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TAIL PROBABILITY OF THE HITTING TIME OF BROWNIAN MOTION TO A SPHERE WITH FIXED HITTING SITES

Yuji Hamana

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Abstract. We consider d-dimensional Brownian motion $\{B_\mu(t)\}_{t\geqq 0}$ with a drift $\mu \in \mathbb{R}^d$ and the first hitting time $\sigma_{r,\mu}^{(d)}$ to the sphere with radius r centered at the origin. This article deals with asymptotic behavior of the probability that both $t < \sigma_{r,\mu}^{(d)} < \infty$ and $B_\mu(\sigma_{r,\mu}^{(d)}) \in A$ occur simultaneously, and we obtain that this probability admits an asymptotic expansion in powers of $1/t$ if $d \geqq 3$ and in that of $1/\log t$ if $d = 2$ for large t. Moreover, we investigate the case of Brownian motion with no drift.

Keywords: Brownian motion, hitting times and sites, asymptotic expansion.

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1. INTRODUCTION AND MAIN RESULTS

In the theory of mathematical finance, random times are significant objects for pricing securities and formulating a strategy for buying and selling. In particular, the hitting time of Ornstein–Uhlenbeck process is applied frequently for the investigate of optimization problems on the pairs trading strategy. Cameron–Martin formula gives that the hitting time to sphere of Ornstein–Uhlenbeck process is represented by the joint distribution of a random time of Brownian motion and a stochastic integral of a suitable function over 0 to the random time. This paper deals with the case that the integrand is a constant and that the random time is the first hitting time to sphere of Brownian motion. Namely, we investigate the joint distribution of the hitting time to sphere of Brownian motion and its arrival site in this article.

We treat a stochastic process $\{B_\mu(t)\}_{t\geqq 0}$ defined by $B_\mu(t) = B(t) + \mu t$ for $t \geqq 0$, where μ is a given vector in $\mathbb{R}^d$ and $\{B(t)\}_{t\geqq 0}$ is a standard Brownian motion on $\mathbb{R}^d$ starting from x. Let $d \geqq 2$ and S_r^{d-1} denote the sphere with radius r in $\mathbb{R}^d$ centered at the origin. We write $\sigma_{r,\mu}^{(d)}$ for the first time that $\{B_\mu(t)\}_{t\geqq 0}$ reaches S_r^{d-1}. The density and the distribution functions of $\sigma_{r,0}^{(d)}$ are obtained in [1, 5, 6, 11]. When the starting point x is inside of S_r^{d-1} but not 0, the joint density function of $\sigma_{r,0}^{(d)}$ and

$B_0(\sigma_{r,0}^{(d)})$ is given in [10] by solving some heat conduction equation under a suitable Dirichlet boundary condition. Since the paper [10] deals with the exit problem, the joint distribution is discussed only for $|x| < r$.

In the case that $\mu \neq 0$, the distribution of $\sigma_{r,\mu}^{(d)}$ is discussed in [15] and the joint density of $\sigma_{r,\mu}^{(d)}$ and $B_\mu(\sigma_{r,\mu}^{(d)})$ is given in [16] for $|x| < r$ with no proofs. Regardless of the location of the starting point, the density function of $\sigma_{r,\mu}^{(d)}$ is deduced in [8] and a formula for the density of $(\sigma_{r,\mu}^{(d)}, B_\mu(\sigma_{r,\mu}^{(d)}))$ is obtained in [9, Theorem 1.2]. The skew-product representation of Brownian motion is useful for deriving the joint Laplace transform of $\sigma_{r,\mu}^{(d)}$ and $B_\mu(\sigma_{r,\mu}^{(d)})$.

For $t > 0$ and a fixed Borel set A in S_r^{d-1} let

$$F_{x,r,A}^{(d)}(t;\mu) = P_x(t < \sigma_{r,\mu}^{(d)} < \infty, B_\mu(\sigma_{r,\mu}^{(d)}) \in A),$$

where P_y denotes the probability law of a Brownian motion starting from y. The leading term of $F_{x,r,A}^{(d)}(t;\mu)$ for large t is provided in [9, Theorem 2.6] when the starting point x is outside of S_r^{d-1}. The purpose of this article is to establish an asymptotic expansion of $F_{x,r,A}^{(d)}(t;\mu)$ as $t \to \infty$ for $|x| > r$.

We first give results for the case that $d \geqq 3$. The notation $\langle \cdot, \cdot \rangle$ will be used for the standard inner product in $\mathbb{R}^d$. Moreover, we put $\rho_t = t/2r^2$ for convenience and simply write $\log^n y$ instead of $(\log y)^n$ for $y > 0$ and an integer $n \geqq 0$ similarly to trigonometric functions.

Theorem 1.1. *Let $\mu \neq 0$, $|x| > r$ and $N \geqq 0$ be a given integer.*

(1) *If d is odd and not less than 3, we have that*

$$F_{x,r,A}^{(d)}(t;\mu) = \frac{2e^{-\langle x,\mu\rangle}}{|\mu|^2} \frac{e^{-|\mu|^2 t/2}}{t^{\nu(d)+1}} \left(\sum_{n=0}^{N} \frac{\kappa_n^{(d)}}{t^n} + O\left[\frac{1}{t^{N+1}}\right]\right) \tag{1.1}$$

as $t \to \infty$ for a suitable sequence $\{\kappa_n^{(d)}\,;\, 0 \leqq n \leqq N\}$ which depends on x, r, μ, A.

(2) *If d is even and not less than 4, we have that*

$$\begin{aligned} F_{x,r,A}^{(d)}(t;\mu) = \frac{2e^{-\langle x,\mu\rangle}}{|\mu|^2} \frac{e^{-|\mu|^2 t/2}}{t^{\nu(d)+1}} \Bigg(&\sum_{j=0}^{N} \sum_{n=0}^{\nu(d)-1} \sum_{k=0}^{j} \frac{\kappa_{n,j,k}^{(d)} \log^k \rho_t}{t^{\nu(d)j+n}} \\ &+ O\left[\frac{\log^{N+1}\rho_t}{t^{\nu(d)(N+1)}}\right]\Bigg) \end{aligned} \tag{1.2}$$

as $t \to \infty$ for a suitable sequence $\{\kappa_{n,j,k}^{(d)}\,;\, 0 \leqq k \leqq j \leqq N, 0 \leqq n \leqq \nu(d)-1\}$ which depends on x, r, μ, A.

It is remarkable that the expansion has many terms of the power of $\log t$ in the even dimensional case. These additional terms do not appear in the odd dimensional case.

We should mention that (1.1) and (1.2) are built on the following formula for the joint density function of $\sigma^{(d)}_{r,\mu}$ and $B_\mu(\sigma^{(d)}_{r,\mu})$ given in [9, Theorem 1.2]: For $t > 0$ and $z \in \mathbb{R}^d$ with $|z| = r$, we have that, if $d \geqq 3$,

$$\begin{aligned} &P_{|x|e}(\sigma^{(d)}_{r,\mu} \in dt, B_\mu(\sigma^{(d)}_{r,\mu}) \in dz) \\ &= e^{-|x|\langle e,\mu\rangle + \langle \mu, z\rangle} e^{-|\mu|^2 t/2} \\ &\quad \times \sum_{n=0}^{\infty} \frac{\nu(d)+n}{\nu(d)} \left(\frac{|x|}{r}\right)^n \phi^{(d+2n)}_{x,r}(t) C^{\nu(d)}_n\left(\frac{\langle e, z\rangle}{r}\right) dt ds^{d-1}_r(z) \end{aligned} \tag{1.3}$$

where $e = (1, 0, \dots 0) \in \mathbb{R}^d$, $\nu(m) = m/2 - 1$ for an integer $m \geqq 1$, $\phi^{(m)}_{x,r}$ denotes the density function of $\sigma^{(m)}_{r,0}$ under the probability measure P_x, s^{d-1}_r is the uniform probability measure on S^{d-1}_r and C^μ_n is Gegenbauer polynomial. The functions $C^\mu_n(x)$ are defined as the coefficients in the Taylor expansion

$$\sum_{n=0}^{\infty} C^\mu_n(x) z^n = \begin{cases} (1 - 2xz - z^2)^{-\mu} & \text{if } \mu > 0, \\ 1 - \log(1 - 2xz + z^2) & \text{if } \mu = 0. \end{cases}$$

For more details, see [12, 13].

We next provide the result in the two dimensional case.

Theorem 1.2. *Let $\mu \neq 0$, $|x| > r$ and $N \geqq 0$ be an integer. If $d = 2$, we have that*

$$F^{(2)}_{x,r,A}(t;\mu) = \frac{4e^{-\langle x,\mu\rangle}}{|\mu|^2} \log\frac{|x|}{r} \frac{e^{-|\mu|^2 t/2}}{t \log^2 \rho_t} \left(\sum_{n=0}^{N} \frac{\kappa^{(2)}_n}{\log^n \rho_t} + O\left[\frac{1}{\log^{N+1} \rho_t}\right]\right) \tag{1.4}$$

as $t \to \infty$ for a suitable sequence $\{\kappa^{(2)}_n\,;\, 0 \leqq n \leqq N\}$ which depends on x, r, μ, A.

Similarly to the case that $d \geqq 3$ Theorem 1.2 is based on the following explicit form of the density of $(\sigma^{(2)}_{r,\mu}, B_\mu(\sigma^{(2)}_{r,\mu}))$ given in [9, Theorem 1.2]:

$$\begin{aligned} &P_{|x|e}(\sigma^{(2)}_{r,\mu} \in dt, B_\mu(\sigma^{(2)}_{r,\mu}) \in dz) \\ &= e^{-|x|\langle e,\mu\rangle + \langle \mu, z\rangle} e^{-|\mu|^2 t/2} \phi^{(2)}_{x,r}(t) dt ds^1_r(z) \\ &\quad + e^{-|x|\langle e,\mu\rangle + \langle \mu, z\rangle} e^{-|\mu|^2 t/2} \\ &\quad\quad \times \sum_{n=1}^{\infty} n \left(\frac{|x|}{r}\right)^n \phi^{(2n+2)}_{x,r}(t) C^0_n\left(\frac{\langle e, z\rangle}{r}\right) dt ds^1_r(z) \end{aligned} \tag{1.5}$$

for $t > 0$ and $z \in \mathbb{R}^2$ with $|z| = r$.

This article is organized as follows. In Section 2 we represent $F^{(d)}_{x,r,A}(t;\mu)$ by the density function of $\sigma^{(m)}_{r,0}$. Section 3 deals with an asymptotic expansion of $F^{(d)}_{x,r,A}(t;\mu)$ for $d \geqq 3$ with the help of an expansion of the density function of $\sigma^{(m)}_{r,0}$ denoted by $\phi^{(m)}_{x,r}(t)$. Section 4 is devoted to a proof of asymptotic expansion of $\phi^{(m)}_{x,r}(t)$ which is used in Section 3. In Section 5 we investigate the two dimensional case and provide an asymptotic expansion of $F^{(2)}_{x,r,A}(t;\mu)$. Section 6 deals with the case of a Brownian motion with no drift, and we establish an asymptotic expansion of $F^{(d)}_{x,r,A}(t;0)$. In addition, throughout this paper, we will use $c_1, c_2, c_3, \ldots, c_{10}$ for suitable constants.

2. REPRESENTATION OF $F^{(d)}_{x,r,A}(t;\mu)$

The purpose of this section is to represent $F^{(d)}_{x,r,A}(t;\mu)$ for $|x| > r$ by means of the density function $\phi^{(m)}_{x,r}$ and the Gegenbauer polynomial C^{ν}_n

Let T be one of orthogonal transformations satisfying that $Tx = |x|e$. The rotational invariance of $\{B(t)\}_{t \geqq 0}$ immediately yields that

$$F^{(d)}_{x,r,A}(t;\mu) = P_{|x|e}(t < \sigma^{(d)}_{r,T\mu} < \infty, B_{T\mu}(\sigma^{(d)}_{r,T\mu}) \in TA). \tag{2.1}$$

In order to give a formula for $F^{(d)}_{x,r,A}(t;\mu)$ we need to prepare two estimates. For $t > 0$, $w \in \mathbb{R}^d$ and an integer $m \geqq 0$ we put

$$H^{(m)}_{x,r}(t;w) = \int_t^\infty e^{-|w|^2 s/2} \phi^{(m)}_{x,r}(s) ds.$$

Since we have that $\phi^{(m)}_{x,r}(t)$ coincides with $\phi^{(m)}_{|x|e,r}(t)$ by the rotational invariance of Brownian motions, it follows that

$$H^{(m)}_{|x|e,r}(t;Tw) = H^{(m)}_{|x|e,r}(t;w) = H^{(m)}_{x,r}(t;w) \tag{2.2}$$

for any $t > 0$. Lemma 2.7 in [9] immediately yields the following lemma.

Lemma 2.1. *Let $m \geqq 3$ be an integer. For $|x| > r$ we have that*

$$H^{(m)}_{x,r}(t;\mu) \leqq \frac{c_1 r^{2\nu(m)}}{\Gamma(\nu(m))} \frac{e^{-|\mu|^2 t/2}}{(2t)^{\nu(m)+1}} \tag{2.3}$$

and the constant c_1 can be chosen independently of m.

We next provide an estimate concerned with Gegenbauer polynomials.

Lemma 2.2. *Let $u, w \in \mathbb{R}^d$ with $u \neq 0$. We have that*

$$\int_A e^{\langle w,z \rangle} \left| C^{\nu(d)}_n \left(\frac{\langle u,z \rangle}{r|u|} \right) \right| ds^{d-1}_r(z) \leqq 2^{d-1} e^{r|w|} n^{d-3} \tag{2.4}$$

for any integer $n \geqq 1$.

Proof. We first consider the cases that $d \geqq 3$. It is known that

$$\max_{|y|\leqq 1} |C_n^\nu(y)| = \frac{\Gamma(n+2\nu)}{\Gamma(2\nu)n!}$$

for $\nu > 0$ (cf. [13, p. 223]) and thus we obtain that, if $d \geqq 4$,

$$\max_{|y|\leqq 1} \left| C_n^{\nu(d)}(y) \right| = \prod_{k=1}^{d-3} \frac{n+k}{k} \leqq (n+1)^{d-3} \leqq (2n)^{d-3}. \tag{2.5}$$

Moreover, since

$$\max_{|y|\leqq 1} \left| C_n^{1/2}(y) \right| = \frac{\Gamma(n+1)}{\Gamma(1)n!} = 1,$$

(2.5) is available in the case that $d = 3$. Hence, it follows from $s_r^{d-1}(A) \leqq 1$ that a bound of the left-hand side of (2.4) is $2^{d-3}e^{r|w|}n^{d-3}$, which yields in particular (2.4) for $d \geqq 3$.

The proof of (2.4) for $d = 2$ is easy. Indeed, since $|C_n^0(y)| \leqq 2/n$ for $|y| \leqq 1$ and $n \geqq 1$ (cf. [13, p. 218]), the left-hand side of (2.4) is dominated by $2e^{r|w|}/n$. This implies (2.4) for $d = 2$. □

For $u, w \in \mathbb{R}^d$ with $u \neq 0$ and an integer $n \geqq 0$ let

$$\Lambda_n^{(d)}(u, r, w, A) = \left(\frac{|u|}{r}\right)^n \int_A e^{\langle w,z\rangle} C_n^{\nu(d)}\left(\frac{\langle u, z\rangle}{r|u|}\right) ds_r^{d-1}(z). \tag{2.6}$$

We are ready to obtain an explicit form of $F_{x,r,A}^{(d)}(t;\mu)$ by using (1.3) and (1.5).

Proposition 2.3. *Let $\mu \neq 0$ and $|x| > r$. We have that, if $d \geqq 3$,*

$$F_{x,r,A}^{(d)}(t;\mu) = e^{-\langle x,\mu\rangle} \sum_{n=0}^{\infty} \frac{\nu(d)+n}{\nu(d)} H_{x,r}^{(d+2n)}(t;\mu) \Lambda_n^{(d)}(x, r, \mu, A) \tag{2.7}$$

and that, if $d = 2$,

$$\begin{aligned} F_{x,r,A}^{(2)}(t;\mu) &= e^{-\langle x,\mu\rangle} H_{x,r}^{(2)}(t;\mu) \Lambda_0^{(2)}(x, r, \mu, A) \\ &\quad + e^{-\langle x,\mu\rangle} \sum_{n=1}^{\infty} n H_{x,r}^{(2n+2)}(t;\mu) \Lambda_n^{(2)}(x, r, \mu, A). \end{aligned} \tag{2.8}$$

Proof. We first consider the case when $d \geqq 3$. Combining (1.3) and (2.1), we have that $F_{x,r,A}^{(d)}(t;\mu)$ is equal to

$$\begin{aligned} &e^{-\langle |x|e,T\mu\rangle} \int_t^\infty ds \int_{TA} ds_r^{d-1}(z) \sum_{n=0}^{\infty} \frac{\nu(d)+n}{\nu(d)} \left(\frac{|x|}{r}\right)^n \\ &\qquad\qquad \times e^{-|T\mu|^2 s/2} \phi_{x,r}^{(d+2n)}(s) e^{\langle T\mu,z\rangle} C_n^{\nu(d)}\left(\frac{\langle e, z\rangle}{r}\right). \end{aligned}$$

For changing the order of the double integrals and the summation it is sufficient to prove that

$$\sum_{n=3}^{\infty} \frac{\nu(d)+n}{\nu(d)} \left(\frac{|x|}{r}\right)^n H_{x,r}^{(d+2n)}(t;\mu) \int_{TA} e^{\langle T\mu, z\rangle} \left| C_n^{\nu(d)}\left(\frac{\langle e, z\rangle}{r}\right)\right| ds_r^{d-1}(z) \tag{2.9}$$

converges. It can be easily shown by (2.3) and (2.4). Indeed, (2.9) is bounded by

$$\frac{c_1 2^{d-1} r^{2\nu(d)} e^{r|\mu|} e^{-|\mu|^2 t/2}}{(2t)^{\nu(d)+1}} \sum_{n=3}^{\infty} \frac{\nu(d)+n}{\nu(d)} \left(\frac{r|x|}{2t}\right)^n \frac{n^{d-3}}{\Gamma(\nu(d)+n)}$$

and then the sum on n converges. Hence, we obtain that

$$F_{x,r,A}^{(d)}(t;\mu) = e^{-\langle x,\mu\rangle} \sum_{n=0}^{\infty} \frac{\nu(d)+n}{\nu(d)} H_{x,r}^{(d+2n)}(t;\mu) \Lambda_n^{(d)}(|x|e, r, T\mu, TA).$$

We next consider the case that $d = 2$ and then deduce from (1.5) and (2.2) that $F_{x,r,A}^{(2)}(t;\mu)$ is equal to $e^{-\langle |x|e, T\mu\rangle}$ multiple of

$$\begin{aligned} &H_{x,r}^{(2)}(t;\mu)\Lambda_0^{(2)}(|x|e, r, T\mu, TA) \\ &+ \int_t^{\infty} ds \int_{TA} ds_r^1(z) \sum_{n=1}^{\infty} n \left(\frac{|x|}{r}\right)^n e^{-|T\mu|^2 s/2} \phi_{x,r}^{(2n+2)}(s) e^{\langle T\mu, z\rangle} C_n^0\left(\frac{\langle e, z\rangle}{r}\right). \end{aligned}$$

We need to change the order of the double integrals and the summation. Similarly to the case that $d \geqq 3$ we should prove the convergence of

$$\sum_{n=3}^{\infty} n \left(\frac{|x|}{r}\right)^n H_{x,r}^{(2n+2)}(t;\mu) \int_{TA} e^{\langle T\mu, z\rangle} \left| C_n^0\left(\frac{\langle e, z\rangle}{r}\right)\right| ds_r^1(z). \tag{2.10}$$

In virtue of (2.3) and (2.4) we have that (2.10) is dominated by

$$\frac{c_1 e^{r|\mu|} e^{-|\mu|^2 t/2}}{t} \sum_{n=3}^{\infty} \left(\frac{r|x|}{2t}\right)^n \frac{1}{\Gamma(n)}$$

and hence obtain that

$$\begin{aligned} F_{x,r,A}^{(2)}(t;\mu) = {} & e^{-\langle x,\mu\rangle} H_{x,r}^{(2)}(t;\mu) \Lambda_0^{(2)}(|x|e, r, T\mu, TA) \\ & + e^{-\langle x,\mu\rangle} \sum_{n=1}^{\infty} n H_{x,r}^{(2n+2)}(t;\mu) \Lambda_n^{(2)}(|x|e, r, T\mu, TA). \end{aligned}$$

It remains to showing that, if $d \geqq 2$,

$$\Lambda_n^{(d)}(|x|e, r, Tw, TA) = \Lambda_n^{(d)}(x, r, w, A) \tag{2.11}$$

for $n \geqq 0$. It follows from (2.6) that

$$\Lambda_n^{(d)}(|x|e, r, Tw, TA) = \left(\frac{|x|}{r}\right)^n \int\limits_{TA} e^{\langle Tw, z\rangle} C_n^{\nu(d)}\left(\frac{\langle |x|e, z\rangle}{r|x|}\right) ds_r^{d-1}(z). \tag{2.12}$$

The change of variables from z to y given by $z = Ty$ yields that the integral in the right-hand side of (2.12) coincides with

$$\int\limits_A e^{\langle Tw, Ty\rangle} C_n^{\nu(d)}\left(\frac{\langle Tx, Ty\rangle}{r|x|}\right) ds_r^{d-1}(y) = \int\limits_A e^{\langle w, y\rangle} C_n^{\nu(d)}\left(\frac{\langle x, y\rangle}{r|x|}\right) ds_r^{d-1}(y).$$

Hence, (2.11) holds. □

3. ASYMPTOTIC EXPANSION OF $F_{x,r,A}^{(d)}(t;\mu)$ FOR $d \geqq 3$

Our goal of this section is to prove Theorem 1.1. Theorem 2.6 in [9] provides that

$$F_{x,r,A}^{(d)}(t;\mu) = \frac{2e^{-\langle x,\mu\rangle}\nu(d)}{|\mu|^2} \zeta_{x,r}^{(d)} \Lambda_0^{(d)}(x, r, \mu, A) \frac{e^{-|\mu|^2 t/2}}{t^{d/2}}(1 + o[1]) \tag{3.1}$$

as $t \to \infty$, where

$$\zeta_{x,r}^{(d)} = \left(\frac{r^2}{2}\right)^{d/2-1} \frac{1}{\Gamma(d/2-1)}\left\{1 - \left(\frac{r}{|x|}\right)^{d-2}\right\}.$$

We shall consider higher terms of $F_{x,r,A}^{(d)}(t;\mu)$. Lemmas 2.1 and 2.2 are effective for an estimate of the error term of $F_{x,r,A}^{(d)}(t;\mu)$ and hence the following lemma holds.

Lemma 3.1. *Let $d \geqq 3$, $|x| > r$ and $\mu \neq 0$. For a given integer $L \geqq 0$ we have that*

$$\begin{aligned} F_{x,r,A}^{(d)}(t;\mu) = e^{-\langle x,\mu\rangle} &\sum_{n=0}^{L} \frac{\nu(d)+n}{\nu(d)} H_{x,r}^{(d+2n)}(t;\mu) \Lambda_n^{(d)}(x, r, \mu, A) \\ &+ O\left[\frac{e^{-|\mu|^2 t/2}}{t^{\nu(d)+L+2}}\right] \end{aligned} \tag{3.2}$$

as $t \to \infty$.

Proof. The formula (2.7) yields that it is sufficient to estimate

$$\sum_{n=L+1}^{\infty} (\nu(d)+n) H_{x,r}^{(d+2n)}(t;\mu) \left|\Lambda_n^{(d)}(x, r, \mu, A)\right|. \tag{3.3}$$

In virtue of (2.3) and (2.4) we have that (3.3) is bounded by a constant multiple of

$$\frac{e^{-|\mu|^2 t/2}}{t^{\nu(d)+L+2}} \sum_{n=L+1}^{\infty} \frac{\nu(d)+n}{\Gamma(\nu(d)+n)} \left(\frac{r|x|}{2}\right)^n n^{d-3}$$

for $t \geqq 1$. Since the summation converges, we conclude that (3.2) holds. □

We can find by (3.2) that asymptotic expansion of $H_{x,r}^{(m)}(t;\mu)$ is quite significant. Recall that $H_{x,r}^{(m)}(t;w)$ has been defined by

$$H_{x,r}^{(m)}(t;w)=\int_t^\infty e^{-|w|^2 s/2}\phi_{x,r}^{(m)}(s)ds$$

in the previous section. In order to investigate behavior of $H_{x,r}^{(m)}(t;\mu)$ we should derive asymptotic expansion of $\phi_{x,r}^{(m)}(t)$ for large t.

Proposition 3.2. *Let $|x|>r$, $m\geqq 3$ and $N\geqq 0$ be an integer.*

(1) *If m is odd, there is a sequence $\{\alpha_n^{(m)}\,;\,0\leqq n\leqq N\}$ of constants, which depend on x and r, such that*

$$\phi_{x,r}^{(m)}(t)=\sum_{n=0}^{N}\frac{\alpha_n^{(m)}}{t^{\nu(m)+n+1}}+O\biggl[\frac{1}{t^{\nu(m)+N+2}}\biggr] \tag{3.4}$$

as $t\to\infty$.

(2) *If m is even, there is a sequence $\{\alpha_{n,j,k}^{(m)}\,;\,0\leqq n\leqq\nu(m)-1, 0\leqq k\leqq j\leqq N\}$ of constants, which depend on x and r, such that*

$$\phi_{x,r}^{(m)}(t)=\sum_{j=0}^{N}\sum_{n=0}^{\nu(m)-1}\sum_{k=0}^{j}\frac{\alpha_{n,j,k}^{(m)}\log^k\rho_t}{t^{\nu(m)(j+1)+n+1}}+O\biggl[\frac{\log^{N+1}\rho_t}{t^{\nu(m)(N+2)+1}}\biggr] \tag{3.5}$$

as $t\to\infty$.

The proof of this proposition is deferred to Section 4. In considering an asymptotic expansion of $H_{x,r}^{(m)}(t;\mu)$ via Proposition 3.2, we need to derive an expansion of $H(t;\alpha,\beta,n)$ which is defined by

$$H(t;\alpha,\beta,n)=\int_t^\infty e^{-\alpha s}s^{-\beta}\log^n\rho_s\,ds$$

for $\alpha\geqq 0$, $\beta>0$, $t>0$ and $n\geqq 0$. It is obvious that $H(t;\alpha,\beta,n)\geqq 0$ if $t\geqq 2r^2$. A change of variables from s to v given by $v=\alpha(s-t)$ yields that

$$\begin{aligned}H(t;\alpha,\beta,n)&=\frac{e^{-\alpha t}}{\alpha}\int_0^\infty e^{-v}\Bigl(t+\frac{v}{\alpha}\Bigr)^{-\beta}\log^n\Bigl\{\frac{1}{2r^2}\Bigl(t+\frac{v}{\alpha}\Bigr)\Bigr\}dv\\&=\frac{e^{-\alpha t}}{\alpha t^\beta}\int_0^\infty e^{-v}\Bigl(1+\frac{v}{\alpha t}\Bigr)^{-\beta}\Bigl\{\log\rho_t+\log\Bigl(1+\frac{v}{\alpha t}\Bigr)\Bigr\}^n dv.\end{aligned}$$

It follows from the binomial theorem that $H(t;\alpha,\beta,n)$ is represented by

$$\frac{e^{-\alpha t}}{\alpha t^\beta}\sum_{k=0}^{n}\binom{n}{k}H_0(t;\alpha,\beta,n-k)\log^k\rho_t,$$

where

$$H_0(t;\alpha,\beta,m)=\int_0^\infty e^{-v}\left(1+\frac{v}{\alpha t}\right)^{-\beta}\log^m\left(1+\frac{v}{\alpha t}\right)dv$$

for an integer $m \geqq 0$.

For an asymptotic expansion of $H(t;\alpha,\beta,n)$ we concentrate on considering behavior of $H_0(t;\alpha,\beta,m)$ for large t. Since $\log(1+y)\leqq y$ for $y>-1$, we have that

$$\int_{\alpha t}^\infty e^{-v}\left(1+\frac{v}{\alpha t}\right)^{-\beta}\log^m\left(1+\frac{v}{\alpha t}\right)dv \leqq \frac{1}{(\alpha t)^m}\int_{\alpha t}^\infty e^{-v}v^m dv = O[e^{-\alpha t}]$$

as $t\to\infty$. Let $L\geqq 1$ be a given integer. The Taylor theorem yields that there exists a sequence $\{a_i^m(\beta); 0\leqq i\leqq L\}$ such that

$$\left|(1+y)^{-\beta}\log^m(1+y)-\sum_{i=0}^L a_i^m(\beta)y^i\right|\leqq c_2 y^{L+1}$$

for $0<y<1$. Hence, we obtain that

$$\int_0^{\alpha t} e^{-v}\left(1+\frac{v}{\alpha t}\right)^{-\beta}\log^m\left(1+\frac{v}{\alpha t}\right)dv=\sum_{i=0}^L\frac{a_i^m(\beta)}{(\alpha t)^i}\int_0^{\alpha t}e^{-v}v^i dv + O\left[\frac{1}{t^{L+1}}\right].$$

Note that

$$\int_0^{\alpha t}e^{-v}v^i dv = \Gamma(i+1)+O[t^i e^{-\alpha t}]$$

and hence we can conclude that

$$H_0(t;\alpha,\beta,m)=\sum_{i=0}^L\frac{a_i^m(\beta)\Gamma(i+1)}{\alpha^i}\frac{1}{t^i}+O\left[\frac{1}{t^{L+1}}\right].$$

Put

$$b_{k,i}^n(\alpha,\beta)=\binom{n}{k}\frac{a_i^{n-k}(\beta)\Gamma(i+1)}{\alpha^i}$$

and then we have the following lemma.

Lemma 3.3. *Let $\alpha>0$ and $\beta>0$. For integers L and n with $0\leqq n\leqq L$ we have that*

$$H(t;\alpha,\beta,n)=\frac{e^{-\alpha t}}{\alpha t^\beta}\sum_{k=0}^n\sum_{i=0}^L\frac{b_{k,i}^n(\alpha,\beta)\log^k\rho_t}{t^i}+O\left[\frac{e^{-\alpha t}\log^n\rho_t}{t^{\beta+L+1}}\right]\tag{3.6}$$

as $t\to\infty$. Especially the following estimate holds:

$$H(t;\alpha,\beta,n)\leqq\frac{c_3 e^{-\alpha t}\log^n\rho_t}{t^\beta}.\tag{3.7}$$

Recall that we consider the case that $\mu \neq 0$. It is easy to see by Proposition 3.2 that, if m is odd and not less than 3,

$$\begin{aligned} H_{x,r}^{(m)}(t;\mu) = \sum_{j=0}^{N} \alpha_j^{(m)} H\Big(t; \frac{1}{2}|\mu|^2, \nu(m)+j+1, 0\Big) \\ + O\Big[H\Big(t; \frac{1}{2}|\mu|^2, \nu(m)+N+2, 0\Big)\Big] \end{aligned} \tag{3.8}$$

and that, if m is even and not less than 4,

$$\begin{aligned} H_{x,r}^{(m)}(t;\mu) = \sum_{j=0}^{N} \sum_{n=0}^{\nu(m)-1} \sum_{k=0}^{j} \alpha_{n,j,k}^{(m)} H\Big(t; \frac{1}{2}|\mu|^2, \nu(m)(j+1)+n+1, k\Big) \\ + O\Big[H\Big(t; \frac{1}{2}|\mu|^2, \nu(m)(N+2)+1, N+1\Big)\Big]. \end{aligned} \tag{3.9}$$

Lemma 3.4. *Let $|x| > r$, $\mu \neq 0$ and $m \geqq 3$.*

(1) *If m is odd, we have that*

$$H_{x,r}^{(m)}(t;\mu) = \frac{2e^{-|\mu|^2 t/2}}{|\mu|^2} \sum_{n=0}^{N} \frac{b_n^{(m)}}{t^{\nu(m)+n+1}} + O\Big[\frac{e^{-|\mu|^2 t/2}}{t^{\nu(m)+N+2}}\Big] \tag{3.10}$$

as $t \to \infty$ for a suitable sequence $\{b_n^{(m)}\,;\, 0 \leqq n \leqq N\}$.

(2) *If m is even, we have that*

$$\begin{aligned} H_{x,r}^{(m)}(t;\mu) = \frac{2e^{-|\mu|^2 t/2}}{|\mu|^2} \sum_{j=0}^{N} \sum_{n=0}^{\nu(m)-1} \sum_{k=0}^{j} \frac{b_{n,j,k}^{(m)} \log^k \rho_t}{t^{\nu(m)(j+1)+n+1}} \\ + O\Big[\frac{e^{-|\mu|^2/t} \log^{N+1} \rho_t}{t^{\nu(m)(N+2)+1}}\Big] \end{aligned} \tag{3.11}$$

as $t \to \infty$ for a suitable sequence $\{b_{n,j,k}^{(m)}\,;\, 0 \leqq k \leqq j \leqq N, 0 \leqq n \leqq \nu(m)-1\}$.

Proof. We first consider the case that m is odd and not lees than 3. Applying (3.6) for $L = N$ and $n = 0$, we obtain by (3.7) and (3.8) that

$$H_{x,r}^{(m)}(t;\mu) = \frac{2e^{-|\mu|^2 t/2}}{|\mu|^2} \sum_{j=0}^{N} \sum_{i=0}^{N} \frac{\alpha_j^{(m)} b_{0,i}^0(|\mu|^2/2, \nu(m)+j+1)}{t^{\nu(m)+j+i+1}} + O\Big[\frac{e^{-|\mu|^2 t/2}}{t^{\nu(m)+N+2}}\Big].$$

We put

$$b_n^{(m)} = \sum_{k=0}^{n} \alpha_k^{(m)} b_{0,n-k}^0(|\mu|^2/2, \nu(m)+k+1)$$

for $n \geqq 0$ and hence can conclude (3.10).

We next consider the case that m is even and not less than 4. Let M be an integer such that $m = 2M + 2$. Then $\nu(m) = M$ and $M \geqq 1$. For convenience, we use M instead of $\nu(m)$. In virtue of (3.7) and (3.9) we obtain that

$$\begin{aligned} H_{x,r}^{(m)}(t;\mu) = \sum_{h=0}^{N}\sum_{l=0}^{M-1}\sum_{n=0}^{h}\alpha_{l,h,n}^{(m)} H\Big(t; \frac{|\mu|^2}{2}, M(h+1)+l+1, n\Big) \\ + O\Big[\frac{e^{-|\mu|^2/t}\log^{N+1}\rho_t}{t^{M(N+2)+1}}\Big]. \end{aligned} \tag{3.12}$$

Hence, (3.6) for $L = M(N+1) - 1$ yields that the first term of the right-hand side of (3.12) is equal to

$$\frac{2e^{-|\mu|^2 t/2}}{|\mu|^2}\sum_{h=0}^{N}\sum_{l=0}^{M-1}\sum_{n=0}^{h}\sum_{k=0}^{n}\sum_{i=0}^{M(N+1)-1}\frac{\alpha_{l,h,n}^{(m)} b_{k,i}^{n}(|\mu|^2/2, M(h+1)+l+1)\log^k\rho_t}{t^{M(h+1)+l+i+1}},$$

which coincides with

$$\begin{aligned} &\frac{2e^{-|\mu|^2 t/2}}{|\mu|^2}\sum_{h=0}^{N}\sum_{l=0}^{M-1}\sum_{k=0}^{h}\sum_{j=0}^{N}\sum_{i=Mj}^{M(j+1)-1}\frac{b_{l,h,k,i}^{(m,1)}\log^k\rho_t}{t^{M(h+1)+l+i+1}} \\ &= \frac{2e^{-|\mu|^2 t/2}}{|\mu|^2}\sum_{h=0}^{N}\sum_{l=0}^{M-1}\sum_{k=0}^{h}\sum_{j=0}^{N}\sum_{i=0}^{M-1}\frac{b_{l,h,k,i-Mj}^{(m,1)}\log^k\rho_t}{t^{M(h+j+1)+l+i+1}}. \end{aligned} \tag{3.13}$$

Here we have put

$$b_{l,h,k,i}^{(m,1)} = \sum_{n=k}^{h}\alpha_{l,h,n}^{(m)} b_{k,i}^{n}(|\mu|^2/2, M(h+1)+l+1).$$

Moreover, it is easy to see that the summation in the right-hand side of (3.13) is equal to

$$\sum_{h=0}^{N}\sum_{l=0}^{M-1}\sum_{k=0}^{h}\sum_{j=h}^{N}\sum_{i=0}^{M-1}\frac{b_{l,h,k,i-M(j-h)}^{(m,1)}\log^k\rho_t}{t^{M(j+1)+l+i+1}} + O\Big[\frac{\log^N\rho_t}{t^{M(N+2)+1}}\Big]. \tag{3.14}$$

Let

$$b_{n,j,k}^{(m,2)} = \sum_{\substack{l+i=n \\ l\geqq 0, n\geqq 0}}\sum_{h=k}^{i} b_{l,h,k,i-M(j-h)}^{(m,1)},$$

and then we have that the first term of (3.14) is equal to

$$\sum_{n=0}^{M-1}\sum_{j=0}^{N}\sum_{k=0}^{j}\frac{b_{n,j,k}^{(m,2)}\log^k\rho_t}{t^{M(j+1)+n+1}} + \sum_{n=M}^{2M-2}\sum_{j=0}^{N}\sum_{k=0}^{j}\frac{b_{n,j,k}^{(m,2)}\log^k\rho_t}{t^{M(j+1)+n+1}}. \tag{3.15}$$

In the case that $M=1$ we remark that the second term of (3.15) should be interpreted as 0. If $M \geqq 2$, the second sum in (3.15) is

$$\sum_{n=0}^{M-2}\sum_{j=1}^{N}\sum_{k=0}^{j}\frac{b_{n,j,k}^{(m,2)}\log^k \rho_t}{t^{M(j+1)+n+1}} + O\left[\frac{\log^{N+2}\rho_t}{t^{M(N+2)+1}}\right].$$

Define a sequence $\{b_{n,j,k}^{(m)}\,;\, 0 \leqq k \leqq j \leqq N, 0 \leqq n \leqq M-1\}$ by

$$b_{n,j,k}^{(m)} = \begin{cases} b_{n,j,k}^{(m,2)} & \text{if } M=1,\ 0 \leqq k \leqq j \leqq n, \\ b_{n,0,0}^{(m,2)} & \text{if } M \geqq 2,\ 0 \leqq n \leqq M-2,\ j=k=0, \\ b_{M-1,j,k}^{(m,2)} & \text{if } M \geqq 2,\ n=M-1,\ 0 \leqq k \leqq j \leqq n, \\ b_{n,j,k}^{(m,2)} + b_{n+M,j-1,k}^{(m,2)} & \text{otherwise.} \end{cases}$$

Hence, we can conclude (3.11). □

We are now ready to provide a proof of Theorem 1.1, which is one of improvements on (3.1), and first consider the odd dimensional case. For simplicity, we write $\Lambda_j^{(d)}$ instead of $\Lambda_j^{(d)}(x,r,\mu,A)$ for $j \geqq 0$. With the help of (3.2) for $L=N$ we obtain that

$$F_{x,r,A}^{(d)}(t;\mu) = e^{-\langle x,\mu\rangle}\sum_{l=0}^{N}\frac{\nu(d)+l}{\nu(d)}\Lambda_l^{(d)}H_{x,r}^{(d+2l)}(t;\mu) + O\left[\frac{e^{-|\mu|^2t/2}}{t^{\nu(d)+N+2}}\right]. \tag{3.16}$$

The main term of the right-hand side of (3.16) is

$$\frac{2e^{-\langle x,\mu\rangle-|\mu|^2t/2}}{|\mu|^2}\sum_{l=0}^{N}\sum_{k=0}^{N}\frac{(\nu(d)+l)\Lambda_l^{(d)}b_k^{(d+2l)}}{\nu(d)}\frac{1}{t^{\nu(d)+l+k+1}} + O\left[\frac{e^{-|\mu|^2t/2}}{t^{\nu(d)+N+2}}\right].$$

Here (3.10) has been applied. We put

$$\kappa_n^{(d)} = \sum_{\substack{l+k=n\\ l,k\geqq 0}}\frac{(\nu(d)+l)\Lambda_l^{(d)}b_k^{(d+2l)}}{\nu(d)}$$

and then obtain (1.1).

We next discuss the case that m is even. Let M be a positive integer satisfying that $d=2M+2$ and thus $\nu(d)=M$. Applying (3.2) for $L=M(N+1)-1$, we have by (3.11) that $F_{x,r,A}^{(d)}(t;\mu)$ is

$$\begin{aligned}\frac{2e^{-\langle x,\mu\rangle-|\mu|^2t/2}}{|\mu|^2}\sum_{l=0}^{M(N+1)-1}\frac{M+l}{M}\sum_{j=0}^{N}\sum_{n=0}^{M-1}\sum_{k=0}^{j}\frac{b_{n,j,k}^{(d+2l)}\log^k\rho_t}{t^{(M+l)(j+1)+n+1}}& \\ + O\left[\frac{e^{-|\mu|^2t/2}\log^{N+1}\rho_t}{t^{M(N+2)+1}}\right]&\end{aligned} \tag{3.17}$$

as $t \to \infty$. In the case that $M = 1$ and $N = 0$, it is obvious that (3.17) is the same as the right-hand side of (1.2). Hence, we shall consider other cases. It is sufficient to see that there exist integers j and n such that $j \geqq 0$, $0 \leqq n \leqq M-1$ and

$$(M + l_0)(j_0 + 1) + n_0 + 1 = M(j + 1) + n + 1 \tag{3.18}$$

for given integers l_0, j_0, n_0 satisfying that $l_0 \geqq 1$, $j_0 \geqq 0$ and $0 \leqq n_0 \leqq M-1$. Indeed, if we can take j and n satisfying (3.18), it follows that

$$\sum_{l=1}^{M(N+1)-1} \sum_{j=0}^{N} \sum_{n=0}^{M-1} \sum_{k=0}^{j} \frac{(M+l) b_{n,j,k}^{(d+2l)} \log^k \rho_t}{t^{(M+l)(j+1)+n+1}}$$

can be represented by

$$\sum_{j=0}^{N} \sum_{n=0}^{M-1} \sum_{k=0}^{j} \frac{\overline{b}_{n,j,k}^{(d)} \log^k \rho_t}{t^{M(j+1)+n+1}} + O\left[\frac{\log^{N+1} \rho_t}{t^{M(N+2)+1}}\right]$$

for a suitable sequence $\{\overline{b}_{n,j,k}^{(d)}\,; 0 \leqq k \leqq j \leqq N, 0 \leqq n \leqq M-1\}$.

It remains to a proof that the pair (j, n) of integers with (3.18) can be chosen uniquely. Let

$$\begin{aligned}\Phi(M, j_0, l_0, n_0) = \{(j, n) \in \mathbb{Z}^2\,; j \geqq 0,\ 0 \leqq n \leqq M-1 \\ \text{and the pair } (j, n) \text{ satisfies (3.18).}\}\end{aligned}$$

and write Φ for simplicity. We put

$$j_1 = j_0 + \left\lfloor \frac{l_0(j_0+1)+n_0}{M} \right\rfloor, \quad n_1 = l_0(j_0+1) + n_0 - M\left\lfloor \frac{l_0(j_0+1)+n_0}{M} \right\rfloor,$$

where $\lfloor y \rfloor$ denotes the largest integer which is not larger than y. It can be easily shown that $(j_1, n_1) \in \Phi$ since the equality $Mj_0 + l_0(j_0+1) + n_0 = Mj + n$ is equivalent to (3.18). Let (j_2, n_2) be another element in Φ and then $M(j_1 - j_2) = n_2 - n_1$ holds. If $j_1 \neq j_2$, it follows that

$$|n_2 - n_1| < M \leqq M|j_1 - j_2|.$$

This contradiction implies that $j_1 = j_2$ and hence (j_2, n_2) coincides with (j_1, n_2). This yields that (1.2) holds.

The proof of Theorem 1.1 is completed.

We should mention the information on coefficients in the asymptotic expansion of $F_{x,r,A}^{(d)}(t;\mu)$. In principle, we can derive them by Theorem 1.1. However, just for the first three terms, we can obtain the coefficients directly by (3.8) and (3.9). The remainder of this section is devoted to these coefficients.

We first give the first three terms of $F_{x,r,A}^{(d)}(t;\mu)$. Theorem 1.1 immediately yields the following behavior:

(1) If d is odd and $d \geqq 3$, then

$$F^{(d)}_{x,r,A}(t;\mu) = \frac{2e^{-\langle x,\mu\rangle - |\mu|^2 t/2}}{|\mu|^2 t^{d/2}} \left(\kappa^{(d)}_0 + \frac{\kappa^{(d)}_1}{t} + \frac{\kappa^{(d)}_2}{t^2} + O\left[\frac{1}{t^3}\right] \right).$$

(2) If d is even and $d \geqq 10$, then

$$F^{(d)}_{x,r,A}(t;\mu) = \frac{2e^{-\langle x,\mu\rangle - |\mu|^2 t/2}}{|\mu|^2 t^{d/2}} \left(\kappa^{(d)}_{0,0,0} + \frac{\kappa^{(d)}_{1,0,0}}{t} + \frac{\kappa^{(d)}_{2,0,0}}{t^2} + O\left[\frac{1}{t^3}\right] \right).$$

(3) If $d = 8$, then

$$F^{(8)}_{x,r,A}(t;\mu) = \frac{2e^{-\langle x,\mu\rangle - |\mu|^2 t/2}}{|\mu|^2 t^4} \left(\kappa^{(8)}_{0,0,0} + \frac{\kappa^{(8)}_{1,0,0}}{t} + \frac{\kappa^{(8)}_{2,0,0}}{t^2} + O\left[\frac{\log \rho_t}{t^3}\right] \right).$$

(4) If $d = 6$, then

$$F^{(6)}_{x,r,A}(t;\mu) = \frac{2e^{-\langle x,\mu\rangle - |\mu|^2 t/2}}{|\mu|^2 t^3} \left(\kappa^{(6)}_{0,0,0} + \frac{\kappa^{(6)}_{1,0,0}}{t} + \frac{\kappa^{(6)}_{0,1,1} \log \rho_t}{t^2} + O\left[\frac{1}{t^2}\right] \right).$$

(5) If $d = 4$, then

$$F^{(4)}_{x,r,A}(t;\mu) = \frac{2e^{-\langle x,\mu\rangle - |\mu|^2 t/2}}{|\mu|^2 t^2} \left(\kappa^{(4)}_{0,0,0} + \frac{\kappa^{(4)}_{0,1,1} \log \rho_t}{t} + \frac{\kappa^{(4)}_{1,0,0}}{t} + O\left[\frac{\log^2 \rho_t}{t^2}\right] \right).$$

A careful calculation shows the following proposition and we omit its proof.

Proposition 3.5. *We simply write $\Lambda^{(d)}_j$ for $\Lambda^{(d)}_j(x,r,\mu,A)$.*

(1) *If d is odd and $d \geqq 3$, then*

$$\kappa^{(d)}_0 = \alpha^{(d)}_0 \Lambda^{(d)}_0, \ \ \kappa^{(d)}_1 = \left(\alpha^{(d)}_1 - \frac{d\alpha^{(d)}_0}{|\mu|^2} \right) \Lambda^{(d)}_0 + \frac{d\alpha^{(d+2)}_0}{d-2} \Lambda^{(d)}_1,$$

$$\kappa^{(d)}_2 = \left(\alpha^{(d)}_2 - \frac{(d+2)\alpha^{(d)}_1}{|\mu|^2} + \frac{d(d+2)\alpha^{(d)}_0}{|\mu|^4} \right) \Lambda^{(d)}_0 + \frac{d}{d-2} \left(\alpha^{(d+2)}_1 - \frac{(d+2)\alpha^{(d+2)}_0}{|\mu|^2} \right) \Lambda^{(d)}_1 + \frac{(d+2)\alpha^{(d+4)}_0}{d-2} \Lambda^{(d)}_2.$$

(2) *If d is even and $d \geqq 8$, the coefficients of first three terms have similar forms to the odd dimensional case. Namely, we have that*

$$\kappa^{(d)}_{0,0,0} = \alpha^{(d)}_{0,0,0} \Lambda^{(d)}_0, \quad \kappa^{(d)}_{1,0,0} = \left(\alpha^{(d)}_{1,0,0} - \frac{d\alpha^{(d)}_{0,0,0}}{|\mu|^2} \right) \Lambda^{(d)}_0 + \frac{d\alpha^{(d+2)}_{0,0,0}}{d-2} \Lambda^{(d)}_1,$$

$$\kappa^{(d)}_{2,0,0} = \left(\alpha^{(d)}_{2,0,0} - \frac{(d+2)\alpha^{(d)}_{1,0,0}}{|\mu|^2} + \frac{d(d+2)\alpha^{(d)}_{0,0,0}}{|\mu|^4} \right) \Lambda^{(d)}_0 + \frac{d}{d-2} \left(\alpha^{(d+2)}_{1,0,0} - \frac{(d+2)\alpha^{(d+2)}_{0,0,0}}{|\mu|^2} \right) \Lambda^{(d)}_1 + \frac{(d+2)\alpha^{(d+4)}_{0,0,0}}{d-2} \Lambda^{(d)}_2.$$

(3) *If $d = 6$, then*

$$\kappa^{(6)}_{0,0,0} = \alpha^{(6)}_{0,0,0}\Lambda^{(6)}_0, \quad \kappa^{(6)}_{1,0,0} = \left(\alpha^{(6)}_{1,0,0} - \frac{6\alpha^{(6)}_{0,0,0}}{|\mu|^2}\right)\Lambda^{(6)}_0 + \frac{3\alpha^{(8)}_{0,0,0}}{2}\Lambda^{(6)}_1,$$
$$\kappa^{(6)}_{0,1,1} = \alpha^{(6)}_{0,1,1}\Lambda^{(6)}_0.$$

(4) *If $d = 4$, then*

$$\kappa^{(4)}_{0,0,0} = \alpha^{(4)}_{0,0,0}\Lambda^{(4)}_0, \quad \kappa^{(4)}_{0,1,1} = \alpha^{(4)}_{0,1,1}\Lambda^{(4)}_0,$$
$$\kappa^{(4)}_{0,1,0} = \left(\alpha^{(4)}_{0,1,0} - \frac{4\alpha^{(4)}_{0,0,0}}{|\mu|^2}\right)\Lambda^{(4)}_0 + 2\alpha^{(6)}_{0,0,0}\Lambda^{(4)}_1.$$

4. ASYMPTOTIC BEHAVIOR OF $\phi^{(m)}_{x,r}(t)$ FOR $m \geqq 3$

The purpose in this section is to establish Proposition 3.2. Note that the first two terms of $\phi^{(m)}_{x,r}(t)$ are given in [2]. We put $c = |x|/r$. For $m \in \mathbb{Z}$ and $y > 0$, let

$$G^{(m)}_{x,r}(y) = \frac{J_{\nu(m)}(y)Y_{\nu(m)}(cy) - J_{\nu(m)}(cy)Y_{\nu(m)}(y)}{J_{\nu(m)}(y)^2 + Y_{\nu(m)}(y)^2},$$

where J_ν and Y_ν denote Bessel functions of the first and the second kinds with index ν, respectively. Since the radial part of m-dimensional Brownian motion starting at x can be regarded as a Bessel process with index $\nu(m)$ starting at $|x|$, we have by (3.9) in [4] that

$$P_x(t < \sigma^{(m)}_{r,0} < \infty) = \frac{c^{-\nu(m)}}{\pi}\int_0^\infty \frac{G^{(m)}_{x,r}(r\sqrt{u})}{u}e^{-tu/2}du \tag{4.1}$$

for $m \geqq 3$ and $t > 0$ in the case that $|x| > r$. We should remark that (4.1) has been originally derived in [2]. Recall that $\rho_t = t/2r^2$. We start to deduce the main part of $\phi^{(m)}_{x,r}(t)$ for large t.

Lemma 4.1. *Let $|x| > r$ and $m \geqq 3$. We have that*

$$\phi^{(m)}_{x,r}(t) = \frac{c^{-\nu(m)}}{2\pi r^2}\int_0^1 G^{(m)}_{x,r}(\sqrt{v})e^{-\rho_t v}dv + O\left[\frac{e^{-\rho_t}}{t}\right] \tag{4.2}$$

as $t \to \infty$.

Proof. It follows from (4.1) that

$$\phi^{(m)}_{x,r}(t) = -\frac{c^{-\nu(m)}}{\pi}\frac{d}{dt}\int_0^\infty \frac{G^{(m)}_{x,r}(r\sqrt{u})}{u}e^{-tu/2}du. \tag{4.3}$$

We aim to exchange the order of the differential with respect to t and the integral of u in the right-hand side of (4.3).

Let $t_0 > 0$ be arbitrarily given. For any $t \in (t_0/2, \infty)$ it follows that

$$\int_0^\infty \left| \frac{d}{dt} \frac{G_{x,r}^{(m)}(r\sqrt{u})}{u} e^{-tu/2} \right| du \leqq \frac{1}{2} \int_0^\infty |G_{x,r}^{(m)}(r\sqrt{u})| e^{-t_0 u/4} du. \tag{4.4}$$

Note that the function $y \mapsto G_{x,r}^{(m)}(y)$ is bounded on $(0, \infty)$ for $m \geqq 3$ (cf. [4, p. 516]). Hence, the right-hand side of (4.4) is dominated by a constant multiple of

$$\int_0^\infty e^{-t_0 u/4} du = \frac{4}{t_0}.$$

This gives that the order of the differential and the integral in (4.3) can be changed, and then we obtain that

$$\phi_{x,r}^{(m)}(t) = \frac{c^{-\nu(m)}}{2\pi} \int_0^\infty G_{x,r}^{(m)}(r\sqrt{u}) e^{-tu/2} du. \tag{4.5}$$

The change of variables from u to v given by $v = r^2 u$ yields that

$$\phi_{x,r}^{(m)}(t) = \frac{c^{-\nu(m)}}{2\pi r^2} \int_0^\infty G_{x,r}^{(m)}(\sqrt{v}) e^{-\rho_t v} dv$$

for $t > 0$. Since $G_{x,r}^{(m)}$ is bounded on $(0, \infty)$, we easily have that

$$\int_1^\infty |G_{x,r}^{(m)}(\sqrt{v})| e^{-\rho_t v} dv \leqq c_4 \int_1^\infty e^{-\rho_t v} dv = c_4 \rho_t^{-1} e^{-\rho_t},$$

which immediately implies (4.2). □

We should mention that the integral representation of $\phi_{x,r}^{(m)}(t)$ is given in [2]. However, no exact proof can be found in [2]. Lemma 4.1 implies that we need asymptotic expansion of $G_{x,r}^{(m)}(y)$ as $y \downarrow 0$ for a proof of Proposition 3.2.

Lemma 4.2. *Let $|x| > r$, $m \geqq 3$ and $N \geqq 0$ be an integer.*

(1) *If m is odd, we can choose a sequence $\{\beta_n^{(m)}; 0 \leqq n \leqq N\}$ which depends on x and r and satisfies that*

$$G_{x,r}^{(m)}(y) = \sum_{n=0}^N \beta_n^{(m)} y^{2\nu(m)+2n} + O[y^{2\nu(m)+2N+2}] \tag{4.6}$$

as $y \downarrow 0$.

(2) *If m is even, there exists a sequence $\{\beta^{(m)}_{n,j,h}; 0 \leqq n \leqq N, 0 \leqq h \leqq j \leqq N\}$, which depends on x and r, such that*

$$\begin{aligned} G^{(m)}_{x,r}(y) = \sum_{j=0}^{N}\sum_{h=0}^{j}\sum_{n=0}^{\nu(m)-1} \beta^{(m)}_{n,j,h} y^{2\nu(m)(j+1)+2n} \log^h y \\ + O[y^{2\nu(m)(N+2)} \log^{N+1} y] \end{aligned} \tag{4.7}$$

as $y \downarrow 0$.

Before proving Lemma 4.2, we will give two notations and a fundamental property on power series of complex numbers. For sequences $a = \{a_n\}_{n=0}^{\infty}$ and $b = \{b_n\}_{n=0}^{\infty}$ we define $a * b = \{(a * b)_n\}_{n=0}^{\infty}$ by

$$(a * b)_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for each integer $n \geqq 0$. In addition, we define a sequence $a/b = \{(a/b)_n\}_{n=0}^{\infty}$ by

$$a_n = (b * (a/b))_n$$

for $n \geqq 0$ inductively. It is obvious that both $(a * b)_0$ and $(a/b)_0$ are not zero if $a_0 \neq 0$ and $b_0 \neq 0$. The following fact is well-known (cf. [3, pp. 18–19]).

Lemma 4.3. *For $z \in \mathbb{C}$ let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

When f and g converge on some disc which is centered at the origin, we have that

$$f(z)g(z) = \sum_{n=0}^{\infty} (a * b)_n z^n$$

on the same disc. Moreover, if $b_0 \neq 0$, we have that

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} (a/b)_n z^n$$

holds on some neighborhood of the origin.

We start to see Lemma 4.2 and first consider the case that $m = 2M + 3$ for an integer $M \geqq 0$. Note that $\nu(m) = M + 1/2$ and thus $2\nu(m)$ is a positive integer. Since

$$Y_{M+1/2}(z) = (-1)^{M+1} J_{-M-1/2}(z) \tag{4.8}$$

(cf. [13, p. 72]), it follows that

$$J_{\nu(m)}(ay)Y_{\nu(m)}(by) = (-1)^{M+1} J_{\nu(m)}(ay)J_{-\nu(m)}(by)$$

for $a, b > 0$. The definition of the Bessel function of the first order gives that, if ν is not a negative integer,

$$J_\nu(ay) = y^\nu \sum_{n=0}^{\infty} p_n^{(\nu)}(a) y^{2n}, \tag{4.9}$$

where the sequence $p^{(\nu)}(a) = \{p_n^{(\nu)}(a)\}_{n=0}^{\infty}$ is

$$p_n^{(\nu)}(a) = \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{a}{2}\right)^{\nu+2n}.$$

Lemma 4.3 yields that

$$J_{\nu(m)}(ay)Y_{\nu(m)}(by) = (-1)^{M+1} \sum_{n=0}^{\infty} (p^{(\nu(m))}(a) * p^{(-\nu(m))}(b))_n y^{2n}.$$

Recall that $c = |x|/r$. We obtain that

$$J_{\nu(m)}(y)Y_{\nu(m)}(cy) - J_{\nu(m)}(cy)Y_{\nu(m)}(y) = \sum_{n=0}^{\infty} p_{1,n}^{(m)} y^{2n},$$

where $p_1^{(m)} = \{p_{1,n}^{(m)}\}_{n=0}^{\infty}$ is a sequence defined by

$$p_{1,n}^{(m)} = (-1)^{M+1}\{(p^{(\nu(m))}(1) * p^{(-\nu(m))}(c))_n - (p^{(\nu(m))}(c) * p^{(-\nu(m))}(1))_n\}$$

for $n \geqq 0$. It follows that $p_{1,0}^{(m)} > 0$ from

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)}$$

(cf. [13, p. 2]).

Applying (4.8) and (4.9) to the denominator of $G_{x,r}^{(m)}(y)$, we obtain that

$$\begin{aligned} J_{\nu(m)}(y)^2 + Y_{\nu(m)}(y)^2 = & y^{2\nu(m)} \sum_{n=0}^{\infty} (p^{(\nu(m))}(1) * p^{(\nu(m))}(1))_n y^{2n} \\ & + y^{-2\nu(m)} \sum_{n=0}^{\infty} (p^{(-\nu(m))}(1) * p^{(-\nu(m))}(1))_n y^{2n}. \end{aligned}$$

Note that $2\nu(m)$ is a positive integer. Let $p_2^{(m)} = \{p_{2,n}^{(m)}\}_{n=0}^{\infty}$ be a sequence given by

$$p_{2,n}^{(m)} = (p^{(-\nu(m))}(1) * p^{(-\nu(m))}(1))_n$$

for $0 \leqq n < 2\nu(m)$ and

$$p_{2,n}^{(m)} = (p^{(-\nu(m))}(1) * p^{(-\nu(m))}(1))_n + (p^{(\nu(m))}(1) * p^{(\nu(m))}(1))_{n-2\nu(m)}$$

for $n \geqq 2\nu(m)$. Hence, we have that

$$J_{\nu(m)}(y)^2 + Y_{\nu(m)}(y)^2 = y^{-2\nu(m)} \sum_{n=0}^{\infty} p_{2,n}^{(m)} y^{2n}.$$

Since $p_{2,0}^{(m)} > 0$, we can apply Lemma 4.3 and obtain that

$$G_{x,r}^{(m)}(y) = y^{2\nu(m)} \sum_{n=0}^{\infty} (p_1^{(m)}/p_2^{(m)})_n y^{2n}$$

for small y, which immediately yields (4.6).

We next consider the case that $m = 2M + 2$ for an integer $M \geqq 1$. The calculation for this case is slightly complicated in comparison with the case that m is odd.

Note that $\nu(m) = M$. We start to give an asymptotic behavior of $Y_M(ay)$ as $y \downarrow 0$ for a given $a > 0$. It is known that

$$\begin{aligned} Y_M(ay) = \frac{2}{\pi} J_M(ay) \log \frac{ay}{2} - \frac{1}{\pi} \sum_{n=0}^{M-1} \frac{(M-n-1)!}{n!} \left(\frac{ay}{2}\right)^{2n-M} \\ - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(M+n)!} \left(\frac{ay}{2}\right)^{2n+M} \{\psi(n+1) + \psi(n+M+1)\}, \end{aligned}$$

where ψ is the logarithmic derivative of the gamma function (cf. [12, p. 107]). This implies that

$$Y_M(ay) = y^{-M} \left(\sum_{n=0}^{\infty} q_n^{(M)}(a) y^{2n} + \frac{2y^M \log y}{\pi} J_M(ay) \right) \tag{4.10}$$

for a suitable sequence $q^{(m)}(a) = \{q_n^{(m)}(a)\}_{n=0}^{\infty}$. Here we note that

$$q_0^{(M)}(a) = -\frac{2^M (M-1)!}{\pi a^M}.$$

We first consider the numerator of $G_{x,r}^{(m)}(y)$. It follows from (4.9) and (4.10) that

$$J_{\nu(m)}(ay) Y_{\nu(m)}(by) = \sum_{n=0}^{\infty} (p^{(M)}(a) * q^{(M)}(b))_n y^{2n} + \frac{2 \log y}{\pi} J_M(ay) J_M(by).$$

This gives that

$$J_{\nu(m)}(y) Y_{\nu(m)}(cy) - J_{\nu(m)}(cy) Y_{\nu(m)}(y) = \sum_{n=0}^{\infty} p_{3,n}^{(m)} y^{2n}, \tag{4.11}$$

where $p_3^{(m)} = \{p_{3,n}^{(m)}\}_{n=0}^{\infty}$ is a sequence defined by

$$p_{3,n}^{(m)} = (p^{(M)}(1) * q^{(M)}(c))_n - (p^{(M)}(c) * q^{(M)}(1))_n$$

for $n \geqq 0$. It is easy to see that

$$p_{3,0}^{(m)} = \frac{c^M - c^{-M}}{\pi M}$$

and thus $p_{3,0}^{(m)}$ is positive since $c = |x|/r > 1$.

The calculation of the denominator of $G_{x,r}^{(m)}(y)$ is slightly different from that of its numerator. It follows from (4.9) and (4.10) that

$$Y_M(ay) = y^{-M}\bigg(\sum_{n=0}^{\infty} q_n^{(M)}(a) y^{2n} + \frac{2y^{2M}\log y}{\pi}\sum_{n=0}^{\infty} p_n^{(M)}(a) y^{2n}\bigg).$$

This yields that the denominator of $G_{x,r}^{(m)}(y)$ is represented by

$$\begin{aligned} y^{-2M}\bigg\{ & \sum_{n=0}^{\infty}(p^{(M)}(1) * p^{(M)}(1))_n y^{2n+4M} \\ & + \sum_{n=0}^{\infty}(q^{(M)}(1) * q^{(M)}(1))_n y^{2n} \\ & + \frac{4y^{2M}\log y}{\pi}\sum_{n=0}^{\infty}(p^{(M)}(1) * q^{(M)}(1))_n y^{2n} \\ & + \frac{4y^{4M}\log^2 y}{\pi}\sum_{n=0}^{\infty}(p^{(M)}(1) * p^{(M)}(1))_n y^{2n}\bigg\}. \end{aligned} \tag{4.12}$$

Since $(q^{(M)}(1) * q^{(M)}(1))_0 > 0$, we can deduce from Lemma 4.3 that (4.12) is

$$y^{-2M}\sum_{n=0}^{\infty} p_{4,n}^{(m)} y^{2n}\bigg(1 - y^{2M}\log y\sum_{n=0}^{\infty} p_{5,n}^{(m)} y^{2n} - y^{4M}\log^2 y\sum_{n=0}^{\infty} p_{6,n}^{(m)} y^{2n}\bigg) \tag{4.13}$$

for small y, where $p_4^{(m)} = \{p_{4,n}^{(m)}\}_{n=0}^{\infty}$, $p_5^{(m)} = \{p_{5,n}^{(m)}\}_{n=0}^{\infty}$, $p_6^{(m)} = \{p_{6,n}^{(m)}\}_{n=0}^{\infty}$ are sequences defined by

$$\begin{aligned} p_{4,n}^{(m)} &= \begin{cases} (q^{(M)}(1) * q^{(M)}(1))_n & \text{if } 0 \leqq n < 2M, \\ (q^{(M)}(1) * q^{(M)}(1))_n + (p^{(M)}(1) * p^{(M)}(1))_{n-2M} & \text{if } n \geqq 2M, \end{cases} \\ p_{5,n}^{(m)} &= -\frac{4}{\pi}((p^{(M)}(1) * q^{(M)}(1))/p_4^{(m)})_n, \\ p_{6,n}^{(m)} &= -\frac{4}{\pi}((p^{(M)}(1) * p^{(M)}(1))/p_4^{(m)})_n. \end{aligned}$$

It is obvious that $p_{4,0}^{(m)} > 0$, $p_{5,0}^{(m)} > 0$ and $p_{6,0}^{(m)} < 0$. We should derive an asymptotic expansion of

$$\frac{1}{1 - y^{2M}\log y\sum_{n=0}^{\infty} p_{5,n}^{(m)} y^{2n} - y^{4M}\log^2 y\sum_{n=0}^{\infty} p_{6,n}^{(m)} y^{2n}} \tag{4.14}$$

as $y \downarrow 0$. Let $N \geqq 1$ be a given integer. It follows that the leading term of (4.14) is

$$\sum_{k=0}^{N}\Bigl(y^{2M}\log y\sum_{n=0}^{\infty}p_{5,n}^{(m)}y^{2n}+y^{4M}\log^2 y\sum_{n=0}^{\infty}p_{6,n}^{(m)}y^{2n}\Bigr)^k \tag{4.15}$$

and its error term is $O[y^{2M(N+1)}\log^{N+1} y]$. Lemma 4.3 shows that (4.15) is equal to

$$\begin{aligned}&\sum_{k=0}^{N}\sum_{l=0}^{k}\binom{k}{l}y^{2M(k+l)}\log^{k+l} y\Bigl(\sum_{n=0}^{\infty}p_{5,n}^{(m)}y^{2n}\Bigr)^{k-l}\Bigl(\sum_{n=0}^{\infty}p_{6,n}^{(m)}y^{2n}\Bigr)^{l}\\ &=\sum_{k=0}^{N}\sum_{l=0}^{k}\binom{k}{l}y^{2M(k+l)}\log^{k+l} y\sum_{n=0}^{\infty}p_{7,n,k,l}^{(m)}y^{2n}\end{aligned} \tag{4.16}$$

for a suitable sequence $\{p_{7,n,k,l}^{(m)}\}$, which gives that the right-hand side of (4.16) and also (4.14) can be represented by

$$\sum_{h=0}^{N}y^{2Mh}\log^h y\sum_{n=0}^{\infty}p_{8,n,h}^{(m)}y^{2n}+O[y^{2M(N+1)}\log^{N+1} y]$$

for some sequence $\{p_{8,n,h}^{(m)}\}$.

In virtue of Lemma 4.3, we consequently obtain that $G_{x,r}^{(m)}(y)$ is equal to

$$\begin{aligned}&y^{2M}\sum_{n=0}^{\infty}(p_3^{(m)}/p_4^{(m)})_n y^{2n}\Bigl(\sum_{h=0}^{N}y^{2Mh}\log^h y\sum_{n=0}^{\infty}p_{8,n,h}^{(m)}y^{2n}\\ &\qquad\qquad+O[y^{2M(N+1)}\log^{N+1} y]\Bigr)\\ &=y^{2M}\Bigl(\sum_{h=0}^{N}\sum_{n=0}^{\infty}p_{9,n,h}^{(m)}y^{2Mh+2n}\log^h y+O[y^{2M(N+1)}\log^{N+1} y]\Bigr)\end{aligned}$$

for small y, where

$$p_{9,n,h}^{(m)}=\sum_{k=0}^{n}(p_3^{(m)}/p_4^{(m)})_k p_{8,n-k,h}^{(m)}.$$

Hence, the following holds:

$$\begin{aligned}G_{x,r}^{(m)}(y)=\sum_{h=0}^{N}\sum_{n=0}^{M(N+1)-1}&p_{9,n,h}^{(m)}y^{2M(h+1)+2n}\log^h y\\ &+O[y^{2M(N+2)}\log^{N+1} y].\end{aligned} \tag{4.17}$$

In addition, the leading term of the right-hand side of (4.17) is equal to

$$\begin{aligned}&\sum_{h=0}^{N}\sum_{j=0}^{N}\sum_{n=Mj}^{M(j+1)-1} p_{9,n,h}^{(m)} y^{2M(h+1)+2n}\log^h y \\ &=\sum_{h=0}^{N}\sum_{j=h}^{N}\sum_{n=0}^{M-1} p_{9,n-M(j-h),h}^{(m)} y^{2M(j+1)+2n}\log^h y + O[y^{2M(N+2)}\log^{N+1} y].\end{aligned}$$

We write $\beta_{n,j,h}^{(m)}$ for $p_{9,n-M(j-h),h}^{(m)}$ and then have that

$$G_{x,r}^{(m)}(y)=\sum_{h=0}^{N}\sum_{j=h}^{N}\sum_{n=0}^{M-1}\beta_{n,j,h}^{(m)} y^{2M(j+1)+2n}\log^h y + O[y^{2M(N+2)}\log^{N+1} y]$$

as $y \downarrow 0$, Recall that $\nu(m)=M$. Changing the order of summations on h and j, we can conclude (4.7). Hence, the proof of Lemma 4.2 is finished.

The remainder of this section is devoted to showing Proposition 3.2. For $q>0$ and an integer $n \geqq 0$ let

$$L(t;q,n)=\int_0^1 e^{-\rho_t v} v^{q-1}\log^n v\,dv.$$

Lemmas 4.1 and 4.2 show that it is sufficient to derive its asymptotic behavior.

Lemma 4.4. *For an integer $n \geqq 0$ we have that*

$$L(t;q,n)=\frac{1}{\rho_t^q}\sum_{k=0}^{n}\gamma_k(q,n)\log^k \rho_t + O\left[\frac{e^{-\rho_t}}{t^{n+1}}\right] \tag{4.18}$$

as $t \to \infty$, where

$$\gamma_k(q,n)=(-1)^k\binom{n}{k}\int_0^\infty e^{-v}v^{q-1}\log^{n-k} v\,dv.$$

In particular,

$$L(t;q,n)=O\left[\frac{\log^n \rho_t}{t^q}\right]. \tag{4.19}$$

Proof. Let

$$L_0(t;q,n)=\int_1^\infty e^{-\rho_t v}v^{q-1}\log^n v\,dv.$$

We first show by induction that

$$L_0(t;q,n)\leqq \frac{n!e^{-\rho_t}}{(\rho_t-q)^{n+1}} \tag{4.20}$$

for $t > 2r^2 q$ and $n \geqq 0$. The integration by parts gives that

$$L_0(t;q,0) \leqq \int_1^\infty e^{-\rho_t v} v^q dv = \frac{e^{-\rho_t}}{\rho_t} + \frac{q}{\rho_t} L_0(t;q,0),$$

which yields that

$$L_0(t;q,n) \leqq \frac{e^{-\rho_t}}{\rho_t - q} \tag{4.21}$$

for $t > 2r^2 q$. Assume (4.20) for $n \geqq 0$. The integration by parts yields that

$$\begin{aligned} L_0(t;q,n+1) &\leqq \int_1^\infty e^{-\rho_t v} v^q \log^{n+1} v\, dv \\ &= \frac{q}{\rho_t} L_0(t;q,n+1) + \frac{n+1}{\rho_t} L_0(t;q,n). \end{aligned}$$

Hence, we have that

$$L_0(t;q,n+1) \leqq \frac{n+1}{\rho_t - q} L_0(t;q,n) \leqq \frac{(n+1)! e^{-\rho_t}}{(\rho_t - q)^{n+2}}$$

for $t > 2r^2 q$. This implies that (4.20) holds for each $n \geqq 0$.

It follows from (4.20) that

$$L(t;q,n) = \int_0^\infty e^{-\rho_t v} v^{q-1} \log^n v\, dv + O\left[\frac{e^{-\rho_t}}{t^{n+1}}\right].$$

It is obvious that the integral in the right-hand side is equal to

$$\frac{1}{\rho_t^q} \int_0^\infty e^{-v} v^{q-1} \log^n \frac{v}{\rho_t}\, dv = \frac{1}{\rho_t^q} \sum_{k=0}^n \gamma_k(q,n) \log^k \rho_t.$$

This implies that (4.18) holds. □

Remark 4.5. We have that

$$\gamma_0(q,0) = \Gamma(q), \quad \gamma_0(q,1) = \Gamma(q)\psi(q), \quad \gamma_1(q,1) = -\Gamma(q)$$

(cf. [3]). Moreover, Lemma 3.4 in [4] provides that

$$\begin{aligned} &L(t;q,0) = \Gamma(q)\rho_t^{-q} + O[t^{-1} e^{-\rho_t}], \\ &L(t;q,1) = \Gamma(q)\rho_t^{-q}\{\psi(q) - \log \rho_t\} + O[t^{-2} e^{-\rho_t}], \\ &L(t;q,2) = O[t^{-q} \log^2 t]. \end{aligned}$$

as $t \to \infty$.

We are ready to prove Proposition 3.2 and start to show (3.4). It follows from (4.2) and (4.6) that

$$\phi_{x,r}^{(m)}(t) = \frac{c^{-\nu(m)}}{2\pi r^2} \sum_{n=0}^{N} \beta_n^{(m)} L(t; \nu(m)+n+1, 0) + O[L(t; \nu(m)+N+2, 0)]$$

for a given integer $N \geqq 0$. In virtue of (4.18) and (4.19) we obtain that the right-hand side is

$$\frac{c^{-\nu(m)}}{2\pi r^2} \sum_{n=0}^{N} \frac{\beta_n^{(m)} \gamma_0(\nu(m)+n+1, 0)}{\rho_t^{\nu(m)+n+1}} + O\left[\frac{1}{\rho_t^{\nu(m)+N+2}}\right].$$

We put

$$\alpha_n^{(m)} = \frac{c^{-\nu(m)}}{\pi} \beta_n^{(m)} \gamma_0(\nu(m)+n+1, 0)(2r^2)^{\nu(m)+n}$$

for $n \geqq 0$ and hence can conclude that (3.4) holds.

We next aim to deduce (3.5). Combining (4.2) and (4.7), we have that

$$\begin{aligned}\phi_{x,r}^{(m)}(t) = {}& \frac{c^{-\nu(m)}}{2\pi r^2} \sum_{j=0}^{N} \sum_{h=0}^{j} \sum_{n=0}^{\nu(m)-1} \frac{\beta_{n,j,h}^{(m)}}{2^h} L(t; \nu(m)(j+1)+n+1, h) \\ & + O[L(t; \nu(m)(N+2)+1, N+2)].\end{aligned}$$

Since (4.18) gives that

$$\begin{aligned}&\sum_{h=0}^{j} \frac{\beta_{n,j,h}^{(m)}}{2^h} L(t; \nu(m)(j+1)+n+1, h) \\ &= \frac{1}{\rho_t^{\nu(m)(j+1)+n+1}} \sum_{k=0}^{j} \sum_{h=k}^{j} \frac{\beta_{n,j,h}^{(m)}}{2^h} \gamma_k(\nu(m)(j+1)+n+1, h) \log^k \rho_t + O\left[\frac{e^{-\rho_t}}{t}\right],\end{aligned}$$

it follows that

$$\phi_{x,r}^{(m)}(t) = \sum_{j=0}^{N} \sum_{k=0}^{j} \sum_{n=0}^{\nu(m)-1} \frac{\alpha_{n,j,k}^{(m)} \log^k \rho_t}{t^{\nu(m)(j+1)+n+1}} + O\left[\frac{\log^{N+1} \rho_t}{t^{\nu(m)(N+2)+1}}\right],$$

where

$$\alpha_{n,h,k}^{(m)} = \frac{c^{-\nu(m)}}{\pi} \sum_{h=k}^{j} \frac{\beta_{n,j,h}^{(m)}}{2^h} \gamma_k(\nu(m)(j+1)+n+1, h)(2r^2)^{\nu(m)(j+1)+n}.$$

Hence, (3.5) holds, and then we complete the proof of Proposition 3.2.

Remark 4.6. In virtue of results in [4] we can determine the signs of coefficients of the first three terms of $\phi_{x,r}^{(d)}(t)$ for $d \geqq 3$. An elementary but slightly complicated calculation gives that the coefficient of the first term is positive and the second term is negative. Moreover, the coefficient of the third term is negative if $d = 4, 5, 6$ and is positive otherwise.

5. ASYMPTOTIC EXPANSION OF $F^{(2)}_{x,r,A}(t;\mu)$

For a given vector $\mu \neq 0$ it is shown that

$$F^{(2)}_{x,r,A}(t;\mu) = \frac{4e^{-\langle x,\mu\rangle}}{|\mu|^2} \log \frac{|x|}{r} \Lambda^{(2)}_0(x,r,\mu,A) \frac{e^{-|\mu|^2 t/2}}{t \log^2 t}(1+o[1]) \tag{5.1}$$

as $t \to \infty$ for $|x| > r$ (cf. [9, Theorem 2.6]). The purpose in this section is to provide a proof of Theorem 1.2, which is improvement of (5.1).

Lemma 5.1. *Let $|x| > r$ and $\mu \neq 0$. We have that*

$$F^{(2)}_{x,r,A}(t;\mu) = e^{-\langle x,\mu\rangle} H^{(2)}_{x,r}(t;\mu) \Lambda^{(2)}_0(x,r,\mu,A) + O\left[\frac{e^{-|\mu|^2 t/2}}{t^2}\right] \tag{5.2}$$

as $t \to \infty$.

Proof. It is sufficient to estimate the second term of the right-hand side of (2.8), and thus we try to see that

$$\sum_{n=1}^{\infty} H^{(2n+2)}_{x,r}(t;\mu) \left|\Lambda^{(2)}_n(x,r,\mu,A)\right| \leqq c_5 t^{-2} e^{-|\mu|^2 t/2} \tag{5.3}$$

for any $t \geqq 1$. Combining (2.3) and (2.4), we have that the left-hand side of (5.3) is dominated by a constant multiple of

$$e^{-|\mu|^2 t/2} \sum_{n=1}^{\infty} \frac{r^{2n+2}}{\Gamma(n)(2t)^{n+1}} \left(\frac{|x|}{r}\right)^n \leqq \frac{r^2 e^{-|\mu|^2 t/2}}{2t^2} \sum_{n=1}^{\infty} \frac{1}{\Gamma(n)} \left(\frac{r|x|}{2}\right)^n$$

for $t \geqq 1$. This yields (5.3) and also (5.2) since the summation in the right-hand side converges. □

Lemma 5.1 yields that we should give an asymptotic expansion of $H^{(2)}_{x,r}(t;\mu)$. Recall that

$$H^{(2)}_{x,r}(t;\mu) = \int_t^{\infty} e^{-|\mu|^2 s/2} \phi^{(2)}_{x,r}(s) ds,$$

and thus we need to investigate large time asymptotic for $\phi^{(2)}_{x,r}(t)$.

Theorem 5.2. *Let $|x| > r$. For an arbitrarily given integer $N \geqq 0$ there exists a sequence $\{\alpha^{(2)}_n\}_{n=0}^N$, which is independent of x and r, such that*

$$\phi^{(2)}_{x,r}(t) = \frac{2\log(|x|/r)}{t \log^2 \rho_t} \left(\sum_{n=0}^{N} \frac{\alpha^{(2)}_n}{\log^n \rho_t} + O\left[\frac{1}{\log^{N+1} \rho_t}\right] \right) \tag{5.4}$$

as $t \to \infty$.

We remark that (5.4) is described in [5, Remark 2] without proof and that its coefficients are not determined exactly. In this article we give its absolute proof and the exact form of each coefficient.

Before proving (5.4), we shall complete the proof of Theorem 1.2. Lemma 5.1 yields that it is sufficient to show that

$$H_{x,r}^{(2)}(t;\mu) = \frac{4e^{-|\mu|^2 t/2}}{|\mu|^2} \frac{\log(|x|/r)}{t\log^2 \rho_t} \left(\sum_{n=0}^{N} \frac{\alpha_n^{(2)}}{\log^n \rho_t} + O\left[\frac{1}{\log^{N+1} \rho_t} \right] \right). \tag{5.5}$$

For $t > 2r^2 e$, $p > 0$ and $n \geqq 1$ let

$$I(t;p,n) = \int_t^\infty \frac{e^{-ps}}{s\log^n \rho_s} ds$$

and note that $I(t;p,n) \geqq 0$. In virtue of (5.4) we have that

$$H_{x,r}^{(2)}(t;\mu) = 2\log\frac{|x|}{r} \sum_{n=0}^{N} \alpha_n^{(2)} I\left(t; \frac{1}{2}|\mu|^2, n+2\right) + O\left[I\left(t; \frac{1}{2}|\mu|^2, N+3\right)\right] \tag{5.6}$$

as $t \to \infty$. The integration by parts yields that

$$I(t;p,n) = \frac{e^{-pt}}{pt\log^n \rho_t} - \frac{1}{p}\int_t^\infty \frac{e^{-pt}}{s^2 \log^n \rho_s}\left(1 + \frac{n}{\log \rho_s}\right) ds, \tag{5.7}$$

which gives that

$$I(t;p,n) \leqq \frac{e^{-pt}}{pt\log^n \rho_t} \tag{5.8}$$

for $t > 2r^2 e$ and $p > 0$. In addition, we have that the integral in the right-hand side of (5.7) is dominated by $(n+1)I(t;p,n)/t$. Combining it with (5.8), we deduce

$$I(t;p,n) = \frac{e^{-pt}}{pt\log^n \rho_t} + O\left[\frac{e^{-pt}}{t^2 \log^n \rho_t} \right]. \tag{5.9}$$

Hence, we can easily deduce (5.5) from (5.6). Putting

$$\kappa_n^{(2)} = \alpha_n^{(2)} \Lambda_0^{(2)}(x,r,\mu,A)$$

for $n \geqq 0$, we can conclude (1.4) and hence the proof Theorem 1.2 is finished.

The remainder of this section is devoted to a proof of Theorem 5.2. Similarly to the case that $d \geqq 3$, we need to give an asymptotic expansion of $G_{x,r}^{(2)}(y)$ as $y \downarrow 0$. Recall that

$$G_{x,r}^{(2)}(y) = \frac{J_0(y)Y_0(cy) - J_0(cy)Y_0(y)}{J_0(y)^2 + Y_0(y)^2}.$$

The definition of Bessel functions gives that

$$J_0(y) = 1 + O[y^2], \quad Y_0(y) = \frac{2}{\pi}(\log y + \gamma - \log 2) + O[y^2 \log y] \tag{5.10}$$

as $y \downarrow 0$, where γ is the Euler constant (cf. [12, p. 107]).

Asymptotic behavior of the numerator of $G_{x,r}^{(2)}(y)$ can be easily derived by (5.10). Indeed, a simple calculation shows that

$$J_0(y)Y_0(cy) - J_0(cy)Y_0(y) = \frac{2\log c}{\pi} + O[y^2 \log y]. \tag{5.11}$$

Moreover, it follows that

$$\begin{aligned} J_0(y)^2 + Y_0(y)^2 &= 1 + \frac{4}{\pi^2}(\log y + \gamma - \log 2)^2 + O[y^2 \log^2 y] \\ &= \frac{4\log^2 y}{\pi^2}\left\{1 - \frac{\xi}{\log y}\left(1 + \frac{\eta}{\log y}\right) + O[y^2]\right\}, \end{aligned}$$

where

$$\xi = 2(\log 2 - \gamma), \quad \eta = -\frac{1}{4}\left(\xi + \frac{\pi^2}{\xi}\right).$$

This gives that

$$\frac{1}{J_0(y)^2 + Y_0(y)^2} = \frac{\pi^2}{4\log^2 y}\left\{\sum_{k=0}^{N} \frac{\xi^k}{\log^k y}\left(1 + \frac{\eta}{\log y}\right)^k + O\left[\frac{1}{\log^{N+1} y}\right]\right\}. \tag{5.12}$$

The binomial theorem yields that the sum on k in the right-hand side of (5.12) is equal to

$$\begin{aligned} \sum_{k=0}^{N}\sum_{n=0}^{k}\binom{k}{n}\frac{\xi^k\eta^n}{\log^{n+k} y} &= \sum_{k=0}^{N}\sum_{n=k}^{2k}\binom{k}{n-k}\frac{\xi^k\eta^{n-k}}{\log^n y} \\ &= \sum_{n=0}^{N}\sum_{k=\lceil n/2\rceil}^{n}\binom{k}{n-k}\frac{\xi^k\eta^{n-k}}{\log^n y} + O\left[\frac{1}{\log^{N+1} y}\right], \end{aligned} \tag{5.13}$$

where $\lceil y\rceil = \min\{n \in \mathbb{Z}\,;\, y \leqq n\}$ for a given $y \in \mathbb{R}$. Change variables from k to m given by $m = n - k$ and then the last sum in (5.13) is represented by

$$\sum_{n=0}^{N}\left\{\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n-m}{m}\xi^{n-m}\eta^m\right\}\frac{1}{\log^n y}.$$

Therefore, it follows that

$$\frac{1}{J_0(y)^2 + Y_0(y)^2} = \frac{\pi^2}{4\log^2 y}\left(\sum_{n=0}^{N}\frac{\beta_n^{(2)}}{\log^n y} + O\left[\frac{1}{\log^{N+1} y}\right]\right), \tag{5.14}$$

where

$$\beta_n^{(2)} = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m}{m} \xi^{n-m} \eta^m$$

for an integer $n \geqq 0$. We remark that $\beta_n^{(2)}$ is independent of x and r for each $n \geqq 0$. Combining (5.11) and (5.14), we obtain the following proposition.

Proposition 5.3. *For a given integer $N \geqq 0$ we have that*

$$G_{x,r}^{(2)}(y) = \frac{\pi}{2} \frac{\log c}{\log^2 y} \left(\sum_{n=0}^{N} \frac{\beta_n^{(2)}}{\log^n y} + O\left[\frac{1}{\log^{N+1} y} \right] \right) \tag{5.15}$$

as $y \downarrow 0$. In particular, we can deduce that

$$\left| G_{x,r}^{(2)}(y) - \frac{\pi}{2} \frac{\log c}{\log^2 y} \sum_{n=0}^{N} \frac{\beta_n^{(2)}}{\log^n y} \right| \leqq \frac{c_6}{|\log y|^{N+3}} \tag{5.16}$$

for any $0 < y \leqq 1/2$.

Remark 5.4. Proposition 5.3 can be found in [14]. However, only outline of its proof is described, and each coefficient is not given in the explicit form. Moreover, the first two terms of $G_{x,r}^{(2)}(y)$ are in [2] with no calculation.

In addition to Proposition 5.3, for a proof of Theorem 5.2, we need to represent the leading term of $\phi_{x,r}^{(2)}(t)$ by an integral of $G_{x,r}^{(2)}$. However, the form of the leading part is slightly different for the higher dimensional cases.

Lemma 5.5. *Let $|x| > r$. We have that*

$$\phi_{x,r}^{(2)}(t) = \frac{1}{\pi t} \int_0^{\sqrt{\rho}_t} G_{x,r}^{(2)}\left(\sqrt{\frac{v}{\rho_t}} \right) e^{-v} dv + O\left[\frac{e^{-\sqrt{\rho}_t}}{t} \right] \tag{5.17}$$

as $t \to \infty$.

Proof. We first give an integral representation of the tail probability of $\sigma_{r,0}^{(2)}$, that is,

$$P_x(\sigma_{r,0}^{(2)} > t) = \frac{1}{\pi} \int_0^{\infty} \frac{G_{x,r}^{(2)}(r\sqrt{u})}{u} e^{-tu/2} du \tag{5.18}$$

for $t > 0$. We remark that (5.18) has been established in [2]. However, that paper is written in Japanese, and we do not find any published papers with the proof of (5.18). Hence, we would like to start with showing (5.18) along the same line as (3.8) in [4].

Since

$$\int_0^{\infty} e^{-\lambda t} P_x(\sigma_{r,0}^{(2)} > t) = \frac{2}{\pi} \int_0^{\infty} \frac{G_{x,r}^{(2)}(r\sqrt{u})}{u(2\lambda + u)} du$$

for $\lambda > 0$ (cf. [4, p. 515]), it is sufficient to prove that

$$\int_0^\infty dt\, e^{-\lambda t} \int_0^\infty \frac{G_{x,r}^{(2)}(r\sqrt{u})}{u} e^{-tu/2} du = 2 \int_0^\infty \frac{G_{x,r}^{(2)}(r\sqrt{u})}{u(2\lambda + u)} du. \tag{5.19}$$

We can conclude (5.19) immediately if we succeed in changing the order of integrals in the left-hand side of (5.19).

In order to apply the Fubini theorem to the left-hand side of (5.19), we concentrate on proving that

$$\int_0^\infty du \int_0^\infty \frac{|G_{x,r}^{(2)}(r\sqrt{u})|}{u} e^{-\lambda t} e^{-tu/2} dt \tag{5.20}$$

converges. It is obvious that (5.20) is equal to

$$2 \int_0^\infty \frac{|G_{x,r}^{(2)}(r\sqrt{u})|}{u(2\lambda + u)} du = 2r^2 \int_0^\infty \frac{|G_{x,r}^{(2)}(\sqrt{v})|}{v(2\lambda r^2 + v)} dv. \tag{5.21}$$

In order to estimate the integral in the right-hand side of (5.21), we divide the interval of the integral into $(0, 1/4)$ and $[1/4, \infty)$.

The estimate of the integral on $(0, 1/4)$ is easy. Indeed, we have by (5.16) that

$$\sup_{0<y\leqq 1/2} (\log^2 y)|G_{x,r}^{(2)}(y)| \leqq c_7 \tag{5.22}$$

and thus it follows that

$$\int_0^{1/4} \frac{|G_{x,r}^{(2)}(\sqrt{v})|}{v(2\lambda r^2 + v)} dv \leqq \frac{c_7}{2\lambda r^2} \int_0^{1/4} \frac{dv}{v \log^2 \sqrt{v}} = \frac{c_7}{\lambda r^2 \log 2}.$$

For the estimate of the integral on $[1/4, \infty)$ we need to derive asymptotic behavior of $G_{x,r}^{(2)}(y)$ as $y \to \infty$. It is known that

$$J_0(y) = \sqrt{\frac{2}{\pi y}} \cos\left(y - \frac{\pi}{4}\right) + O\left[\frac{1}{y^{3/2}}\right], \quad Y_0(y) = \sqrt{\frac{2}{\pi y}} \sin\left(y - \frac{\pi}{4}\right) + O\left[\frac{1}{y^{3/2}}\right]$$

as $y \to \infty$ (cf. [12, pp. 122–123]) and hence $G_{x,r}^{(2)}(y)$ is asymptotically equal to a constant multiple of $\sin(c-1)y$ for large y. This immediately yields that

$$\sup_{y\geqq 1/2} |G_{x,r}^{(2)}(y)| \leqq c_8, \tag{5.23}$$

and then we can obtain that

$$\int_{1/4}^\infty \frac{|G_{x,r}^{(2)}(\sqrt{v})|}{v(2\lambda r^2 + v)} dv \leqq c_8 \int_{1/4}^\infty \frac{dv}{v^2} = 4c_8.$$

Hence, we conclude that (5.21) and also (5.20) converge, which yields that (5.18) holds.

We next try to show (5.17). The argument used to prove (4.5) can be applied for showing that

$$\phi_{x,r}^{(2)}(t) = \frac{1}{2\pi}\int_0^\infty G_{x,r}^{(2)}(r\sqrt{u})e^{-tu/2}du. \tag{5.24}$$

Indeed, we have by (5.22) and (5.23) that

$$\begin{aligned}\int_{1/4r^2}^\infty |G_{x,r}^{(2)}(r\sqrt{u})|e^{-t_0u/4}du &\leqq c_8\int_{1/4r^2}^\infty e^{-t_0u/4}du = \frac{4c_8}{t_0}e^{-t_0/16r^2},\\ \int_0^{1/4r^2} |G_{x,r}^{(2)}(r\sqrt{u})|e^{-t_0u/4}du &\leqq c_7\int_0^{1/4r^2}\frac{du}{\log^2(r\sqrt{u})}\\ &= \frac{2c_7}{r^2}\int_0^{1/2}\frac{v}{\log^2 v}dv \leqq \frac{c_7}{2r^2\log 2}\end{aligned}$$

for an arbitrarily given $t_0 > 0$. Therefore, by differentiating both hand sides of (5.18) with respect to t, we can conclude that (5.24) holds for any $t > 0$.

The change of variables in (5.24) from u to v given by $v = tu/2$ yields that

$$\phi_{x,r}^{(2)}(t) = \frac{1}{\pi t}\int_0^\infty G_{x,r}^{(2)}\left(\sqrt{\frac{v}{\rho_t}}\right)e^{-v}dv.$$

Since (5.22) and (5.23) give that $G_{x,r}^{(2)}$ is bonded on $(0,\infty)$, we have that

$$\int_{\sqrt{\rho_t}}^\infty \left|G_{x,r}^{(2)}\left(\sqrt{\frac{v}{\rho_t}}\right)\right|e^{-v}dv \leqq c_9\int_{\sqrt{\rho_t}}^\infty e^{-v}dv = c_9e^{-\sqrt{\rho_t}},$$

which immediately yields (5.17). □

Let $t > 32r^2$ and $N \geqq 0$ be a given integer. We next aim to estimate the error term of the integral in the right-hand side of (5.17), which is

$$\int_0^{\sqrt{\rho_t}}\left|G_{x,r}^{(2)}\left(\sqrt{\frac{v}{\rho_t}}\right) - \frac{\pi\log c}{2}\sum_{n=0}^N\frac{\beta_n^{(2)}}{(\log^{n+2}\sqrt{v/\rho_t})}\right|e^{-v}dv. \tag{5.25}$$

Since $\sqrt{v/\rho_t} \leqq 1/2$ for $0 < v \leqq \sqrt{\rho_t}$, we can apply (5.16) and thus (5.25) is dominated by a constant multiple of

$$\int_0^{\sqrt{\rho_t}}\frac{e^{-v}}{|\log^{N+3}\sqrt{v/\rho_t}|}dv = 2^{N+3}\int_0^{\sqrt{\rho_t}}\frac{e^{-v}}{(\log\rho_t - \log v)^{N+3}}dv. \tag{5.26}$$

Note that

$$\log \rho_t - \log v \geqq \frac{1}{2} \log \rho_t$$

for $0 < v \leqq \sqrt{\rho_t}$. The right-hand side of (5.26) is less than or equal to

$$\frac{4^{N+3}}{\log^{N+3} \rho_t} \int_0^{\sqrt{\rho_t}} e^{-v} dv \leqq \frac{4^{N+3}}{\log^{N+3} \rho_t}.$$

Therefore, (5.26) and also (5.25) are of order $1/\log^{N+3} \rho_t$. It follows from (5.17) that

$$\phi_{x,r}^{(2)}(t) = \frac{\log c}{t} \sum_{n=0}^{N} 2^{n+1} \beta_n^{(2)} \int_0^{\sqrt{\rho_t}} \frac{e^{-v}}{(\log v - \log \rho_t)^{n+2}} dv + O\left[\frac{1}{\log^{N+3} \rho_t}\right]. \tag{5.27}$$

In order to calculate the first term of the right-hand side of (5.27), we give three lemmas.

Lemma 5.6. *For an integer $m \geqq 1$ we have that*

$$\int_0^{\sqrt{\rho_t}} e^{-y} \log^m y \, dy = \int_0^{\infty} e^{-y} \log^m y \, dy + O\left[e^{-\sqrt{\rho_t}} \log^m \rho_t\right] \tag{5.28}$$

as $t \to \infty$.

Proof. Let $t > 8m^2 r^2$. The integration by parts yields that

$$\int_{\sqrt{\rho_t}}^{\infty} e^{-y} \log^m y dy = e^{-\sqrt{\rho_t}} \log^m \sqrt{\rho_t} + m \int_{\sqrt{\rho_t}}^{\infty} \frac{e^{-y}}{y} \log^{m-1} y dy,$$

which is dominated by

$$\frac{1}{2^m} e^{-\sqrt{\rho_t}} \log^m \rho_t + \frac{m}{\sqrt{\rho_t}} \int_{\sqrt{\rho_t}}^{\infty} e^{-y} \log^m y dy.$$

Hence, we have that

$$\int_{\sqrt{\rho_t}}^{\infty} e^{-y} \log^m y dy \leqq \frac{1}{2^m} e^{-\sqrt{\rho_t}} \log^m \rho_t + \frac{1}{2} \int_{\sqrt{\rho_t}}^{\infty} e^{-y} \log^m y dy$$

and can conclude (5.28). $\square$

Lemma 5.7. *For any integer $m \geqq 1$ we have that*

$$\int_0^\infty e^{-y}|\log y|^m dy \leqq 2m!. \tag{5.29}$$

Proof. We have that

$$\int_0^1 e^{-y}\log^m \frac{1}{y}dy \leqq \int_0^1 \log^m \frac{1}{y}dy = m!.$$

Moreover, the integration by parts implies that

$$\int_1^\infty e^{-y}\log^k ydy = k\int_1^\infty \frac{e^{-y}\log^{k-1} y}{y}dy \leqq k\int_1^\infty e^{-y}\log^{k-1} ydy$$

for any integer $k \geqq 1$, which yields that

$$\int_1^\infty e^{-y}\log^m ydy \leqq m!\int_1^\infty e^{-y}dy \leqq m!$$

for any $m \geqq 1$. Hence, we conclude that (5.29) holds. $\square$

Lemma 5.8. *Let $m \geqq 1$ and $N \geqq 0$ be fixed integers. For $y \leqq 1/2$ we have that*

$$\left|(1-y)^{-m} - \sum_{k=0}^N \binom{m+k-1}{m-1}y^k\right| \leqq c_{10}|y|^{N+1}. \tag{5.30}$$

Proof. Let

$$R_{m,N}(y) = (1-y)^{-m} - \sum_{k=0}^N \binom{m+k-1}{m-1}y^k.$$

The Taylor theorem gives that

$$R_{m,N}(y) = \binom{m+N}{m-1}(1-\theta y)^{-m-N-1}y^{N+1}$$

for some $\theta \in [0,1]$.

We have that $1-\theta y \geqq 1/2$ if $0 \leqq y \leqq 1/2$ and that $1-\theta y \geqq 1$ if $y<0$. Hence, $|R_{m,N}(y)|$ is dominated by a constant multiple of $|y|^{N+1}$. $\square$

For given integers $m \geqq 1$ and $N \geqq 0$ it follows from (5.29) and (5.30) that

$$\int_0^{\sqrt{\rho_t}}\left|\left(1-\frac{\log v}{\log \rho_t}\right)^{-m} - \sum_{n=0}^N \binom{m+k-1}{m-1}\left(\frac{\log v}{\log \rho_t}\right)\right|e^{-v}dv$$

$$\leqq \frac{c_{10}}{\log^{N+1}\rho_t}\int_0^{\sqrt{\rho_t}} e^{-v}|\log v|^{N+1}dv \leqq \frac{2c_{10}(N+1)!}{\log^{N+1}\rho_t}.$$

Combining it with (5.28), we obtain that

$$\int_0^{\sqrt{\rho}_t} \frac{e^{-v}}{(\log v - \log \rho_t)^m} dv = \frac{(-1)^m}{\log^m \rho_t} \sum_{n=0}^{N} \binom{m+k-1}{m-1} \frac{\lambda_k}{\log^k \rho_t} + O\Big[\frac{1}{\log^{N+1} \rho_t}\Big],$$

where

$$\lambda_k = \int_0^\infty e^{-y} \log^k y dy$$

for an integer $k \geqq 0$. This yields that

$$\begin{aligned}\phi_{x,r}^{(2)}(t) &= \frac{\log c}{t \log^2 \rho_t} \Big\{ \sum_{n=0}^{N} \sum_{k=0}^{N} \binom{n+k+1}{n+1} \frac{(-1)^n 2^{n+1} \beta_n^{(2)} \lambda_k}{\log^{n+k} \rho_t} + O\Big[\frac{1}{\log^{N+1} \rho_t}\Big] \Big\} \\ &= \frac{\log c}{t \log^2 \rho_t} \Big\{ \sum_{m=0}^{N} \sum_{n=0}^{m} \binom{m+1}{n+1} \frac{(-1)^n 2^{n+1} \beta_n^{(2)} \lambda_{m-n}}{\log^m \rho_t} + O\Big[\frac{1}{\log^{N+1} \rho_t}\Big] \Big\}.\end{aligned}$$

We put

$$\alpha_m^{(2)} = \sum_{n=0}^{m} (-1)^n 2^n \beta_n^{(2)} \lambda_{m-n} \binom{m+1}{n+1}$$

and thus have (5.4). This completes the proof of Theorem 5.2.

Remark 5.9. A simple calculation shows that

$$\alpha_0^{(2)} = 1, \quad \alpha_1^{(2)} = 2\gamma_0, \quad \alpha_2^{(2)} = 3\gamma_0^2 - \frac{1}{2}\pi^2, \quad \alpha_3^{(2)} = 2\gamma_0(\gamma_0^2 - \pi^2) + 8\zeta(3),$$

where $\gamma_0 = \gamma - 2\log 2$. In addition, we compute them by "Mathematica", and have that

$$\alpha_1^{(2)} = -1.61815\cdots, \quad \alpha_2^{(2)} = -2.07097\cdots, \quad \alpha_3^{(2)} = 23.4685\cdots.$$

6. THE CASE OF BROWNIAN MOTION WITHOUT DRIFT

This section deals with the joint distribution of $\sigma_{r,0}^{(d)}$ and $B(\sigma_{r,0}^{(d)})$. Recall that

$$F_{x,r,A}^{(d)}(t;0) = P_x(t < \sigma_{r,0}^{(d)} < \infty, B(\sigma_{r,0}^{(d)}) \in A)$$

for $t > 0$ and a Borel set $A \subset S_r^{d-1}$. Since (1.3) and (1.5) hold for $\mu = 0$, we obtain in the same way as Proposition 2.3 that

$$F_{x,r,A}^{(d)}(t;0) = \sum_{n=0}^{\infty} \frac{\nu(d)+n}{\nu(d)} H_{x,r}^{(d+2n)}(t;0) \Lambda_n^{(d)}(x,r,0,A) \tag{6.1}$$

if $d \geqq 3$ and that

$$F^{(2)}_{x,r,A}(t;0) = H^{(2)}_{x,r}(t;0)\Lambda^{(2)}_0(x,r,0,A) + \sum_{n=1}^{\infty} nH^{(2n+2)}_{x,r}(t;0)\Lambda^{(2)}_n(x,r,0,A) \tag{6.2}$$

if $d = 2$. Moreover, Theorem 2.6 in [9] provides the leading term of $F^{(d)}_{x,r,A}(t;0)$ for large t. The purpose of this section is to deduce higher terms of $F^{(d)}_{x,r,A}(t;0)$.

We first provide the result for $d \geqq 3$.

Theorem 6.1. *Let $|x| > r$ and $N \geqq 0$ be a given integer.*

(1) *If d is odd and not less than 3, we have that*

$$F^{(d)}_{x,r,A}(t;0) = \frac{1}{t^{\nu(d)}}\left(\sum_{n=0}^{N} \frac{\eta^{(d)}_n}{t^n} + O\left[\frac{1}{t^{N+1}}\right]\right)$$

as $t \to \infty$ for a suitable sequence $\{\eta^{(d)}_n\,;\, 0 \leqq n \leqq N\}$ which depends on x, r, A.

(2) *If d is even and not less than 4, we have that*

$$F^{(d)}_{x,r,A}(t;0) = \frac{1}{t^{\nu(d)}}\left(\sum_{j=0}^{N}\sum_{n=0}^{\nu(d)-1}\sum_{k=0}^{j} \frac{\eta^{(d)}_{n,j,k}\log^k \rho_t}{t^{\nu(d)j+n}} + O\left[\frac{\log^{N+1}\rho_t}{t^{\nu(d)(N+1)}}\right]\right)$$

as $t \to \infty$ for a suitable sequence $\{\eta^{(d)}_{n,j,k}\,;\, 0 \leqq k \leqq j \leqq N, 0 \leqq n \leqq \nu(d)\}$ which depends on x, r, A.

We can prove Theorem 6.1 along the same line as the proof of Theorem 1.1 with the help of Lemma 2.4 in [9] and

$$H(t;0,q,n) = \sum_{k=0}^{n} \frac{n!}{(q-1)^{n-k+1}k!}\frac{\log^k \rho_t}{t^{q-1}}.$$

for $q > 1$ and $n \geqq 0$. Hence, the proof is omitted.

Remark 6.2. Similarly to $F^{(d)}_{x,r,A}(t;\mu)$ we can represent coefficients of the first three terms of $F^{(d)}_{x,r,A}(t;0)$.

(1) If d is odd and $d \geqq 3$, then

$$\begin{aligned}
\eta^{(d)}_0 &= \frac{2\alpha^{(d)}_0}{d-2}s^{d-1}_r(A), \quad \eta^{(d)}_1 = \frac{2\alpha^{(d)}_1}{d}s^{d-1}_r(A) + \frac{2\alpha^{(d+2)}_0}{d-2}\Lambda^{(d)}_1(x,r,0,A),\\
\eta^{(d)}_3 &= \frac{2\alpha^{(d)}_2}{d+2}s^{d-1}_r(A) + \frac{2d\alpha^{(d+2)}_1}{(d-2)(d+2)}\Lambda^{(d)}_1(x,r,0,A) + \frac{2\alpha^{(d+4)}_0}{d-2}\Lambda^{(d)}_2(x,r,0,A).
\end{aligned}$$

(2) If d is even and $d \geqq 8$, then

$$\begin{aligned}
\eta^{(d)}_{0,0,0} &= \frac{2\alpha^{(d)}_{0,,0,0}}{d-2} s^{d-1}_r(A), \quad \eta^{(d)}_{1,0,0} = \frac{2\alpha^{(d)}_{1,0,0}}{d} s^{d-1}_r(A) + \frac{2\alpha^{(d+2)}_{0,0,0}}{d-2} \Lambda^{(d)}_1(x,r,0,A), \\
\eta^{(d)}_{3,0,0} &= \frac{2\alpha^{(d)}_{2,0,0}}{d+2} s^{d-1}_r(A) \\
&\quad + \frac{2d\alpha^{(d+2)}_{1,0,0}}{(d-2)(d+2)} \Lambda^{(d)}_1(x,r,0,A) + \frac{2\alpha^{(d+4)}_{0,0,0}}{d-2} \Lambda^{(d)}_2(x,r,0,A).
\end{aligned}$$

(3) If $d = 6$, then

$$\begin{aligned}
\eta^{(6)}_{0,0,0} &= \frac{\alpha^{(6)}_{0,0,0}}{2} s^5_r(A), \quad \eta^{(6)}_{1,0,0} = \frac{\alpha^{(6)}_{1,0,0}}{3} s^5_r(A) + \frac{\alpha^{(8)}_{0,0,0}}{2} \Lambda^{(6)}_1(x,r,0,A), \\
\eta^{(6)}_{0,1,1} &= \frac{\alpha^{(6)}_{0,1,1}}{4} s^5_r(A).
\end{aligned}$$

(4) If $d = 4$, then

$$\begin{aligned}
\eta^{(4)}_{0,0,0} &= \alpha^{(4)}_{0,0,0} s^3_r(A), \quad \eta^{(4)}_{0,1,1} = \frac{\alpha^{(4)}_{0,1,1}}{2} s^3_r(A), \\
\eta_{0,1,0} &= \left(\frac{\alpha^{(4)}_{0,1,1}}{4} + \frac{\alpha^{(4)}_{0,1,0}}{2} \right) s^3_r(A) + \alpha^{(6)}_{0,0,0} \Lambda^{(4)}_1(x,r,0,A).
\end{aligned}$$

We next consider the case that $d = 2$. In virtue of (2.4) and Lemma 2.4 in [9] we have that the summation in (6.2) is of order $1/t$. Note that $\Lambda^{(2)}_0(x,r,0,A) = s^1_r(A)$. Thus, (6.2) gives that

$$F^{(2)}_{x,r,A}(t;0) = s^{d-1}_r(A) \int_t^\infty \phi^{(2)}_{x,r}(s)ds + O\left[\frac{1}{t}\right] \tag{6.3}$$

as $t \to \infty$. Since

$$\int_t^\infty \frac{ds}{s \log^{m+1} \rho_s} = \frac{1}{m \log^m \rho_t}$$

for each integer $m \geqq 1$, Theorem 5.2 shows that $F^{(2)}_{x,r,A}(t;0)$ admits an asymptotic expansion in powers of $1/\log \rho_t$.

Theorem 6.3. *Let $|x| > r$ and $N \geqq 0$. We have that*

$$F^{(2)}_{x,r,A}(t;0) = \frac{2s^1_r(A)\log(|x|/r)}{\log \rho_t} \left(\sum_{n=0}^{N} \frac{\alpha^{(2)}_n}{n+1} \frac{1}{\log^n \rho_t} + O\left[\frac{1}{\log^{N+1} \rho_t}\right] \right) \tag{6.4}$$

as $t \to \infty$.

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Yuji Hamana
hamana@math.tsukuba.ac.jp
https://orcid.org/0000-0002-9997-3114

University of Tsukuba
Department of Mathematics
1-1-1 Tennodai, Tsukuba 305–8571, Japan

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