us define

$$D_p(u) := \sum_{i=n-r+1}^{n} |u_{x_i}|^p.$$

It is sufficient to prove the inequality for $\phi \in C^\infty(\Omega_\ell)$ such that $\phi = 0$ on $\ell\omega_1 \times \partial\omega_2$, as it is a dense subspace of $V_p(\Omega_\ell)$. Since ω_2 is a bounded subset of $\mathbb{R}^{n-r}$, we can use the usual Poincaré inequality to obtain

$$C \int\limits_{\omega_2} |\phi|^p dX_2 \leq \int\limits_{\omega_2} |\nabla_{X_2}\phi|^p dX_2.$$

Then using the inequality $D_p(\phi) \geq |\nabla_{X_2}\phi|^p$ together with the previous inequality, we get

$$C \int\limits_{\omega_2} |\phi|^p dX_2 \leq \int\limits_{\omega_2} D_p(\phi) dX_2.$$

Integrating both sides over the set $\ell^{''}\omega_1 \setminus \ell'\omega_1$ finishes the proof of the lemma. □

Lemma 2.2. *There exist a constant $C > 0$, independent of ℓ,*

$$\int_{\Omega_\ell} |\nabla(u_\ell - W)|^{q_i} dx \leq C\ell^{n-r} \quad \text{for } i = 1, \dots, m.$$

Proof. Taking $v = u_\ell$ in (1.3), we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \int_{\Omega_\ell} |(u_\ell)_{x_i}|^{q_j} dx = \int_{\Omega_\ell} f u_\ell dx.$$

For a fixed $j \in \{1, \dots, m\}$, using Hölder's inequality, we have

$$\sum_{i=1}^{n} \int_{\Omega_\ell} |(u_\ell)_{x_i}|^{q_j} dx \leq \int_{\Omega_\ell} f u_\ell dx \leq \left(\int_{\Omega_\ell} f^{q_j'} dx \right)^{1/q_j'} \left(\int_{\Omega_\ell} u_\ell^{q_j} dx \right)^{1/q_j},$$

where $1/q_j + 1/q_j' = 1$. In the last inequality, we dropped $m-1$ terms of the summation, keeping only the j-th term. Now, using the uniform Poincaré inequality in Lemma 2.1 with $p = q_j, \ell'' = \ell$ and $\ell' = 0$ we have

$$\int_{\Omega_\ell} \sum_{i=1}^{n} | \, (u_\ell)_{x_i}|^{q_j} dx \leq C|f|^{q_j}_{L^{q_j}(\omega_2)} |\omega_1| \ell^{n-r},$$

for some C independent of ℓ. Finally,

$$\int_{\Omega_\ell} |\nabla(u_\ell - W)|^{q_j} dx \leq C \sum_{i=1}^{n} \int_{\Omega_\ell} |(u_\ell)_{x_i}|^{q_j} + |W_{x_i}|^{q_j} dx \leq C\ell^{n-r}$$

This finishes the proof of the lemma. □

Lemma 2.3. *If $p \geq 2$, then there exists a constant $C_p > 0$ such that*

$$\sum_{i=1}^{n} \left(|x_i|^{p-2}x_i - |y_i|^{p-2}y_i\right)(x_i - y_i) \geq C_p|x-y|^p$$

for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Proof. See [18]. □

Lemma 2.4. *The function $V_\ell(X_1, X_2) := W(X_2) \in V_{q_1}(\Omega_\ell)$, where W is as in (1.4), satisfies the following equation weakly, for each $\ell > 0$:*

$$\begin{cases} -\sum_{i=1}^{n} \left(\sum_{j=1}^{m} |(V_\ell)_{x_i}|^{q_j-2}(V_\ell)_{x_i} \right)_{x_i} = f(X_2) & \text{in } \Omega_\ell, \\ V_\ell = 0 & \text{on } \ell\omega_1 \times \partial\omega_2, \\ V_\ell = W & \text{on } \partial(\ell\omega_1) \times \omega_2. \end{cases}$$

Proof. For any $v \in V_{q_1}(\Omega_\ell)$, we have to prove the following:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int\limits_{\Omega_\ell} |(V_\ell)_{x_i}|^{q_j-2}(V_\ell)_{x_i} v_{x_i} dx = \int\limits_{\Omega_\ell} f(X_2) v dx.$$

Using Fubini's theorem, we get

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m}\int\limits_{\Omega_\ell} |(V_\ell)_{x_i}|^{q_j-2}(V_\ell)_{x_i} v_{x_i} dx \\
&= \int\limits_{\ell\omega_1} \left(\sum_{i=n-r+1}^{n} \sum_{j=1}^{m} \int\limits_{\omega_2} |(W)_{x_i}(_, X_2)|^{q_j-2} W_{x_i}(_, X_2) v_{x_i}(_, X_2) dX_2 \right) dX_1.
\end{aligned}$$

Now, using the weak formulation of equation (1.2) and applying Fubini's theorem again, we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m}\int\limits_{\Omega_\ell} |(V_\ell)_{x_i}|^{q_j-2}(V_\ell)_{x_i} v_{x_i} dx &= \int\limits_{\ell\omega_1} \left(\int\limits_{\omega_2} f(X_2) v(_, X_2) dX_2 \right) dX_1 \\
&= \int\limits_{\Omega_\ell} f(X_2) v dx.
\end{aligned}$$

This finishes the proof of the lemma. □

3. PROOFS OF THE THEOREMS

We now turn to the proof of our first result – Theorem 1.1.

Proof of Theorem 1.1. Since u_ℓ satisfies (1.1) weakly, this means for any $v \in W_0^{1,q_1}(\Omega_\ell)$ one has

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int\limits_{\Omega_\ell} |(u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i} v_{x_i} dx = \int\limits_{\Omega_\ell} f(X_2) v dx.$$

This together with Lemma 2.4 gives for all $v \in W_0^{1,q_1}(\Omega_\ell)$,

$$\sum_{j=1}^{k}\sum_{i=1}^{m}\int\limits_{\Omega_\ell} ((u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i} - |W_{x_i}|^{q_j-2} W_{x_i}) v_{x_i} dx = 0.$$

For $\ell' \in (0, \ell-1)$, let $\rho_{\ell'}$ be a function, whose precise properties will be specified later, such that $v(= v_\ell) := \rho_\ell(u_\ell - W) \in W_0^{1,q_1}(\Omega_\ell)$. Substituting this v into the previous equation yields

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{i=1}^{n}\int\limits_{\Omega_\ell} \left(|(u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i} - | W_{x_i}|^{q_j-2} W_{x_i}\right) \{(u_\ell - W)(\rho_\ell)_{x_i} \\
+ \rho_\ell (u_\ell - W)_{x_i}\} dx = 0.
\end{aligned}$$

Hence, we have

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\int\limits_{\Omega_\ell}\left(|(u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i}-|\,W_{x_i}|^{q_j-2}W_{x_i}\right)(u_\ell-W)_{x_i}\rho_\ell dx$$
$$=-\sum_{j=1}^{m}\sum_{i=1}^{n}\int\limits_{\Omega_\ell}\left(|(u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i}-|\,W_{x_i}|^{q_j-2}W_{x_i}\right)(u_\ell-W)(\rho_\ell)_{x_i}dx.$$

Now, applying the inequality from Lemma 2.3, we obtain

$$C\sum_{j=1}^{m}\int\limits_{\Omega_\ell}\rho_\ell|\nabla(u_\ell-W)|^{q_j}dx$$
$$\leq\sum_{j=1}^{m}\sum_{i=1}^{n}\int\limits_{\Omega_\ell}\left(|(u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i}-|\,W_{x_i}|^{q_j-2}W_{x_i}\right)(W-u_\ell)(\rho_\ell)_{x_i}dx.$$

Taking modulus on both sides of the equation above and using triangle inequality together making the choice of $\rho_{\ell'}$, a function of X_1 satisfying the following properties:

$$0\leq\rho_{\ell'}\leq 1,\;\rho_{\ell'}=1\text{ on }\Omega_{\ell'},\;\rho_{\ell'}=0\text{ outside }\Omega_{\ell'+1}\text{ and }|\nabla_{X_1}\rho_{\ell'}|\leq 1,$$

we get

$$C\sum_{j=1}^{m}\int\limits_{\Omega_\ell}\rho_\ell|\nabla(u_\ell-W)|^{q_j}dx$$
$$\leq\sum_{j=1}^{m}\sum_{i=1}^{n}\int\limits_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}}\left||(u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i}-|W_{x_i}|^{q_j-2}W_{x_i}\right||u_\ell-W|\,|(\rho_\ell)_{x_i}|\,dx.$$

Now, $|(\rho_\ell)_{x_i}|\leq 1$, so

$$C\sum_{j=1}^{m}\int\limits_{\Omega_\ell}\rho_\ell|\nabla(u_\ell-W)|^{q_j}dx\leq\sum_{j=1}^{m}\sum_{i=1}^{n-r}\int\limits_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}}|(u_\ell-W)_{x_i}|^{q_j-1}|u_\ell-W|dx.$$

Using Hölder's inequality and Poincaré inequality, we have

$$C\sum_{j=1}^{m}\int\limits_{\Omega_\ell}\rho_\ell|\nabla(u_\ell-W)|^{q_j}dx$$
$$\leq\sum_{j=1}^{m}\sum_{i-1}^{n-r}\left(\int\limits_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}}|(u_\ell-W)_{x_i}|^{q_j}dx\right)^{\frac{q_j-1}{q_j}}\left(\int\limits_{\Omega_{\ell'+1}\setminus\Omega_{\ell'}}|u_\ell-W|^{q_j}dx\right)^{\frac{1}{q_j}}.$$

Now, using the uniform Poincaré inequality (Lemma 2.1), we have

$$|(u_\ell - W)_{x_i}|^q \leq |\nabla(u_\ell - W)|^q.$$

Substituting this into the right-hand side of the above expression, we obtain, for some constant $C > 0$,

$$\sum_{j=1}^{m} \int_{\Omega_{\ell'}} |\nabla(u_\ell - W)|^{q_j} dx \leq \frac{C}{C+1} \left(\sum_{j=1}^{m} \int_{\Omega_{\ell'+1}} |\nabla(u_\ell - W)|^{q_j} dx \right).$$

Now, by iterating the above inequality with the choice $\ell' = \frac{\ell}{2}, \frac{\ell}{2}+1, \frac{\ell}{2}+2, \dots, \frac{\ell}{2}+[\frac{\ell}{2}]$, where $[\frac{\ell}{2}]$ denotes the greatest integer less than or equal to $\frac{\ell}{2}$, we obtain

$$\begin{aligned} \sum_{j=1}^{m} \int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - W)|^{q_j} dx &\leq \left(\frac{C}{C+1}\right)^{[\ell/2]} \left\{ \int_{\Omega_{\ell/2+[\ell/2]}} \sum_{j=1}^{m} |\nabla(u_\ell - W)|^{q_j} dx \right\} \\ &\leq \left(\frac{C}{C+1}\right)^{[\ell/2]} \left\{ \int_{\Omega_\ell} \sum_{j=1}^{m} |\nabla(u_\ell - W)|^{q_j} dx \right\}. \end{aligned}$$

Rewriting the above equation differently, we obtain

$$\sum_{j=1}^{k} \int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - W)|^{q_j} dx \leq e^{[\ell/2]\log(\frac{C}{C+1})} \left\{ \sum_{j=1}^{m} \int_{\Omega_\ell} |\nabla(u_\ell - W)|^{q_j} dx \right\}.$$

The proof of the theorem for $\alpha = \frac{1}{2}$ follows by applying Lemma 2.2 and noting that $\log(\frac{C}{C+1}) < 0$. The result follows by appropriately choosing the number of iterations used earlier. □

Now we present the proof of Theorem 1.2

Proof of Theorem 1.2. Consider the sequences of test function $\psi_\ell \in W_0^{1,q_1}(\Omega_\ell)$ defined as

$$\psi_\ell(X_2) := \frac{1}{\mu_{n-r}(\ell\omega_1)} \int_{\ell\omega_1} u_\ell(\_, X_2) dX_1.$$

Since we have

$$J_{\omega_2}(W) = \inf_{u \in W_0^{1,q_1}(\omega_2)} J_{\omega_2}(u),$$

this implies that for $\ell > 0$,

$$J_{\omega_2}(W) \leq J_{\omega_2}(\psi_\ell) = \sum_{j=1}^{m} \frac{1}{q_j} \left(\sum_{i=n-r+1}^{n} \int_{\omega_2} |(\psi_\ell)_{x_i}|^{q_j} dX_2 \right) - \int_{\omega_2} f\psi_\ell dX_2$$
$$= \sum_{j=1}^{m} \frac{1}{q_j} \left(\sum_{i=n-r+1}^{n} \int_{\omega_2} \left| \frac{1}{\mu_{n-r}(\ell\omega_1)} \int_{\ell\omega_1} (u_\ell)_{x_i}(\_, X_2) dX_1 \right|^{q_j} dX_2 \right)$$
$$- \int_{\omega_2} \frac{f(X_2)}{\mu_{n-r}(\ell\omega_1)} \left(\int_{\ell\omega_1} u_\ell(\_, X_2) dX_1 \right) dX_2.$$

Now using Jensen's inequality for the integrals, one has

$$\mu_{n-r}(\ell\omega_1) J_{\omega_2}(W) \leq \sum_{j=1}^{m} \frac{1}{q_j} \left(\sum_{i=n-r+1}^{n} \int_{\omega_2} \int_{\ell\omega_1} |(u_\ell)_{x_i}(\_, X_2)|^{q_j} dX_1 dX_2 \right)$$
$$- \int_{\omega_2} \int_{\ell\omega_1} f u_\ell(X_1, X_2) dX_1 dX_2$$
$$\leq \sum_{j=1}^{m} \frac{1}{q_j} \left(\sum_{i=1}^{n} \int_{\Omega_\ell} |(u_\ell)_{x_i}|^{q_j} dx \right) - \int_{\Omega_\ell} f u_\ell dx = J_\ell(\Omega_\ell).$$

For the second inequality, first we consider a Lipschitz continuous cutoff function $\rho_\ell = \rho_\ell(X_1)$, $0 \leq \rho_\ell \leq 1$, $|\nabla_{X_1} \rho_\ell| \leq C$. We also further assume that $\rho_\ell = 1$ on $(\ell - 1)\omega_1$ and $\rho_\ell = 0$ on $\partial(\ell\omega_1)$. Since the function $\rho_\ell(X_1)W(X_2) \in W_0^{1,q_1}(\Omega_\ell)$, we have $J_\ell(u_\ell) \leq J_\ell(\rho_\ell W)$. Now, estimating the right-hand side, we have

$$J_\ell(\rho_\ell W) = J_{\ell-1}(W) - \int_{\Omega_\ell \setminus \Omega_{\ell-1}} f\rho_\ell W dx$$
$$+ \sum_{j=1}^{m} \frac{1}{q_j} \left(\sum_{i=1}^{n} \int_{\Omega_\ell \setminus \Omega_{\ell-1}} |(\rho_\ell W) x_i|^{q_j} dx \right)$$
$$= J_\ell(W)$$
$$+ \left\{ \sum_{j=1}^{m} \frac{1}{q_j} \left(\sum_{i=1}^{n} \int_{\Omega_\ell \setminus \Omega_{\ell-1}} |(\rho_\ell W) x_i|^{q_j} dx \right) - \int_{\Omega_\ell \setminus \Omega_{\ell-1}} (f\rho_\ell W - fW) dx \right\}$$
$$- \left\{ \sum_{j=1}^{m} \frac{1}{q_j} \left(\sum_{i=1}^{n} \int_{\Omega_\ell \setminus \Omega_{\ell-1}} |W_{x_i}|^{q_j} dx \right) \right\}$$
$$= \mu_{n-r}(\ell\omega_1) J_{\omega_2}(W) + \mathcal{A}_\ell.$$

We estimate the first term of $\mathcal{A}_\ell$ and the rest term can be treated in a similar manner:

$$\sum_{j=1}^{m}\frac{1}{q_j}\left(\sum_{i=1}^{n}\int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|(\rho_\ell W)x_i|^{q_j}\,dx\right)$$
$$\leq\sum_{j=1}^{m}\frac{C}{q_j}\left(\sum_{i=1}^{n}\int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|\rho_\ell W_{x_i}|^{q_j}+|W(\rho_\ell)_{x_i}|^{q_j}\,dx\right).$$

Using the properties of ρ_ℓ, we can further estimate and, for some other constant $D_W>0$, obtain

$$\mathcal{A}_\ell\leq\sum_{j=1}^{m}\frac{C}{q_j}\left(\sum_{i=1}^{n}\int\limits_{\Omega_\ell\setminus\Omega_{\ell-1}}|W_{x_i}|^{q_j}+|W|^p dx\right)=\frac{D_W}{p}\mu_{n-r}(\ell\omega_1\setminus(\ell-1)\omega_1)=C\ell^{n-r-1}.$$

Combining the above estimates, we get

$$\frac{J_\ell(u_\ell)}{\mu_{n-r}(\ell\omega_1)}\leq J_{\omega_2}(W)+\frac{C}{\ell}.$$

This finishes the proof of the theorem. □

4. APPENDIX

We will show the following claim:

$$J_\ell(u_\ell)=\inf_{u\in W_0^{1,q_1}(\Omega_\ell)}J_\ell(u).$$

Proof. Fix $\ell>0$. First we have to show that

$$\inf_{u\in W_0^{1,q_1}(\Omega_\ell)}J_\ell(u)>-\infty.$$

By definition, for $u\in W_0^{1,q_1}(\Omega_\ell)$,

$$J_\ell(u)=\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{q_j}\int\limits_{\Omega_\ell}|u_{x_i}|^{q_j}\,dx-\int\limits_{\Omega_\ell}fudx\geq\sum_{i=1}^{n}\frac{1}{q_1}\int\limits_{\Omega_\ell}|u_{x_i}|^{q_1}\,dx-\int\limits_{\Omega_\ell}fudx.$$

Now using Young's inequality, for $a,b\in\mathbb{R}$ and $\epsilon>0,p>1$,

$$ab\leq\epsilon a^p+\frac{1}{\epsilon}b^{p'},$$

we get

$$J_\ell(u)\geq\sum_{i=1}^{n}\frac{1}{q_1}\int\limits_{\Omega_\ell}|u_{x_i}|^{q_1}\,dx-\frac{1}{\epsilon}\int\limits_{\Omega}|f|^{q_1'}dx-\epsilon\int\limits_{\Omega}|u|^{q_1}\,dx.$$

Now using (2.2), we have for some constant $C > 0$,

$$J_\ell(u) \geq (C-\epsilon)\int\limits_{\Omega} |u|^{q_1}\,dx - \frac{1}{\epsilon}\int\limits_{\Omega} |f|^{q_1'}\,dx \geq -\frac{1}{\epsilon}\int\limits_{\Omega} |f|^{q_1'}\,dx.$$

The last inequality is ensured by choosing small $\epsilon > 0$. Now the next step is to show that the infimum value is achieved by a unique function $u_0 \in W^{1,q_1}(\Omega_\ell)$. Let u_n be a minimizing sequence, that is,

$$J_\ell(u_n) \to \inf_{u \in W_0^{1,q_1}(\Omega_\ell)} J_\ell(u).$$

This implies that

$$J_\ell(u_n) = \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{1}{q_j}\int\limits_{\Omega_\ell} |(u_n)_{x_i}|^{q_j}\,dx - \int\limits_{\Omega_\ell} f u_n dx \leq M.$$

Dropping all the terms in the double summation, except for the term containing the exponent q_1, we have

$$\sum_{i=1}^{n} \frac{1}{q_1}\int\limits_{\Omega_\ell} |(u_n)_{x_i}|^{q_1}\,dx \leq M + \int\limits_{\Omega_\ell} f u_n dx. \tag{4.1}$$

Now, using Lemma (2.2) together with Young's inequality, one can show that the sequence u_n is uniformly bounded in $W_0^{1,q_1}(\Omega_\ell)$ and hence weakly converges to some function u_ℓ. By the lower semicontinuity of the functional J_ℓ, it follows that

$$J_\ell(u_\ell) = \inf_{u \in W_0^{1,q_1}(\Omega_\ell)} J_\ell(u).$$

The uniqueness of the function u_ℓ follows from the strict convexity of the functional J_ℓ. Therefore, it remains to show that u_ℓ is a weak solution of (1.1). This part is standard and can be obtained by differentiating the real-valued function

$$f(t) := J_\ell(u_\ell + t\phi)$$

at $t = 0$, where ϕ is an arbitrary function in $C_c^\infty(\Omega_\ell)$, and using the fact that f attains its minimum at $t = 0$.

Let u_ℓ and v_ℓ be two different solutions of (1.1). Then we have

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\int\limits_{\Omega_\ell} |(u_\ell)_{x_i}|^{q_j-2}\,(u_\ell)_{x_i} v_{x_i} dx$$
$$-\sum_{j=1}^{m}\sum_{i=1}^{n}\int\limits_{\Omega_\ell} |(v_\ell)_{x_i}|^{q_j-2}\,(v_\ell)_{x_i} v_{x_i} dx = 0, \quad \forall v \in W_0^{1,q_1}(\Omega_\ell).$$

That is,

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\int_{\Omega_\ell}\left\{|(u_\ell)_{x_i}|^{q_j-2}(u_\ell)_{x_i}-|(v_\ell)_{x_i}|^{q_j-2}(v_\ell)_{x_i}\right\}v_{x_i}dx=0,\quad \forall v\in W_0^{1,q_1}(\Omega_\ell).$$

Choosing $v=u_\ell-v_\ell$ and using (2.3), we obtain

$$\sum_{j=1}^{m}C_{p_j}\int_{\Omega_\ell}|\nabla(u_\ell-v_\ell)|^{p_j}dx=0.$$

Hence, $u_\ell=v_\ell$ almost everywhere. □

Acknowledgements
The author would like to thank Prof. Michel Chipot for his valuable comments on an earlier version of this article. The author also thanks the editor, J.I. Díaz, for prompt communication.

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Received: April 25, 2025.
Revised: June 30, 2025.
Accepted: July 6, 2025.