Opuscula Math. 45, no. 3 (2025), 339–350
https://doi.org/10.7494/OpMath.2025.45.3.339 Opuscula Mathematica

UPPER DISTANCE-TWO DOMINATION

Jason T. Hedetniemi, Stephen T. Hedetniemi, and Thomas M. Lewis

Communicated by Dalibor Fronček

Abstract. Let $G = (V, E)$ be a graph with vertex set V and edge set E. A set $S \subset V$ is a 2-packing in G if for any two vertices $u, v \in S$, the distance between them satisfies $d(u, v) > 2$. The upper 2-packing number $P_2(G)$ is the maximum cardinality of a 2-packing in G. A set $S \subset V$ is a dominating set for G if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. A set $S \subset V$ is a distance-2 dominating set if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that $d(u, v) \leq 2$. The upper distance-2 domination number $\Gamma_{\leq 2}(G)$ is the maximum cardinality of a minimal distance-2 dominating set in G. In this paper we establish two families of graphs G for which $P_2(G) = \gamma(G) = \Gamma_{\leq 2}(G)$, which extend several well-known equalities of the form $P_2(G) = \gamma(G)$.

Keywords: 2-packing, domination, distance-2 domination.

Mathematics Subject Classification: 05C69, 05C85.

1. INTRODUCTION

In this paper, we present theorems concerning two families of graphs, block graphs and coronas, for every graph within which, the upper 2-packing number, the domination number, and the upper distance-2 domination number are all equal. We will need to introduce some terminology and notation in order to present these results.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *distance* between a pair of vertices u and v, denoted by $d(u, v)$, is the number of edges in a shortest path between u and v. Given a positive integer k and a vertex v, the *open k-neighborhood* of v is the set

$$N_k(v) = \{u \in V : 1 \leq d(u, v) \leq k\},$$

and the *closed k-neighborhood* of v is the set $N_k[v] = N_k(v) \cup \{v\}$.

Given a set $S \subset V$, the *closed k-neighborhood* of S is the set $N_k[S] = \bigcup_{v \in S} N_k[v]$. A set S is called a *distance-k dominating set* (or a dk-set) in G provided that $N_k[S] = V$. A set S is called a *k-packing* of G if for any two vertices $u, v \in S$, $d(u, v) > k$. A 1-packing is often called an *independent set*, or equivalently, a set in which no two vertices are adjacent.

There are several parameters related to domination and packing. A minimum cardinality dk-set is called a *lower dk-set*, and the *lower distance-k domination number* $\gamma_{\leq k}(G)$ is the cardinality of a lower dk-set in G. The parameter $\gamma_{\leq k}(G)$ is well-studied and has natural applications, since the vertices in a lower dk-set in G can serve as a most cost-effective, distance-k monitoring set of all vertices in G. An *upper dk-set* is a minimal dk-set of maximum cardinality, and the *upper distance-k domination number* $\Gamma_{\leq k}(G)$ is the cardinality of an upper dk-set in G. This parameter is less intuitive since we are placing in G a largest possible number of monitors which can minimally monitor, within distance-k, all vertices in G.

A *lower k-packing* is a maximal k-packing of minimum cardinality and the *lower k-packing number* $p_k(G)$ is the cardinality of a lower k-packing in G. A maximum cardinality k-packing is called an *upper k-packing*, and the *upper k-packing number* $P_k(G)$ is the cardinality of an upper k-packing in G.

For $k = 1$, many of these parameters have different names and notations. For example, $\gamma(G) = \gamma_1(G)$, $\Gamma(G) = \Gamma_{\leq 1}(G)$ and $\alpha(G) = P_1(G)$ are called, respectively, the *domination number*, the *upper domination number*, and the *independence number* of G. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set in G. The following inequality chain is well known: for any graph G,

$$p_2(G) \leq P_2(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G). \tag{1.1}$$

For a comprehensive treatment of these and other graph parameters, see [3].

In this paper, we are primarily interested in $P_2(G)$, $\gamma(G)$, and $\Gamma_{\leq 2}(G)$. Some of the possible relationships between these parameters are exhibited in Figure 1. For G_1, all three parameters are equal: the filled squares are an upper d2-set and 2-packing, and the filled discs are a lower d1-set. For G_2, all three parameters are equal: the filled squares are an upper d2-set, an upper 2-packing, and a lower d1-set. For G_3, $P_2(G_3) = \gamma(G_3) = 2$ (filled discs) but $\Gamma_{\leq 2}(G_3) = n$ (filled squares). Finally, for G_4, all three parameters are different: the two gray discs are an upper 2-packing, the three filled discs are a lower d1-set, and the four filled squares are an upper d2-set.

In 1975, Meir, and Moon [5] proved that for any tree T, $P_{k+1}(T) = \gamma_{\leq k}(T)$. This result was generalized in 1988 by Domke, Hedetniemi, Laskar, and Allan [2], who showed that for any connected block graph G, $P_{k+1}(G) = \gamma_{\leq k}(G)$. A *block graph* (sometimes called a *clique tree*) is a graph in which every biconnected component is a complete graph; see Figure 2. Our first theorem, to be proved subsequently, is a natural extension of the Domke *et al.* equality for $k = 1$, adding a third parameter, the upper distance-2 domination number, to the equality chain.

Theorem 1.1. *For any connected block graph G, $P_2(G) = \gamma(G) = \Gamma_{\leq 2}(G)$.*

It would be interesting to see if the equalities of Theorem 1.1 are true for other classes of graphs. A promising entry point for such an investigation is the paper of Rubalcaba, Schneider, and Slater [6], which surveys classes of graphs with equal domination and 2-packing numbers. Efficient graphs are one such class with this property. A graph is *efficient* provided it has a set of vertices whose closed neighborhoods form a partition of the vertex set; such a set is called an *efficient dominating set*. The graph G_3 in Figure 1 is efficient, the set $S = \{a, c\}$ is an efficient dominating set, but it does not satisfy the equalities in Theorem 1.1. Graphs that have equal domination and upper 2-packing numbers are members of the class of BG-graphs, a family of graphs that is notable for its connection to Vizing's conjecture; see [1].

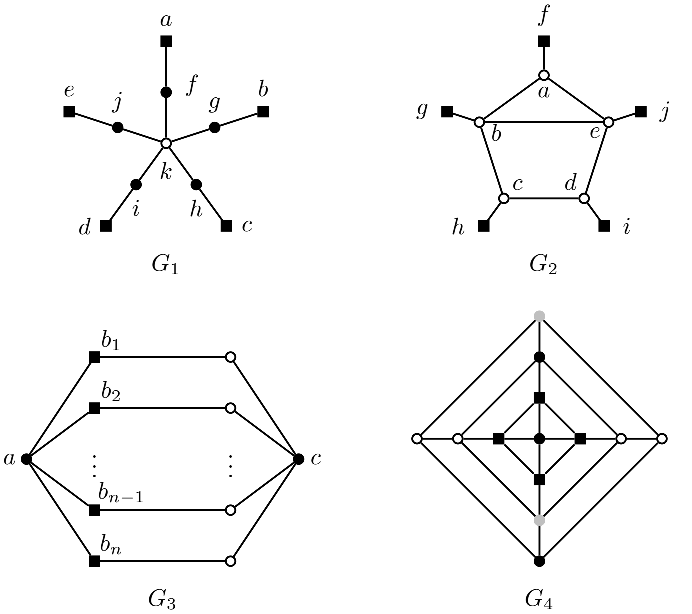

Fig. 1. The graphs G_1 through G_4 exhibit some of the possible relationships between the parameters P_2, γ, and $\Gamma_{\leq 2}$. For G_1, all three parameters are equal: the filled squares are an upper d2-set and 2-packing, and the filled discs are a lower d1-set. For G_2, all three parameters are equal: the filled squares are an upper d2-set, an upper 2-packing, and a lower d1-set. For G_3, $P_2(G_3) = \gamma(G_3) = 2$ (filled discs) but $\Gamma_{\leq 2}(G_3) = n$ (filled squares). Finally, for G_4, all three parameters are different: the two gray discs are an upper 2-packing, the three filled discs are a lower d1-set, and the four filled squares are an upper d2-set

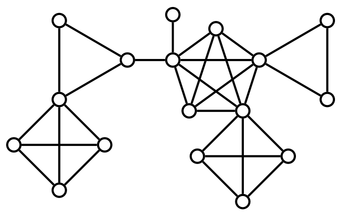

Fig. 2. A block graph

One particular family of efficient graphs that satisfies the equalities in Theorem 1.1 can be described as follows. Let H be a connected graph and let $G = H \odot K_1$ be the graph obtained by attaching a leaf to every vertex of H; this graph is called the *corona* of H and K_1.

One can observe by inspection that the set of leaves of $G = H \odot K_1$ is an upper 2-packing, a lower d1-set, and an upper d2-set and an efficient dominating set; thus, $P_2(G) = \gamma(G) = \Gamma_{\leq 2}(G)$. The graph G_2 in Figure 1 is the corona of the *house graph* H, with vertex set $V = \{a, b, c, d, e\}$, and K_1. Our next theorem asserts that graphs in this family are extremal.

Theorem 1.2. *Let G be a connected graph of order $n \geq 2$ and let S be an upper d2-set for G. Then $|S| \leq \lfloor n/2 \rfloor$. If $|S| = n/2$, then there is a connected subgraph H of G such that $G = H \odot K_1$. Moreover, $P_2(G) = \gamma(G) = \Gamma_{\leq 2}(G)$.*

The remainder of this paper is organized as follows. In Section 2 we introduce some additional terminology and show that every upper k-packing of a graph is a minimal dk-set of the graph; thus, $P_k(G) \leq \Gamma_{\leq k}(G)$. In Section 3 we prove a sequence of lemmas, culminating in the proof of Theorem 1.1. In Section 4, we show that any minimal d2-set of a graph induces a natural partition of its vertex set, which we use to prove Theorem 1.2.

2. SOME PROPERTIES OF k-PACKINGS AND DISTANCE-k DOMINATING SETS

In 1991, Henning, Oellermann and Swart [4] proved the following characterization of minimal dk-sets.

Theorem 2.1 ([4])**.** *A set $S \subset V$ in a graph $G = (V, E)$ is minimal dk-set if and only if S is a dk-set and for every vertex $v \in S$ at least one of the following two conditions hold:*

(1) *there exists a vertex $w \in V - S$ such that $N_k(w) \cap S = \{v\}$, or*
(2) *$N_k[v] \cap S = \{v\}$.*

This theorem suggests the following terminology. Let S be a dk-set for the graph G. An element of S is called a *dominator*. Given $x \in S$, we say that a vertex $y \in V - S$ is a *private-k neighbor of x* provided that $N_k[y] \cap S = \{x\}$. A dominator x is called *k-remote* provided that $N_k[x] \cap S = \{x\}$. Theorem 2.1 can be restated as follows: a set $S \subset V$ is a minimal dk-set in a graph $G = (V, E)$ if and only if $N_k[S] = V$, and each $x \in S$ is either k-remote or has a private-k neighbor $y \in V - S$.

To illustrate these terms, consider the graphs G_1, G_2, and G_4 in Figure 1. In G_1 the dominator a is 2-remote and vertex f is a private-2 neighbor of a. In G_2, the dominator f is 2-remote but has no private-2 neighbors. In G_4, the dominator a is not 2-remote and vertex b is a private-2 neighbor of a.

Our first result relates k-packings and minimal dk-sets.

Proposition 2.2. *Every maximal k-packing in a graph G is a minimal dk-set.*

Proof. Let S be a maximal k-packing in a graph G. Then, by maximality, for every $v \in V$, $N_k[v] \cap S \neq \emptyset$, which shows that S is a dk-set. Since S is k-packing, each vertex in S is k-remote; thus, by condition (2) of Theorem 2.1, S is a minimal dk-set. □

The converse of Proposition 2.2 is false: a minimal dk-set in a graph is not necessarily a k-packing. Consider, for example, the graph G_3 in Figure 1: the set $\{b_1, \ldots, b_n\}$ is a minimal d2-set but not a 2-packing. This example can be extended. If we subdivide each of the paths from a to b with k vertices, then the set $\{b_1, \ldots, b_n\}$ is a minimal dk-set but not a k-packing.

The following corollary follows from Proposition 2.2 and can be seen as a distance-2 analog of the well-known inequality chain $\gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G)$ from equation (1.1).

Corollary 2.3. *For any graph G, $\gamma_{\leq 2}(G) \leq p_2(G) \leq P_2(G) \leq \Gamma_{\leq 2}(G)$.*

3. MINIMAL $d2$-SETS AND 2-PACKINGS OF BLOCK GRAPHS

The primary aim of this section is to provide a proof of Theorem 1.1. Throughout this section, let $G = (V, E)$ be a connected block graph. Recall that for these graphs, Domke, Hedetniemi, Laskar, and Allan [2] showed that $\gamma_{\leq k}(G) = P_{2k}(G)$. In particular, for $k = 1$, we obtain:

Corollary 3.1. *For any connected block graph G, $\gamma(G) = P_2(G)$.*

Their proof of this result relies on the following two lemmas, which we will also use.

Lemma 3.2 ([2])**.** *Let $G = (V, E)$ be a connected block graph. If $u, v \in V$, $d(u, v) > 1$, and $u\, x_1\, x_2\, \ldots\, x_n\, v$ is any shortest u-v path, then the vertices $x_1, x_2, \ldots, x_n$ are cut vertices in G.*

Lemma 3.3 ([2])**.** *Let $G = (V, E)$ be any connected block graph. If $u, v \in V$, where $u \neq v$, there is a unique shortest u-v path.*

Let S be a minimal d2-set in G and let $x \in S$. A vertex w is a *p2n1 vertex* of x if w is a private-2 neighbor of x and $d(x, w) = 1$. Likewise, w is a *p2n2 vertex* of x if w is a private-2 neighbor of x and $d(x, w) = 2$. Let w be a *p2n2* vertex of x. By Lemma 3.3, there exists a unique cut vertex $y \in V$ such that $x\, y\, w$ is the unique shortest $u - v$ path in G. We say that y is a *mediator* for x and w or, simply, that y *mediates* x and w.

A key step in the proof of Theorem 1.1 is the following Algorithm 1. Let S be a 2-packing and minimal d2-set in G, and suppose there exists a vertex $r \in S$ that has at least one private-2 neighbor. We will show that Algorithm 1 transforms the set S into a set S' such that (a) $r \notin S'$, (b) $|S'| \geq |S|$, and (c) S' is a 2-packing and a minimal d2-set in G.

Algorithm 1 Improving S

$S' = S - \{r\}$
if $\mathscr{C}_1 \neq \emptyset$ **then**
  Select a component $\Gamma \in \mathscr{C}_1$.
  Select one *p2n1* vertex w of r from Γ.
  $S' \leftarrow S' \cup \{w\}$.
end if
while $\mathscr{C}_2 \neq \emptyset$ **do**
  Select a component $\Gamma \in \mathscr{C}_2$.
  for v in the root block of Γ **do**
    if v mediates a *p2n2* vertex of r **then**
      Select a *p2n2* vertex x of r mediated by v.
      $S' \leftarrow S' \cup \{x\}$.
    end if
    $\mathscr{C}_2 \leftarrow \mathscr{C}_2 - \{\Gamma\}$.
  end for
end while

We need the following terminology to explain Algorithm 1. Let $B_1, \ldots, B_k$ be the blocks of G containing the vertex r; vertex r in Figure 3 is contained in three blocks. The subgraph induced by $V - \{r\}$ will have k connected components $\Gamma_1, \ldots, \Gamma_k$, each *rooted* at the corresponding blocks $B_1 - \{r\}, \ldots B_k - \{r\}$. We will classify these components as follows: a component is of *type I* if it contains a *p2n1* vertex of r but no *p2n2* vertices of r; a component is of *type II* if it contains a *p2n2* vertex of r; a component is of *type III*, otherwise. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be the collections of type I and type II components of $G - \{r\}$, respectively.

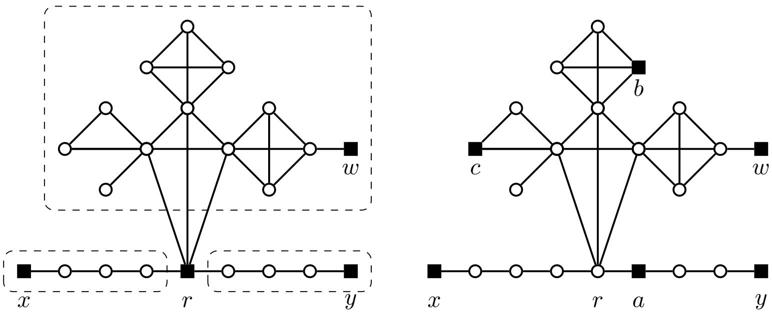

Fig. 3. The set S (filled squares on the left) is transformed to S' (filled squares on the right) after running Algorithm 1

Consider, for example, the graph on the left in Figure 3. The set $S = \{r, w, x, y\}$ is a 2-packing and a minimal d2-set of the graph. Vertex r has several p2n1 and p2n2 vertices. After deleting r, the graph has three components (enclosed in dashed rectangles). The components to the east and west are of type I and are rooted by K_1's. The component to the north is of type II and is rooted by a K_3. The graph on the right shows the result of running Algorithm 1. At the start, $S' = \{w, x, y\}$. Since $\mathscr{C}_1 \neq \emptyset$, a single vertex, a, is joined to S' in the first stage of the algorithm. During the second stage, vertices b and c are joined to S', yielding $S' = \{w, x, y, a, b, c\}$.

Lemma 3.4. *Let $G = (V, E)$ be a connected block graph and let $r \in V$. Let $S \subset V$ be a 2-packing and a minimal d2-set of G. If $r \in S$ and r has a private-2 neighbor, then there exists a set $S' \subset V$ with the following properties:* (a) *$r \notin S'$,* (b) *$|S'| \geq |S|$, and* (c) *S' is a 2-packing and a minimal d2-set of G.*

Proof. Let S' be the output of Algorithm 1. Properties (a) and (b) are clearly satisfied by S'; thus, we are left to show that S' satisfies property (c).

First, we will show that S' is a d2-set. In effect, we must show that r and its private-2 neighbors are dominated by S'. Notice that each vertex added to S' is a private-2 neighbor of r; thus, $N_2[r] \cap S' \neq \emptyset$. Let y be a private-2 neighbor of r and let us suppose y is in component Γ. There are three cases: (1) $\Gamma \in \mathscr{C}_1$, (2) $\Gamma \in \mathscr{C}_2$ and y is a *p2n2* vertex of r, and (3) $\Gamma \in \mathscr{C}_2$ and y is a *p2n1* vertex of r.

(1) Let us suppose $\Gamma \in \mathscr{C}_1$. Then y is necessarily a *p2n1* vertex of r and hence $d(r, y) = 1$. Since $\mathscr{C}_1 \neq \emptyset$, according to Algorithm 1, a component $\Gamma' \in \mathscr{C}_1$ is selected (line 3) and a *p2n1* vertex w of r from Γ' is selected (line 4) and added to S' (line 5). Either $w = y$ or $d(w, y) \leq 2$. In either case, $N_2[y] \cap S' \supset \{w\}$.
(2) Let us suppose $\Gamma \in \mathscr{C}_2$ and y is a *p2n2* vertex of r. In this case, there is a vertex v in the root block of Γ that mediates y and r. According to Algorithm 1, a *p2n2* vertex x of r mediated by v is selected (line 11) and joined to S' (line 12). Either $y = x$ or $d(x, y) \leq 2$. In either case, $N_2[y] \cap S' \supset \{x\}$.
(3) Finally, let us suppose $\Gamma \in \mathscr{C}_2$ and y is a *p2n1* vertex of r. Because $\Gamma \in \mathscr{C}_2$, there is at least one vertex v in the root block of Γ that mediates a *p2n2* vertex of r. According to Algorithm 1, a *p2n2* vertex x of r mediated by v is selected (line 11) and joined to S' (line 12). Since y and v are in the root block of Γ, $d(x, y) = 2$ and $N_2[y] \cap S' \supset \{x\}$.

Next, we will show that S' is a 2-packing. In what follows let $\overline{S} = V - S$. Let $x, y \in S'$. There are three cases. (1) $x, y \in S' \cap S$, (2) $x, y \in S' \cap \overline{S}$, and (3) $x \in S' \cap S$ and $y \in S' \cap \overline{S}$.

(1) Let $x, y \in S' \cap S$. Since S is a 2-packing, $d(x, y) > 2$.
(2) Let $x, y \in S' \cap \overline{S}$. Thus, x and y were added to $S - \{r\}$ at some stage of Algorithm 1. Suppose x and y are from components Γ_1 and Γ_2 of $G - \{r\}$. (For example, consider vertices a and b in Figure 3). If $\Gamma_1 \neq \Gamma_2$, then, since at most one of x or y can be at distance 1 from r, $d(x, y) > 2$. If $\Gamma_1 = \Gamma_2$, then x and y are both *p2n2* vertices of r mediated by vertices v_1 and v_2 from the root block of their component, $v_1 \neq v_2$. (For example, consider vertices b and c in Figure 3). Thus, xv_1v_2y is the shortest $x - y$ path in G and $d(x, y) = 3$.
(3) Finally, let $x \in S' \cap S$ and $y \in S' \cap \overline{S}$. Only private-2 neighbors are joined to $S - \{r\}$ by Algorithm 1. Consequently, $N_2[y] \cap S = \{r\}$ and $d(x, y) > 2$.

Finally, since S' is a d2-set and a 2-packing, it is a minimal d2-set, as was to be shown. □

Recall that a vertex x in a minimal d2-set S is 2-remote provided that $N_2[x] \cap S = \{x\}$. Let $\nu(S)$ denote the set of non-2-remote vertices in S, or equivalently, the set of *p2n1* and *p2n2* vertices. Observe that S is also a 2-packing if and only if $\nu(S) = \emptyset$.

Lemma 3.5. *Let S be a minimal d2-set in a connected block graph G with $|\nu(S)| > 0$. Then there exists a minimal d2-set S' in G satisfying: (a) $|S'| \geq |S|$, and (b) $|\nu(S')| < |\nu(S)|$.*

Proof. Fix a vertex $x \in \overline{S} = V - S$. Let r be a vertex in $\nu(S)$ at maximum distance from x. Our first step is to cut the vertex set V into two parts, one containing r and one containing x. Let $r\, u_1 \cdots u_n\, x$ be the shortest path from r to x. Since r is not a 2-remote vertex, the set $N_2(r) \cap S \neq \emptyset$, and since r is at maximum distance from x, it follows that $N_2(r) \cap S \subset N_1[u_1]$. Our partition of V falls into two cases:

(1) If any elements of $N_2(r) \cap S$ are in the block determined by r and u_1, then cut each of the edges adjacent to r in this block. The induced subgraph has components denoted by $G_r = (V_r, E_r)$ and $G_x = (V_x, E_x)$, the former containing r and the latter containing x.
(2) Otherwise, cut each of the edges adjacent to u_1 in the block determined by r and u_1. The induced subgraph has components denoted by $G_r = (V_r, E_r)$ and $G_x = (V_x, E_x)$, the former containing r and the latter containing x.

Consider, for example, the block graph in Figure 4. If any of a, b, c, or u_1 is in S, then we observe cut 1; otherwise, at least one of d, e, f, g, h, i, j, or u_2 is in S, in which case we observe cut 2.

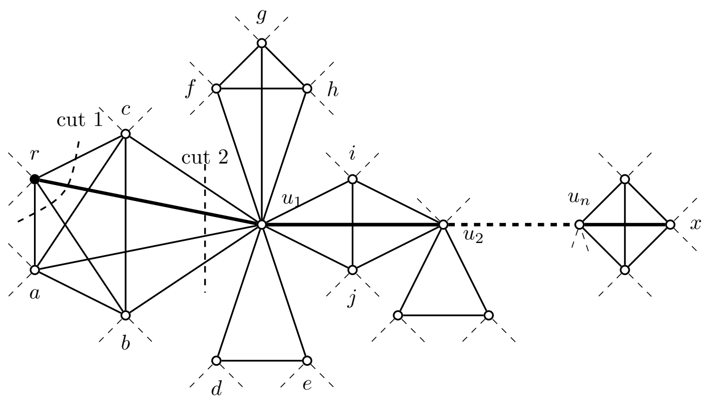

Fig. 4. The block graph G is split into two rooted subgraphs, G_r on the left and G_x on the right

Let $S_r = S \cap V_r$ and let $S_x = S \cap V_x$. By our choice of cuts, all private neighbors of S_r are in V_r and all private neighbors of S_x are in V_x. In other words, each of S_r and S_x is a minimal d2-set for its respective block subgraph. Moreover, S_r is a 2-packing of G_r.

Let S'_r be the 2-packing and minimal d2-set on G_r obtained in accord with Lemma 3.4, and let $S' = S'_r \cup S_x$. Then S' is a minimal d2-set for G and $|S'| \geq |S|$. Finally, since we have replaced r with 2-remote vertices in G_r, $|\nu(S')| < |\nu(S)|$. □

We are now prepared to prove Theorem 1.1.

Proof of Theorem 1.1. Let S be an upper d2-set for G. Apply Lemma 3.5 repeatedly until $\nu(S') = \emptyset$, at which point S' is an upper d2-set and a 2-packing. Thus, $\Gamma_{\leq 2}(G) \leq P_2(G)$. Combining this with Corollary 3.1 and Corollary 2.3, we arrive at

$$P_2(G) = \gamma(G) = \Gamma_{\leq 2}(G),$$

as was to be shown. □

4. A PARTITIONING PROOF OF THEOREM 1.2

The main goal of this section is to provide a proof of Theorem 1.2. Our proof relies on a particular partitioning of the vertex set of a connected graph. Let S be a minimal d2-set for a connected graph G. Given $x \in S$, let

$$P_x = \{y \in V : N_2[y] \cap S = \{x\}\}.$$

Thus, P_x contains the private 2-neighbors of x. When x is 2-remote, $x \in P_x$. Let $P = \bigcup_{x \in S} P_x$.

A vertex $z \in V - (S \cup P)$ is *intrinsic to* $x \in S$ provided that $x \in N_1(z)$ and $N_1(z) \cap P_x \neq \emptyset$. Let I_x denote the set of intrinsic vertices of x. It is easy to show that if $x, w \in S$, $x \neq w$, then $I_x \cap I_w = \emptyset$. Let $I = \bigcup_{x \in S} I_x$ denote the intrinsic vertices of G. Finally, let $C = V - (S \cup P \cup I)$. An element of C is called a *contested* vertex.

For $x \in S$, the *domain* of x is the set $D_x = \{x\} \cup P_x \cup I_x$. The domain of a dominator x is a set of vertices that can be unambiguously associated with x. We offer the following lemma without proof.

Lemma 4.1. *Let S be a minimal d2-set for a connected graph G. If $x, y \in S$, $x \neq y$, then $D_x \cap D_y = \emptyset$.*

Figure 5 depicts these four vertex types. The set $S = \{v, w, x, y, z\}$ is a minimal d2-set for the graph. The dominators are marked by black squares, their private-2 neighbors by black discs, and their intrinsic vertices by gray discs. White discs mark the contested vertices. The dashed rectangles enclose the domains of the dominators.

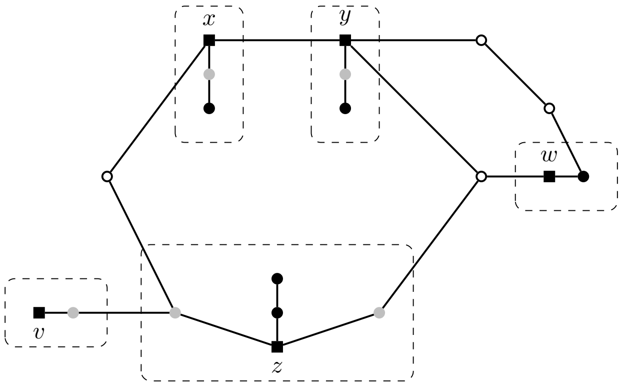

Fig. 5. An illustration of the four vertex types. Filled squares denote the d2-set, black discs the private neighbors, gray discs the intrinsic vertices, and open circles the contested vertices. The dashed rectangles enclose the domains of the dominators

A key result of this section is Lemma 4.3. Our next result is a technical lemma that will be used to prove Lemma 4.3.

Lemma 4.2. *Let S be a minimal d2-set for a connected graph G and let $x \in S$.*

(1) *Then $N_1(x) \cap I \subset I_x$.*
(2) *If $y \in N_1(x) \cap P_x$, then $N_1(y) \cap I \subset I_x$.*

Proof. To prove (1), let $u \in N_1(x) \cap I$ and suppose u is intrinsic to $w \in S$. Then $w \in N_1(u)$ and $N_1(u) \cap P_w \neq \emptyset$. Let $v \in N_1(u) \cap P_w$. Since v is 2-private to w, $N_2[v] \cap S = \{w\}$. However, $N_2[v] \cap S \supset \{x\}$. Thus, $w = x$, as was to be shown.

To prove (2), let $u \in N_1(y) \cap I$ and suppose u is intrinsic to $w \in S$. Then $w \in N_1(u)$ hence $N_2[y] \cap S \supset \{w\}$. Since $y \in P_x$, $N_2[y] \cap S = \{x\}$. Thus, $w = x$, as was to be shown. □

Unless a connected graph consists of a single node, a domain of a dominator must contain at least two vertices. Our next lemma addresses the structure of a graph when a domain has size two. The two cases addressed in this lemma can be seen in Figure 5: the vertex w is in case (1) and the vertex v is in case (2).

Lemma 4.3. *Let S be a minimal d2-set for a connected graph G, let $x \in S$, and let $|D_x| = 2$. There are two cases:*

(1) *If $y \in P_x$ and $y \neq x$, then $P_x = \{y\}$, $y \in N_1(x)$, and each member of $N_1(x) \cup N_1(y) - \{x, y\}$ is a contested vertex.*
(2) *If x is 2-remote, then x is a leaf.*

Proof. To prove case (1), let $y \in P_x$ and $y \neq x$. Since $D_x = \{x\} \cup P_x \cup I_x$ and $|D_x| = 2$, then $P_x = \{y\}$, and $I_x = \emptyset$.

For the sake of a contradiction, suppose that $d(x, y) = 2$. Then there exists a vertex w such that $d(x, w) = d(w, y) = 1$, which implies $w \in I_x$. But this contradicts the conclusion that $I_x = \emptyset$. Hence, $y \in N_1(x)$.

Finally, let $z \in N_1(x) \cup N_1(y) - \{x, y\}$; we will show, by a process of elimination, that z is a contested vertex. First, since $z \in N_2[y]$ and since $N_2[y] \cap S = \{x\}$, it follows that $z \notin S$. Second, since $N_2[z] \cap S \supset \{x\}$, z could be private only to x. But y is the only private-2 neighbor of x; thus, $z \notin P$. Finally, let us suppose that $z \in I$. By either (1) or (2) of Lemma 4.2, $z \in I_x$. But this contradicts the conclusion that $I_x = \emptyset$; thus, $z \notin I$. In summary, z is a contested vertex, as was to be shown.

We are left to prove case (2). Let x be a 2-remote dominator. Then $D_x \supset N_1[x]$, since each vertex adjacent to x is either 2-private or intrinsic to x. Since $|D_x| = 2$, it follows that $|N_1[x]| = 2$, which shows that x is a leaf. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The domains of the dominators and the contested vertices form a partition of the vertex set: $V = C \cup (\cup_{x \in S} D_x)$. Since each domain contains at least two vertices, $n \geq |C| + 2|S|$ hence $|S| \leq n/2$.

If $|S| = n/2$, then $|D_x| = 2$ for each $x \in S$ and $|C| = 0$. Since there are no contested vertices, by Lemma 4.3, each member of S is 2-remote and a leaf. Let H be the subgraph of G induced by $V - S$. Then H is connected and $G = H \odot K_1$. Finally, since S is a maximum 2-packing and a minimum dominating set, $P_2(G) = \gamma(G) = \Gamma_{\leq 2}(G)$. □

REFERENCES

[1] B. Brešar, P. Dorbec, W. Goddard, B.L. Hartnell, M.A. Henning, S. Klavžar, D.F. Rall, *Vizing's conjecture: a survey and recent results*, J. Graph Theory **69** (2012), no. 1, 46–76.

[2] G.S. Domke, S. Hedetniemi, R. Laskar, R. Allan, *Generalized packings and coverings of graphs*, Congr. Numer. **62** (1988), 259–270, Seventeenth Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, MB, 1987).

[3] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Monographs and Textbooks in Pure and Applied Mathematics, 208, Marcel Dekker, Inc., New York, 1998.

[4] M.A. Henning, O.R. Oellermann, H.C. Swart, *Bounds on distance domination parameters*, J. Combin. Inform. System Sci. **16** (1991), 11–18.

[5] A. Meir, J.W. Moon, *Relations between packing and covering numbers of a tree*, Pacific J. Math. **61** (1975), no. 1, 225–233.

[6] R.R. Rubalcaba, A. Sneider, P.J. Slater, *A survey on graphs which have equal domination and closed neighborhood packing numbers*, AKCE Int. J. Graphs Comb. **3** (2006), no. 2, 93–114.

Jason T. Hedetniemi
jhedetniemi@fau.edu

Florida Atlantic University
Harriet L. Wilkes Honors College
Boca Raton, FL, USA

Stephen T. Hedetniemi
hedet@clemson.edu

Clemson University
Emeritus College
Clemson, SC, USA

Thomas M. Lewis (corresponding author)
tom.lewis@furman.edu
https://orcid.org/0000-0001-8366-2867

Furman University
Department of Mathematics
Greenville, SC, USA

Received: October 26, 2023.
Revised: February 2, 2025.
Accepted: February 24, 2025.