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BLOCK JACOBI MATRICES AND TITCHMARSH–WEYL FUNCTION

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Abstract. We collect some results and notions concerning generalizations for block Jacobi matrices of several concepts, which have been important for spectral studies of the simpler and better known scalar Jacobi case. We focus here on some issues related to the matrix Titchmarsh–Weyl function, but we also consider generalizations of some other tools used by subordinacy theory, including the matrix orthogonal polynomials, the notion of finite cyclicity, a variant of a notion of nonsubordinacy, as well as Jitomirskaya–Last type semi-norms.

The article brings together some issues already known, our new concepts, and also improvements and strengthening of some results already existing. We give simpler proofs of some known facts, or we add details usually omitted in the existing literature. The introduction contains a separate part devoted to a brief review of the main spectral analysis methods used so far for block Jacobi operators.

Keywords: block Jacobi matrix, matrix measures, Titchmarsh–Weyl function, Liouville–Ostrogradsky formulae.

Mathematics Subject Classification: 47B36.

1. INTRODUCTION

Jacobi matrices (JM) and the appropriate Jacobi operators (JO) in the standard Hilbert space ℓ^2 of square summable scalar sequences have been classical objects of interest to mathematicians for years, especially to those involved in spectral analysis. Therefore, the mathematical literature on JO is currently very rich.

Block Jacobi matrices (BJM) and operators (BJO) are generalizations of those scalar ones with the scalar coefficients replaced by square matrices. They were originally introduced by Kreĭn in [52, 53] in 1940s, but the number of strict mathematical results for them is still small, compared to those for scalar JM-s and JO-s.

Their basic properties are discussed in a monograph of Berezanskiĭ [4, Section VII.§2]. Other good references are [19] and [39, Chapter 12 and 13].

The interest in block Jacobi operators comes from their close relation to the matrix moment problem (see, e.g., [6, 30]) as well as from the theory of matrix orthogonal

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polynomials on the real line, see, e.g., [19]. They are also useful for analysis of difference equations of finite order, see [31]. Some types of block Jacobi operators are related to random walks and level dependent quasi-birth-death processes, see, e.g., [20]. In [2] BJO found some applications in mathematical physics. Some extensions of BJO are useful in studying the scalar periodic JO (see, e.g., [70, Chapter 8]), various moment problems (see [5]) and orthogonal polynomials of several variables (see [28, Section 3.4]). Finally, let us mention that BJO on $\mathbb{Z}$ can be conveniently modeled as BJO on $\mathbb{N}_0$ by doubling the dimension of the blocks (see, e.g., [4, Chapter VII.§3]). For further applications we refer to [72].

Before discussing the subject of this paper, we present here a brief overview of several spectral analysis methods for BJO used so far.

1.1. SHORT REVIEW ON SPECTRAL ANALYSIS METHODS FOR BLOCK JACOBI OPERATORS

The spectral analysis of block Jacobi operators has already been partially developed in several directions, using methods that are, in general, adaptations (often far from being trivial) of methods previously used for JO.

Let us mention first some non-strictly spectral studies, related to the self-adjointness problem. Papers [51] and [33] present results corresponding to the maximal possible deficiency indices for the so-called *completely indeterminate* case. Moreover, in [32] it is shown that minimal block Jacobi operators can have non-equal deficiency indices.

Below we briefly sketch the main approaches to spectral studies of BJO in the mathematical literature, organized according to the type of the methods used.

1.1.1. Perturbations theory for block Toeplitz operators

It is one of the first methods which has been applied to get some "strictly spectral" results for BJO-s. Its goal is to control the number of eigenvalues in spectral gaps (open intervals outside the essential spectrum) of the operator. The method consists in using some abstract perturbation theory results (see Cojuhari: [12, Sections 1–3] and [15, Section 2]) to study perturbations of matrix Toeplitz operators, called also matrix Wiener–Hopf discrete operators (see, e.g., the monograph [7]). Recall that BJO with constant coefficients are bounded self-adjoint matrix Toeplitz operators with a simple formula for the symbol (see, e.g., [14, Section 2]), so their spectral properties could be precisely described thanks to the general theory (see, e.g., [37]). The above general idea is a "matrix adaptation" of previous "scalar version" used for JO (see, e.g., [13]). The method can be used for "small perturbations" of BJO-s with constant blocks – see, e.g., [14]. The results of [12, 15] refer to perturbations of scalar JO with periodic coefficients, which can be treated as the appropriate BJO with constant blocks. In such a case, if the period is at least two, the off-diagonal blocks are singular, so the methods using transfer matrices for BJO (e.g., in the present paper, where we assume (3.1)) are useless in this case.

1.1.2. Combes–Thomas type estimates

It is well-known (see, e.g., [68, Corollary 2.4]) that the entries of the resolvent of *bounded* BJO decay exponentially with respect to the distance between the entries. Since a similar conclusion was earlier shown by Combes–Thomas in [17] for the resolvent of multiparticle system Schrödinger operators, estimates of this kind for the resolvent are sometimes called *Combes–Thomas estimates* (see, e.g., [10, Proposition 2.3], [9, Theorem A.1]).

By extending the method of Combes–Thomas in the articles Janas–Naboko–Stolz [46] and Janas–Naboko [42] some analogues of such estimates for possibly *unbounded* JO were established for some regions of the resolvent set. Later in Janas–Naboko–Silva [43, 44] these methods were adapted to BJO. Finally, in Naboko–Simonov [59], such estimates for BJO were proved for the whole resolvent set.

1.1.3. Estimates of quadratic forms

Estimates of the quadratic form of JO have been a popular topic in the literature, see, e.g., [22, 23, 25–27, 45, 58, 78]. Such estimates allowed to discover gaps in the essential spectrum of some classes of JO. In the article Kupin–Naboko [54], by extending some methods of [18], some estimates of the quadratic form for a quite general class of BJO were established. In a different style, in [75], by extending the techniques developed for JO from [73, 77], estimates for some quadratic forms of two consecutive indices of generalized eigenvectors (the so-called Turán determinants) were established for a wide class of BJO. As a consequence some asymptotic estimates for generalized eigenvectors were established leading to continuity of the spectrum. Note that, at the same time, this asymptotic information turned out to be crucial to get important example for the main result of [76].

1.1.4. Commutator methods

For scalar JO various commutator methods are quite popular, see, e.g., [24, 62, 73] (extensions of Putnam–Kato's method), [65] (Mourre's method) and [8] (a double commutator method).

In Janas [40, Theorem 3] emptiness of the point spectrum of some BJO was established by extending [62]. Later, in [75, Theorem 6], this result was partially generalized, and the continuous spectrum was identified. In Sahbani [66] the Mourre's commutator method was applied to study compact perturbations of constant coefficients for BJO. Under some regularity hypotheses it was shown there that the singular continuous spectrum is empty.

1.1.5. Compactness of the resolvent

For scalar JO it is sometimes possible to prove directly that the resolvent is a compact operator, which leads to the conclusion that the essential spectrum of JO is empty, see, e.g., [41, Section 4], [56, Theorem 3.1(iv)], [16], [65, Theorem 2.1]. By exploiting a connection with BJO with singular off-diagonals this approach was also implemented in [74, Theorem B(c)]. In [11] compactness of the resolvent for a class of BJO was studied.

1.1.6. Matrix Titchmarsh–Weyl function and subordinacy methods

The scalar Titchmarsh–Weyl function (called also Weyl–Titchmarsh or simply Weyl function) has been one of useful tools for spectral studies of some differential (see, e.g., the monograph [3]) and difference operators (the "scalar" ones), including Jacobi operators. The Weyl function has been generalized also for BJO case as the matrix Weyl function (see, e.g., [4, Section VII.§2.10], [1] and see subsections 1.2, 5.1 for some more explanations), however, its use in spectral analysis of BJO has been difficult and papers on such applications have only recently begun to appear.

Some scalar methods from [47, 55] using among other things the Weyl function has been recently adapted in Oliveira–Carvalho [60] to BJO case (see also [61] for other use of matrix Weyl function in spectral analysis for BJM considered on $\mathbb{Z}$).

It seems that the switch from the scalar to the $d > 1$ case is especially problematic for the theory of subordinacy (also see Subsection 1.2), being one of the main spectral methods based on the Weyl function. As far as we know, there is still no understanding of how a complete analog of subordinacy theory for BJO might look like. Nevertheless, the present paper contains a kind of introduction to some possible spectral methods based on the matrix Weyl function, including one of "non-subordinacy type" notions and a simple "subordinacy theory type" spectral result for BJO. Moreover, in our parallel article [76], we have obtained some interesting "subordinacy-type" results. Namely, we have found a sufficient condition for absolute continuity of BJO in terms of our new notion of barrier nonsubordinacy.

1.2. THE PRESENT WORK

This work is devoted to generalizations of several concepts, tools and "small theories", which turned out to be important for spectral studies of scalar Jacobi matrices (JM) to the general case of block Jacobi matrices (BJM) with $d \times d$ blocks for arbitrary finite dimension d. We focus here especially on the results related to the matrix Titchmarsh–Weyl function for BJM, which will be often named here simply matrix Weyl function for short, similarly to the scalar function case. We also pay attention to generalizations of some other tools used by the so-called subordinacy theory (see, e.g., [36, 47, 50]).

The matrix Weyl function is a generalization of the scalar Weyl (Titchmarsh–Weyl) function and was introduced much earlier for many kinds of self-adjoint operators (see, e.g., [3] for a general treatment). The scalar one was very useful, e.g., for Jacobi matrices, and from our perspective here, it was crucial as one of the fundamental objects enabling proofs of main theorems of subordinacy theory for JM. But its history is much longer, see, e.g., [34, Section 8.2] and [3]. The matrix Weyl function naturally appeared before in the study of spectral properties of higher order ordinary differential operators, see [80].

The matrix Weyl function for BJM is closely related to the so-called orthogonal matrix polynomials P and Q, to which we also pay some attention here, and which are generalizations of the appropriate scalar orthogonal polynomials p and q.

We hope that in the future some of the generalizations considered here will also prove useful in the spectral analysis of BJM. This hope is not groundless: the present work was written in parallel to the paper [76], where some of these generalizations have already turned out to be useful exactly for new spectral results "of subordinacy style" for BJM, which have been missing in spectral studies for BJM so far.

This article brings together both the results already known, at least in part, and our new results, improvements and strengthening of those already existing. We extend some known results and sometimes give other proofs, more ingenious and more elementary, while trying to make this work largely self-sufficient. We add many details often omitted in the existing literature. We also try to somewhat unify the various notations and terminologies used in the rather scattered literature on this subject, sometimes suggesting slightly different approaches to certain concepts.

Let us now recall some basic issues concerning Jacobi (scalar) matrices.

A *Jacobi matrix* (see, e.g., [4, Chapter VII.§1]) is a complex semi-infinite tridiagonal Hermitian matrix of the form

$$\mathcal{J} = \begin{pmatrix} b_0 & a_0 & & \\ \overline{a_0} & b_1 & a_1 & \\ & \overline{a_1} & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \tag{1.1}$$

with $a_n \neq 0$ and $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0 = \{0,\ 1,\ 2, \ldots\}$. The action of $\mathcal{J}$ is well-defined on the linear space $\ell(\mathbb{N}_0, \mathbb{C})$ of all complex-valued sequences treated as column vectors, and it is natural to define the operator J as the "restriction" of $\mathcal{J}$ to the standard Hilbert space $\ell^2(\mathbb{N}_0, \mathbb{C})$ of square summable sequences. Namely, we define $Jx = \mathcal{J}x$ on the domain consisting of all such $x \in \ell^2(\mathbb{N}_0, \mathbb{C})$ that also $\mathcal{J}x \in \ell^2(\mathbb{N}_0, \mathbb{C})$. Such operator J is called *the maximal Jacobi operator*, and here we shall usually call it simply *Jacobi operator*[1)] or JO for short. It need not be either bounded or self-adjoint, in general. But if it is self-adjoint, then one can define a Borel probability measure μ (*the scalar spectral measure of* J) on the real line by the formula

$$\mu(G) = \langle E_J(G)\delta_0(1), \delta_0(1)\rangle_{\ell^2}, \quad G \in \mathrm{Bor}(\mathbb{R}), \tag{1.2}$$

where E_J is the projection-valued spectral measure of J and $\delta_0(1)$ is a sequence with 1 on 0th position and 0 elsewhere. Then one can prove that J is unitary equivalent to the operator acting by multiplication by the identity function on $L^2(\mu)$, see, e.g., [67, Theorem 5.14 and Theorem 6.16].

1) Note, however, that also some non-maximal operators related to the matrix $\mathcal{J}$ are considered, and the name "Jacobi operator/matrix" is often used for them, too.

The interest in Jacobi operators comes from their close relation to the classical moment problem as well as the theory of orthogonal polynomials on the real line, see, e.g., [69]. As every self-adjoint operator having a "non-degenerate" cyclic vector is unitary equivalent to a maximal or non-maximal Jacobi operator, those maximal ones are quite typical "basic building blocks" of self-adjoint operators. Some types of Jacobi operators are related to random walks and birth–death processes, see, e.g., [48, 49]. Finally, Jacobi matrices are very useful in numerical analysis in the construction of Gaussian quadratures, see, e.g., [38].

A method of spectral analysis, the *theory of subordinacy*, due to Gilbert–Pearson [36] and later, due to Khan–Pearson in its Jacobi variant (see [50]), started to be more and more prominent during the last three decades. Given $\lambda \in \mathbb{C}$ a sequence $u = (u_n)_{n\in\mathbb{N}_0}$ is called *generalized eigenvector* (associated with λ), $u \in \mathrm{GEV}(\lambda)$, if it satisfies the recurrence relation

$$(\mathcal{J}u)_n = \lambda u_n, \quad n \geq 1,$$

with some initial conditions (u_0, u_1). A non-zero sequence $u \in \mathrm{GEV}(\lambda)$ is *subordinate* if for any linearly independent $v \in \mathrm{GEV}(\lambda)$

$$\lim_{n\to\infty} \frac{\|u\|_{[0,n]}}{\|v\|_{[0,n]}} = 0, \tag{1.3}$$

where for a sequence $x \in \ell(\mathbb{N}_0, \mathbb{C})$ and each $n \geq 0$ the seminorm $\|\cdot\|_{[0,n]}$ is given by

$$\|x\|_{[0,n]} := \sqrt{\sum_{k=0}^{n} |x_k|^2}. \tag{1.4}$$

Let us decompose the measure μ as $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}} = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}}$, where μ_{ac}, μ_{sing}, μ_{sc} and μ_{pp} denote the absolutely continuous, the singular, the singular continuous and the pure point part of μ with respect to the Lebesgue measure.

The main result, [50, Theorem 3], says that the set

$$S_{\mathrm{ac}} = \{\lambda \in \mathbb{R} : \text{no non-zero } u \in \mathrm{GEV}(\lambda) \text{ is subordinate}\}$$

is a minimal support of μ_{ac} with respect to the Lebesgue measure and

$$S_{\mathrm{sing}} = \{\lambda \in \mathbb{R} : \text{a non-zero } u \in \mathrm{GEV}(\lambda) \\ \text{such that also } (\mathcal{J}u)_0 = \lambda u_0 \text{ holds, is subordinate}\}$$

is a support of μ_{sing} and its Lebesgue measure is zero. Since we often have some idea about asymptotic behaviour of generalized eigenvectors, this theory turned out to be very successful in spectral analysis of various classes of Jacobi matrices, see, e.g., [55]. Note that similar theories exist also for some other classes of operators, e.g., for continuous one-dimensional Schödinger operators on the real half-line (see [36]).

There exist two main approaches to the subordinacy theory of J in the spectral literature. The first is the original one, contained in [36] and [50]. And the second, Jitomirskaya–Last's approach (see [47]), is based on a continuous interpolation of the discrete family of semi-norms (1.4). As was mentioned earlier, in both approaches, the key object enabling the transition from subordination issues to the properties of the spectral measure for J is precisely the Weyl function. It is related to ℓ^2 generalized eigenvectors for J and complex, non-real λ-s and to two special non-ℓ^2 generalized eigenvectors p and q for J – the orthogonal polynomials of the first and of the second kind. We recall the detailed definition of the Weyl function in Subsection 5.1, both in the scalar case and in the general matrix case for BJM with blocks of dimension d. As one can see, the block case is just the direct generalization of the definition in the scalar case (for JM), which is simply the block case with $d = 1$.

The Block Jacobi Matrices and Block Jacobi Operators are important generalizations of those scalar ones (see, e.g., [4, Chapter VII.§2]). Recall here that BJM is an analog of JM (1.1) with matrix "blocks" being its terms instead of scalars, namely it is a semi-infinite block-tridiagonal Hermitian matrix of the form

$$\mathcal{J} = \begin{pmatrix} B_0 & A_0 & & \\ A_0^* & B_1 & A_1 & \\ & A_1^* & B_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{1.5}$$

where A_n and B_n are $d \times d$ complex matrices with all the A_n invertible and Hermitian B_n. And this, together with the induced maximal Block Jacobi Operator (BJO) (its detailed definition is placed in Section 3), are the main objects of all our generalizations in this paper.

The article is organized as follows.

In Section 2 we fix our notation, and we collect some rather elementary but convenient facts that we need in subsequent sections. One of novelties here (see Subsection 2.5) is the analog for the general block case of the $d = 1$ Jitomirskaya–Last's continuous interpolation from [47] of a discrete family of semi-norms (1.4) from Khan–Pearson [50] original version of subordinacy theory.

In Section 3 we start from recalling the definition of the block Jacobi operator J (the maximal one), and then we prove the finite-cyclicity of J. Note that finite-cyclicity is a natural generalization of the notion of cyclicity, being the well-known property of J in the scalar $d = 1$ case. In Subsection 3.2, for our main case when J is self-adjoint, we recall the notion of the matrix measure M of J. We also recall the representation of J as the multiplication operator in the L^2- matrix measure space $L^2(M)$ of $\mathbb{C}^d$-valued functions, being an analog of the classical $L^2(\mu)$ Hilbert space for the spectral measure μ of J from the case $d = 1$. Next, we "compute" in detail a general example 3.7 of the matrix measure for the "simplest" block Jacobi operators with all the blocks being diagonal.

Section 4 is devoted to generalized eigenvectors and to transfer matrices in the block case. We study here two types of generalized eigenvectors: "the usual", i.e., with $\mathbb{C}^d$ terms, and the matrix generalized eigenvectors – with $d \times d$ matrix terms (for the left side multiplication by the blocks of J). We give here also a strict approach to their extensions to the "initial values" at -1 and 0 instead of 0 and 1. Then, in Subsection 4.2, we recall the matrix orthogonal polynomials P and Q, and we prove a result being a block case generalization (Proposition 4.6) of a result from [47]. It shows a possibility of a kind of "a continuous control" of the size of Jitomirskaya–Last type semi-norms for both matrix orthogonal polynomials P and Q also for the block case. Next, in Subsection 4.3 we study transfer matrices for BJM. We show here how to use transfer matrices to get a simple alternative proof of the classical algebraic results concerning P and Q, called Liouville–Ostrogradsky formulae (see, e.g., [6, Theorem 5.2]).

Section 5 is the central section of this paper. We recall and study here the matrix Weyl function W and its relation to properties of M and thus, to spectral properties of J. The main results of Subsection 5.1 are Theorem 5.6 and Theorem 5.7. Both are generalizations of the scalar case results, and both seems to be slightly more general than the results known from the literature: In the first case see [79, formula (2.4)] for scalar Jacobi matrices; in the second case, see [73, Proposition 1] for scalar Jacobi matrices, and see [80, Theorem 11.1] for higher order ordinary differential operators. The first one gives some important formulae and estimates for the matrix Weyl function and the so-called Weyl matrix solution. The second gives an upper estimate of the dimension of the space of all the ℓ^2 generalized eigenvectors, and it works also for the non-self-adjoint case of J. In the next subsections we briefly recall the fact that the Cauchy transform of the spectral matrix measure of BJO is equal to its matrix Weyl function (see Fact 5.8) and we use this to study the relations of boundary limits of the matrix Weyl function with the properties of the spectral matrix measure. This leads us to Theorem 5.11, being one of the important spectral results for BJO, analogic to the appropriate well-known result for $d = 1$ (see, e.g., [79, Lemma 3.11]). This result with its rigorous proof seems to be absent in the spectral literature so far.

The last Section 6 is devoted to the "simplest form" of the notion of nonsubordinacy. The main issue here is Theorem 6.5. It seems to be quite a strong, simple and convenient spectral result having at the same time a short and "almost elementary" proof. It should be mentioned here also, that the general idea of nonsubordinacy and its spectral consequences for J are developed more deeply in our parallel paper [76], where we consider a more sophisticated notion of barrier nonsubordinacy.

In Appendix A we collect basic notions and simple facts concerning general vector measures and matrix measures, used in the previous sections.

2. PRELIMINARIES

Here we collect and fix some general notation and terminology for the paper, and we also introduce several convenient tools, which will be important in the main sections.

2.1. ABBREVIATIONS AND SYMBOLS

We use here also some "more or less common" abbreviations and symbols like:

iff:	if and only if,
TFCAE:	the following conditions are (mutually) equivalent,
w.r.t.:	with respect to,
s.a.:	self-adjoint (for operators),
a.c.:	absolutely continuous (for operators, measures etc.),
sing.:	singular (as above),
a.e.:	almost everywhere,
JM, BJM:	Jacobi matrix, block Jacobi matrix, respectively,
JO, BJO:	Jacobi operator, block Jacobi operator, respectively,
$\operatorname{lin} Y$:	the linear subspace generated by a subset Y of a linear space,
$F\restriction_Y$:	the restriction of function F to the subset Y of the domain,
$F(Y)$:	the image of subset Y with respect to function F,
$\mathrm{Dom}(A)$:	the domain of linear operator A,
$\mathrm{Dom}(A^\infty)$:	the intersection of all $\mathrm{Dom}(A^n)$ for $n \in \mathbb{N}$.

2.2. INTRODUCTORY NOTATION AND NOTIONS

We use here the following symbols for some sets of scalars:

$$\begin{aligned} \mathbb{C}_+ &:= \{z \in \mathbb{C} : \Im(z) > 0\}, \\ \mathbb{R}_+ &:= \{t \in \mathbb{R} : t > 0\}, \\ \mathbb{N}_k &:= \{n \in \mathbb{Z} : n \geq k\} \quad \text{for } k \in \mathbb{Z}, \end{aligned}$$

so, e.g.,

$$\mathbb{N} = \mathbb{N}_1, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad \mathbb{N}_{-1} = \mathbb{N} \cup \{-1, 0\}.$$

Let us fix here some $d \in \mathbb{N}$. The vectors of the standard base in $\mathbb{C}^d$ are denoted by $e_1, \ldots, e_d$. By $M_d(\mathbb{C})$ we denote the space of all $d \times d$ complex matrices, with the usual matrix/operator norm. We identify any $A \in M_d(\mathbb{C})$ with the appropriate linear transformation of $\mathbb{C}^d$ induced by matrix A. In particular, we shall use alternatively both the operator and the matrix notation for the action of A on vectors from $\mathbb{C}^d$, namely, for $v \in \mathbb{C}^d$ we typically use Av, but sometimes we also write Av^{T}. For $i, j \in \{1, \ldots, d\}$ the term of A from its ith row and jth column is denoted as usual by $A_{i,j}$ and v_j is the jth term of v. The symbol $A^{\{j\}}$ denotes the jth column of A (usually we treat columns as $\mathbb{C}^d$-vectors and not as one-column matrices). Similarly, for $A_{\{i\}}$ – the ith row of A. Moreover, for vectors $v^{(1)}, \ldots, v^{(d)} \in \mathbb{C}^d$ the matrix A with $A^{\{j\}} = v^{(j)}$ for any j is denoted by $[v^{(1)}, \ldots, v^{(d)}]$.

If X is a linear space, then by $\ell(\mathbb{N}_k, X)$ we denote the linear space of **all** the sequences $x = (x_n)_{n \in \mathbb{N}_k}$ with terms in X and $\ell_{\mathrm{fin}}(\mathbb{N}_k, X)$ is the subspace of $\ell(\mathbb{N}_k, X)$ consisting of all the "finite" sequences, i.e., of such x that $x_n = 0$ for n sufficiently large.

For sequences of vectors: if $u^{(1)}, \ldots, u^{(d)} \in \ell(\mathbb{N}_k, \mathbb{C}^d)$ with $u^{(j)} = (u_n^{(j)})_{n\in\mathbb{N}_k}$ for $j = 1, \ldots, d$, then the symbol $[u^{(1)}, \ldots, u^{(d)}]$, used already above for vectors and not for sequences of vectors, denotes the matrix sequence $U \in \ell(\mathbb{N}_k, M_d(\mathbb{C}))$ with $U_n := [u_n^{(1)}, \ldots, u_n^{(d)}]$ for any $n \geq k$.

The symbols $\|\cdot\|_X$ and $\langle\cdot,\cdot\rangle_X$ denote here the norm in a normed space X and the scalar product in a Hilbert space X, respectively, but we sometimes simplify the subscript X for X-s having some longer symbols, and we generally omit it for all operator norms, which we use by default for the bounded operators (mainly for matrices from $M_d(\mathbb{C})$), if no other choice is made.

For a linear operator $A : X \longrightarrow X$ in a normed space $X \neq \{0\}$ we define its *minimum modulus* by

$$\rfloor\!\! A \!\!\lfloor := \inf_{\|x\|_X=1} \|Ax\|_X. \tag{2.1}$$

Obviously, if $\dim X = 1$, then $\rfloor\!\! A \!\!\lfloor = \|A\|$. Recall that if A is invertible, then

$$\rfloor\!\! A \!\!\lfloor = \frac{1}{\|A^{-1}\|}, \tag{2.2}$$

and for $0 < \dim X < +\infty$ A is invertible iff $\rfloor\!\! A \!\!\lfloor > 0$.

We use the symbols

$$\operatorname{cl} G, \quad \overline{G}^{\mathrm{e}}$$

for the "usual" (topological) closure of a subset G of a topological space and for the essential closure of a Borel set $G \subset \mathbb{R}$, respectively (and $\overline{\lambda}$ is used here for the complex conjugation of $\lambda \in \mathbb{C}$). Recall, that

$$\overline{G}^{\mathrm{e}} := \{t \in \mathbb{R} : \forall_{\varepsilon>0} |G \cap (t-\varepsilon; t+\varepsilon)| > 0\} \tag{2.3}$$

for $G \in \mathrm{Bor}(\mathbb{R})$ and $|\cdot|$ is here the standard 1-dimensional Lebesgue measure on $\mathrm{Bor}(\mathbb{R})$ but, as usual, it will denote also the absolute value.

If μ is a measure[2)] on $\mathfrak{M}$ – a σ-algebra of subsets of some Ω, and $p \in [1, +\infty)$, then $L^p(\mu)$ (without the "universum" Ω and the σ-algebra for the measure, for short) denotes the standard L^p Banach space of the classes of the appropriate complex functions on the "universum" for the measure μ. We need sometimes to distinguish here $L^p(\mu)$ from the appropriate space of functions (and not classes) denoted here by $\mathcal{L}^p(\mu)$. For $p = 1$ and $X := \mathbb{R}^d$ we also use $\mathcal{L}^1_X(\mu)$ to denote the space of integrable functions from Ω into X in the standard coordinatewise sense of the integral and the integrability.

We need also some basic terminology concerning vector and matrix measures important for this paper. It is quite long, so it is collected in Appendix A, instead of here. Moreover, for a matrix measure M by $L^2(M)$ we denote the appropriate

2) Here the name "measure", without any extra adjectives / names of the type vector, matrix, complex, real, spectral etc., denotes always a classical measure with values in $[0, +\infty]$, without necessity of adding "non-negative". And the remaining ones, all "adjective (of the above type) + measures", belong to a wide class of vector measures for an appropriate vector space (real or complex). See Appendix A.

L^2-Hilbert space induced by this matrix measure (see Section 3.2 and, e.g., [29, 57] for more details).

If X is a normed space, then

$$\ell^2(\mathbb{N}_0, X) := \left\{ x \in \ell(\mathbb{N}_0, X) : \sum_{n=0}^{+\infty} \|x_n\|_X^2 < \infty \right\},$$

is a normed space with the norm defined for $x \in \ell^2(\mathbb{N}_0, X)$ by

$$\|x\|_{\ell^2} := \sqrt{\sum_{n=0}^{+\infty} \|x_n\|_X^2}\,.$$

If, moreover, X is a Hilbert space, then $\ell^2(\mathbb{N}_0, X)$ is a Hilbert space with the scalar product given for $x, y \in \ell^2(\mathbb{N}_0, X)$ by

$$\langle x, y\rangle_{\ell^2} := \sum_{n=0}^{+\infty} \langle x_n, y_n\rangle_X. \tag{2.4}$$

Here, the most important case for us is the Hilbert space $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$. As *the standard orthonormal basis of* $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$ we consider

$$\big\{\delta_n(e_i)\big\}_{(i,n)\in\{1,\ldots,d\}\times\mathbb{N}_0},$$

where for any vector $v \in \mathbb{C}^d$ and $n \in \mathbb{N}_0$ we define the sequence $\delta_n(v) \in \ell_{\mathrm{fin}}(\mathbb{N}_0, \mathbb{C}^d)$ by

$$(\delta_n(v))_m := \begin{cases} v & \text{if } m = n, \\ 0 & \text{otherwise}, \end{cases} \qquad m \in \mathbb{N}_0. \tag{2.5}$$

Moreover, we have

$$\ell_{\mathrm{fin}}(\mathbb{N}_0, \mathbb{C}^d) = \mathrm{lin}\{\delta_n(v) : v \in \mathbb{C}^d, n \in \mathbb{N}_0\}, \text{ and } \mathrm{cl}\,\ell_{\mathrm{fin}}(\mathbb{N}_0, \mathbb{C}^d) = \ell^2(\mathbb{N}_0, \mathbb{C}^d). \tag{2.6}$$

If $\mathcal{H}$ is a Hilbert space and A – a self-adjoint operator (possibly unbounded) in $\mathcal{H}$, then the projection-valued spectral measure ("the resolution of identity") of (for) A is denoted by E_A. In particular $E_A : \mathrm{Bor}(\mathbb{R}) \longrightarrow \mathcal{B}(\mathcal{H})$, where $\mathrm{Bor}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$ and $\mathcal{B}(\mathcal{H})$ denotes, as usual, the space of bounded operators on $\mathcal{H}$. If $x, y \in \mathcal{H}$, then $E_{A,x,y}$ denotes *the spectral measure for A, x and y*, i.e. the complex measure given by

$$E_{A,x,y}(\omega) := \langle E_A(\omega)x, y\rangle_{\mathcal{H}}\,, \quad \omega \in \mathrm{Bor}(\mathbb{R}), \tag{2.7}$$

and $E_{A,x} := E_{A,x,x}$ (*the spectral measure for A and x*). Denote also

$$\mathcal{H}_{\mathrm{ac}}(A) := \{x \in \mathcal{H} : E_{A,x} \text{ is a.c. with respect to the Lebesgue measure on } \mathrm{Bor}(\mathbb{R})\}.$$

If and $G \in \mathrm{Bor}(\mathbb{R})$, then the symbol $(A)_G$ denotes the part of A in the (invariant reducing) subspace

$$\mathcal{H}_G(A) := \mathrm{Ran}\, E_A(G).$$

Recall one of the important spectral notions:

Definition 2.1. *A is absolutely continuous*[3] *in G* iff $\mathcal{H}_G(A) \subset \mathcal{H}_{\mathrm{ac}}(A)$.
A is absolutely continuous iff A is absolutely continuous in $\mathbb{R}$.

Some further notation and terminology will be successively introduced in next subsections.

2.3. SOME MATRIX FORMULAE AND INEQUALITIES

Let us recall some notions related in particular to $d \times d$ matrices. For $A \in M_d(\mathbb{C})$:

– its *Hilbert–Schmidt norm* is defined by

$$\|A\|_{\mathrm{HS}} := \left(\sum_{i=1}^{d} \|Ae_i\|^2_{\mathbb{C}^d} \right)^{1/2}. \tag{2.8}$$

– its *real and imaginary parts* $\Re(A)$ and $\Im(A)$ (in the adjoint, and not in the complex conjugation sense) are given by

$$\Re(A) := \tfrac{1}{2}(A + A^*), \quad \Im(A) := \tfrac{1}{2i}(A - A^*). \tag{2.9}$$

For $v \in \mathbb{C}^d$ we shall use the symbol E^v to denote the matrix / operator $[v, 0, \ldots, 0] \in M_d(\mathbb{C})$, i.e.,

$$E^v(e_j) = \begin{cases} v & \text{for } j = 1 \\ 0 & \text{for } j > 1, \end{cases} \qquad j = 1, \ldots, d. \tag{2.10}$$

Proposition 2.2. *Let $A \in M_d(\mathbb{C})$ and $v \in \mathbb{C}^d$. Then:*

(i) $\|A\| \leq \|A\|_{\mathrm{HS}}$.
(ii) $\|\Im(A)\| \leq \|A\|$, $\|\Re(A)\| \leq \|A\|$.
(iii) $\Re \langle \nu, A\nu \rangle_{\mathbb{C}^d} = \langle \nu, (\Re A)\nu \rangle_{\mathbb{C}^d}$, $\quad \Im \langle \nu, A\nu \rangle_{\mathbb{C}^d} = - \langle \nu, (\Im A)\nu \rangle_{\mathbb{C}^d}$.
(iv) $\|AE^v\| = \|Av\|_{\mathbb{C}^d}$.
(v) *If $A \geq 0$, then $\|A\| \leq \operatorname{tr} A \leq d\,\|A\|$.*

Proof. Part (i) is a classical result (easy to obtain by the Schwarz inequality), and (ii) follows from $\|A^*\| = \|A\|$ and the triangle inequality. Part (iii) follows directly from the definitions of $\Re, \Im$ for complex numbers and for matrices – in (2.9).

To get (iv) observe first that $AE^v = E^{Av}$ by (2.10), so it suffices to consider $A = I$. But using (i) we get $\|E^v\| \leq \|v\|_{\mathbb{C}^d}$ and $\|E^v\| \geq \|E^v e_1\|_{\mathbb{C}^d} = \|v\|_{\mathbb{C}^d}$, so the equality holds.

[3] Note that we fix it because, unfortunately, there is no common unique terminology for the property defined here. Several various names are also popular in spectral theory for the same or for somewhat “similar” property. E.g., *A is purely absolutely continuous in...* or *A has purely absolutely continuous spectrum in...* The version used here (for the case $G = \mathbb{R}$) has a long tradition in spectral theory and comes from Putnam – see, e.g., [63].

Suppose that $A \geq 0$ and let $\lambda_1 \ldots, \lambda_d$ be all the eigenvalues of A repeated according to their multiplicities. We thus get (v) by

$$\|A\| = \max_{1\leq i\leq d} \lambda_i \leq \sum_{i=1}^{d} \lambda_i = \operatorname{tr} A \leq d \max_{1\leq i\leq d} \lambda_i = d\,\|A\|. \qquad \square$$

2.4. ASYMPTOTIC SYMBOLS AND THE AFFINE INTERPOLATION

Let S be an arbitrary set and $f, g : S \longrightarrow \mathbb{C}$. We define the symbol $\asymp$ of "*asymptotic similarity*" of functions:

$$f \asymp g \iff \exists_{c,C\in\mathbb{R}_+} \forall_{s\in S}\ c|g(s)| \leq |f(s)| \leq C|g(s)| \tag{2.11}$$

(note the presence of the absolute value in this definition). We shall use also alternative notation: $f(s) \asymp_s g(s)$.

If $S = \mathbb{N}_k$ for some $k \in \mathbb{Z}$, then f is just a sequence from $\ell(\mathbb{N}_k, \mathbb{C})$, i.e. $f = (f(n))_{n\in\mathbb{N}_k} = (f_n)_{n\in\mathbb{N}_k}$, and we shall consider its *affine interpolation* $\operatorname{aff}(f) : [k, +\infty) \longrightarrow \mathbb{C}$, uniquely defined by the conditions:

(a) $\operatorname{aff}(f)\restriction_{\mathbb{N}_k} = f$,
(b) for each $n \in \mathbb{N}_k$ $\operatorname{aff}(f)\restriction_{[n,n+1]}$ is an affine function, i.e. a function of the form $[n, n+1] \ni t \mapsto at + c \in \mathbb{C}$ for some $a, c \in \mathbb{C}$ (depending here also on n).

In particular $\operatorname{aff}(f)$ is a continuous interpolation of f, and one can easily check that it can be also given by the explicit formula:

$$\operatorname{aff}(f)(t) = f_{\lfloor t\rfloor} + \{t\}\left(f_{\lfloor t\rfloor+1} - f_{\lfloor t\rfloor}\right), \quad t \in [0, \infty) \tag{2.12}$$

where $\{t\} = t - \lfloor t\rfloor$ is the fractional part of t.

The following result joining the asymptotic similarity of sequences and of their affine interpolations, will be convenient in the proof of our main result.

Let us prove first the following result on quotients of two affine functions.

Lemma 2.3. *Suppose that* $\alpha, \beta, a_1, c_1, a_2, c_2 \in \mathbb{R}$, $\alpha < \beta$ *and* $a_2 t + c_2 \neq 0$ *for any* $t \in [\alpha, \beta]$. *Let* $\varphi : [\alpha, \beta] \longrightarrow \mathbb{R}$ *be given by*

$$\varphi(t) := \frac{a_1 t + c_1}{a_2 t + c_2}, \quad t \in [\alpha, \beta].$$

Then φ *is monotonic and, in particular, its maximal and minimal value is attained on the set* $\{\alpha, \beta\}$.

Proof. We have $\varphi'(t) := \frac{a_1c_2 - a_2c_1}{(a_2t+c_2)^2}$ and thus, we have only two cases:

(i) $a_1c_2 - a_2c_1 = 0$, hence φ is constant;
(ii) $a_1c_2 - a_2c_1 \neq 0$, so φ' has no zeros.

In both cases our assertion holds. $\square$

2.5. ANALOGS OF "J–L CONTINUOUS INTERPOLATION" OF A DISCRETE FAMILY OF SEMI-NORMS

The main idea of the Jitomirskaya–Last's new approach in [47] was "technically" based on a continuous interpolation of the discrete family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}_0}$ in $\ell(\mathbb{N}_0,\mathbb{C})$ to the family $\{\|\cdot\|_t\}_{t\in[0,+\infty)}$.

For our purposes we shall extend this notion in two ways. Namely, let V be a normed space, $n_0 \in \mathbb{Z}$ and assume that $X \in \ell(\mathbb{N}_{n_0}, V)$.

For any $(n_1,t) \in \mathbb{Z}\times\mathbb{R}$ such that $n_0 \le n_1 \le t$, we define

$$\|X\|_{[n_1,t]} := \left(\sum_{k=n_1}^{\lfloor t\rfloor} \|X_k\|_V^2 + \{t\} \big\|X_{\lfloor t\rfloor+1}\big\|_V^2 \right)^{1/2}, \tag{2.13}$$

where $\lfloor t\rfloor$ and $\{t\}$ are the integer and the fractional part of t, respectively, and for $t=\infty$

$$\|X\|_{[n_1,\infty]} := \left(\sum_{k=n_1}^{+\infty} \|X_k\|_V^2 \right)^{1/2} \tag{2.14}$$

(which can possibly be $+\infty$).

Moreover, if $V = M_d(\mathbb{C})$) for some $d \ge 1$, then similarly to (2.13) we define a continuous family based on minimum modulus (see (2.1)) instead of matrix norm. So, for any $(n_1,t)\in\mathbb{Z}\times\mathbb{R}$ such that $n_0 \le n_1 \le t$ we consider

$$\rfloor X \lfloor_{[n_1,t]} = \left(\sum_{k=n_1}^{\lfloor t\rfloor} \rfloor X_k\lfloor^2 + \{t\} \rfloor X_{\lfloor t\rfloor+1}\lfloor^2 \right)^{1/2}. \tag{2.15}$$

By (2.12) we can relate the above constructions with the notion of affine extension introduced in the previous subsection. The squares of the "new objects" are just the affine extensions of their discrete counterparts (see (2.16) and (2.17) below), which are somewhat more natural and simpler for the context of operators "acting on sequences" considered in this paper.

Observation 2.4. *The notions defined by* (2.13) *and* (2.15) (*i.e., for* $t < +\infty$) *satisfy respectively*

$$\|X\|^2_{[n_1,t]} = \mathrm{aff}\,(S_{X,n_1})\,(t)$$

and

$$\rfloor X\lfloor^2_{[n_1,t]} = \mathrm{aff}\left(\check{S}_{X,n_1}\right)(t)$$

for $t \ge n_1$*, where* $S_{X,n_1},\ \check{S}_{X,n_1} : \mathbb{N}_{n_1} \longrightarrow \mathbb{R}$ *are given for* $n \ge n_1$ *by*

$$S_{X,n_1}(n) := \|X\|^2_{[n_1,n]} = \sum_{k=n_1}^{n} \|X_k\|^2, \tag{2.16}$$

$$\check{S}_{X,n_1}(n) := \rfloor X\lfloor^2_{[n_1,n]} = \sum_{k=n_1}^{n} \rfloor X_k\lfloor_V^2 . \tag{2.17}$$

Note also that in the scalar case $d=1$ for $X \in \ell(\mathbb{N}_{n_0}, M_d(\mathbb{C}))$ we simply have $\|X\|_{[n_1,t]} = \rfloor X\lfloor_{[n_1,t]}$ and $S_{X,n_1} = \check{S}_{X,n_1}$.

Consider a second sequence $Y \in \ell(\mathbb{N}_{n_0}, V)$. Using the above observation and Lemma 2.3 we get:

Corollary 2.5. *Suppose that* $n_1 \in \mathbb{Z}$, $t \in \mathbb{R}$, $n_0 \le n_1 \le t$. *Let* $n := \lfloor t \rfloor$ *and assume that* $\|Y\|_{[n_1,n]} \neq 0$. *Then there exist such* $\underline{n}, \overline{n} \in \{n, n+1\}$ *that*

$$\frac{\|X\|_{[n_1,\underline{n}]}}{\|Y\|_{[n_1,\underline{n}]}} \le \frac{\|X\|_{[n_1,t]}}{\|Y\|_{[n_1,t]}} \le \frac{\|X\|_{[n_1,\overline{n}]}}{\|Y\|_{[n_1,\overline{n}]}}.$$

If $V = M_d(\mathbb{C})$ *then the analogous result with all the "*$\|\cdot\|$*" replaced by "*$\|\cdot\|\!\lfloor$*" is also true.*

Proof. It suffices to use Observation 2.4 and then Lemma 2.3 to the function φ given on $[n, n+1]$ by the formula $\varphi(t) := \left(\frac{\|X\|_{[n_1,t]}}{\|Y\|_{[n_1,t]}}\right)^2$. □

The above result is covenient when we consider two variants of definitions of "subordination type" notions: the first one – based on the ratio of the seminorms only for the "discrete $n \in \mathbb{N}$", and the second – based on the ratio of the seminorms for all the "continuous $t > 0$". It often allows to prove the equivalence of such two approaches (see, e.g., Subsection 6.1).

3. BLOCK JACOBI OPERATOR. THE SPECTRAL MATRIX MEASURE AND A SPECTRAL REPRESENTATION

We denote "the size of the block" for BJM by d, $d \in \mathbb{N}$, and for the whole paper we assume that $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are sequences of matrices from $M_d(\mathbb{C})$ such that

$$\det A_n \neq 0, \quad B_n = B_n^*, \quad n \in \mathbb{N}_0. \tag{3.1}$$

Define a *block Jacobi matrix*

$$\mathcal{J} = \begin{pmatrix} B_0 & A_0 & & \\ A_0^* & B_1 & A_1 & \\ & A_1^* & B_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

The pair of the sequences $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ will be called *Jacobi parameters of* $\mathcal{J}$. In fact, we mean that $\mathcal{J}$ is the linear operator (*"the formal block Jacobi operator"*) acting on the linear space $\ell(\mathbb{N}_0, \mathbb{C}^d)$ of all the $\mathbb{C}^d$ sequences, and the action of $\mathcal{J}$ is well-defined via formal matrix multiplication:

$$\mathcal{J} : \ell(\mathbb{N}_0, \mathbb{C}^d) \longrightarrow \ell(\mathbb{N}_0, \mathbb{C}^d),$$

i.e., for any $u \in \ell(\mathbb{N}_0, \mathbb{C}^d)$

$$(\mathcal{J}u)_n := \begin{cases} B_0 u_0 + A_0 u_1 & \text{for } n = 0, \\ A_{n-1}^* u_{n-1} + B_n u_n + A_n u_{n+1} & \text{for } n \in \mathbb{N}. \end{cases} \tag{3.2}$$

Recall now two important operators related to $\mathcal{J}$ and acting in the Hilbert space $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$: $J_{\min}$ and J (which will coincide in most of our further considerations). The operator $J_{\min}$ (*the minimal block Jacobi operator*) is simply the closure in $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$ of the operator in $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$ being the restriction: $\mathcal{J}\restriction_{\ell_{\mathrm{fin}}(\mathbb{N}_0,\mathbb{C}^d)}$. And to define J (*the maximal block Jacobi operator*) we first choose its domain in a usual way for "maximal-type" operators:

$$\mathrm{Dom}(J) := \left\{u \in \ell^2(\mathbb{N}_0, \mathbb{C}^d) : \mathcal{J}u \in \ell^2(\mathbb{N}_0, \mathbb{C}^d)\right\}, \tag{3.3}$$

and then we define $J := \mathcal{J}\restriction_{\mathrm{Dom}(J)}$.

When both $(A_n)_{n\in\mathbb{N}_0}$ and $(B_n)_{n\in\mathbb{N}_0}$ are bounded sequences, then obviously we have only one operator $J_{\min} = J \in \mathcal{B}(\ell^2(\mathbb{N}_0, \mathbb{C}^d))$, which is symmetric, so self-adjoint. But here we consider also unbounded cases, so let us recall the following result

Fact 3.1. $(J_{\min})^* = J$. *Moreover, TFCAE:*

(i) $J_{\min}$ *is s.a.,*
(ii) J *is s.a.;*

and if one of the above conditions holds, then $J_{\min} = J$.

See [4, Chapter VII.§2.5] for[4)] $J^*_{\min} = J$, and the remaining part follows from this by [71, formula (7.1.22)].

For the main results of the theory presented here the maximality of the domain is crucial. Moreover, we study mainly "the self-adjoint case", so (by Fact 3.1) the best choice here is to focus just only on the operator J, later on[5)].

By (3.3) and (3.2) we get

$$\ell_{\mathrm{fin}}(\mathbb{N}_0, \mathbb{C}^d) \subset \mathrm{Dom}(J), \quad J\left(\ell_{\mathrm{fin}}(\mathbb{N}_0, \mathbb{C}^d)\right) \subset \ell_{\mathrm{fin}}(\mathbb{N}_0, \mathbb{C}^d), \tag{3.4}$$

so in particular J is densely defined by (2.6). Moreover, by (3.2) we compute

$$J\left(\delta_n(v)\right) = \delta_{n-1}(A_{n-1}v) + \delta_n(B_n v) + \delta_{n+1}(A_n^* v), \quad v \in \mathbb{C}^d,\ n \in \mathbb{N}_0, \tag{3.5}$$

where we additionally denote

$$\delta_{-1}(v) := 0, \quad v \in \mathbb{C}^d.$$

3.1. THE FINITE-CYCLICITY AND THE CANONICAL CYCLIC SYSTEM

In the scalar $d = 1$ case Jacobi operator J is cyclic with a cyclic vector $\varphi := \delta_0(e_1)$ (*the canonical cyclic vector for* J), which means that the space

$$\mathrm{lin}\{J^n\varphi : n \in \mathbb{N}_0\} \tag{3.6}$$

[4)] Actually, in [4, Chapter VII.§2.5] only the case $A_n = A_n^*$ for all $n \geq 0$ was considered, but the proof in the general case is similar.

[5)] So, it means in particular that we do not study here any other possible s.a. extensions of $J_{\min}$ than J. Note that the same is important for the classical ($d = 1$) subordinate theory – see [50].

is dense in $\ell^2(\mathbb{N}_0, \mathbb{C})$ (here, for $d = 1$, we simply have $e_1 = 1 \in \mathbb{C}$). Indeed, this is known that the above space is just equal to $\ell_{\mathrm{fin}}(\mathbb{N}_0, \mathbb{C})$ for such φ. Cyclicity plus self-adjointness provides a very simple spectral representation of the operator.

So, suppose now, that "the scalar" J is s.a. and consider: $\mathbb{x}$ – the identity function on $\mathbb{R}$, $\mathbb{x}(t) = t$ for $t \in \mathbb{R}$, and μ – "the scalar" spectral measure for J and φ, i.e. $\mu = E_{J,\varphi}$ (see Section 2.2 for the spectral notation).

By the well-known spectral result, J, as a cyclic s.a. operator, is unitary equivalent to the operator of the multiplication by $\mathbb{x}$ in the space $L^2(\mu)$.

This one-dimensional result has also its analog for the block-Jacobi case with arbitrary d. First of all, instead of cyclicity, we consider here the so-called *finite-cyclicity* notion (see [57]). It means that for some $k \in \mathbb{N}$ there exists *a cyclic system* $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k)$ *for* J, i.e., a system of such vectors from $\mathrm{Dom}(J^\infty)$, that

$$\mathrm{lin}\{J^n \varphi_j : n \in \mathbb{N}_0, \quad j = 1, \ldots, k\} \tag{3.7}$$

is dense in $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$. In our general d-dimensional case the choice of $\vec{\varphi}$ can be done in an analogical way, as it was for $d = 1$, namely define

$$\vec{\varphi} := (\varphi_1, \ldots, \varphi_d), \quad \varphi_j := \delta_0(e_j), \quad j = 1, \ldots, d. \tag{3.8}$$

The following result is true.

Proposition 3.2. *The system* $\vec{\varphi}$ *is a cyclic system for* J*.*

Proof. For each $n \in \mathbb{N}_0$ denote[6)]

$$X_n := \left\{x \in \ell^2(\mathbb{N}_0, \mathbb{C}^d) : \forall_{m \neq n} x_m = 0\right\}, \quad Y_n := \sum_{k=0}^{n} X_k. \tag{3.9}$$

So $X_n, Y_n \subset \ell_{\mathrm{fin}}(\mathbb{N}_0, \mathbb{C}^d)$ and

$$X_n = \left\{\delta_n(w) : w \in \mathbb{C}^d\right\} = \mathrm{lin}\left\{\delta_n(e_j) : j = 1, \ldots, d\right\}, \quad n \in \mathbb{N}_0. \tag{3.10}$$

In particular

$$X_0 = \mathrm{lin}\left\{\varphi_j : j = 1, \ldots, d\right\}. \tag{3.11}$$

Using (3.10) and (3.5) we get

$$J(X_n) \subset Y_{n+1}, \quad n \in \mathbb{N}_0,$$

so also

$$J(Y_n) \subset Y_{n+1}, \quad n \in \mathbb{N}_0. \tag{3.12}$$

Using this by the obvious induction we obtain

$$J^n(X_0) \subset Y_n, \quad n \in \mathbb{N}_0. \tag{3.13}$$

6) Here we use the standard operation of the sum of subsets of a linear space, i.e., $\sum_{k=0}^{n} D_k := \{\sum_{k=0}^{n} x_k : \ x_k \in D_k$ for any $k = 0, \ldots n\}$ for subsets $D_0, \ldots D_n$.

Denote $Y_{-1} := \{0\}$ and for $k \in \mathbb{N}_0$ denote also

$$C_k := \begin{cases} (A_0 \cdot \dots \cdot A_{k-1})^* & \text{for } k > 0, \\ I & \text{for } k = 0, \end{cases} \tag{3.14}$$

so in particular C_k is invertible. To continue the proof, we shall first prove the following two results.

Lemma 3.3. *For any* $k \in \mathbb{N}_0$

$$\forall_{w \in \mathbb{C}^d} \exists_{u \in Y_{k-1}} J^k \delta_0(w) = u + \delta_k(C_k w). \tag{3.15}$$

Proof. For $k = 0$ and $w \in \mathbb{C}^d$ we have $J^k \delta_0(w) = \delta_0(w) = 0 + \delta_0(C_0 w)$, so (3.15) holds for $k = 0$. Suppose now that it holds for some $k \in \mathbb{N}_0$. Then by (3.15) for k and by (3.5), for $w \in \mathbb{C}^d$

$$J^{k+1}\delta_0(w) = J(J^k \delta_0(w)) = J(u) + J\left(\delta_k(C_k w)\right) = J(u) + u' + \delta_{k+1}(A_k^* C_k w),$$

with some $u \in Y_{k-1}$, $u' \in Y_k$. Finally, by (3.12) and (3.14)

$$J^{k+1}\delta_0(w) = u'' + \delta_{k+1}(A_k^* C_k w) = u'' + \delta_{k+1}(C_{k+1} w),$$

with some $u'' \in Y_k$. So (3.15) holds for $k + 1$, and we obtain our assertion by induction. □

We shall use this lemma to prove the result below.

Fact 3.4. *For any* $n \in \mathbb{N}_0$

$$Y_n = \sum_{k=0}^{n} J^k(X_0). \tag{3.16}$$

Proof. We get "⊃" from (3.13). Let us prove "⊂" by induction. For $n = 0$ the assertion is obvious. Now consider $n \in \mathbb{N}_0$ and suppose that $Y_n \subset \sum_{k=0}^{n} J^k(X_0)$.

Then

$$\operatorname{lin}\left(Y_n \cup J^{n+1}(X_0)\right) \subset \sum_{k=0}^{n+1} J^k(X_0),$$

by the linearity of the RHS. Thus, by (3.9), to get $Y_{n+1} \subset \sum_{k=0}^{n+1} J^k(X_0)$, it suffices to prove

$$X_{n+1} \subset \operatorname{lin}\left(Y_n \cup\ J^{n+1}(X_0)\right). \tag{3.17}$$

Indeed, if $w \in \mathbb{C}^d$, then by Lemma 3.3

$$\delta_{n+1}(w) = v + J^{n+1}\delta_0(C_{n+1}^{-1} w),$$

with some $v \in Y_n$. Therefore, by (3.10) we get (3.17). □

Let us continue the proof of Proposition 3.2. Using again the standard argumentation concerning the linear spaces generated by a subset, we see by (3.11) that

$$\operatorname{lin}\{J^k\varphi_j : k = 0, \dots, n, j = 1, \dots, d\} = \sum_{k=0}^{n} J^k(X_0), \quad n \in \mathbb{N}_0. \tag{3.18}$$

So, by Fact 3.4 and (3.18)

$$\begin{aligned}\operatorname{lin}\{J^n\varphi_j : n \in \mathbb{N}_0, j = 1, \dots, d\} &= \bigcup_{n=0}^{+\infty} \operatorname{lin}\{J^k\varphi_j : k = 0, \dots, n, j = 1, \dots, d\} \\ &= \bigcup_{n=0}^{+\infty} Y_n = \ell_{\text{fin}}(\mathbb{N}_0, \mathbb{C}^d),\end{aligned}$$

and the density of $\ell_{\text{fin}}(\mathbb{N}_0, \mathbb{C}^d)$ in $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$ finishes the proof. □

Surely, this choice of a cyclic system for J is not unique. The particular $\vec{\varphi}$ defined by (3.8) is called *the canonical cyclic system for J*.

3.2. THE SPECTRAL MATRIX MEASURE M AND THE REPRESENTATION OF J AS THE MULTIPLICATION OPERATOR

The next notions which should be generalized, when we are moving from the scalar to the block case, is the classical L^2-measure type Hilbert space with "the scalar non-negative" spectral measure μ for JO J, when J is s.a. The spectral measure μ is defined on Borel subsets of $\mathbb{R}$, and it should be replaced by a less-known object – the so-called *spectral matrix measure M* for BJO J, if we assume that it is s.a. And so, the classical $L^2(\mu)$ Hilbert space will be replaced by a new type Hilbert space – the L^2- matrix measure space. The spectral matrix measure for BJO J is a particular example of the general notion of matrix measure (see Appendix A). Similarly to the "scalar" spectral measure for the JO case, it is defined on $\operatorname{Bor}(\mathbb{R})$, but the values of M are non-negative $d \times d$ matrices, instead of non-negative numbers. Is tightly related to J, as in the JO case. E.g., J can be recovered from its spectral matrix measure up to a unitary equivalence, as follows from the representation Theorem 3.6 below.

Let us assume for the rest of this subsection that J is s.a. We use here the following terminology. Let $\vec{\varphi} = (\varphi_1, \dots, \varphi_k)$ be the canonical cyclic system for J given by (3.8).

Definition 3.5. *The spectral matrix measure $M = M_J = E_{J,\vec{\varphi}}$* [7] *for J* is given by

$$M : \operatorname{Bor}(\mathbb{R}) \to M_d(\mathbb{C}), \quad M(\omega) := \Big(E_{J,\varphi_j,\varphi_i}(\omega)\Big)_{i,j=1,\dots,d} \in M_d(\mathbb{C}), \quad \omega \in \operatorname{Bor}(\mathbb{R}) \tag{3.19}$$

(recall that $E_{J,\varphi_j,\varphi_i}(\omega) = \langle E_J(\omega)\varphi_j, \varphi_i\rangle_{\ell^2}$ by (2.7)).

[7] See also [57]; note the more general notation $E_{A,\vec{\psi}}$ used there for the spectral matrix measure of any s.a. finitely cyclic operator A and a cyclic system $\vec{\psi}$ for A.

We use rather M and not M_J in this paper to denote the spectral matrix measure for J, when J is fixed.

The analog of the scalar-Jacobi unitary representation result, mentioned before, can be formulated for our block-Jacobi case in a short way, as follows.

Theorem 3.6. *J is unitary equivalent to the operator of the multiplication by $\mathbb{x}$ in the space $L^2(M)$.*

For the full formulation and the proof (in the general finitely cyclic case) see [57, $\mathbb{x}$MUE Theorem][8].

In particular, the above result means that the spectral matrix measure M of J "contains" all the important spectral information about J, similarly to the spectral measure μ in the scalar Jacobi case. Thus, properties of M are important for spectral studies in many concrete examples. For instance, for such typically studied spectral information, as: the absolute continuity (or the singularity) of J in some subset of $\mathbb{R}$, the location of the spectrum and of some particular parts of spectra of J, etc.

On the other hand, "the nice properties" of the trace measure tr_M (see A.4) for matrix measure M suggest that instead of dealing with spectral matrix measure, somewhat sophisticated at times, it could be more useful to deal with its trace measure, being just a classical object – a finite Borel measure. We can see it, e.g., when we try to "control" the above mentioned spectral properties of J related, to the absolute continuity or singularity. In Fact A.5, Fact A.6, and Lemma A.7 from Appendix A we discussed this problem in more details for its abstract (vector) measure theory aspect. Their main "spectral operator theory" consequences for J are Proposition 5.11 and Proposition 5.10 in Section 5.

At the end of this subsection, to better illustrate the notion of the spectral matrix measure for $d > 1$, we compute M in a general case of "simplest" block Jacobi operators, namely in the case of diagonal parameters of the block Jacobi matrix.

Example 3.7. Consider $d \geq 1$ and diagonal blocks defining $d \times d$ block Jacobi operator J (the maximal one, as usual here):

$$A_n = \begin{pmatrix} a_n^{(1)} & & \\ & \ddots & \\ & & a_n^{(d)} \end{pmatrix}, \quad B_n = \begin{pmatrix} b_n^{(1)} & & \\ & \ddots & \\ & & b_n^{(d)} \end{pmatrix},$$

where $a^{(i)} = (a_n^{(i)})_{n \in \mathbb{N}_0}$, $b^{(i)} = (b_n^{(i)})_{n \in \mathbb{N}_0}$ are Jacobi parameters of d scalar Jacobi matrices $\mathcal{J}^{(i)}$ for $i = 1, \dots d$. Let $J^{(i)}$ be the appropriate (maximal) Jacobi operators, and assume that they are all s.a. Define the unitary transformation

$$U : \ell^2(\mathbb{N}_0, \mathbb{C}^d) \longrightarrow \bigoplus_{j=1}^{d} \ell^2(\mathbb{N}_0, \mathbb{C})$$

[8] Also, the detailed definition of the multiplication by a function operator in L^2-matrix measure spaces is presented there.

by

$$(Uu)_i := ((u_n)_i)_{n\in\mathbb{N}_0}\,, \quad i = 1, \dots d, \tag{3.20}$$

for $u = (u_n)_{n\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, \mathbb{C}^d)$.

Using the maximality of all the $d + 1$ operators J, $J^{(1)}, \dots J^{(d)}$ one can easily check that

$$J = U^{-1}\left(\bigoplus_{j=1}^{d} J^{(s)}\right)U. \tag{3.21}$$

Hence, in particular, also J is s.a. Moreover, by the general properties of projection-valued spectral measures for s.a. operators (for direct sums of operators and for the unitary transfer of operator, in both cases by the uniqueness of the projection-valued spectral measure) we have

$$E_J(\omega) = U^{-1}\left(\bigoplus_{j=1}^{d} E_{J^{(j)}}(\omega)\right)U, \quad \omega \in \mathrm{Bor}(\mathbb{R}). \tag{3.22}$$

The above formulae allow us to express the spectral matrix measure M of our block Jacobi operator J directly by the d "scalar" spectral measures $\mu^{(i)}$ for $J^{(i)}$, $i = 1, \dots d$. Recall that

$$\mu^{(i)}(\omega) := \langle E_{J^{(i)}}(\omega)\delta_0(1), \delta_0(1)\rangle_{\ell^2(\mathbb{N}_0,\mathbb{C})}\,, \quad \omega \in \mathrm{Bor}(\mathbb{R}), \quad i = 1, \dots d \tag{3.23}$$

and that $\vec{\varphi}$ used in (3.19) is the canonical cyclic system for J in $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$ given by $\varphi_i := \delta_0(e_i)$ (see Subsection 3.1). Therefore, by the unitarity of U and by (3.22)

$$\begin{aligned}(M(\omega))_{ij} &= \langle E_J(\omega)\delta_0(e_j), \delta_0(e_i)\rangle_{\ell^2(\mathbb{N}_0,\mathbb{C}^d)} \\ &= \left\langle \left(\bigoplus_{s=1}^{d} E_{J^{(s)}}(\omega)\right)U\delta_0(e_j), U\delta_0(e_i)\right\rangle_{\bigoplus_{s=1}^{d}\ell^2(\mathbb{N}_0,\mathbb{C})} \\ &= \sum_{s=1}^{d}\left\langle E_{J^{(s)}}(\omega)\left(U\delta_0(e_j)\right)_s, \left(U\delta_0(e_i)\right)_s\right\rangle_{\ell^2(\mathbb{N}_0,\mathbb{C})}\end{aligned}$$

for any $\omega \in \mathrm{Bor}(\mathbb{R})$ and $i, j = 1, \dots d$. But by (3.20)

$$\left(U\delta_0(e_j)\right)_s = \begin{cases}\delta_0(1) & \text{for } s = j,\\ 0 & \text{for } s \neq j,\end{cases} \quad s = 1, \dots d,$$

so

$$(M(\omega))_{ij} = \begin{cases}\langle E_{J^{(i)}}(\omega)\delta_0(1), \delta_0(1)\rangle_{\ell^2(\mathbb{N}_0,\mathbb{C})} & \text{for } i = j,\\ 0 & \text{for } i \neq j,\end{cases}$$

and finally, by (3.23), we get the diagonal form of our spectral matrix measure

$$M(\omega) = \begin{pmatrix}\mu^{(1)}(\omega) & & \\ & \ddots & \\ & & \mu^{(d)}(\omega)\end{pmatrix}, \quad \omega \in \mathrm{Bor}(\mathbb{R}). \tag{3.24}$$

4. THE ASSOCIATED DIFFERENCE EQUATIONS AND TRANSFER MATRICES

4.1. THE TWO ASSOCIATED DIFFERENCE EQUATIONS AND "THE SOLUTION EXTENSIONS TO -1"

For any $z \in \mathbb{C}$ let us consider two difference equations tightly related to $\mathcal{J}$. The first – "the vector" one – is the infinite system of equations for a sequence $u = (u_n)_{n\in\mathbb{N}_0} \in \ell(\mathbb{N}_0, \mathbb{C}^d)$:

$$(\mathcal{J}u)_n = zu_n, \quad n \geq 1. \tag{4.1}$$

Each such a vector sequence u is called *generalized eigenvector* (*for* J *and* z)[9] – *"gev"* for short. By (3.2) equivalently its explicit form can be written

$$A^*_{n-1}u_{n-1} + B_n u_n + A_n u_{n+1} = zu_n, \quad n \geq 1. \tag{4.2}$$

The second difference equation – "the matricial" one – is the analog equation (with the right-side multiplication choice[10]) for a matrix sequence $U = (U_n)_{n\in\mathbb{N}_0} \in \ell(\mathbb{N}_0, M_d(\mathbb{C}))$:

$$A^*_{n-1}U_{n-1} + B_n U_n + A_n U_{n+1} = zU_n, \quad n \geq 1, \tag{4.3}$$

and each such a matrix sequence U is called *matrix generalized eigenvector* (*for* J *and* z) – *"mgev"* for short.

Having our BJM $\mathcal{J}$ fixed, for $z \in \mathbb{C}$ we denote

$$\mathrm{GEV}(z) := \{u \in \ell(\mathbb{N}_0, \mathbb{C}^d) : u \text{ is a gev for } J \text{ and } z\}$$

and parallelly

$$\mathrm{MGEV}(z) := \{U \in \ell(\mathbb{N}_0, M_d(\mathbb{C})) : U \text{ is a mgev for } J \text{ and } z\},$$

being obviously liner subspaces of $\ell(\mathbb{N}_0, \mathbb{C}^d)$ and $\ell(\mathbb{N}_0, M_d(\mathbb{C}))$, respectively. By (3.1), both recurrence relations are of degree 2, in the sense that for any initial condition $\big(C_0, C_1\big) \in (M_d(\mathbb{C}))^2$ there is a unique sequence U satisfying (4.3) with $U_0 = C_0$, $U_1 = C_1$, and analogously for (4.2). More precisely, one can easily check the following.

Fact 4.1. *For any* $z \in \mathbb{C}$ *the map* $\mathrm{Ini}_{z;0,1} : \mathrm{GEV}(z) \longrightarrow \big(\mathbb{C}^d\big)^2$, *given by*

$$\mathrm{Ini}_{z;0,1}(u) = (u_0, u_1), \quad u \in \mathrm{GEV}(z),$$

is a linear isomorphism. The analogous result is true for $\mathrm{MGEV}(z)$ *and* $(M_d(\mathbb{C}))^2$. *In particular* $\dim \mathrm{GEV}(z) = 2d$ *and* $\dim \mathrm{MGEV}(z) = 2d^2$.

[9] Note here, that it is "generalized" for two reasons; the first, because u may not belong to $\mathrm{Dom}(J)$, not even $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$, and the second, since we do not require the equality for $n = 0$ above.

[10] The left-side one is also possible and used for several reasons, but we shall not consider it here.

However, it is often more convenient to use another kind of "initial conditions", namely "at -1 and 0" instead of 0 and 1. To formulate this properly, we shall define first the appropriate extension of each solution (in both, vector and matrix cases), which is tightly related to the "a priori choice" of A_{-1}:

$$A_{-1} := -\,\mathrm{I}\,. \tag{4.4}$$

The informal idea of the extension is simply to "extend to $n = 0$" the system (4.3) (and (4.2) analogously) and to "compute the "value at -1", using our choice made in (4.4), i.e., we get "$U_{-1} := (B_0 - z\,\mathrm{I})U_0 + A_0U_1$" for the matrix case. Unfortunately, as one can see, such a definition seems to depend explicitly on the parameter z, and not only on U. So, at the first sight it seems that the notation for the extension "to -1" of a solution U has to contain always this parameter, which would be not very convenient. And it is true, that we really have this problem, extending in such a way **any** sequence $U \in \ell(\mathbb{N}_0, M_d(\mathbb{C}))$ (similarly for the vector version). So we define first the following family $\{{}_{z\bullet}\}_{z\in\mathbb{C}}$ of "extending transformations" ${}_{z\bullet} : \ell(\mathbb{N}_0, M_d(\mathbb{C})) \longrightarrow \ell(\mathbb{N}_{-1}, M_d(\mathbb{C}))$ given for $U \in \ell(\mathbb{N}_0, M_d(\mathbb{C}))$ and $z \in \mathbb{C}$ simply by

$$({}_{z\bullet}U)_n := \begin{cases} (B_0 - z\,\mathrm{I})U_0 + A_0U_1 & \text{for } n = -1, \\ U_n & \text{for } n \in \mathbb{N}_0. \end{cases} \tag{4.5}$$

We shall use here the same notation for the vector sequences without any risk of confusion, i.e., we shall also write ${}_{z\bullet}u$ for $u \in \ell(\mathbb{N}_0, \mathbb{C}^d)$ and $z \in \mathbb{C}$ with the analogous meaning

$$({}_{z\bullet}u)_n := \begin{cases} (B_0 - z\,\mathrm{I})u_0 + A_0u_1 & \text{for } n = -1, \\ u_n & \text{for } n \in \mathbb{N}_0. \end{cases} \tag{4.6}$$

Therefore, for any $z \in \mathbb{C}$ both kinds of transformations are linear, and moreover:

$${}_{z\bullet}(UC) = ({}_{z\bullet}U)C, \quad \text{for any } U \in \ell(\mathbb{N}_0, M_d(\mathbb{C})),\ C \in M_d(\mathbb{C}). \tag{4.7}$$

Fortunately, the situation is much simpler if restricted to the subspace of all the (M)GEV-s. Namely, we have:

Fact 4.2. *If $z, w \in \mathbb{C}$, $z \neq w$, then*[11)]

$$\mathrm{GEV}(z) \cap \mathrm{GEV}(w) = \{0\}, \quad \mathrm{MGEV}(z) \cap \mathrm{MGEV}(w) = \{0\}.$$

Proof. For the matrix case, consider U satisfying both (4.3), and its analog for w. Then, subtracting, we get

$$0 = (z - w)U_n, \quad n \geq 1,$$

hence $U_n = 0$ for any $n \in \mathbb{N}$, but now again by (4.3) used only for $n = 1$ we get also $U_0 = 0$, i.e., $U = 0$. □

[11)] Below 0 denotes the zero sequence both for the $\mathbb{C}^d$-vector and for the matrix case.

This means, that for any non-zero vector or matrix solution of our equations, the parameter 'z' is in fact "coded in the solution". On the other hand, independently of z, the value of the extension ${}_{z}\bullet$ for the zero sequence is obviously also the zero sequence (on $\mathbb{N}_{-1}$ already) by the linearity. Hence, denote

$$\mathrm{GEV} := \bigcup_{z\in\mathbb{C}} \mathrm{GEV}(z)\,, \quad \mathrm{MGEV} := \bigcup_{z\in\mathbb{C}} \mathrm{MGEV}(z)\,,$$

and, thanks to Fact 4.2, for any $U \in \mathrm{MGEV}\setminus\{0\}$ $(u \in \mathrm{GEV}\setminus\{0\})$ we can define $\mathrm{Par}(U)$ $(\mathrm{Par}(u))$ as the unique number $z \in \mathbb{C}$ satisfying

$$U \in \mathrm{MGEV}(z) \;\; (u \in \mathrm{GEV}(z)).$$

Finally, we can simplify our notation and omit the parameter 'z', defining

$$\bullet : \mathrm{MGEV} \longrightarrow \ell(\mathbb{N}_{-1}, M_d(\mathbb{C}))$$

given for $U \in \mathrm{MGEV}$ simply by the formula

$$\bullet U := \begin{cases} {}_{\mathrm{Par}(U)}\bullet U & \text{for } U \neq 0, \\ 0 & \text{for } U = 0, \end{cases} \tag{4.8}$$

and analogically for $u \in \mathrm{GEV}$.

Now, taking (4.4) into account, let us consider "extensions" of the systems (4.2) and (4.3):

$$A^*_{n-1}u_{n-1} + B_n u_n + A_n u_{n+1} = z u_n, \quad n \geq 0 \tag{4.9}$$

for sequences $u = (u_n)_{n\in\mathbb{N}_{-1}} \in \ell(\mathbb{N}_{-1}, \mathbb{C}^d)$ and

$$A^*_{n-1}U_{n-1} + B_n U_n + A_n U_{n+1} = z U_n, \quad n \geq 0 \tag{4.10}$$

for sequences $U = (U_n)_{n\in\mathbb{N}_{-1}} \in \ell(\mathbb{N}_{-1}, M_d(\mathbb{C}))$. Their solutions will be called *extended generalized eigenvectors* and *extended matrix generalized eigenvectors*, respectively, *(for J and z) – "egev"* and *"emgev"* for short.

We denote also

$$\mathrm{GEV}_{-1}(z) := \big\{u \in \ell(\mathbb{N}_{-1}, \mathbb{C}^d) : u \text{ is an egev for } J \text{ and } z\big\}$$

and

$$\mathrm{MGEV}_{-1}(z) := \big\{U \in \ell(\mathbb{N}_{-1}, M_d(\mathbb{C})) : U \text{ is an emgev for } J \text{ and } z\big\}.$$

Let us formulate explicitly some simple relations between all the above "extended" and "non-extended" notions and the $\bullet$ transformation.

Fact 4.3. *For any $z \in \mathbb{C}$ the following assertions hold:*

(i) $\bullet\restriction_{\mathrm{GEV}(z)} = {}_{z}\bullet\restriction_{\mathrm{GEV}(z)}$,

(ii) $\bullet\restriction_{\mathrm{GEV}(z)}\colon \mathrm{GEV}(z) \longrightarrow \mathrm{GEV}_{-1}(z)$ *is a linear isomorphism between* $\mathrm{GEV}(z)$ *and* $\mathrm{GEV}_{-1}(z)$,

(iii) $\left(\bullet\restriction_{\mathrm{GEV}(z)}\right)^{-1} u = u\restriction_{\mathbb{N}_0}$ *for any* $u \in \mathrm{GEV}_{-1}(z)$,
(iv) *for any* $(c_{-1}, c_0) \in (\mathbb{C}^d)^2$ *there is a unique* $u \in \mathrm{GEV}_{-1}(z)$ *with* $u_{-1} = c_{-1}$, $u_0 = c_0$,

and their obvious reformulations for the matrix sequences variants are also true.

Proof. Let us check, e.g., the $\mathbb{C}^d$ vector version. Observe that (i) is in fact the definition of $\bullet$. Using this and taking $u \in \mathrm{GEV}(z)$, $v \in \mathrm{GEV}_{-1}(z)$, we get obviously $(\bullet u)\restriction_{\mathbb{N}_0} = u$ by (4.6), but we get also $\bullet(v\restriction_{\mathbb{N}_0}) = v$, because v satisfies (4.9) – in particular for $n = 0$. Linearity is clear by the definition, so (ii) and (iii) hold. Part (iv) follows directly from (4.9) and (3.1). □

It can be easily checked that (e)mgev-s and (e)gev-s are mutually related in the following simple ways.

Fact 4.4. *If* $U \in \ell(\mathbb{N}_{-1}, M_d(\mathbb{C}))$, $z \in \mathbb{C}$, *then:*

(i) U *is an emgev for* J *and* z *iff for any* $j = 1, \ldots, d$ *the vector sequence* $U^{\{j\}} := \left(U_n^{\{j\}}\right)_{n\in\mathbb{N}_{-1}} \in \ell(\mathbb{N}_{-1}, \mathbb{C}^d)$ *is an egev for* J *and* z;
(ii) *If* U *is an emgev for* J *and* z *then for any* $v \in \mathbb{C}^d$ *the vector sequence* $Uv := (U_n v)_{n\in\mathbb{N}_{-1}} \in \ell(\mathbb{N}_{-1}, \mathbb{C}^d)$ *is an egev for* J *and* z.

The analog result holds for sequences $U \in \ell(\mathbb{N}_0, M_d(\mathbb{C}))$ *and mgev-s and gev-s.*

4.2. MATRIX ORTHOGONAL POLYNOMIALS AND JITOMIRSKAYA–LAST TYPE SEMI-NORMS

According to Fact 4.3 (iv) for the matrix case, for any $z \in \mathbb{C}$ choose $Q(z), P(z) \in \mathrm{MGEV}_{-1}(z)$ corresponding to

$$\begin{cases} Q_{-1}(z) = \mathrm{I}, \\ Q_0(z) = 0, \end{cases} \qquad \begin{cases} P_{-1}(z) = 0, \\ P_0(z) = \mathrm{I}, \end{cases} \tag{4.11}$$

with the following general notation: for any sequence $U(p) = ((U(p))_n)_{n\in\mathbb{N}_k}$ depending on an extra "function variable type parameter" p:

$$U_n(p) := (U(p))_n, \quad n \in \mathbb{N}_k \tag{4.12}$$

for any p. The two sequences of functions Q, P are the so-called *second* and *first kind matrix orthogonal polynomials*[12].

We can also use "the conditions in $0, 1$" instead of those "in $-1, 0$":

$$\begin{cases} Q_0(z) = 0, \\ Q_1(z) = A_0^{-1}, \end{cases} \qquad \begin{cases} P_0(z) = \mathrm{I}, \\ P_1(z) = A_0^{-1}(z\,\mathrm{I} - B_0). \end{cases} \tag{4.13}$$

These two special solutions $Q(z)$ and $P(z)$ play an important algebraic role in the linear space of all matrix solutions $\mathrm{MGEV}_{-1}(z)$.

[12] More precisely, this name and the orthogonality property belong to the appropriate two sequences $(Q_n)_{n\in\mathbb{N}}$, $(P_n)_{n\in\mathbb{N}_0}$ of matrix valued polynomial functions Q_n, P_n on $\mathbb{R}$ or on $\mathbb{C}$ with the values at each z given by above defined $Q_n(z), P_n(z)$.

Fact 4.5. *Let $z \in \mathbb{C}$.*

(i) *If $U \in \mathrm{MGEV}_{-1}(z)$, then for any $V \in M_d(\mathbb{C})$ also $UV = (U_n V)_{n\in\mathbb{N}_0} \in \mathrm{MGEV}_{-1}(z)$.*

(ii) *Each $U \in \mathrm{MGEV}_{-1}(z)$ has the form*

$$U = P(z)S + Q(z)T, \tag{4.14}$$

for a unique pair (S, T) of matrices from $M_d(\mathbb{C})$. This pair is given by

$$\begin{cases} S := U_0, \\ T := U_{-1} = (B_0 - z\,\mathrm{I})U_0 + A_0 U_1. \end{cases} \tag{4.15}$$

(iii) *For any $S \in M_d(\mathbb{C})$ the matrix sequence $H := (P(z)S)\restriction_{\mathbb{N}_0}$ satisfies "the formal matrix eigenequation for $\mathcal{J}$ and z", namely: $H \in \mathrm{MGEV}(z)$ and*

$$B_0 H_0 + A_0 H_1 = z H_0. \tag{4.16}$$

So, for any $v \in \mathbb{C}^d \setminus \{0\}$ the $\mathbb{C}^d$-sequence $h := (P(z)v)\restriction_{\mathbb{N}_0}$ is an eigenvector of the formal operator $\mathcal{J}$ for z:

$$\mathcal{J}h = zh, \quad h \neq 0. \tag{4.17}$$

Proof. Part (i) is obvious. Hence, using it, by linearity, by (4.11) and by the unicity from Fact 4.3 (iv), we get (4.14) with S and T given by (4.15). – I.e., we can simply assume that some S and T are given by (4.15) and then we see that the initial conditions of the solution on the RHS of (4.14) are just the pair (U_{-1}, U_0), which proves (4.14) by the unicity. So, to finish (ii) we should check that the choice of the pair (S, T) is also unique. By the linearity, it suffices to check only that if $P(z)S + Q(z)T$ is the zero solution, then $S = T = 0$. Indeed, in this case we have $0 = P_0(z)S + Q_0(z)T = S$ and $0 = P_{-1}(z)S + Q_{-1}(z)T = T$. Now, to get (iii), we can first use (i) with (ii) for U of the form (4.14) with $T = 0$, so, using also Fact 4.3 (the matrix version), we get $H \in \mathrm{MGEV}(z)$ with (4.16) obtained by (4.15) for $T = 0$. Now we obtain $(\mathcal{J}h)_n = zh_n$ by (4.3) for $n \geq 1$ and separately for $n = 0$ from (4.16) with $S = \mathrm{I}$. Finally, $h \neq 0$, because by (4.11) $h_0 = (P(z)v)_0 = P_0 v = v \neq 0$. □

Assume now that J is s.a. We show here a result being a block case analog of the appropriate scalar case one result from [47]. It makes use of the special choice of Jitomirskaya–Last type semi-norms (see Subsection 2.5) and, similarly as in the case $d = 1$, it can be useful to "control the boundary limits of the matrix Weyl function" (see Definition 5.5).

Proposition 4.6. *Suppose that J is self-adjoint. For any $\lambda \in \mathbb{R}$ there exists a unique function $\ell_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\|P(\lambda)\|_{[0,\ell_\lambda(\epsilon)]} \, \|Q(\lambda)\|_{[0,\ell_\lambda(\epsilon)]} = \frac{1}{2\epsilon}, \quad \epsilon > 0. \tag{4.18}$$

Moreover, ℓ_λ is a strictly decreasing continuous function and satisfies

$$\lim_{\epsilon \to 0^+} \ell_\lambda(\epsilon) = +\infty, \quad \lim_{\epsilon \to +\infty} \ell_\lambda(\epsilon) = 0. \tag{4.19}$$

Consequently, its inverse ℓ_λ^{-1} is also strictly decreasing continuous function and satisfies

$$\lim_{t\to 0^+} \ell_\lambda^{-1}(t) = +\infty, \quad \lim_{t\to +\infty} \ell_\lambda^{-1}(t) = 0. \tag{4.20}$$

Proof. Define a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ by the formula

$$f(t) = \|P(\lambda)\|_{[0,t]} \, \|Q(\lambda)\|_{[0,t]}.$$

By (2.13) this function is continuous, non-negative and weakly increasing. Moreover, by (4.11) it is positive in fact. Let us observe that it is strictly increasing. Indeed, because if not, then there would exist $n \in \mathbb{N}_0$ such that both $\|P(\lambda)\|_{[0,t]}$ and $\|Q(\lambda)\|_{[0,t]}$ were constant for $t \in (n, n+1)$. It would mean that $\|P_{n+1}(\lambda)\| = \|Q_{n+1}(\lambda)\| = 0$. Which by (4.27) would imply that $R_n(\lambda)$ was singular. This contradicts (4.34). Next, observe that

$$\lim_{t\to 0^+} f(t) = 0, \quad \lim_{t\to +\infty} f(t) = +\infty.$$

Indeed, the first limit follows from $\|Q_0(\lambda)\| = 0$. The second follows from the fact that if $\|P(\lambda)\|_{[0,+\infty]} < +\infty$ and $\|Q(\lambda)\|_{[0,+\infty]} < +\infty$, then the operator J is not self-adjoint, see [33, Theorem 1.3]. Thus, we have shown that f is continuous, strictly increasing and surjective. So its inverse $f^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ has the same properties. Consider the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g(\epsilon) = 1/(2\epsilon)$. This function is strictly decreasing and surjective. Thus, if ℓ_λ function exists it is a solution of the equation

$$f\big(\ell_\lambda(\epsilon)\big) = g(\epsilon).$$

This equation has a unique solution given by

$$\ell_\lambda(\epsilon) = f^{-1}\big(g(\epsilon)\big).$$

It is immediate that this defines a function satisfying (4.18). From this representation it is immediate that $\ell_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous strictly decreasing surjective function. It implies (4.19). Since again ℓ_λ^{-1} has analogous properties, we also obtain (4.20). □

The unique function ℓ_λ described above will be called *J-L function* (*for J and λ*). Note that ℓ_λ and Proposition 4.6 were essentially used to prove the main result of our paper [76].

4.3. TRANSFER MATRICES AND THE LIOUVILLE–OSTROGRADSKY FORMULAE

In the scalar case $d = 1$ the transfer matrix sequences turned out to be a very useful tool for describing spectral properties of the operator J. As we shall see, this is the case of general dimension d.

Let us fix here $z \in \mathbb{C}$. Our basic difference equations: the generalized eigenequation (4.1), its matrix analog (4.3), as well as their extended variants, can be written in equivalent forms with the use of the so-called (one step) *transfer matrices* (*for J and z*).

The nth transfer matrix $T_n(z) \in M_{2d}(\mathbb{C})$ has the block form, with blocks in $M_d(\mathbb{C})$:

$$T_n(z) := \begin{pmatrix} 0 & \mathrm{I} \\ -A_n^{-1}A_{n-1}^* & A_n^{-1}(z\,\mathrm{I} - B_n) \end{pmatrix}, \quad n \geq 0 \tag{4.21}$$

(for $n = 0$ recall that $A_{-1} = -\mathrm{I}$ by (4.4)). Hence, obviously, (4.2) (or (4.9)) is equivalent to

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = T_n(z) \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}, \quad n \geq 1 (\text{or } \geq 0). \tag{4.22}$$

Similarly, (4.3) (or (4.10)) is equivalent to

$$\begin{pmatrix} U_n \\ U_{n+1} \end{pmatrix} = T_n(z) \begin{pmatrix} U_{n-1} \\ U_n \end{pmatrix}, \quad n \geq 1 (\text{or } \geq 0). \tag{4.23}$$

Let us observe that $T_n(z)$ is invertible and

$$\big(T_n(z)\big)^{-1} = \begin{pmatrix} \big(A_{n-1}^*\big)^{-1}(z\,\mathrm{I} - B_n) & -\big(A_{n-1}^*\big)^{-1}A_n \\ \mathrm{I} & 0 \end{pmatrix}, \quad n \geq 0, \tag{4.24}$$

which is clear by direct multiplying (or by expressing u_{n-1} by u_n and u_{n-1} from (4.9)).

Moreover, we define *the n-step transfer matrix* by

$$R_n(z) = T_{n-1}(z) \ldots T_0(z), \quad n \geq 1. \tag{4.25}$$

This name is justified, e.g., by the property

$$\begin{pmatrix} U_{n-1} \\ U_n \end{pmatrix} = R_n(z) \begin{pmatrix} U_{-1} \\ U_0 \end{pmatrix}, \quad n \geq 1, \tag{4.26}$$

which we obtain from (4.23). Hence, by (4.26) and (4.11) we get

$$R_n(z) = \begin{pmatrix} Q_{n-1}(z) & P_{n-1}(z) \\ Q_n(z) & P_n(z) \end{pmatrix}, \quad n \geq 1, \tag{4.27}$$

being simply a direct consequence of $R_n(z) = R_n(z) \begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix}$.

Presently we shall derive a formula for the inverse of $R_n(z)$ expressing it explicitly in terms of $R_n(\overline{z})$. Let us set

$$K_n := \begin{pmatrix} A_n^* & 0 \\ 0 & \mathrm{I} \end{pmatrix}, \quad n \geq -1 \tag{4.28}$$

and

$$\tilde{T}_n(z) := K_n T_n(z) K_{n-1}^{-1}, \quad n \geq 0. \tag{4.29}$$

So by (4.21)

$$\tilde{T}_n(z) = \begin{pmatrix} 0 & A_n^* \\ -A_n^{-1} & A_n^{-1}(z\,\mathrm{I} - B_n) \end{pmatrix}, \quad n \geq 0.$$

Therefore, defining

$$\Omega := \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix},$$

we verify by direct computations that

$$\Omega = \big(\tilde{T}_n(\overline{z})\big)^* \Omega \tilde{T}_n(z), \quad n \geq 0. \tag{4.30}$$

By using (4.29) we get

$$\begin{aligned} R_n(z) &= (K_{n-1}^{-1}\tilde{T}_{n-1}(z)K_{n-2})(K_{n-2}^{-1}\tilde{T}_{n-2}(z)K_{n-3})\dots(K_0^{-1}\tilde{T}_0(z)K_{-1}) \\ &= K_{n-1}^{-1}\tilde{R}_n(z)K_{-1}, \end{aligned} \tag{4.31}$$

where

$$\tilde{R}_n(z) := \tilde{T}_{n-1}(z)\tilde{T}_{n-2}(z)\dots\tilde{T}_0(z), \quad n \geq 1. \tag{4.32}$$

We claim that

$$\Omega = \big(\tilde{R}_n(\overline{z})\big)^* \Omega \tilde{R}_n(z), \quad n \geq 1. \tag{4.33}$$

We shall prove it inductively. By (4.32) we have $\tilde{R}_1(z) = \tilde{T}_0(z)$ for any $z \in \mathbb{C}$. Thus, in view of (4.30) the formula (4.33) holds true for $n = 1$. Next, if (4.33) holds for some $n \geq 1$, then by (4.30) we have

$$\Omega = \big(\tilde{R}_n(\overline{z})\big)^* \Omega \tilde{R}_n(z) = \big(\tilde{R}_n(\overline{z})\big)^* \Big(\big(\tilde{T}_n(\overline{z})\big)^* \Omega \tilde{T}_n(z)\Big) \tilde{R}_n(z) = \big(\tilde{R}_{n+1}(\overline{z})\big)^* \Omega \tilde{R}_{n+1}(z),$$

where in the last equality we have used (4.32). It ends the inductive step in the proof of (4.33). Thus, by multiplying both sides of (4.33) by Ω^{-1} on the left and then by $\big(\tilde{R}_n(z)\big)^{-1}$ on the right we arrive at

$$\big(\tilde{R}_n(z)\big)^{-1} = \Omega^{-1}\big(\tilde{R}_n(\overline{z})\big)^* \Omega.$$

Consequently, using (4.31) twice we can derive

$$\big(R_n(z)\big)^{-1} = K_{-1}^{-1}\Omega^{-1}\big(K_{-1}^{-1}\big)^*\big(R_n(\overline{z})\big)^* K_{n-1}^* \Omega K_{n-1},$$

so, by (4.28), finally

$$\big(R_n(z)\big)^{-1} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \big(R_n(\overline{z})\big)^* \begin{pmatrix} 0 & A_{n-1} \\ -A_{n-1}^* & 0 \end{pmatrix}, \quad z \in \mathbb{C}. \tag{4.34}$$

Note that (4.34) has been obtained by using the same argument in [40, formula (5)] for real z and with real A_n-s and B_n-s.

The following result is well-known, see, e.g., [6, Theorem 5.2] and [61, Lemma 2.4], however our proof seems to be new and, even taking the above preparatory calculations into account, it is relatively short.

Theorem 4.7 (Liouville–Ostrogradsky)**.** *For any $w \in \mathbb{C}$ one has*

$$Q_k(w)\big(P_k(\overline{w})\big)^* = P_k(w)\big(Q_k(\overline{w})\big)^*, \quad k \geq 0 \tag{4.35}$$

$$Q_k(w)\big(P_{k-1}(\overline{w})\big)^* - P_k(w)\big(Q_{k-1}(\overline{w})\big)^* = A_{k-1}^{-1}, \quad k \geq 1. \tag{4.36}$$

Proof. By (4.34) and (4.27) we have

$$\begin{aligned}\begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix} &= R_k(w) R_k^{-1}(w) \\ &= \begin{pmatrix} -P_{k-1}(w) & Q_{k-1}(w) \\ -P_k(w) & Q_k(w) \end{pmatrix} \begin{pmatrix} -\big(Q_k(\overline{w})\big)^* A_{k-1}^* & \big(Q_{k-1}(\overline{w})\big)^* A_{k-1} \\ -\big(P_k(\overline{w})\big)^* A_{k-1}^* & \big(P_{k-1}(\overline{w})\big)^* A_{k-1} \end{pmatrix}.\end{aligned}$$

Thus, the formulas (4.35) and (4.36) follows from computing the last row. □

Now we are ready to solve the non-homogeneous version of the matrix recurrence relation (4.10). The fact that despite the general non-commutativity of matrices of size $d > 1$ we obtain the formula for the solution, which is similar to the one from the scalar ($d = 1$) case, is a kind of a miracle. And the reason for this miracle lies precisely in the Liouville–Ostrogradsky formulae. This result is also not new, see, e.g., [51, Section 2], but we present our proof simpler and more detailed than other proofs we found in the literature.

Proposition 4.8. *Let $F \in \ell(\mathbb{N}_0, M_d(\mathbb{C}))$. The unique solution $S = (S_n)_{n \in \mathbb{N}_{-1}} \in \ell(\mathbb{N}_{-1}, M_d(\mathbb{C}))$ of the recurrence relation*

$$A_n S_{n+1} + B_n S_n + A_{n-1}^* S_{n-1} = z S_n + F_n, \quad n \geq 0. \tag{4.37}$$

with the initial conditions $S_{-1} = S_0 = 0$ is equal to

$$S_n := \sum_{k=0}^{n-1} \Big(Q_n(z) \big(P_k(\overline{z})\big)^* - P_n(z) \big(Q_k(\overline{z})\big)^* \Big) F_k, \quad n \geq -1. \tag{4.38}$$

Proof. Since all A_k are invertible it is clear that the solution of (4.37) with given initial conditions S_{-1} and S_0 is unique. It remains to prove that it is satisfied by the sequence defined by (4.38).

It is immediate from (4.38) that $S_{-1} = S_0 = 0$. Moreover, we get

$$S_1 = \Big(Q_1(z) \big(P_0(\overline{z})\big)^* - P_1(z) \big(Q_0(\overline{z})\big)^* \Big) F_0 = A_0^{-1} F_0$$

which is in agreement with (4.37) for $n = 0$. So let us assume that $n \geq 1$. Since both $P(z)$ and $Q(z)$ satisfy (4.3) we get

$$\begin{aligned} A_n S_{n+1} &= A_n \sum_{k=0}^{n} \Big(Q_{n+1}(z) \big(P_k(\overline{z})\big)^* - P_{n+1}(z) \big(Q_k(\overline{z})\big)^* \Big) F_k \\ &= (z\,\mathrm{I} - B_n) \sum_{k=0}^{n} \Big(Q_n(z) \big(P_k(\overline{z})\big)^* - P_n(z) \big(Q_k(\overline{z})\big)^* \Big) F_k \\ &\quad - A_{n-1}^* \sum_{k=0}^{n} \Big(Q_{n-1}(z) \big(P_k(\overline{z})\big)^* - P_{n-1}(z) \big(Q_k(\overline{z})\big)^* \Big) F_k. \end{aligned}$$

Thus,

$$A_n S_{n+1} = (z\,\mathrm{I} - B_n) S_n - A_{n-1}^* S_{n-1} + \tilde{F}_n,$$

where

$$\begin{aligned}\tilde{F}_n = (z\,\mathrm{I} - B_n)\Big(Q_n(z)\big(P_n(\overline{z})\big)^* - P_n(z)\big(Q_n(\overline{z})\big)^*\Big)F_n \\ - A_{n-1}^*\Big(Q_{n-1}(z)\big(P_n(\overline{z})\big)^* - P_{n-1}(z)\big(Q_n(\overline{z})\big)^*\Big)F_n \\ - A_{n-1}^*\Big(Q_{n-1}(z)\big(P_{n-1}(\overline{z})\big)^* - P_{n-1}(z)\big(Q_{n-1}(\overline{z})\big)^*\Big)F_{n-1}.\end{aligned} \tag{4.39}$$

It remains to prove that $\tilde{F}_n = F_n$. To do so, let us apply (4.35) for $w = z$ with $k = n$ and $k = n - 1$. Then we get that on the right-hand side of (4.39) the first and the third lines are equal to 0. By considering (4.36) for $w = \overline{z}$ with $k = n$ and taking the adjoint of both sides we get

$$\begin{aligned}\big(A_{n-1}^{-1}\big)^* &= \Big(Q_n(\overline{z})\big(P_{n-1}(z)\big)^* - P_n(\overline{z})\big(Q_{n-1}(z)\big)^*\Big)^* \\ &= P_{n-1}(z)\big(Q_n(\overline{z})\big)^* - Q_{n-1}(z)\big(P_n(\overline{z})\big)^*,\end{aligned}$$

which results in the second line on the right-hand side of (4.39) being equal to F_n, and consequently, $\tilde{F}_n = F_n$. It ends the proof. □

5. THE WEYL FUNCTION

Similarly to the scalar Jacobi case, Weyl coefficient (being a matrix for the block case), is the main object in the method of subordinacy, which gives the link between generalized eigenvectors and the absolute continuous and the singular part of the spectral measure. And consequently – the absolutely continuous and the singular spectrum of J.

5.1. ℓ^2 MATRIX SOLUTIONS AND THE MATRIX WEYL FUNCTION W

In the scalar Jacobi case "the scalar orthogonal polynomials" are used, with the common notation $p(z) := P(z), q(z) := Q(z)$ for $z \in \mathbb{C}$. That is, since $d = 1$, we treat complex numbers as elements of $M_d(\mathbb{C})$ and also as $\mathbb{C}^d$-vectors, and $p(z), q(z)$ are solutions of both (4.9) and (4.10), being now just the same equation.

It is also well-known for this case that if J is s.a. and $z \in \mathbb{C} \setminus \mathbb{R}$, then neither $p(z)\restriction_{\mathbb{N}_0}$ nor $q(z)\restriction_{\mathbb{N}_0}$ belong to the Hilbert space $\ell^2(\mathbb{N}_0, \mathbb{C})$ in which Jacobi operator J acts, and there exists exactly one $w(z) \in \mathbb{C}$ such that $\big(w(z)p(z) + q(z)\big)\restriction_{\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, \mathbb{C})$. Surely, instead of making the restriction to $\mathbb{N}_0$, we can equivalently just claim here that $p(z), q(z) \notin \ell^2(\mathbb{N}_{-1}, \mathbb{C})$, and $w(z)p(z) + q(z) \in \ell^2(\mathbb{N}_{-1}, \mathbb{C})$, respectively. The above unique $w(z)$ is called *the Weyl coefficient (for J and z)*, and the appropriate function $w : \mathbb{C} \setminus \mathbb{R} \longrightarrow \mathbb{C}$ is called *the Weyl function for J*.

Let us recall here some less known generalisations of the above results and definitions for block Jacobi case. Hence, assume temporarily that J is s.a. and $z \in \mathbb{C} \setminus \mathbb{R}$, until we give up these assumptions in the last part of this subsection. Thus, $z \notin \sigma(J)$, and we can define

$$u^{(j)}(z) := (J - z\,\mathrm{I})^{-1}\delta_0(e_j), \quad j = 1, \ldots, d. \tag{5.1}$$

In particular, for any j we have $u^{(j)}(z) \in \mathrm{Dom}(J) \subset \ell^2(\mathbb{N}_0, \mathbb{C}^d)$ and

$$\mathcal{J}u^{(j)}(z) = zu^{(j)}(z) + \delta_0(e_j), \quad j = 1, \ldots, d.$$

Considering the terms $n \geq 1$ of the above equality of sequences we see that each $u^{(j)}(z)$ is a gev for J and z. Moreover, when $n = 0$ we get

$$e_j = \Big((\mathcal{J} - z\,\mathrm{I})u^{(j)}(z)\Big)_0 = (B_0 - z\,\mathrm{I})u_0^{(j)}(z) + A_0 u_1^{(j)}(z). \tag{5.2}$$

So, defining matrix sequences

$$\tilde{U}(z) := [u^{(1)}(z), \ldots, u^{(d)}(z)] \quad \text{and} \qquad U(z) := \;_\bullet(\tilde{U}(z)), \tag{5.3}$$

we have

$$\tilde{U}(z) \in \ell^2(\mathbb{N}_0, M_d(\mathbb{C})), \quad U(z) \in \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C})) \tag{5.4}$$

and by Fact 4.4 (the version for gev-s and mgev-s) together with Fact 4.3 we see that $\tilde{U}(z)$ is an mgev for J and z and $U(z)$ is an emgev for J and z.

Definition 5.1. If J is s.a. and $z \in \mathbb{C} \setminus \mathbb{R}$, then $U(z)$ is *the Weyl matrix solution for J and z.*

By (4.5) and (5.2) we obtain

$$\mathrm{I} = (B_0 - z\,\mathrm{I})U_0(z) + A_0 U_1(z) = U_{-1}(z). \tag{5.5}$$

For each $z \in \mathbb{C} \setminus \mathbb{R}$ and $\nu \in \mathbb{C}^d$ denote

$$u(\nu, z) := \tilde{U}(z)\nu. \tag{5.6}$$

Soon it will be convenient for us to use the following simple results.

Lemma 5.2. *If J is s.a., then for any $\nu \in \mathbb{C}^d$ and $z \in \mathbb{C} \setminus \mathbb{R}$*

$$u(\nu, z) \in D(J), \tag{5.7}$$

$$u(\nu, z) = (J - zI)^{-1}\delta_0(\nu), \tag{5.8}$$

$$u(e_j, z) = u^{(j)}(z), j = 1, \ldots, d, \tag{5.9}$$

$$\langle \nu, u_0(\nu, z)\rangle_{\mathbb{C}^d} = \langle Ju(\nu, z), u(\nu, z)\rangle_{\ell^2} - z\,\|u(\nu, z)\|^2_{\ell^2}. \tag{5.10}$$

Proof. By the linearity of $\tilde{U}(z)$ and by (5.3)

$$u(\nu, z) := \tilde{U}(z)\nu = \sum_{j=1}^{d} \nu_j \tilde{U}(z)e_j = \sum_{j=1}^{d} \nu_j u^{(j)}(z)$$

which gives (5.7), because each $u^{(j)}(z)$ is in $\mathrm{Dom}(J)$. Now, by (5.1) and by the linearity of $(J - zI)^{-1}$ we have

$$\sum_{j=1}^{d} \nu_j u^{(j)}(z) = \sum_{j=1}^{d} \nu_j (J - zI)^{-1}\delta_0(e_j) = (J - zI)^{-1}\delta_0\left(\sum_{j=1}^{d} \nu_j e_j\right) = (J - zI)^{-1}\delta_0(\nu),$$

hence we obtain (5.8), which gives also (5.9), by (5.1). Using (5.8) we get

$$\delta_0(\nu) = (J - zI)u(\nu, z) = Ju(\nu, z) - zu(\nu, z),$$

and taking the scalar product of booth sides with $u(\nu, z)$ we obtain

$$\langle Ju(\nu, z), u(\nu, z)\rangle_{\ell^2} - z\,\|u(\nu, z)\|^2_{\ell^2} = \langle \delta_0(\nu), u(\nu, z)\rangle_{\ell^2} = \langle \nu, u_0(\nu, z)\rangle_{\mathbb{C}^d},$$

by (2.5), so (5.10) is proved. □

We can now formulate the expected result on "matrix ℓ^2 solutions". For some special cases it is not new – for the case $A_n = A_n^*$ see, e.g., [4, Theorem VII.2.8] and for the case $A_n \equiv \mathrm{I}$ see [1, Section 2]. We give here a simple proof for the general case, for the sake of self-sufficiency.

Proposition 5.3. *Suppose that J is s.a. and $z \in \mathbb{C} \setminus \mathbb{R}$. Then there exists exactly one $W(z) \in M_d(\mathbb{C})$ such that*

$$P(z)W(z) + Q(z) \in \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C})). \tag{5.11}$$

Moreover, with the above unique $W(z)$:

(i) *$P(z)W(z) + Q(z)$ is the Weyl matrix solution for J and z:*

$$P(z)W(z) + Q(z) = U(z), \tag{5.12}$$

(ii) *$W(z) = \tilde{U}_0(z) = U_0(z)$,*
(iii) *$\det W(z) \neq 0$.*

Proof. Fix $z \notin \mathbb{R}$ and consider the Weyl matrix solution $U(z)$ for J and z. By Fact 4.5 (ii) $U(z)$ has the form (4.14), where $S = U_0(z) = \tilde{U}_0(z)$ and $T = U_{-1}(z) = \mathrm{I}$, by (5.5). So (5.4) proves the "exists"– part of the assertion and, assuming that we already have the uniqueness, we will also get (i) and (ii). Hence, let us now prove the uniqueness. Suppose that for some $z \notin \mathbb{R}$ there exist two different matrices "$W(z)$" satisfying (5.11). Then, subtracting, we get a non-zero $C \in M_d(\mathbb{C})$ such that $P(z)C \in \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C}))$. Now, choosing $w \in \mathbb{C}^d$ such that $v := Cw \neq 0$ we get $P(z)\nu \in \ell^2(\mathbb{N}_{-1}, \mathbb{C}^d)$. Thus, using Fact 4.5 (iii), for $h := (P(z)\nu)\restriction_{\mathbb{N}_0}$ we get $\mathcal{J}h = zh \in \ell^2(\mathbb{N}_0, \mathbb{C}^d)$, which means that $h \in \mathrm{Dom}(J)$. Moreover, $h \neq 0$, because $h_0 = P_0(z)\nu = \mathrm{I}\,\nu = \nu \neq 0$. Thus, h is an eigenvector of J with the eigenvalue $z \notin \mathbb{R}$ – a contradiction with the assumption, that J is s.a.

To prove (iii), i.e., that $W(z)$ is invertible, consider $\nu \in \operatorname{Ker} W(z)$, and the vector $w := u(\nu, z)$. By (5.6) and by (ii), just proved, we get

$$w_0 = U_0(z)\nu = W(z)\nu = 0,$$

which together with (5.10) gives

$$0 = \langle \nu, w_0\rangle_{\mathbb{C}^d} = \langle Jw, w\rangle_{\ell^2} - z\,\|w\|^2_{\ell^2}.$$

Therefore, using the s.a. of J, we get $z\,\|w\|^2_{\ell^2} = \langle Jw, w\rangle_{\ell^2} \in \mathbb{R}$. But since $z \notin \mathbb{R}$, w has to be the 0-vector. Now, by (5.8)

$$\delta_0(\nu) = (J - zI)w = 0,$$

so $\nu = 0$. □

It is a good moment to note the following result related to (5.11).

Corollary 5.4. *Suppose that J is s.a. and $z \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$P(z), Q(z) \notin \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C})).$$

Proof. We have $Q(z) = P(z)0 + Q(z)$, so if $Q(z) \in \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C}))$ then $W(z) = 0$ by the uniqueness from Proposition 5.3. But it contradicts the condition $\det W(z) \neq 0$ from (iii).

By (5.11) and (iii) of Proposition 5.3 we get

$$P(z) + Q(z)(W(z))^{-1} \in \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C})).$$

Thus, if $P(z) \in \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C}))$ then also

$$Q(z)(W(z))^{-1} \in \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C})).$$

But then also

$$Q(z) = \left(Q(z)(W(z))^{-1}\right) W(z) \in \ell^2(\mathbb{N}_{-1}, M_d(\mathbb{C})),$$

which contradicts the part just proved. □

Thanks to the results of Proposition 5.3, the notion of Weyl coefficient can be generalized from the "scalar" Jacobi operator case to all the block Jacobi operators with any (finite) dimension d of blocks.

Definition 5.5. Let J be s.a. For fixed $z \in \mathbb{C} \setminus \mathbb{R}$ such $W(z)$, that (5.11) holds is called *the matrix Weyl coefficient* (*for J and z*), and the appropriate function $W : \mathbb{C} \setminus \mathbb{R} \longrightarrow M_d(\mathbb{C})$ is called *the matrix Weyl function* (*for J*).[13)]

We usually omit here the dependence on J in the notation, and we write simply W, assuming that we consider a fixed J.

Theorem 5.6. *Assume that J is s.a. and $z \in \mathbb{C} \setminus \mathbb{R}$. For any $\nu \in \mathbb{C}^d$*

$$\frac{1}{\Im z}\left\langle\left(\Im W(z)\right)\nu, \nu\right\rangle_{\mathbb{C}^d} = \left\|\tilde{U}(z)\nu\right\|^2_{\ell^2} = \|U(z)\nu\|^2_{[0,+\infty]} \geq \|W(z)\nu\|^2. \tag{5.13}$$

Moreover, $\Im W(z)$ is strictly positive for $\Im z > 0$ and strictly negative for $\Im z < 0$ and

$$\left\|\tilde{U}(z)\right\|^2_{\ell^2(\mathbb{N}_0, M_d(\mathbb{C}))} = \|U(z)\|^2_{[0,+\infty]} \leq \frac{\operatorname{tr}\left(\Im W(z)\right)}{\Im z}. \tag{5.14}$$

13) Using the argumentation from the proof of Proposition 5.3 and from the beginning of this subsection one can easily see that in fact it suffices here to assume that $z \in \mathbb{C} \setminus \sigma(J)$ to properly define the matrix Weyl coefficient for J and z. But note also that the invertibility of $W(z)$ from property (iii) is guaranteed only for $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. The equality $\left\|\tilde{U}(z)\nu\right\|_{\ell^2}^2 = \|U(z)\nu\|_{[0,+\infty]}^2$ is obvious by the definition of $U(z)$ (see (5.3)) and by (2.14). Taking the imaginary parts of both sides of (5.10) and using Proposition 5.3 (ii), (5.6) and the s.a. of J we get

$$\Im\big\langle \nu, W(z)\nu \big\rangle_{\mathbb{C}^d} = \Im\big\langle \nu, \tilde{U}_0(z)\nu \big\rangle_{\mathbb{C}^d} = \Im\big\langle \nu, u_0(\nu,z) \big\rangle_{\mathbb{C}^d} = -\Im z\, \|u(\nu,z)\|_{\ell^2}^2 = -\Im z \left\|\tilde{U}(z)\nu\right\|_{\ell^2}^2. \tag{5.15}$$

By Proposition 2.2 (iii) we have also

$$\Im\big\langle \nu, W(z)\nu \big\rangle_{\mathbb{C}^d} = -\big\langle \nu, (\Im W(z))\nu \big\rangle_{\mathbb{C}^d} = -\big\langle (\Im W(z))\nu, \nu \big\rangle_{\mathbb{C}^d},$$

thus by (5.15) we obtain the equality from (5.13), and the inequality follows from Proposition 5.3 (ii).

To get the assertion on the strict positivity/negativity we use (5.13) and the fact that $W(z)\nu = 0$ only when $\nu = 0$, which follows from Proposition 5.3 (iii).

Now, apply (5.13) to $\nu \in \{e_1, e_2, \ldots, e_d\}$. Summing them up we get

$$\sum_{i=1}^{d} \|U(z)e_i\|_{[0,+\infty]}^2 = \frac{1}{\Im z} \sum_{i=1}^{d} \big\langle \big(\Im W(z)\big)e_i, e_i \big\rangle = \frac{1}{\Im z} \operatorname{tr}\big(\Im W(z)\big).$$

By Proposition 2.2 (i) and (2.8) we get

$$\sum_{i=1}^{d} \|U(z)e_i\|_{[0,+\infty]}^2 = \sum_{k=0}^{+\infty}\sum_{i=1}^{d} \|(U(z)e_i)_k\|_{\mathbb{C}^d}^2 = \sum_{k=0}^{+\infty} \|(U(z))_k\|_{\mathrm{HS}}^2 \geq \sum_{k=0}^{+\infty} \|(U(z))_k\|^2 = \|U(z)\|_{[0,+\infty]}^2.$$

Hence,

$$\|U(z)\|_{[0,+\infty]}^2 \leq \frac{1}{\Im z} \operatorname{tr}\big(\Im W(z)\big).$$

from which the result follows. □

Now we present also a result being a stronger version of [75, Proposition 3]. Let us stress that we **do not assume the self-adjointness of** J at this moment.

For $z \in \mathbb{C}$ denote

$$\mathrm{GEV}_{\ell^2}(z) := \mathrm{GEV}(z) \cap \ell^2(\mathbb{N}_0, \mathbb{C}^d). \tag{5.16}$$

Recall that by Fact 4.1 we have

$$\dim\big(\mathrm{GEV}(z)\big) = 2d,$$

and let us think about the dimension of its subspace $\mathrm{GEV}_{\ell^2}(z)$. To find it in some cases, consider also $\mathrm{EV}(z)$ – the eigenspace for J and z, which in the case of arbitrary $z \in \mathbb{C}$ is defined by

$$\mathrm{EV}(z) := \{u \in \mathrm{Dom}(J) : Ju = zu\}$$

(and so, it is just the trivial zero space if z is not an eigenvalue of J). Since J is the maximal block Jacobi operator (see (3.3)), we have

$$\mathrm{EV}(z) = \{u \in \mathrm{GEV}_{\ell^2}(z) : ((J-z)u)_0 = 0\}. \tag{5.17}$$

Indeed, the above equality follows directly from

$$\mathrm{GEV}_{\ell^2}(z) \subset \mathrm{Dom}(J),$$

which holds, because for $u \in \mathrm{GEV}_{\ell^2}(z)$ we have $u \in \ell^2(\mathbb{N}_0, \mathbb{C}^d)$, so also $zu \in \ell^2(\mathbb{N}_0, \mathbb{C}^d)$, but $\mathcal{J}u$ and zu differ at most at the zero term, hence $\mathcal{J}u \in \ell^2(\mathbb{N}_0, \mathbb{C}^d)$.

Theorem 5.7. *If $z \in \mathbb{C} \setminus \sigma_p(J)$, then $\dim(\mathrm{GEV}_{\ell^2}(z)) \le d$. If, moreover, $z \in \mathbb{C} \setminus \sigma(J)$, then $\dim(\mathrm{GEV}_{\ell^2}(z)) = d$.*

Proof. Consider first an arbitrary $z \in \mathbb{C}$. and define $\Psi : \mathrm{GEV}_{\ell^2}(z) \longrightarrow \mathbb{C}^d$ by

$$\Psi(u) := \left((J - z\,\mathrm{I})u\right)_0, \quad u \in \mathrm{GEV}_{\ell^2}(z)\,.$$

It is a linear transformation and by (5.17) $\mathrm{Ker}\,\Psi = \mathrm{EV}(z)$. So, by the standard linear algebra result,

$$\dim\left(\mathrm{GEV}_{\ell^2}(z)\right) = \dim\left(\mathrm{EV}(z)\right) + \dim\left(\mathrm{Ran}\,\Psi\right).$$

Hence, if $z \in \mathbb{C} \setminus \sigma_p(J)$, then $\dim(\mathrm{GEV}_{\ell^2}(z)) = \dim(\mathrm{Ran}\,\Psi) \le d$. But, if moreover $z \in \mathbb{C} \setminus \sigma(J)$, then $\mathrm{Ran}(J - z\,\mathrm{I}) = \ell^2(\mathbb{N}_0, \mathbb{C}^d)$, so in particular, for any $\nu \in \mathbb{C}^d$ we have $\delta_0(\nu) \in \mathrm{Ran}(J - z\,\mathrm{I})$. Therefore, for some $u \in \mathrm{Dom}(J)$

$$Ju - zu = \delta_0(\nu),$$

thus $u \in \mathrm{GEV}_{\ell^2}(z)$ and $\Psi(u) = \nu$ for such u. So, $\mathrm{Ran}\,\Psi = \mathbb{C}^d$ and $\dim(\mathrm{GEV}_{\ell^2}(z)) = \dim(\mathrm{Ran}\,\Psi) = d$. □

This result gives in particular $\dim(\mathrm{GEV}_{\ell^2}(z)) = d$, when J is s.a. and $z \in \mathbb{C} \setminus \mathbb{R}$. On the other hand, one can easily see that for such z

$$\mathrm{GEV}_{\ell^2}(z) = \mathrm{lin}\{u^{(j)}(z) : j = 1, \ldots, d\},$$

where $u^{(j)}(z)$ are defined by (5.1), and they are just the successive jth column sequences ($j = 1, \ldots, d$) of the matrix sequence $\tilde{U}(z)$, being the restriction to $\mathbb{N}_0$ of the Weyl matrix solution $U(z)$ for J and z (see (5.3)).

5.2. THE CAUCHY TRANSFORM OF THE SPECTRAL MATRIX MEASURE AND THE MATRIX WEYL FUNCTION

Assume here that J is s.a. The Cauchy transform of the spectral matrix measure $M := E_{J,\vec{\varphi}}$ from (3.19) of J is defined as $\mathcal{C}_J : \mathbb{C} \setminus \mathbb{R} \longrightarrow M_d(\mathbb{C})$, with

$$\mathcal{C}_J(z) := \int_{\mathbb{R}} \frac{1}{\lambda - z}\, \mathrm{d}M(\lambda), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{5.18}$$

where the above integral is understood in the sense of (A.11). Consequently, (5.18) means just

$$\mathcal{C}_J(z) := \left(\int\limits_{\mathbb{R}} \frac{1}{\lambda - z} \,\mathrm{d}M_{i,j}(\lambda) \right)_{i,j=1,\ldots,d}, \tag{5.19}$$

with

$$M_{i,j} = E_{J,\varphi_j,\varphi_i}, \quad i,j = 1,\ldots,d,$$

and φ_j given by (3.8).

Hence, by spectral calculus for s.a. operators, for $z \in \mathbb{C} \setminus \mathbb{R}$ we get

$$\mathcal{C}_J(z) := \left(\int\limits_{\mathbb{R}} \frac{1}{\lambda - z} \,\mathrm{d}E_{J,\varphi_j,\varphi_i}(\lambda) \right)_{i,j=1,\ldots,d} = \left(\left\langle (J - z\,\mathrm{I})^{-1} \delta_0(e_j), \delta_0(e_i) \right\rangle_{\ell^2} \right)_{i,j=1,\ldots,d}. \tag{5.20}$$

Now, by (5.1), (2.4) and (5.3) for any $i,j = 1,\ldots,d$

$$(\mathcal{C}_J(z))_{i,j} = \left\langle u^{(j)}(z), \delta_0(e_i) \right\rangle_{\ell^2} = \left\langle (u^{(j)}(z))_0, e_i \right\rangle_{\mathbb{C}^d} = (U_0(z))_{i,j}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where $u^{(j)}(z)$ is given by (5.1) and $U(z)$ is Weyl matrix solution for J and z. Finally, by Proposition 5.3, we get the following fact.

Fact 5.8. *If J is s.a., then the Cauchy transform of the spectral matrix measure of J is equal to the matrix Weyl function for J and moreover, for any $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\mathcal{C}_J(z) = W(z) = U_0(z) = \left(\left\langle (J - z\,\mathrm{I})^{-1} \delta_0(e_j), \delta_0(e_i) \right\rangle_{\ell^2} \right)_{i,j=1,\ldots,d}.$$

5.3. THE BOUNDARY LIMITS OF THE MATRIX WEYL FUNCTION AND THE PROPERTIES OF THE SPECTRAL TRACE MEASURE

Assume here the self-adjointness of J, as before, and let us use the notation from the previous subsection, including $M := E_{J,\vec{\varphi}}$.

Fact 5.8 implies that W is a holomorphic matrix-valued function. Moreover, $\Im W$ is a strictly positive matrix on $\mathbb{C}_+$ from Theorem 5.6. Thus, we see that the restriction of W to $\mathbb{C}_+$ is a matrix Herglotz function. Denote

$$L(W) := \left\{ \lambda \in \mathbb{R} : \lim_{\epsilon \to 0^+} W(\lambda + i\epsilon) \text{ exists}^{14)} \right\},$$

and

$$W(\lambda + i0) := \lim_{\epsilon \to 0^+} W(\lambda + i\epsilon), \quad \lambda \in L(W),$$

[14] As the limit in $M_d(\mathbb{C})$, i.e., in particular the limit must belong to $M_d(\mathbb{C})$.

and define the following sets:

$$S_{\mathrm{ac,r}} := \big\{\lambda \in L(W) : \operatorname{rank}\big(\Im W(\lambda + i0)\big) = r\big\}, \quad 1 \leq r \leq d, \tag{5.21}$$

$$S_{\mathrm{ac}} := \bigcup_{r=1}^{d} S_{\mathrm{ac,r}}, \tag{5.22}$$

$$S_{\mathrm{sing}} := \Big\{\lambda \in \mathbb{R} : \lim_{\epsilon \to 0^+} \Im\big(\operatorname{tr} W(\lambda + i\epsilon)\big) = +\infty\Big\}. \tag{5.23}$$

In particular, it follows that

$$S_{\mathrm{sing}} \subset \mathbb{R} \setminus L(W). \tag{5.24}$$

Referring to (5.23), it is worth to note that for any $A \in M_d(\mathbb{C})$

$$\operatorname{tr}(\Im A) = \Im(\operatorname{tr} A).$$

It is important that (5.22) means simply

$$S_{\mathrm{ac}} = \big\{\lambda \in L(W) : \Im W(\lambda + i0) \neq 0\big\}. \tag{5.25}$$

Let us recall now the crucial result, joining the above defined sets with properties of M_{ac} and M_{sing} – the a.c. and the sing. parts of the spectral matrix measure M of J w.r.t. the Lebesgue measure $|\cdot|$ on $\operatorname{Bor}(\mathbb{R})$ (see Fact A.6). This theorem is obtained just as the direct use of the abstract result [35, Theorem 6.1] to the spectral matrix measure M.

Define $D : L(W) \longrightarrow M_d(\mathbb{C})$ by

$$D(\lambda) := \frac{1}{\pi} \Im W(\lambda + i0), \quad \lambda \in L(W). \tag{5.26}$$

Theorem 5.9.

(i) *S_{sing} is a support of M_{sing},*
(ii) *$|\mathbb{R} \setminus L(W)| = 0$,*
(iii) *D is a density of M_{ac} on $L(W)$ w.r.t. $|\cdot|$.*

So, this theorem shows that controlling of the boundary limits of the matrix Weyl function allows to get a lot of detailed information about the spectral matrix measure of J and of its sing. and a.c. parts.

Combining the theorem with Lemma A.7 we get the following proposition.

Proposition 5.10. *S_{ac} is a minimal support of $(\operatorname{tr}_M)_{\mathrm{ac}}$ with respect to $|\cdot|$. Moreover, S_{sing} is a support of $(\operatorname{tr}_M)_{\mathrm{sing}}$ and $|S_{\mathrm{sing}}| = 0$. Moreover, $S_{\mathrm{ac}} \cup \delta$ is also a minimal support of $(\operatorname{tr}_M)_{\mathrm{ac}}$ with respect to $|\cdot|$ for any Borel $\delta \subset \mathbb{R}$ with $|\delta| = 0$.*

Proof. We use Lemma A.7 taking:

$$\nu := |\cdot| \text{ so, also } \Omega := \mathbb{R} \text{ and } \mathfrak{M} := \operatorname{Bor}(\mathbb{R}),\ S_a := S_{\mathrm{ac}},\ S_s := S_{\mathrm{sing}},\ F := D\restriction_{S_{\mathrm{ac}}}$$

and by (5.24) with (5.25) we see that $S_a \cap S_s = \emptyset$, that is, the assumption (ii) of the lemma holds. Now, (ii) of Theorem 5.9 shows that $L(W)$ is a support of M_{ac}, since this matrix measure, by definition, is a.c. w.r.t. $|\cdot|$. But (iii) of this theorem with (5.25) and (5.26) mean, that the restriction of M_{ac} to $L(W)\setminus S_{\mathrm{ac}}$ is the zero matrix measure. Therefore, S_{ac} is also a support of M_{ac}. This together with (i) of Theorem 5.9 prove that the assumption (i) of the lemma holds. The assumption (iii) also holds, by the fact that $S_{\mathrm{ac}}\subset L(W)$ and by (iii) of Theorem 5.9, again. And (iv) of the lemma is obvious by (5.25). Therefore, Lemma A.7 yields the first two assertions and $|S_{\mathrm{sing}}|=0$ by (5.24) with (iii) of the theorem. The last assertion is obvious just by the definition of minimal support. □

5.4. THE BOUNDARY LIMITS AND SPECTRAL CONSEQUENCES FOR J

Assume the self-adjointness of J, and let us hold the notation as above.

Here we "translate" Theorem 5.9 into the spectral operator language via Proposition 5.10. Namely, we show:

Theorem 5.11. *Suppose that J is self-adjoint and $G\in \mathrm{Bor}(\mathbb{R})$.*

(i) *If $G\subset\mathbb{R}\setminus S_{\mathrm{sing}}$, then J is absolutely continuous in G.*
(ii) *If $G\subset S_{\mathrm{ac}}\cup(\mathbb{R}\setminus(L(W)\cup S_{\mathrm{sing}}))$, then J is absolutely continuous in G and $\overline{G}^{\mathrm{e}}\subset\sigma_{\mathrm{ac}}(J)$. So, if moreover G is open, or if G is a sum of an arbitrary family of connected non-singletons in $\mathbb{R}$, then $\mathrm{cl}\,G\subset\sigma_{\mathrm{ac}}(J)$.*

Proof. First of all, by Proposition 3.2, J is s.a. finitely-cyclic operator, with $\vec{\varphi}$ being a cyclic system for J and with M being the spectral matrix measure of J and $\vec{\varphi}$. Hence, the initial assumptions of [57, Theorem C.2] hold for J. So, using its assertion (2), we obtain our (i), because, by Proposition 5.10, $(\mathrm{tr}_M)_{\mathrm{sing}}(G)=0$ for $G\subset\mathbb{R}\setminus S_{\mathrm{sing}}$.

By Theorem 5.9 (iii) we get $|\mathbb{R}\setminus(L(W)\cup S_{\mathrm{sing}})|=0$. Hence, $S_{\mathrm{ac}}\cup\big(\mathbb{R}\setminus(L(W)\cup S_{\mathrm{sing}})\big)$ is a minimal support of $(\mathrm{tr}_M)_{\mathrm{ac}}$ with respect to $|\cdot|$ by Proposition 5.10. Moreover, $(\mathrm{tr}_M)_{\mathrm{sing}}(G)=0$, again because

$$S_{\mathrm{ac}}\cup\big(\mathbb{R}\setminus(L(W)\cup S_{\mathrm{sing}})\big)\subset\mathbb{R}\setminus S_{\mathrm{sing}}$$

and S_{sing} is a support of $(\mathrm{tr}_M)_{\mathrm{sing}}$ by Theorem 5.9 (i).

Now, by the assertion (3) of [57, Theorem C.2], the assertion for the special kinds of G follows from the property $\overline{G}^{\mathrm{e}}=\mathrm{cl}\,G$, which holds for those G (see, e.g., [57, Fact C.3]). □

6. ON SOME IDEAS OF NONSUBORDINACY FOR THE BLOCK CASE

6.1. THE NONSUBORDINACY AND THE VECTOR NONSUBORDINACY

Let us define a notion, which seems quite natural from the context of the crucial notion of *subordinated solutions* from Gilbert–Pearson–Khan subordination theory (see [36, 50]), concerning the $d=1$ case. Recall that the "interpolated" semi-norms $\|\cdot\|_{[0,t]}$ were introduced here in Section 2.5.

Definition 6.1. We say that J *satisfies vector nonsubordinacy* (*condition*) *for* $\lambda \in \mathbb{R}$ iff[15] for each pair of non-zero $u, v \in \mathrm{GEV}_{-1}(\lambda)$

$$\liminf_{t\to+\infty} \frac{\|u\|_{[0,t]}}{\|v\|_{[0,t]}} < +\infty. \tag{6.1}$$

We can also consider "another" notion, using the analogical matrix (emgev) terms formulation.

Definition 6.2. J satisfies *nonsubordinacy* condition for $\lambda \in \mathbb{R}$ iff for each pair of non-zero $U, V \in \mathrm{MGEV}_{-1}(\lambda)$

$$\liminf_{t\to+\infty} \frac{\|U\|_{[0,t]}}{\|V\|_{[0,t]}} < +\infty. \tag{6.2}$$

Note here that the choice of "liminf-s over $t > 0$", instead of more original Khan and Pearson's like "liminf-s over $n \in \mathbb{N}$", does not matter in both definitions, i.e., in both of them it is equivalent to this "over $n \in \mathbb{N}$". One direction of the implication follows directly from the definition of $\liminf$, and the other can be immediately obtained by Corollary 2.5. As we shall see soon, also the distinction between these two kinds "vector nonsubordinacy" / "nonsubordinacy" notions is not very important here.

We can use the symmetry w.r.t. u and v or U and V, respectively, and we get:

Fact 6.3. *J satisfies vector nonsubordinacy for $\lambda \in \mathbb{R}$ iff for each pair of non-zero $u, v \in \mathrm{GEV}_{-1}(\lambda)$*

$$\limsup_{t\to+\infty} \frac{\|u\|_{[0,t]}}{\|v\|_{[0,t]}} > 0. \tag{6.3}$$

And analogically in the nonsubordinacy case.

As we already announced, the above two definitions are equivalent, so we shall rather use only the name "nonsubordinacy" and not "vector nonsubordinacy".

Proposition 6.4. *Let $\lambda \in \mathbb{R}$. J satisfies nonsubordinacy for λ iff it satisfies vector nonsubordinacy for λ.*

Proof. ($\Rightarrow$) Take any non-zero $u, v \in \mathrm{GEV}_{-1}(\lambda)$ and view them as a sequence of column vectors. Let us define

$$U_n := E^{u_n}, \quad V_n := E^{v_n}, \quad n \geq -1,$$

cf. (2.10), i.e. the first column of U_n is equal to the column vector u_n and the rest is zero, analogously for V_n. By Fact 4.4 both $U, V \in \mathrm{MGEV}_{-1}(\lambda)$. By Proposition 2.2 for any $n \geq -1$ we have $\|U_n\| = \|u_n\|$ and $\|V_n\| = \|v_n\|$. Consequently, $\|U\|_{[0,t]} = \|u\|_{[0,t]}$ and $\|V\|_{[0,t]} = \|v\|_{[0,t]}$ for any $t \geq 0$. Thus, the condition (6.2) implies (6.1).

[15] Note that we consider here the function given by the fraction $\|u\|^2_{[0,t]} / \|v\|^2_{[0,t]}$ for $t > 0$ only, and the denominator is positive since here sequences are not the zero sequence and they belong to GEV; Similarly for the definition below.

($\Leftarrow$) Let us observe that for any $X \in \mathrm{MGEV}_{-1}(\lambda)$ and any $w \in \mathbb{C}^d$ such that $\|w\| = 1$ we have

$$\|Xw\|^2_{[0,t]} \leq \|X\|^2_{[0,t]} \leq \sum_{i=1}^{d} \|Xe_i\|^2_{[0,t]}, \quad t > 0. \tag{6.4}$$

Let $U, V \in \mathrm{MGEV}_{-1}(\lambda)$ be non-zero. Then there exists $w \in \mathbb{C}^d$ such that $\|w\| = 1$ and Vw is non-zero. Then by (6.4) we have

$$\frac{\|U\|^2_{[0,t]}}{\|V\|^2_{[0,t]}} \leq \sum_{i=1}^{d} \frac{\|Ue_i\|^2_{[0,t]}}{\|Vw\|^2_{[0,t]}}, \quad t > 0. \tag{6.5}$$

Now, by Fact 4.4, $Ue_i \in \mathrm{GEV}_{-1}(\lambda)$ for $i = 1, \ldots, d$ and $Vw \in \mathrm{GEV}_{-1}(\lambda)$. Thus, by (6.1) we get

$$\liminf_{t \to +\infty} \frac{\|Ue_i\|^2_{[0,t]}}{\|Vw\|^2_{[0,t]}} < +\infty, \quad i = 1, \ldots, d,$$

which together with (6.5) implies (6.2). $\square$

6.2. SOME "FAST" SPECTRAL CONSEQUENCE OF NONSUBORDINACY

In fact, this "negative form" of subordination type notions is not only "natural", as it was mentioned before, but it is also related to some spectral results for J in the block case which are a bit like the absolute continuity.

Recall that $\mathrm{GEV}_{\ell^2}(\lambda)$ was defined in (5.16).

Theorem 6.5. *Suppose that J is self-adjoint and it satisfies nonsubordinacy for some $\lambda \in \mathbb{R}$. Then* $\dim(\mathrm{GEV}_{\ell^2}(\lambda)) = 0$ *and* $\lambda \in \sigma(J) \setminus \sigma_{\mathrm{p}}(J)$.

Proof. Let $u, v \in \mathrm{GEV}_{-1}(\lambda)$ be non-zero. By Fact 6.3 there exists a constant $c > 0$ and a sequence $(t_k)_{k \in \mathbb{N}}$ tending to $+\infty$ such that

$$\|u\|_{[0,t_k]} \geq c\,\|v\|_{[0,t_k]}.$$

Consequently, taking the limit we get

$$\frac{1}{c}\,\|u\|_{[0,+\infty]} \geq \|v\|_{[0,+\infty]}. \tag{6.6}$$

Thus, if there existed a non-trivial $u \in \ell^2(\mathbb{N}_0, \mathbb{C}^d)$, then all v would be also in $\ell^2(\mathbb{N}_0, \mathbb{C}^d)$. But then the operator J would not be self-adjoint, by [33, Theorem 1.3]. Thus, each non-zero $u \in \mathrm{GEV}_{-1}(\lambda)$ is not square-summable. Therefore, by (5.17) we get $\lambda \notin \sigma_{\mathrm{p}}(A)$. Finally, Proposition 5.7 yields the last assertion. $\square$

6.3. ON FURTHER NONSUBORDINATION IDEAS

However, this concept of nonsubordinacy turns out to be rather insufficient for the purposes of true absolute continuity. In our parallel paper [76] we develop this idea deeper, and we consider more sophisticated notion called *barrier nonsubordinacy*. This new condition can already guarantee the spectral absolute continuity property for the block case of J. And the proof of it is based on arguments somewhat analogical to those used in the proofs of subordination theory for $d = 1$, including the generalization for the block case of Weyl theory presented in the previous sections.

Let us only mention here, that the barrier nonsubordinacy controls two things: a bound on $\frac{\|U\|_{[0,t]}}{\|V\|_{[0,t]}}$ for any fixed "large" t and $\lambda \in G$, but joint for all such $U, V \in \mathrm{MGEV}_{-1}(\lambda)$ which are "normalized" in a certain sense, and also "the size" of the above bound as a function of t.

APPENDIX A. VECTOR AND MATRIX MEASURES – SELECTED BASIC NOTIONS

For self-consistency and some self-sufficiency of the paper we collect here selected definitions of some basic notions and some properties related to matrix measures and – more generally – vector measures. We omit here the more sophisticated construction of the appropriate L^2-type space for the matrix measure, referring the reader to the literature (see, e.g., [80, Section 8] or [57][16)])

We start from the general definition of vector measure.

Consider a set Ω with $\mathfrak{M}$ – a σ-algebra of subsets of Ω and a certain norm space X. Let $V : \mathfrak{M} \longrightarrow X$

Definition A.1. *V is a vector measure (in X)* iff V is countably additive in the norm sense in X.

Now, let us consider a special case $X := M_d(\mathbb{C})$ for some $d \in \mathbb{N}$ (with a standard norm, say). And let $M : \mathfrak{M} \longrightarrow M_d(\mathbb{C})$.

Definition A.2. *M is a $(d \times d)$ matrix measure* iff

(a) M is a vector measure,
(b) $M(\omega) \geq 0$ for any $\omega \in \mathfrak{M}$.[17)]

So, in particular, each matrix measure is a vector measure, but despite its name and due to the extra non-negativity property (b), matrix measure is "much more" than a vector measure in $M_d(\mathbb{C})$.

The above Ω, $\mathfrak{M}$ and d are "fixed" below.

We have to precise here some terminology (choosing it from various versions in literature) and to recall several basic facts related to vector measures, matrix

[16)] The second position contains the detailed definition of the Hilbert space $L^2(M)$ for matrix measures and the details of the abstract spectral theory for finitely cyclic s.a. operators, based on the matrix measure approach and multiplication by function operators in $L^2(M)$ spaces.

[17)] So, in particular $M(\omega)$ is s.a.

measures and measures. Recall that some "measure" terminology was already fixed in Introduction – see footnote 2 page 360.

Suppose that ν is a measure on $\mathfrak{M}$ and $V : \mathfrak{M} \longrightarrow X$ is a vector measure, where X is a norm space.

In several cases below we will also need to assume additionally that $X = \mathbb{C}^k$ for some k, including possible obvious identifications, as, e.g., $M_d(\mathbb{C}) \equiv \mathbb{C}^{(d^2)}$, to provide the clear and standard sense of the integral and of $\mathcal{L}^1_X(\nu)$ functions (see the appropriate part of Section 2.2).

For any $G \in \mathfrak{M}$ denote $\mathfrak{M}_G := \{\omega \in \mathfrak{M} : \omega \subset G\}$. Surely $\mathfrak{M}_G$ is a σ-algebra of subsets of G and $V\restriction_{\mathfrak{M}_G}$ is a vector measure on $\mathfrak{M}_G$ ("on G"). We denote it by V_G, i.e.

$$V_G := V\restriction_{\mathfrak{M}_G},$$

and we call it *the restriction of V to G*. Analogous situation (and notation, and terminology) is well known and valid here for measures.

Suppose that $H : \Omega \longrightarrow X = \mathbb{C}^k$ is a measurable function w.r.t. $\mathfrak{M}$. If, moreover, $H \in \mathcal{L}^1_X(\nu)$, then we define a new function from $\mathfrak{M}$ into X by the formula:

$$\int_\omega H \,\mathrm{d}\nu, \quad \omega \in \mathfrak{M}. \tag{A.1}$$

It is obviously a vector measure, and we denote it by

$$H \,\mathrm{d}\nu,$$

analogously as in the case of measures (when H should be a scalar non-negative measurable function, instead of $H \in \mathcal{L}^1_X(\nu)$). But in our main case $X = M_d(\mathbb{C})$ (identified with $\mathbb{C}^{(d^2)}$) the situation is somewhat similar and one easily checks the following result.

Fact A.3. *If $H \in \mathcal{L}^1_X(\nu)$ and $H(t) \geq 0$ for ν-a.e. $t \in \Omega$, then the vector measure $H \,\mathrm{d}\nu$ is a matrix measure.*

Definition A.4. If a vector measure V is such that $V = H \,\mathrm{d}\nu$ with some $H \in \mathcal{L}^1_X(\nu)$, then we call H *the density*[18] *of V with respect to ν.*

Let $G \in \mathfrak{M}$. We also define the following notions:

the density on G w.r.t.: *the density of V on G w.r.t. ν* means: any density of V_G w.r.t. ν_G;

a support: *G is a support of V* iff $V_{\Omega\setminus G}$ is the zero vector measure (on $\mathfrak{M}_{\Omega\setminus G}$)[19];

a minimal support w.r.t.: *G is a minimal support of V w.r.t. ν* iff G is a support of V and for any support G' of V included in G

$$\nu(G \setminus G') = 0;$$

[18] However, it can be not unique, as a function from $\mathcal{L}^1_X(\nu)$.

[19] Generally, it is not sufficient here (contrary to measures) that $V(\Omega \setminus G) = 0$ because the property of having zero measure is not inheritable into measurable subsets. However, for matrix measures it **is** sufficient by non-negativity.

a.c. w.r.t.: V *is absolutely continuous* (*abbrev.: a.c.*) *w.r.t.* ν iff for any $\omega \in \mathfrak{M}$ if $\nu(\omega) = 0$, then $V(\omega) = 0$;

sing. w.r.t.: V *is singular* (*abbrev.: sing.*) *w.r.t.* ν iff there exists such a support $S \in \mathfrak{M}$ of V that $\nu(S) = 0$;

the a.c./sing. part w.r.t.: if $V_1, V_2 : \mathfrak{M} \longrightarrow X$ are two vector measures, such that:

(i) V_1 is a.c. w.r.t. ν and V_1 is sing. w.r.t. ν,

(ii) $V = V_1 + V_2$,

then we call V_1 *the a.c. part of* V *w.r.t.* ν and we denote it by

$$V_{\mathrm{ac},\nu},$$

and we call V_2 *the sing. part of* V *w.r.t.* ν, and we denote it by

$$V_{\mathrm{sing},\nu}.$$

Note here, that above two notions are well(uniquely)-defined, since the above decomposition, if exists, is unique[20];

a.c. on G w.r.t.: V *is a.c. on* G *w.r.t.* ν iff V_G is a.c. w.r.t. ν_G;

sing. on G w.r.t.: V *is sing. on* G *w.r.t.* ν iff V_G is sing. w.r.t. ν_G.

We adopt all the above definitions and names also for any measure μ instead of a vector measure V (recall that measure may be not a vector measure) just by interchanging the symbols μ and V, including the notation

$$\mu_{\mathrm{ac},\nu}, \quad \mu_{\mathrm{sing},\nu},$$

however they are most commonly known in the case of measures.

We adopt here also the convention, that the part "w.r.t. ν", as well as "$,\nu$" in the appropriate symbols, as e.g., $V_{\mathrm{ac},\nu}$, $V_{\mathrm{sing},\nu}$, can be omitted to shorten the notation to, e.g.,

$$V_{\mathrm{ac}}, V_{\mathrm{sing}}, \mu_{\mathrm{ac}}, \mu_{\mathrm{sing}}$$

etc., in the case when $\Omega = \mathbb{R}$, $\mathfrak{M} = \mathrm{Bor}(\mathbb{R})$ and ν is the Lebesgue measure $|\cdot|$.

For a $d \times d$ matrix measure M and $i, j = 1, \ldots, d$ let us define $M_{ij} : \mathfrak{M} \to \mathbb{C}$ by

$$M_{ij}(\omega) := \big(M(\omega)\big)_{ij}, \quad \omega \in \mathfrak{M}. \tag{A.2}$$

By the point (a) of Definition A.2, each of the M_{ij} is a complex measure on $\mathfrak{M}$. Moreover, non-negativity from (b) of a matrix means also its self-adjointness, so we have

$$M_{j,i} = \overline{M_{i,j}} \quad i, j = 1, \ldots, d. \tag{A.3}$$

By non-negativity, defining $\mathrm{tr}_M : \mathfrak{M} \to \mathbb{C}$ by the formula

$$\mathrm{tr}_M(\omega) := \mathrm{tr}(M(\omega)), \quad \omega \in \mathfrak{M}, \tag{A.4}$$

[20] Because any linear combination of vector measures being a.c. (sing.) w.r.t. ν is also a.c. (sing.) w.r.t. ν; and a vector measure which is both a.c. and sing. w.r.t. ν is the zero measure.

we get in fact a finite measure tr_M, called *trace measure of M*. This "classical" measure is much a simpler mathematical object than the matrix measure M, but it contains a lot of important information on M.

The results below are proved in [57, Section III.1].

Fact A.5. *For M being a matrix measure as above:*

(i) *M, as well as each M_{ij} for $i, j = 1, \ldots, d$, are absolutely continuous with respect to* tr_M*,*

(ii)
$$M(\omega) = 0 \iff \mathrm{tr}_M(\omega) = 0, \qquad \omega \in \mathfrak{M}, \tag{A.5}$$

(iii)
$$0 \le M(\omega) \le \mathrm{tr}_M(\omega)\,\mathrm{I} \qquad \omega \in \mathfrak{M}, \tag{A.6}$$

(iv) *There exists a density $D : \Omega \longrightarrow M_d(\mathbb{C})$ of M w.r.t.* tr_M*, such that for any $t \in \Omega$*

$$0 \le D(t) \le \mathrm{I}\ , \text{[21]} \quad t \in \Omega.$$

Note that each density D of M w.r.t. tr_M is determined only up to tr_M-a.e. equality, and each one is called *the trace density of M*. The set of all the densities of M w.r.t. tr_M is denoted here by $\mathbb{D}_M$, and by $\mathbb{D}_M^{\bullet}$ we denote the set of those D, which satisfy conditions from the point (iv) above.

By the definition of density we have

$$M(\omega) = \int_\omega D \,\mathrm{d}\,\mathrm{tr}_M, \quad \omega \in \mathfrak{M}, D \in \mathbb{D}_M. \tag{A.7}$$

We assume here, **to the end if this section**, that $M : \mathfrak{M} \longrightarrow M_d(\mathbb{C})$ is a matrix measure and $\nu : \mathfrak{M} \longrightarrow [0, +\infty]$ is a σ-finite measure.

There exist several ways to get a decomposition of **some** vector measures V into its a.c. and sing. parts w.r.t. a measure ν. All they are based somehow on the Lebesgue–Radon–Nikodym Theorem (see, e.g. [64]) for a complex measure "w.r.t. a σ-finite measure" version. We need it only for our matrix measure M, and it will be convenient to make it *via* the appropriate decomposition of tr_M onto parts $(\mathrm{tr}_M)_{\mathrm{ac},\nu}$ and $(\mathrm{tr}_M)_{\mathrm{sing},\nu}$, which exist by the Lebesgue–Radon–Nikodym Theorem. So, we just consider any $D \in \mathbb{D}_M$ and two matrix measures:

$$D\,\mathrm{d}(\mathrm{tr}_M)_{\mathrm{ac},\nu}, \quad D\,\mathrm{d}(\mathrm{tr}_M)_{\mathrm{sing},\nu}$$

(see notation (A.1))

Fact A.6. *The a.c. and the sing. parts of M w.r.t. ν exist, and they satisfy*

$$M_{\mathrm{ac},\nu} = D\,\mathrm{d}(\mathrm{tr}_M)_{\mathrm{ac},\nu}, \quad M_{\mathrm{sing},\nu} = D\,\mathrm{d}(\mathrm{tr}_M)_{\mathrm{sing},\nu}, \tag{A.8}$$

[21] In particular, $D(t)$ is self-adjoint.

where D is an arbitrary density from $\mathbb{D}_M$. Moreover, both $M_{\mathrm{ac},\nu}, M_{\mathrm{sing},\nu}$ are matrix measures[22]*, and*

$$\mathrm{tr}_{M_{\mathrm{ac},\nu}} = (\mathrm{tr}_M)_{\mathrm{ac},\nu}, \quad \mathrm{tr}_{M_{\mathrm{sing},\nu}} = (\mathrm{tr}_M)_{\mathrm{sing},\nu}. \tag{A.9}$$

In particular, if $S \in \mathfrak{M}$, then TFCAE:

(i) *S is a support (version 2.: minimal support w.r.t. ν) of $M_{\mathrm{ac},\nu}$,*
(ii) *S is a support (version 2.: minimal support w.r.t. ν) of $\mathrm{tr}_{M_{\mathrm{ac},\nu}}$,*
(iii) *S is a support (version 2.: minimal support w.r.t. ν) of $(\mathrm{tr}_M)_{\mathrm{ac},\nu}$,*

and analogically for $M_{\mathrm{sing},\nu}$, $\mathrm{tr}_{M_{\mathrm{sing},\nu}}$, $(\mathrm{tr}_M)_{\mathrm{sing},\nu}$.

Proof. Using "the short theory" presented above, one immediately checks that the pair of vector measures $Dd(\mathrm{tr}_M)_{\mathrm{ac},\nu}$, $Dd(\mathrm{tr}_M)_{\mathrm{sing},\nu}$ satisfies the conditions from Definition A.4 of the parts $M_{\mathrm{ac},\nu}$ $M_{\mathrm{sing},\nu}$. So, by the uniqueness of the decomposition, we get (A.8). Now, by the non-negativity of D and by Fact A.3 we see that $M_{\mathrm{ac},\nu}, M_{\mathrm{sing},\nu}$ are matrix measures.

To get the assertion (A.9), observe that the equality $M = M_{\mathrm{ac},\nu} + M_{\mathrm{sing},\nu}$ yields $\mathrm{tr}_M = \mathrm{tr}_{M_{\mathrm{ac},\nu}} + \mathrm{tr}_{M_{\mathrm{sing},\nu}}$ by the definition of the trace measure. But $M_{\mathrm{ac},\nu}, M_{\mathrm{sing},\nu}$ are a.c. or, respectively, sing. w.r.t. ν, hence also $\mathrm{tr}_{M_{\mathrm{ac},\nu}}, \mathrm{tr}_{M_{\mathrm{sing},\nu}}$ are a.c. or, respectively, sing. w.r.t. ν, just by the use of the property (A.5) for both those matrix measures. So, we get the result just by the definitions of a.c. and sing. parts. And the last part follows directly from (A.9), (A.5) and by the observation that both notions: of support, as well as of minimal support w.r.t. a measure, are determined by zero vector measure sets, only. □

Now we turn to a "technical" result concerning the notion of the minimal support.

Lemma A.7. *Consider a matrix measure $M : \mathfrak{M} \longrightarrow M_d(\mathbb{C})$, a measure $\nu : \mathfrak{M} \longrightarrow [0, +\infty]$, sets $S_a, S_s \in \mathfrak{M}$ and a non-negative function $F : S_a \longrightarrow M_d(\mathbb{C})$. If*

(i) *S_a is a support of $M_{\mathrm{ac},\nu}$ and S_s is a support of $M_{\mathrm{sing},\nu}$,*
(ii) *$S_a \cap S_s = \emptyset$,*
(iii) *F is a density of $M_{\mathrm{ac},\nu}$ on S_a w.r.t. ν,*
(iv) *$\nu(\{t \in S_a : F(t) = 0\}) = 0$,*

then S_s is a support of $(tr_M)_{\mathrm{sing},\nu}$ and S_a is a minimal support of $(tr_M)_{\mathrm{ac},\nu}$ w.r.t. ν.

Proof. From Fact A.6 we immediately see that S_s is a support of $(\mathrm{tr}_M)_{\mathrm{sing},\nu}$ and S_a is support of $(\mathrm{tr}_M)_{\mathrm{ac},\nu}$. To prove the minimality, consider any $S' \subset S_a$ which is also a support of $(\mathrm{tr}_M)_{\mathrm{ac},\nu}$. Hence, again by Fact A.6

$$0 = (\mathrm{tr}_M)_{\mathrm{ac},\nu}(S_a \setminus S') = \mathrm{tr}_{M_{\mathrm{ac},\nu}}(S_a \setminus S'). \tag{A.10}$$

[22] By the definition, they are "only" vector measures in $M_d(\mathbb{C})$.

On the other hand, by the assumption (iii), using $(S_a \setminus S') \subset S_a$ and the non-negativity of $\operatorname{tr} F(t)$ for any $t \in S_a$, we have

$$\operatorname{tr}_{M_{\mathrm{ac},\nu}}(S_a \setminus S') = \operatorname{tr}\big(M_{\mathrm{ac},\nu}(S_a \setminus S')\big) = \operatorname{tr}\left(\int\limits_{(S_a \setminus S')} F \,\mathrm{d}\nu\right) = \int\limits_{(S_a \setminus S')} \operatorname{tr} F(t)\ \mathrm{d}\nu(t).$$

Thus, by (A.10) we have $\operatorname{tr} F(t) = 0$ for ν-a.e. $t \in (S_a \setminus S')$. Moreover, by Proposition 2.2(v), $\operatorname{tr} F(t) = 0 \iff F(t) = 0$. So, by the assumption (iv), we get $\operatorname{tr} F(t) \neq 0$ also for ν-a.e. $t \in (S_a \setminus S')$. Thus, $\nu(S_a \setminus S') = 0$. $\square$

At the end of this section let us recall the definition of the integral of the scalar function w.r.t. a vector measure for some simplest case, but sufficient for our goals. Consider a vector measure $V : \mathfrak{M} \longrightarrow X$, where $X = \mathbb{C}^k$ (e.g. – a matrix measure, with $k = d^2$) and $f : \Omega \longrightarrow \mathbb{C}$ – a bounded $\mathfrak{M}$-measurable function. Then for any $s = 1, \ldots, k$ the function $V_s : \mathfrak{M} \longrightarrow \mathbb{C}$ given by

$$V_s(\omega) := \big(V(\omega)\big)_s, \quad \omega \in \mathfrak{M},$$

is a complex measure and therefore the integral

$$\int\limits_\Omega f \,\mathrm{d}V_s$$

is well-defined (see, e.g., [21, Section I.1]). Thus, we define simply:

$$\int\limits_\Omega f \,\mathrm{d}V := \left(\int\limits_\Omega f \,\mathrm{d}V_1, \ldots, \int\limits_\Omega f \,\mathrm{d}V_k\right) \in \mathbb{C}^k. \tag{A.11}$$

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