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SYMMETRIC 2×2 MATRIX FUNCTIONS WITH ORDER PRESERVING PROPERTY

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Abstract. It is known that the discrete matrix Riccati equation has the order preserving property under some assumptions. In this paper we formulate and prove the converse statement for the case when the dimensions of the matrices are 2×2 and the order preserving property holds for all such symmetric matrices.

Keywords: Riccati matrix equation, order preserving property, symmetric matrix function.

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1. INTRODUCTION AND MOTIVATION

In the whole paper we denote by $\mathbf{S}$ the set of real symmetric $n \times n$ matrices and by $\mathbf{S_2}$ the set of real symmetric 2×2 matrices. For any two symmetric matrices $Q, \hat{Q} \in \mathbf{S}$, by the inequality $Q \leq \hat{Q}$ we mean that the symmetric matrix $\hat{Q} - Q$ is non-negative definite. The $2n \times 2n$ matrix $\mathcal{S} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ with block entries $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ is *symplectic* if

$$\mathcal{A}^T\mathcal{C} = \mathcal{C}^T\mathcal{A}, \quad \mathcal{B}^T\mathcal{D} = \mathcal{D}^T\mathcal{B}, \quad \mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B} = I,$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are $n \times n$ matrices.

A function $f\colon \mathbf{S} \to \mathbf{S}$ has the *order preserving property* on a set $M \subseteq \mathbf{S}$ if

$$Q \leq \hat{Q} \Leftrightarrow f(Q) \leq f(\hat{Q}) \quad \text{for all } Q, \hat{Q} \in M \subseteq \mathbf{S}. \tag{1.1}$$

There are known results about the order preserving property of the discrete matrix Riccati equation,

$$R[Q]_k := Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) = 0,$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, and Q_k are real $n \times n$ matrices, Q_k are symmetric and the $2n \times 2n$ matrices $\mathcal{S}_k$ with block entries $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ are symplectic. See e.g. [5]. The following more general form of this result is formulated in [8].

Proposition 1.1 ([8, Corollary 2.7])**.** *Let $S = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a symplectic $2n \times 2n$ matrix and $Q, \hat{Q} \in \mathbf{S}$ be such that both inverses $(\mathcal{A} + \mathcal{B}Q)^{-1}$ and $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}$ exist and both inequalities $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \leq 0, (\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \leq 0$ hold, or both inequalities $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0, (\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \geq 0$ hold. Then*

$$Q \leq \hat{Q} \quad \Leftrightarrow \quad (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \leq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}.$$

The question is, if there exist symmetric matrix functions with the order preserving property that are not of this Riccati type, i.e. not defined as

$$f(Q) = (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1}.$$

A corresponding result in the continuous case was formulated in [6] by Stokes. It says that if a symmetric matrix differential equation has the order preserving property and the matrix dimension is $n \geq 2$, then this equation is the continuous matrix Riccati equation. Converse results for the continuous case were published by Reid in [4] and Coppel in [1].

In the discrete case, there is a result regarding this problem, published in [7], but modified assumptions are required there.

By the symmetric matrix function $f\colon \mathbf{S} \to \mathbf{S}$ we mean here any function mapping $\mathbf{S}$ to $\mathbf{S}$. Another commonly used meaning of the term "matrix function" is a function derived from a scalar function by applying this function to the eigenvalues of the matrix. Such types of functions are studied *e.g.* in [2]. There are many results about monotonicity of such matrix functions (called also operator monotone), which were studied already by Löwner in [3]. The order preserving property is often formulated only as an implication,

$$Q \leq \hat{Q} \Rightarrow f(Q) \leq f(\hat{Q}), \quad \text{for all } Q, \hat{Q} \in M \subseteq \mathbf{S}, \tag{1.2}$$

compared to the order preserving property (1.1), which is an equivalence. Löwner showed in [3, p. 187] that if the matrix dimension $n = 2$, the only function such that it has property (1.2) together with its inverse on a suitable set M, is the function derived from the scalar function $f(q) = \frac{c+dq}{a+bq}$, where a, b, c, d, q are real numbers.

In this paper we formulate and prove the converse of Proposition 1.1 for the case when $\mathcal{B} = 0$ and $n = 2$.

1.1. PROBLEM FORMULATION

First, we rewrite Proposition 1.1 in a more suitable form.

Proposition 1.2. *Let $S = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a symplectic $2n \times 2n$ matrix and let function $f : \mathbf{S} \to \mathbf{S}$ be defined as $f(Q) = (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1}$. Then f has property* (1.1) *on M^+ and f has property* (1.1) *on M^-, where*

$$\begin{aligned} M^+ &= \{Q \in \mathbf{S}\colon (\mathcal{A} + \mathcal{B}Q)^{-1} \textit{ exists}, (\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0\}, \\ M^- &= \{Q \in \mathbf{S}\colon (\mathcal{A} + \mathcal{B}Q)^{-1} \textit{ exists}, (\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \leq 0\}. \end{aligned}$$

Now, if $\mathcal{B} = 0$, then $M^+ = M^- = \mathbf{S}$, and further, if the matrix $S = [\begin{smallmatrix} \mathcal{A} & 0 \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}]$ is symplectic, then

$$\mathcal{A}^T\mathcal{C} = \mathcal{C}^T\mathcal{A}, \quad \mathcal{A}^T\mathcal{D} = I$$

and

$$(\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} = (\mathcal{C} + \mathcal{A}^{T-1}Q)A^{-1} = \mathcal{C}\mathcal{A}^{-1} + \mathcal{A}^{T-1}Q\mathcal{A}^{-1}.$$

The matrix $\mathcal{C}\mathcal{A}^{-1}$ is symmetric and the matrix $\mathcal{A}^{-1}$ is invertible. Proposition 1.2 becomes the following.

Proposition 1.3. *Let $f\colon \mathbf{S} \to \mathbf{S}$ be defined as $f(Q) = K + L^TQL$, where K is a symmetric $n \times n$ matrix and L is an invertible $n \times n$ matrix. Then f has property* (1.1) *on the whole* $\mathbf{S}$.

Proof of this proposition is trivial. The converse of this proposition may be formulated as follows.

Hypothesis 1.4. Let $n \geq 2$ and $f\colon \mathbf{S} \to \mathbf{S}$ be a continuous function such that f has property (1.1) on the whole $\mathbf{S}$. Then f is of the form $f(Q) = K + L^TQL$, where K is a symmetric $n \times n$ matrix and L is an invertible $n \times n$ matrix.

In this paper we present the proof of this hypothesis for the case when $n = 2$. To prove this, we use geometric approach, where any symmetric 2×2 matrix represents a point in 3-dimensional space and the set of all positive definite matrices is a cone in this space.

Notation 1.5. Let $A \in \mathbf{S}$. Denote

$$\begin{aligned} K^+(A) &:= \{Q \in \mathbf{S} : Q \geq A\}, \quad K_0^+(A) := \{Q \in K^+(A) : \det(Q - A) = 0\}, \\ K^-(A) &:= \{Q \in \mathbf{S} : Q \leq A\}, \quad K_0^-(A) := \{Q \in K^-(A) : \det(Q - A) = 0\}, \\ K(A) &:= K^+(A) \cup K^-(A), \quad K_0(A) := K_0^+(A) \cup K_0^-(A). \end{aligned}$$

2. AUXILIARY LEMMAS

In this section is $n = 2$ and $\mathbf{S} = \mathbf{S_2}$ is the set of real symmetric 2×2 matrices. This section contains auxiliary lemmas, statements of which are mostly obvious from a geometric point of view, they follow from the basic theory of linear algebra and geometry. To ensure the correctness of all proofs, we present them all here in our non-geometric notation. These lemmas will be used in Section 3, which deals with the properties of the function f.

Lemma 2.1. *Let $Q \in \mathbf{S_2}$ be such that $\det(Q) = 0$ and $Q \neq 0$. Then there exist unique $\varphi \in [0, 2\pi)$ and $r \in \mathbb{R}$ such that*

$$Q = r\left(I + \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}\right). \tag{2.1}$$

Proof. If $Q \in \mathbf{S_2}$, $\det(Q) = 0$ and $Q \neq 0$, then there exist unique $x, z, \sigma \in \mathbb{R}$ such that $Q = \sigma \begin{bmatrix} x^2 & xz \\ xz & z^2 \end{bmatrix}$ and $\sigma = \pm 1$, where $x^2 + z^2 \neq 0$. We take $r := \sigma \frac{x^2+z^2}{2}$ and $\varphi \in [0, 2\pi)$ such that $\cos \frac{\varphi}{2} = \frac{x}{\sqrt{2|r|}}$, $\sin \frac{\varphi}{2} = \frac{z}{\sqrt{2|r|}}$. The relation (2.1) follows from the basic trigonometric identities. □

We will now introduce a possible geometric view, that is useful for a better idea of all this. Each symmetric 2×2 matrix Q with zero determinant can be uniquely represented by a point in 3-dimensional space. We take the point with the spherical coordinates r, φ and $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ (radial distance is r, azimuthal angle is φ, and polar angle is $\frac{\pi}{4}$ or $\frac{3\pi}{4}$, depending on the sign of the diagonal elements of the matrix), where r, φ are given from relation (2.1). The matrix, which is represented by the point with spherical coordinates 1, φ and $\frac{\pi}{4}$, we denote $U(\varphi)$.

Notation 2.2. Let $\varphi \in [0, 2\pi)$. Denote

$$U(\varphi) := I + \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}.$$

Remark 2.3. The point, which is represented by the sum $U(\varphi) + U(\varphi + \pi) = 2I$, lies on the z-axis. The point, which is represented by the difference $U(\varphi) - U(\varphi + \pi) = 2(U(\varphi) - I)$, is the projection of $2U(\varphi)$ to the xy-plane.

We will further work with the following two types of sets.

Notation 2.4. Let $A \in \mathbf{S_2}$ and $\varphi, \psi, \theta \in [0, 2\pi)$. Denote

$$\begin{aligned} l_\varphi(A) &:= \{A + tU(\varphi) : t \in \mathbb{R}\}, \\ p_{\psi,\theta}(A) &:= \{A + rU(\psi) + sU(\theta) : r, s \in \mathbb{R}\}. \end{aligned}$$

In our geometric view, the set $l_\varphi(A)$ is represented by the straight line passing through the point A, with the polar angle of the direction vector $\frac{\pi}{4}$ and azimuthal angle φ. We will refer to this type of line as $\frac{\pi}{4}$-line. The $\frac{\pi}{4}$-lines $l_\varphi(A)$ and $l_{\varphi+\pi}(A)$ are perpendicular.

The set $p_{\psi,\theta}(A)$ is then the plane given by the point A and the $\frac{\pi}{4}$-lines $l_\psi(A)$, $l_\theta(A)$. We will refer to this type of plane as $\frac{\pi}{4}$-plane. The $\frac{\pi}{4}$-plane $p_{\psi,\psi+\pi}(A)$ is vertical.

The set $K_0(A)$ from Notation 1.5 is (for $n = 2$) the right circular double cone with angle $\frac{\pi}{4}$ and with vertex at the point A. We will refer to this surface simply as the cone with vertex at the point A, since no other types of cones appear in this text. The set $K_0^+(A)$ is the upper half of this cone and the set $K_0^-(A)$ is its lower half.

The following three lemmas contain auxiliary identities that will be used later in the proofs.

Lemma 2.5. *Let $\varphi, \alpha, \beta \in [0, 2\pi)$. The following identities hold:*

$$\det (U(\varphi) - I) = -1, \tag{2.2}$$

$$[U(\alpha) - I][U(\alpha) - I] = I, \tag{2.3}$$

$$[U(\alpha) - I][U(\beta) - I] = \begin{bmatrix} \cos(\alpha - \beta) & -\sin(\alpha - \beta) \\ \sin(\alpha - \beta) & \cos(\alpha - \beta) \end{bmatrix}, \tag{2.4}$$

$$[U(\alpha) - I]U(\varphi)[U(\alpha) - I] = U(2\alpha - \varphi). \tag{2.5}$$

Proof. The identities can be proven directly with use of the basic trigonometric identities. □

Lemma 2.6. *Let $\varphi \in [0, 2\pi)$. The following identities hold:*

$$\begin{aligned} \left[U\left(\tfrac{\varphi}{2}\right) - I\right] U(\varphi) \left[U\left(\tfrac{\varphi}{2}\right) - I\right] &= U(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \\ \left[U\left(\tfrac{\varphi}{2}\right) - I\right] U(\varphi + \pi) \left[U\left(\tfrac{\varphi}{2}\right) - I\right] &= U(-\pi) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \\ \left[U\left(\tfrac{\varphi}{2}\right) - I\right] \left[U(\varphi + \tfrac{\pi}{2}) - I\right] \left[U\left(\tfrac{\varphi}{2}\right) - I\right] &= U\left(-\tfrac{\pi}{2}\right) - I = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \end{aligned} \tag{2.6}$$

Proof. The identities follow from identities (2.3), (2.5) in Lemma 2.5. □

Lemma 2.7. *Let $\alpha_1, \ldots, \alpha_k, \beta \in [0, 2\pi)$, and $a_1, \ldots, a_k, b \in \mathbb{R}$. Then the identity*

$$2b \sum_{i=1}^{k} a_i (1 - \cos(\beta - \alpha_i)) = -\det \sum_{i=1}^{k} a_i U(\alpha_i)$$

holds if and only if there exist $\gamma \in [0, 2\pi)$, $c \in \mathbb{R}$ such that

$$a_1 U(\alpha_1) + a_2 U(\alpha_2) + \ldots + a_k U(\alpha_k) + bU(\beta) = cU(\gamma). \tag{2.7}$$

Proof. We multiply equation (2.7) by the matrix $U\left(\frac{\beta}{2}\right) - I$ from both the left and the right and then use identity (2.5) to obtain the equivalent equation

$$\sum_{i=1}^{k} a_i U(\beta - \alpha_i) + bU(0) = cU(\beta - \gamma).$$

Hence, there exist $\gamma \in [0, 2\pi)$, $c \in \mathbb{R}$ such that the equation holds if and only if the determinant of the matrix on the left is zero, that is

$$\begin{aligned} 0 &= \det \left[\sum_{i=1}^{k} a_i U(\beta - \alpha_i) + bU(0) \right] \\ &= \det \sum_{i=1}^{k} a_i U(\alpha_i) + 2b \sum_{i=1}^{k} a_i (1 - \cos(\beta - \alpha_i)), \end{aligned}$$

where we used that $\det \sum_{i=1}^{k} a_i U(\beta - \alpha_i) = \det \sum_{i=1}^{k} a_i U(\alpha_i)$, which follows from identity (2.2). This completes the proof. □

The lemmas, that are presented in the rest of this section, have obvious geometric meanings. Sometimes it is more convenient to consider a representation of a matrix not as a point, but as a vector starting at 0 and ending at a given point. The next two lemmas say, that any vector in 3-dimensional space can be written as a linear combination of 3 mutually perpendicular vectors $U(\varphi)$, $U(\varphi+\pi)$ and $U(\varphi+\frac{\pi}{2})-I$ for any φ, and any vector can be written as a linear combination of 2 mutually perpendicular vectors $U(\varphi)$, $U(\varphi+\pi)$ for some φ. Then φ or $\varphi+\pi$ is the azimuthal angle of this vector.

Lemma 2.8. *Let $A \in \mathbf{S_2}$ and $\varphi \in [0, 2\pi)$. Then there exist $a, b, c \in \mathbb{R}$ such that*

$$A = aU(\varphi) + bU(\varphi+\pi) + c\left(U(\varphi+\tfrac{\pi}{2}) - I\right). \tag{2.8}$$

Moreover, equation (2.8) *is equivalent with*

$$\begin{bmatrix} 2a & -c \\ -c & 2b \end{bmatrix} = \left[U\left(\tfrac{\varphi}{2}\right) - I\right] A \left[U\left(\tfrac{\varphi}{2}\right) - I\right]. \tag{2.9}$$

Proof. Equation (2.9) is equivalent with

$$\begin{aligned} \left[U\left(\tfrac{\varphi}{2}\right) - I\right] A \left[U\left(\tfrac{\varphi}{2}\right) - I\right] &= aU(0) + bU(-\pi) + c\left(U\left(-\tfrac{\pi}{2}\right) - I\right), \\ A &= \left[U\left(\tfrac{\varphi}{2}\right) - I\right]\left[aU(0) + bU(-\pi) + c\left(U\left(-\tfrac{\pi}{2}\right) - I\right)\right]\left[U\left(\tfrac{\varphi}{2}\right) - I\right]. \end{aligned}$$

The conclusion follows from identities (2.3), (2.6). □

Lemma 2.9. *Let $A \in \mathbf{S_2}$. Then there exists $\varphi \in [0, 2\pi)$ and $a, b \in \mathbb{R}$ such that*

$$A = aU(\varphi) + bU(\varphi+\pi). \tag{2.10}$$

Moreover, if $A > 0$, then $a > 0, b > 0$.

Proof. By Lemma 2.8, we have that it suffices to show that there exists such $\varphi \in [0, 2\pi)$ that the matrix $\left[U\left(\frac{\varphi}{2}\right) - I\right] A \left[U\left(\frac{\varphi}{2}\right) - I\right]$ is diagonal. The matrix $U\left(\frac{\varphi}{2}\right) - I$ is orthonormal and can be found by eigendecomposition of the matrix A. The coefficients $2a, 2b$ are the eigenvalues of the matrix A. □

Lemma 2.10. *Let $\varphi, \delta \in [0, 2\pi)$ and $a, b, c, d \in \mathbb{R}$ be such that*

$$aU(\varphi) + bU(\varphi+\pi) + c\left(U(\varphi+\tfrac{\pi}{2}) - I\right) = dU(\delta). \tag{2.11}$$

Then

$$aU(\varphi) + bU(\varphi+\pi) - c\left(U(\varphi+\tfrac{\pi}{2}) - I\right) = dU(2\varphi-\delta). \tag{2.12}$$

Moreover, for all $\varphi \in [0, 2\pi)$ it holds that $c^2 = 4ab$ if and only if there exist $\delta \in [0, 2\pi)$ and $d \in \mathbb{R}$ such that (2.11) *holds.*

Proof. We multiply both sides of equation (2.11) by the matrix $U(\varphi) - I$ and then, using identity (2.5), we obtain

$$\begin{aligned} aU(\varphi) + bU(\varphi-\pi) + c\left(U\left(\varphi-\tfrac{\pi}{2}\right) - I\right) &= dU(2\varphi-\delta), \\ aU(\varphi) + bU(\varphi+\pi) - c\left(U(\varphi+\tfrac{\pi}{2}) - I\right) &= dU(2\varphi-\delta). \end{aligned}$$

Further, Lemma 2.8, (2.9) yields that equation (2.11) is equivalent with

$$\begin{bmatrix} 2a & -c \\ -c & 2b \end{bmatrix} = \left[U\left(\tfrac{\varphi}{2}\right) - I\right] dU(\delta) \left[U\left(\tfrac{\varphi}{2}\right) - I\right],$$

and this implies that $0 = \det \left[\begin{smallmatrix} 2a & -c \\ -c & 2b \end{smallmatrix}\right] = 4ab - c^2$. Conversely, if $\det \left[\begin{smallmatrix} 2a & -c \\ -c & 2b \end{smallmatrix}\right] = 0$, then there exist $\delta \in [0, 2\pi)$ and $d \in \mathbb{R}$ such that (2.11) holds. □

The geometric meaning of the next lemma is, that if a linear combination of two vectors with the polar angle $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ is again a vector with the polar angle $\frac{\pi}{4}$ or $\frac{3\pi}{4}$, then one of these vectors is a multiple of the other. This lemma has also an algebraic meaning, namely that if the sum of two nonzero symmetric 2×2 matrices with zero determinants has also zero determinant, then one of these matrices is a multiple of the other.

Lemma 2.11. *Let $\alpha, \beta, \gamma \in [0, 2\pi)$, and $a, b, c \in \mathbb{R}$ be such that*

$$aU(\alpha) + bU(\beta) + cU(\gamma) = 0. \tag{2.13}$$

If $a \neq 0, b \neq 0$, then $\alpha = \beta$. If $\alpha \neq \beta \neq \gamma \neq \alpha$, then $a = b = c = 0$.

Proof. Lemma 2.7 applied to (2.13) yields that

$$2ab(1 - \cos(\beta - \alpha)) = -\det aU(\alpha) = 0$$

and this implies that if $a \neq 0, b \neq 0$, then $\alpha = \beta$ and if $\alpha \neq \beta$, then either $a = 0$ or $b = 0$. The same statements hold for the pairs α, γ and β, γ, which together with (2.13) implies that if $\alpha \neq \beta \neq \gamma \neq \alpha$, then $a = b = c = 0$. □

Lemma 2.12. *Let $\alpha_1, \alpha_2 \in [0, 2\pi)$, $\alpha_1 < \alpha_2$ and $a_1, a_2, c \in \mathbb{R}$, $c \neq 0$ be such that*

$$a_1 U(\alpha_1) + a_2 U(\alpha_2) = cI. \tag{2.14}$$

Then $\alpha_2 = \alpha_1 + \pi$ and $a_1 = a_2 = \frac{c}{2}$.

Proof. We write (2.14) as

$$a_1 \left[U(\alpha_1) - I\right] + a_2 \left[U(\alpha_2) - I\right] = (c - a_1 - a_2)I,$$

and this equation holds if and only if both its sides are zero matrices, thus $c = a_1 + a_2$. By substituting this into (2.14) we further get

$$a_1 U(\alpha_1) + a_2 U(\alpha_2) = (a_1 + a_2)I = \frac{a_1 + a_2}{2}(U(\alpha_1) + U(\alpha_1 + \pi)),$$

$$\frac{a_1 - a_2}{2} U(\alpha_1) + a_2 U(\alpha_2) - \frac{a_1 + a_2}{2} U(\alpha_1 + \pi) = 0.$$

Lemma 2.11 applied to this equation yields that $a_1 = a_2$, because $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq \alpha_1 + \pi$. Then $a_2 U(\alpha_2) - a_2 U(\alpha_1 + \pi) = 0$, thus $\alpha_2 = \alpha_1 + \pi$. □

Lemma 2.13. *Let $\alpha_1, \ldots, \alpha_k, \beta_1, \beta_2, \gamma \in [0, 2\pi)$, $\beta_1 \neq \beta_2$ and $a_1, \ldots, a_k \in \mathbb{R}$, $b, c \in \mathbb{R}$ be such that*

$$a_1 U(\alpha_1) + a_2 U(\alpha_2) + \ldots + a_k U(\alpha_k) + b(U(\beta_1) + U(\beta_2)) = cU(\gamma). \tag{2.15}$$

If $0 \leq a_i$, $i = 1, \ldots, k$, then $b \leq 0$.

Proof. We will suppose that $b \neq 0$. Similarly, as in the proof of Lemma 2.12, we get from (2.15) that $c = 2b + \sum_{i=1}^{k} a_i$ and

$$\sum_{i=1}^{k} a_i \left[U(\alpha_i) - I\right] + b \left[U(\beta_1) - I\right] + b \left[U(\beta_2) - I\right] = c \left[U(\gamma) - I\right]. \tag{2.16}$$

Since $b \neq 0$, $\beta_1 \neq \beta_2$ and $0 \leq a_i$, $i = 1, \ldots, k$, equation (2.16) yields $|c| < \sum_{i=1}^{k} a_i + 2|b|$. From this, by substituting $\sum_{i=1}^{k} a_i = c - 2b$, we further get $|c| < c - 2b + 2|b|$, which implies that $b < 0$. $\square$

Lemma 2.14. *Let $\alpha_1, \ldots, \alpha_k \in [0, 2\pi)$, $a_1, \ldots, a_k \in \mathbb{R}$. Then there exists $\gamma \in [0, 2\pi)$ and $c \in \mathbb{R}$, $c \geq 0$ such that*

$$a_1 \left[U(\alpha_1) - I\right] + a_2 \left[U(\alpha_2) - I\right] + \ldots + a_k \left[U(\alpha_k) - I\right] = c \left[U(\gamma) - I\right], \tag{2.17}$$

and $c^2 = -\det \sum_{i=1}^{k} a_i \left[U(\alpha_i) - I\right]$.

Proof. There exist $a, b \in \mathbb{R}$ such that

$$\sum_{i=1}^{k} a_i \left[U(\alpha_i) - I\right] = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \sqrt{a^2 + b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{-a}{\sqrt{a^2+b^2}} \end{bmatrix},$$

$$-\det \sum_{i=1}^{k} a_i \left[U(\alpha_i) - I\right] = a^2 + b^2.$$

Further, there exist $\gamma \in [0, 2\pi)$ and $c \in \mathbb{R}$ such that $\cos\gamma = \frac{a}{\sqrt{a^2+b^2}}, \sin\gamma = \frac{b}{\sqrt{a^2+b^2}}$, $c = \sqrt{a^2 + b^2}$, $c \geq 0$. $\square$

Lemma 2.15. *Let $A, D \in \mathbf{S_2}$, $\alpha \in [0, 2\pi)$ be such that $l_\alpha(A) \cap K_0(D) = \emptyset$. Then the following propositions hold.*

(i) *For all $t \in \mathbb{R}$, $t \neq 0$ and all $\delta \in [0, 2\pi)$, $\delta \neq \alpha$, the intersection $l_\alpha(A) \cap K_0(D + tU(\delta))$ is nonempty.*
(ii) *For all $\delta \in [0, 2\pi), \delta \neq \alpha$, except one, the intersection $l_\delta(A) \cap K_0(D)$ is nonempty.*

Proof. Let $\delta \in [0, 2\pi), \delta \neq \alpha$. The intersection $l_\alpha(A) \cap K_0(D + tU(\delta))$ is nonempty if and only if there exist $a, b \in \mathbb{R}$, $\beta \in [0, 2\pi)$ such that

$$A + aU(\alpha) = D + tU(\delta) + bU(\beta). \tag{2.18}$$

Let $\gamma \in [0, 2\pi)$, $c, d \in \mathbb{R}$ be such that

$$A - D = cU(\gamma) + dU(\gamma + \pi).$$

Existence of such constants follows from Lemma 2.9. Then (2.18) is equivalent with

$$aU(\alpha) + cU(\gamma) + dU(\gamma + \pi) - tU(\delta) - bU(\beta) = 0.$$

Lemma 2.7 applied to this equation yields that (2.18) holds for some $a, b \in \mathbb{R}$, $\beta \in [0, 2\pi)$ if and only if

$$\begin{aligned} &2a\big[c(1 - \cos(\alpha - \gamma)) + d(1 - \cos(\alpha - \gamma - \pi)) - t(1 - \cos(\alpha - \delta))\big] \\ &= -\det\left[cU(\gamma) + dU(\gamma + \pi) - tU(\delta)\right] \end{aligned} \tag{2.19}$$

holds for some $a \in \mathbb{R}$. Denote the coefficient at a as

$$k(t) := c(1 - \cos(\alpha - \gamma)) + d(1 - \cos(\alpha - \gamma - \pi)) - t(1 - \cos(\alpha - \delta)).$$

Since $l_\alpha(A) \cap K_0(D) = \emptyset$, we get that $k(0) = 0$, hence $k(t) = -t(1 - \cos(\alpha - \delta))$. If $t \neq 0$, then $k(t) \neq 0$, because $\delta \neq \alpha$, and there exists a such that (2.19) holds. Then, by Lemma 2.7, there also exist b, β such that (2.18) holds.

Now we prove (ii). We again have that $k(0) = 0$. Thus if $\delta \neq \alpha$, $\delta \neq 2\gamma - \alpha + 2k\pi$, $k \in \mathbb{Z}$, we have that

$$c(1 - \cos(\delta - \gamma)) + d(1 - \cos(\delta - \gamma - \pi)) \neq 0,$$

therefore there exists a such that

$$2a\left[c(1 - \cos(\delta - \gamma)) + d(1 - \cos(\delta - \gamma - \pi))\right] = -\det\left[cU(\gamma) + dU(\gamma + \pi)\right],$$

and then, by Lemma 2.7, there also exist b, β such that

$$aU(\delta) + cU(\gamma) + dU(\gamma + \pi) - bU(\beta) = 0,$$

thus (ii) holds. □

Lemma 2.16. *Let $A \in \mathbf{S_2}, A > 0$. Then there exist $\alpha, \beta, \gamma, \delta \in [0, 2\pi)$, all different, and $a, b, c, d \in \mathbb{R}$ such that*

$$A = aU(\alpha) + bU(\beta) = cU(\gamma) + dU(\delta).$$

Proof. We have by Lemma 2.9 that there exist $\alpha, \beta \in [0, 2\pi)$ and $a, b \in \mathbb{R}$ such that $\beta = \alpha + \pi$, $A = aU(\alpha) + bU(\beta)$. Since $A > 0$, we have that $a \neq 0$, $b \neq 0$. Now let $\gamma \in [0, 2\pi)$ be such that it is different from α, β. Now we need to show that there exist c, d, δ such that $aU(\alpha) - cU(\gamma) = dU(\delta) - bU(\beta)$. This holds if the set $l_\gamma(aU(\alpha)) \cap K_0(-bU(\beta))$ is nonempty. In the case this set is empty, then by Lemma 2.15, there exist infinitely many $\bar{\gamma} \in [0, 2\pi)$ such that the set $l_{\bar{\gamma}}(aU(\alpha)) \cap K_0(-bU(\beta))$ is nonempty, and we can take any of these $\bar{\gamma}$ that is different from α, β.

Thus, there exist c, d, γ, δ such that γ is different from α, β and $aU(\alpha) - cU(\gamma) = dU(\delta) - bU(\beta)$. Now, since $a \neq 0$, $b \neq 0$, then, by Lemma 2.11, $\delta \neq \alpha$, $\delta \neq \beta$, $\delta \neq \gamma$. □

Lemma 2.17. *Let $D \in \mathbf{S_2}$, $a, b \in \mathbb{R}$, $\alpha, \beta, \gamma \in [0, 2\pi)$ be such that $a \neq 0, b \neq 0$, $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$ and denote*

$$A = D + aU(\alpha) + bU(\beta), \quad B = D + (a+b)U(\alpha), \quad C = D + (a+b)U(\gamma).$$

Then there exist $V, W \in \mathbf{S_2}$ such that $V < W$ and $A, B, C \in K_0^+(V) \cap K_0^-(W)$.

Proof. First we show that there exists $v \in \mathbb{R}$ such that

$$\begin{aligned}&\det\left(v\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - (a+b)\left[U(\alpha) - I\right]\right)\\ &= \det\left(v\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - (a+b)\left[U(\gamma) - I\right]\right),\end{aligned} \tag{2.20}$$

$$\begin{aligned}&\det\left(v\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - (a+b)\left[U(\alpha) - I\right]\right)\\ &= \det\left(v\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - a\left[U(\alpha) - I\right] - b\left[U(\beta) - I\right]\right).\end{aligned} \tag{2.21}$$

Identities (2.5), (2.2) yield that (2.20) holds for all $v \in \mathbb{R}$. Now we multiply matrices in (2.21) by $\left[U\left(\frac{\alpha+\gamma}{2}\right) - I\right]$ from the right, and use identity (2.5) to get equivalent equation

$$\begin{aligned}&\det\left(vI - (a+b)\left[U(\alpha) - I\right]\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right]\right)\\ &= \det\left(vI - a\left[U(\alpha) - I\right]\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - b\left[U(\beta) - I\right]\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right]\right).\end{aligned}$$

The matrices, by identity (2.4), are of the form

$$\begin{aligned} vI - (a+b)\left[U(\alpha) - I\right]\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] &= \begin{bmatrix} p+v & q \\ -q & p+v \end{bmatrix},\\ vI - a\left[U(\alpha) - I\right]\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - b\left[U(\beta) - I\right]\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] &= \begin{bmatrix} r+v & s \\ -s & r+v \end{bmatrix},\end{aligned}$$

where $p, q, r, s \in \mathbb{R}$ and $p - r = b\left(\cos\left(\frac{\alpha-\gamma}{2}\right) - \cos\left(\frac{\alpha+\gamma-2\beta}{2}\right)\right)$. We have supposed that $b \neq 0$, $\beta \neq \gamma$ and $\beta \neq \alpha$, hence $p \neq r$ and there exists such v that

$$v^2 + 2pv + p^2 + q^2 = v^2 + 2rv + r^2 + s^2, \quad \det\left[\begin{smallmatrix} p+v & q \\ -q & p+v \end{smallmatrix}\right] = \det\begin{bmatrix} r+v & s \\ -s & r+v \end{bmatrix}.$$

Now, by Lemma 2.14, there exists $d \in \mathbb{R}$, $d \geq 0$ and $\delta_1, \delta_2, \delta_3 \in [0, 2\pi)$ such that

$$\begin{aligned} v\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - a\left[U(\alpha) - I\right] - b\left[U(\beta) - I\right] &= d\left[U(\delta_1) - I\right],\\ v\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - (a+b)\left[U(\alpha) - I\right] &= d\left[U(\delta_2) - I\right],\\ v\left[U\left(\tfrac{\alpha+\gamma}{2}\right) - I\right] - (a+b)\left[U(\gamma) - I\right] &= d\left[U(\delta_3) - I\right].\end{aligned}$$

We denote $w = d - v + a + b$ and obtain the following relations from the above

$$\begin{aligned} vU\left(\tfrac{\alpha+\gamma}{2}\right) - aU(\alpha) - bU(\beta) &= dU(\delta_1) - wI = -dU(\delta_1 + \pi) - (w - 2d)I,\\ vU\left(\tfrac{\alpha+\gamma}{2}\right) - (a+b)U(\alpha) &= dU(\delta_2) - wI = -dU(\delta_2 + \pi) - (w - 2d)I,\\ vU\left(\tfrac{\alpha+\gamma}{2}\right) - (a+b)U(\gamma) &= dU(\delta_3) - wI = -dU(\delta_3 + \pi) - (w - 2d)I.\end{aligned}$$

Since $\alpha \neq \gamma$, Lemma 2.12 yields that $d > 0$. Now we take $V, W \in \mathbf{S_2}$ as

$$W = D + vU\left(\frac{\alpha+\gamma}{2}\right) + wI \quad \text{and} \quad V = D + vU\left(\frac{\alpha+\gamma}{2}\right) + (w-2d)I,$$

and we get

$$\begin{aligned}
W - A &= vU\left(\tfrac{\alpha+\gamma}{2}\right) - aU(\alpha) - bU(\beta) + wI = dU(\delta_1),\\
W - B &= vU\left(\tfrac{\alpha+\gamma}{2}\right) - (a+b)U(\alpha) + wI = dU(\delta_2),\\
W - C &= vU\left(\tfrac{\alpha+\gamma}{2}\right) - (a+b)U(\gamma) + wI = dU(\delta_3),\\
V - A &= vU\left(\tfrac{\alpha+\gamma}{2}\right) - aU(\alpha) - bU(\beta) + (w-2d)I = -dU(\delta_1+\pi),\\
V - B &= vU\left(\tfrac{\alpha+\gamma}{2}\right) - (a+b)U(\alpha) + (w-2d)I = -dU(\delta_2+\pi),\\
V - C &= vU\left(\tfrac{\alpha+\gamma}{2}\right) - (a+b)U(\gamma) + (w-2d)I = -dU(\delta_3+\pi).
\end{aligned}$$

Thus, $A, B, C \in K_0^+(V) \cap K_0^-(W)$. Further, $W - V = 2dI > 0$. □

Lemma 2.18. *Let $V, W \in \mathbf{S_2}$, $V < W$, $\psi, \theta \in [0, 2\pi)$, $\psi < \theta$. Let $A, B, C \in \mathbf{S_2}$ be such that $A, B, C \in p_{\psi,\theta}(A) \cap K_0^+(V) \cap K_0^-(W)$. Then $B = C$ or $B = A$ or $C = A$.*

Proof. Since $A, B, C \in p_{\psi,\theta}(A) \cap K_0^+(V) \cap K_0^-(W)$, there exist $a, b, c \in \mathbb{R}$, $a > 0$, $b > 0$, $c > 0$ and $\alpha, \beta, \gamma \in [0, 2\pi)$ such that

$$\begin{aligned}
A - V = aU(\alpha), \quad B - V = bU(\beta), \quad C - V = cU(\gamma),\\
A - B = aU(\alpha) - bU(\beta), \quad A - C = aU(\alpha) - cU(\gamma).
\end{aligned} \tag{2.22}$$

Similarly, there exist $\bar{a}, \bar{b}, \bar{c} \in \mathbb{R}$, $\bar{a} < 0$, $\bar{b} < 0$, $\bar{c} < 0$ and $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in [0, 2\pi)$ such that

$$\begin{aligned}
A - W = \bar{a}U(\bar{\alpha}), \quad B - W = \bar{b}U(\bar{\beta}), \quad C - W = \bar{c}U(\bar{\gamma}),\\
A - B = \bar{a}U(\bar{\alpha}) - \bar{b}U(\bar{\beta}), \quad A - C = \bar{a}U(\bar{\alpha}) - \bar{c}U(\bar{\gamma}).
\end{aligned} \tag{2.23}$$

First suppose that there exists $m \in \mathbb{R}$ such that

$$A - C = m(U(\psi) + U(\theta)).$$

Then (2.22) yields

$$aU(\alpha) - m(U(\psi) + U(\theta)) = cU(\gamma).$$

Using Lemma 2.13 and the fact that $a > 0$, we obtain $m \geq 0$. Similarly, (2.23) yields

$$-\bar{a}U(\bar{\alpha}) + m(U(\psi) + U(\theta)) = -\bar{c}U(\bar{\gamma}),$$

and since $-\bar{a} > 0$, we obtain $m \leq 0$, again by Lemma 2.13. Hence, $m = 0$ and $A = C$.

If

$$A - C \neq m(U(\psi) + U(\theta))$$

for all $m \in \mathbb{R}$, then since $B, C \in p_{\psi,\theta}(A)$, there exist $k, l \in \mathbb{R}$ such that

$$A - B + k(U(\psi) + U(\theta)) = l(A - C).$$

This relation is equivalent with

$$C - B + k(U(\psi) + U(\theta)) = (1-l)(C-A)$$

and if $l \neq 0$ then also with

$$A - C - \frac{k}{l}(U(\psi) + U(\theta)) = \frac{1}{l}(A-B).$$

We may suppose that

$$A - B + k(U(\psi) + U(\theta)) = l(A-C), \quad l \geq 1. \tag{2.24}$$

Otherwise, the rest would be same, just we would exchange B with C or A with C.

By substituting from (2.22) into (2.24), we get

$$aU(\alpha) - bU(\beta) + k(U(\psi) + U(\theta)) = l(aU(\alpha) - cU(\gamma)),$$

$$a(l-1)U(\alpha) + bU(\beta) - k(U(\psi) + U(\theta)) = clU(\gamma).$$

Since $a > 0, (l-1) \geq 0, b > 0$, Lemma 2.13 yields that $k \geq 0$. Now, by substituting from (2.23) into (2.24) we similarly get

$$-\bar{a}(l-1)U(\bar{\alpha}) - \bar{b}U(\bar{\beta}) + k(U(\psi) + U(\theta)) = -\bar{c}lU(\bar{\gamma}).$$

Since $-\bar{a} > 0, (l-1) \geq 0, -\bar{b} > 0$, Lemma 2.13 yields that $k \leq 0$. The only possibility is that $k = 0$, and

$$a(l-1)U(\alpha) + bU(\beta) = clU(\gamma), \quad -\bar{a}(l-1)U(\bar{\alpha}) - \bar{b}U(\bar{\beta}) = -\bar{c}lU(\bar{\gamma}).$$

Then Lemma 2.11 implies that either $l = 1$ or $\alpha = \beta, \bar{\alpha} = \bar{\beta}$. If $l = 1$, then $B = C$. If $\alpha = \beta$, $\bar{\alpha} = \bar{\beta}$, then we obtain from (2.22), (2.23) that

$$A - B = (a-b)U(\alpha) = (\bar{a} - \bar{b})U(\bar{\alpha})$$

and hence either $A = B$ or $\alpha = \bar{\alpha}$. But since $V < W$, we have that

$$0 \neq \det[W - V] = \det[aU(\alpha) - \bar{a}U(\bar{\alpha})]$$

and $\alpha \neq \bar{\alpha}$. Thus, $B = C$ or $A = B$. $\square$

Lemma 2.19. *Let $\psi \in [0, 2\pi)$, $u_1, u_2 \in \mathbb{R}$, $0 < u_1, 0 < u_2$ and let $A, V, W \in \mathbf{S_2}$ be such that*

$$W = V + 2u_1U(\psi) + 2u_2U(\psi + \pi)$$

and $p_{\psi,\psi+\pi}(A) \cap K_0(V) \cap K_0(W) = \{A\}$. Then A is of the form

$$A = \tfrac{V+W}{2} \pm 2\sqrt{u_1u_2}\left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right].$$

Proof. By Lemma 2.8, there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$A = \tfrac{V+W}{2} + a_1 U(\psi) + a_2 U(\psi+\pi) + a_3 \left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right].$$

Since $A \in K_0(V) \cap K_0(W)$, there exist $a, b \in \mathbb{R}$ and $\alpha, \beta \in [0, 2\pi)$ such that $a = a_1 + a_2 + u_1 + u_2$, $b = -a_1 - a_2 + u_1 + u_2$ and

$$\begin{aligned} A - V = aU(\alpha) &= \tfrac{W-V}{2} + a_1 U(\psi) + a_2 U(\psi+\pi) + a_3 \left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right] \\ &= (a_1 + u_1) U(\psi) + (a_2 + u_2) U(\psi+\pi) + a_3 \left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right], \end{aligned} \tag{2.25}$$

$$\begin{aligned} W - A = bU(\beta) &= \tfrac{W-V}{2} - a_1 U(\psi) - a_2 U(\psi+\pi) - a_3 \left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right] \\ &= (-a_1 + u_1) U(\psi) + (-a_2 + u_2) U(\psi+\pi) - a_3 \left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right]. \end{aligned} \tag{2.26}$$

Now let $\bar{A} \in \mathbf{S_2}$ be such that

$$\bar{A} = A - 2a_1 U(\psi) - 2a_2 U(\psi+\pi).$$

Then

$$\bar{A} - V = (-a_1 + u_1) U(\psi) + (-a_2 + u_2) U(\psi+\pi) + a_3 \left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right], \tag{2.27}$$

$$W - \bar{A} = (a_1 + u_1) U(\psi) + (a_2 + u_2) U(\psi+\pi) - a_3 \left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right], \tag{2.28}$$

and Lemma 2.10 applied to (2.25), (2.28) and to (2.26), (2.27) yields that

$$\bar{A} - V = bU(2\psi - \beta), W - \bar{A} = aU(2\psi - \alpha),$$

hence $\bar{A} \in p_{\psi,\psi+\pi}(A) \cap K_0(V) \cap K_0(W)$ and $\bar{A} = A$. This implies that $a_1 = a_2 = 0$. Further, by Lemma 2.10 we also have that $a_3^2 = 4(a_1 + u_1)(a_2 + u_2)$, which yields $a_3 = \pm 2\sqrt{u_1 u_2}$. □

Lemma 2.20. *Let $\psi \in [0, 2\pi)$, $z \in \mathbb{R}$, $z > 0$ and $A, B, V, W \in \mathbf{S_2}$ be such that $A = B + z\left[U\left(\psi + \frac{\pi}{2}\right) - I\right]$, $V = B - zI, W = B + zI$. Then*

$$p_{\psi,\psi+\pi}(A) \cap K_0(V) \cap K_0(W) = \{A\}.$$

Proof. It suffices to prove this for $B = 0$, because $p_{\psi,\psi+\pi}(A) \cap K_0(V) \cap K_0(W) = \{A\}$ if and only if $p_{\psi,\psi+\pi}(A + B) \cap K_0(V + B) \cap K_0(W + B) = \{A + B\}$.

$$A - V = zU\left(\psi + \tfrac{\pi}{2}\right), \quad W - A = zI - z\left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right] = zU\left(\psi + \tfrac{3\pi}{2}\right),$$

hence $A \in K_0(V) \cap K_0(W)$. It remains to show, that if $\bar{A} \in p_{\psi,\psi+\pi}(A) \cap K_0(V) \cap K_0(W)$, then $\bar{A} = A$. Let $\bar{A}$ be such matrix. Then there exist $a, b, c, d \in \mathbb{R}$, $\alpha, \beta \in [0, 2\pi)$ such that

$$W - \bar{A} = zI - \bar{A} = aU(\alpha), \quad \bar{A} - V = \bar{A} + zI = bU(\beta), \tag{2.29}$$

$$\begin{aligned} \bar{A} &= A + cU(\psi) + dU(\psi+\pi) \\ &= z\left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right] + cU(\psi) + dU(\psi+\pi). \end{aligned} \tag{2.30}$$

We have from (2.29) that $W - \bar{A} + \bar{A} - V = 2zI = aU(\alpha) + bU(\beta)$, and Lemma 2.12 yields $a = b = z$. Now from (2.29), we have that $\bar{A} = zI - zU(\alpha)$. This with (2.30) gives us

$$z\left[U\left(\psi + \tfrac{\pi}{2}\right) - I\right] + cU(\psi) + dU(\psi + \pi) = zI - zU(\alpha),$$

thus $c = -d$. From identities (2.3), (2.5), (2.9) we further get

$$\begin{bmatrix} 2c & -z \\ -z & -2c \end{bmatrix} = \left[U\left(\tfrac{\psi}{2}\right) - I\right][zI - zU(\alpha)]\left[U\left(\tfrac{\psi}{2}\right) - I\right] = zI - zU(\psi - \alpha),$$

$$-4c^2 - z^2 = z^2 \det[I - U(\psi - \alpha)] = -z^2,$$

hence $c = d = 0$ and $\bar{A} = A$. □

Lemma 2.21. *Let $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$ and let $\alpha, \beta, \gamma, \delta \in [0, 2\pi)$ be all different and such that*

$$\begin{aligned} a_1U(\alpha) + b_1U(\beta) &= c_1U(\gamma) + d_1U(\delta), \\ a_2U(\alpha) + b_2U(\beta) &= c_2U(\gamma) + d_2U(\delta). \end{aligned} \tag{2.31}$$

Then there exists t such that $a_1 = tb_1$, $a_2 = tb_2$.

Proof. Lemma 2.11 implies that either none of a_1, b_1, c_1, d_1 is zero, or all of them are. We will further suppose that the first case is true. From equations (2.31) we get that

$$(a_1b_2 - a_2b_1)U(\alpha) = (c_1b_2 - c_2b_1)U(\gamma) + (d_1b_2 - d_2b_1)U(\delta),$$

and Lemma 2.11 implies that $a_1b_2 - a_2b_1 = 0$. Since $b_1 \neq 0$, there exists t such that $a_1 = tb_1$. Then $a_2b_1 = a_1b_2 = tb_1b_2$ and since $b_1 \neq 0$, we get that $a_2 = tb_2$. □

Lemma 2.22. *Let $A, B \in \mathbf{S_2}$, $A > 0$ and B be such that $B \notin \{At, t \in \mathbb{R}\}$. Then there exist $t_1, t_2 \in \mathbb{R}$, $t_1 \neq t_2$ and $\alpha_1, \alpha_2 \in [0, 2\pi)$, $a_1, a_2 \in \mathbb{R}$ such that $At_1 + B = a_1U(\alpha_1)$, $At_2 + B = a_2U(\alpha_2)$.*

Proof. Let $\varphi \in [0, 2\pi)$ and $a, b, c, d, e \in \mathbb{R}$ be such that

$$\begin{aligned} A &= aU(\varphi) + bU(\varphi + \pi), \\ B &= cU(\varphi) + dU(\varphi + \pi) + e\left(U(\varphi + \tfrac{\pi}{2}) - I\right). \end{aligned}$$

Such constants exist by Lemma 2.8 and Lemma 2.9, and since $A > 0$, then $a > 0, b > 0$. Then

$$At + B = (at + c)U(\varphi) + (bt + d)U(\varphi + \pi) + e\left(U(\varphi + \tfrac{\pi}{2}) - I\right).$$

Now, by Lemma 2.10, if $e^2 = 4(at + c)(bt + d)$, then the matrix $At + B$ has zero determinant. And since $a > 0, b > 0$ and $B \notin \{At, t \in \mathbb{R}\}$, the equation $e^2 = 4(at + c)(bt + d)$ has two solutions t_1, t_2. □

Lemma 2.23. *Let $A, B, C \in \mathbf{S_2}$, $B > C$ and $A \notin \{C + t(B - C) : t \in \mathbb{R}\}$. Then there exist $\alpha, \beta \in [0, 2\pi)$ such that $A, B \in p_{\alpha,\beta}(C)$.*

Proof. By Lemma 2.22, there exist $t_1 \neq t_2$ and α, β, a, b such that

$$A - C + t_1(B - C) = aU(\alpha), \quad A - C + t_2(B - C) = bU(\beta).$$

From this,

$$B - C = \frac{a}{t_1 - t_2}U(\alpha) - \frac{b}{t_1 - t_2}U(\beta), \quad A - C = \frac{-t_2 a}{t_1 - t_2}U(\alpha) + \frac{t_1 b}{t_1 - t_2}U(\beta),$$

which implies that $A, B \in p_{\alpha,\beta}(C)$. □

3. LEMMAS ABOUT FUNCTIONS WITH ORDER PRESERVING PROPERTY

In this section, statements about functions with the order preserving property (1.1) are derived. The first four lemmas in this section (Lemmas 3.1–3.4) are proven for any dimension $n \in \mathbb{N}$. The first two of them are about the injectivity of such functions and the order preserving property of their inverse. If $f\colon \mathbf{S} \to \mathbf{S}$ has property (1.1) on the whole set $\mathbf{S}$, then its surjectivity is also expected, but not obvious, and we will not be able to prove it, until we have the final form of f. Therefore, we will work with the surjective function $f\colon M \to f(M)$ or $f\colon \mathbf{S} \to f(\mathbf{S})$.

Lemma 3.1. *Let $M \subseteq \mathbf{S}$ and $f\colon M \to f(M)$ have property* (1.1) *on M. Then f is injective on M and its inverse $f^{-1}\colon f(M) \to M$ has property* (1.1) *on $f(M)$,*

$$Q \le \hat{Q} \Leftrightarrow f^{-1}(Q) \le f^{-1}(\hat{Q}) \quad \textit{for all } Q, \hat{Q} \in f(M) \subseteq \mathbf{S}. \tag{3.1}$$

Proof. We have from property (1.1) that if $Q_1, Q_2 \in M$, then $f(Q_1) = f(Q_2)$ implies both $Q_1 \le Q_2$ and $Q_1 \ge Q_2$. This gives us that $Q_1 = Q_2$. Hence, f is injective on M and its inverse exists there. The property (3.1) follows directly from property (1.1). □

Lemma 3.2. *Let $f\colon \mathbf{S} \to \mathbf{S}$ have property* (1.1) *on $\mathbf{S}$. Then f is injective on $\mathbf{S}$ and $f^{-1}\colon f(\mathbf{S}) \to \mathbf{S}$ has property* (1.1) *on $f(\mathbf{S})$.*

Proof. It follows from Lemma 3.1, we have $M = \mathbf{S}$. □

3.1. IMAGES AND PREIMAGES OF CONES, $\frac{\pi}{4}$-LINES AND $\frac{\pi}{4}$-PLANES

We show successively, that if f has property (1.1) on the whole set $\mathbf{S_2}$, then the image/preimage of a subset of any (upper/lower) cone is again a subset of (upper/lower) cone. Further, this implies, that the image/preimage of a subset of any $\frac{\pi}{4}$-line is again a subset of a $\frac{\pi}{4}$-line and this further implies that the image/preimage of a subset of any $\frac{\pi}{4}$-plane is a subset of a $\frac{\pi}{4}$-plane.

If f has property (1.1) only on a subset $M \subset \mathbf{S_2}$, then the statements about cones and $\frac{\pi}{4}$-lines hold as well, but the preimage of a subset of a $\frac{\pi}{4}$-plane is not always a subset of a $\frac{\pi}{4}$-plane for such f.

Lemma 3.3. *Let $M \subseteq \mathbf{S}$ and let $f\colon M \to f(M)$ be continuous and have property* (1.1) *on M. Let $A \in M$. Then the following sets are equal:*

(i) $f(K_0(A) \cap M) = K_0(f(A)) \cap f(M)$,
(ii) $f(K_0^+(A) \cap M) = K_0^+(f(A)) \cap f(M)$,
(iii) $f(K_0^-(A) \cap M) = K_0^-(f(A)) \cap f(M)$.

Proof. If the matrix dimension $n = 1$, then we have $K_0(A) = K_0^+(A) = K_0^-(A) = \{A\}$ and the sets in (i)–(iii) are all equal to $\{f(A)\}$.

Further we suppose that $n \geq 2$. If $Q \in K^+(A) \cap M$, then $Q \geq A$ and property (1.1) implies that $f(Q) \geq f(A)$. That means $f(Q) \in K^+(f(A))$. Similarly, if $Q \in K^-(A) \cap M$, then $f(Q) \in K^-(f(A))$. And thus, if $Q \in K(A) \cap M$, then $f(Q) \in K(f(A))$.

We have that $K_0(A)$ is a limit of $K(A)$, i.e. if $Q \in K_0(A)$, then in any neighborhood of Q there exists a matrix Q_1 such that both $Q_1 \ngeq A$ and $Q_1 \nleq A$. Then also $K_0(A) \cap M$ is a limit of $K(A) \cap M$. From continuity of f it follows that if $Q \in K_0(A) \cap M$, then $f(Q) \in K_0(f(A))$, and hence

$$f(K_0(A) \cap M) \subseteq K_0(f(A)) \cap f(M)$$

holds.

Now if

$$Q \in K_0^+(A) \cap M = K^+(A) \cap K_0(A) \cap M,$$

then we have shown that $f(Q) \in K_0(f(A))$ and also $f(Q) \in K^+(f(A))$, hence $f(Q) \in K_0^+(f(A))$, and

$$f(K_0^+(A) \cap M) \subseteq K_0^+(f(A)) \cap f(M)$$

holds. Similarly, we can prove that

$$f(K_0^-(A) \cap M) \subseteq K_0^-(f(A)) \cap f(M)$$

holds.

Now we use Lemma 3.1, property (3.1) and obtain that for any $f(A) \in f(M)$ the inclusion

$$f^{-1}(K_0(f(A)) \cap f(M)) \subseteq K_0(A) \cap M$$

holds, which is equivalent with

$$K_0(f(A)) \cap f(M) \subseteq f(K_0(A) \cap M),$$

hence (i) holds. Statements (ii) and (iii) can be proven in analogical way. $\square$

Lemma 3.4. *Let $f\colon \mathbf{S} \to \mathbf{S}$ be continuous and have property* (1.1) *on $\mathbf{S}$. Let $A \in \mathbf{S}$. Then:*

(i) $f(K_0(A)) = K_0(f(A)) \cap f(\mathbf{S})$,
(ii) $f(K_0^+(A)) = K_0^+(f(A)) \cap f(\mathbf{S})$,
(iii) $f(K_0^-(A)) = K_0^-(f(A)) \cap f(\mathbf{S})$.

Proof. It follows from Lemma 3.3, we have $M = \mathbf{S}$. □

Now throughout the rest of this section we again suppose that $n = 2$.

Lemma 3.5. *Let $M \subseteq \mathbf{S_2}$ and let $f\colon M \to f(M)$ be continuous and have property* (1.1) *on M. Let $\varphi \in [0, 2\pi)$, $Q \in M$. Then there exist $\psi, \theta \in [0, 2\pi)$ such that*

$$l_\varphi(f(Q)) \cap f(M) = f(l_\psi(Q) \cap M), \quad f(l_\varphi(Q) \cap M) = l_\theta(f(Q)) \cap f(M).$$

Proof. From Lemma 3.3 we have that the following equivalences

$$Q_1, Q_2 \in K_0(Q) \cap M \;\Leftrightarrow\; f(Q_1), f(Q_2) \in K_0(f(Q)), \tag{3.2}$$

$$Q_1 \in K_0(Q_2) \cap M \;\Leftrightarrow\; f(Q_1) \in K_0(f(Q_2)) \tag{3.3}$$

hold for all $Q_1, Q_2 \in \mathbf{S_2}$. Let $\varphi \in \mathbb{R}$ and $Q_1 \in l_\varphi(Q) \cap M$, $Q_1 \neq Q$. Since $l_\varphi(Q) \subseteq K_0(Q)$, we get from (3.2) that $f(Q_1) \in K_0(f(Q))$ and there exists $\theta \in [0, 2\pi)$ such that $f(Q_1) - f(Q) = tU(\theta)$, where $t \neq 0$.

Let $Q_2 \in l_\varphi(Q) \cap M$, $Q_2 \neq Q$. Since $l_\varphi(Q) \subseteq K_0(Q)$ and $Q_1 \in l_\varphi(Q_2) \subseteq K_0(Q_2)$, we get from (3.2), (3.3) that $f(Q_2) \in K_0(f(Q))$, $f(Q_1) \in K_0(f(Q_2))$ and there exist $\alpha, \beta \in [0, 2\pi)$, $a, b \in \mathbb{R}$, $a \neq 0$, such that

$$f(Q_2) - f(Q) = aU(\alpha), \quad f(Q_1) - f(Q_2) = bU(\beta).$$

Since $tU(\theta) - aU(\alpha) = bU(\beta)$, we have by Lemma 2.11 that $\alpha = \theta$ and thus $f(Q_2) \in l_\theta(f(Q))$.

We have proven that for all $\varphi \in [0, 2\pi)$ there exists $\theta \in [0, 2\pi)$ such that the inclusion

$$f(l_\varphi(Q) \cap M) \subseteq l_\theta(f(Q)) \cap f(M)$$

holds. By Lemma 3.1, property (3.1), we analogically get that for all $\theta \in [0, 2\pi)$ there exists $\omega \in [0, 2\pi)$ such that the inclusion

$$l_\theta(f(Q)) \cap f(M) \subseteq f(l_\omega(Q) \cap M)$$

holds. Together we have that for all $\varphi \in [0, 2\pi)$ there exist $\theta, \omega, \nu \in [0, 2\pi)$ such that

$$f(l_\varphi(Q) \cap M) \subseteq l_\theta(f(Q)) \cap f(M) \subseteq f(l_\omega(Q) \cap M) \subseteq l_\nu(f(Q)) \cap f(M),$$

which implies that $\theta = \nu$, $\omega = \varphi$, and

$$f(l_\varphi(Q) \cap M) = l_\theta(f(Q)) \cap f(M).$$

The proof of the second equality is analogical. □

Lemma 3.6. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi \in [0, 2\pi)$, $Q \in \mathbf{S_2}$. Then there exist $\psi, \theta \in [0, 2\pi)$ such that*

$$l_\varphi(f(Q)) \cap f(\mathbf{S_2}) = f(l_\psi(Q)), \quad f(l_\varphi(Q)) = l_\theta(f(Q)) \cap f(\mathbf{S_2}).$$

Proof. It follows from Lemma 3.5, we have $M = \mathbf{S_2}$. □

Lemma 3.7. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \bar{\varphi} \in [0, 2\pi)$, $\varphi \neq \bar{\varphi}$ and $Q \in \mathbf{S_2}$. Then there exist $\psi, \theta \in [0, 2\pi)$ such that*

$$l_\varphi(f(Q) \cap f(\mathbf{S_2}) = f(l_\psi(Q)), \quad l_{\bar{\varphi}}(f(Q)) \cap f(\mathbf{S_2}) = f(l_\theta(Q)),$$
$$p_{\varphi,\bar{\varphi}}(f(Q)) \cap f(\mathbf{S_2}) \subseteq f(p_{\psi,\theta}(Q)).$$

Proof. Let $\psi, \theta \in [0, 2\pi)$ be such that

$$l_\varphi(f(Q)) \cap f(\mathbf{S_2}) = f(l_\psi(Q)), \quad l_{\bar{\varphi}}(f(Q)) \cap f(\mathbf{S_2}) = f(l_\theta(Q)).$$

Such ψ, θ exist by Lemma 3.6, and $\psi \neq \theta$ because $\varphi \neq \bar{\varphi}$. Let $D \in \mathbf{S_2}$ be such that $D \in p_{\varphi,\bar{\varphi}}(f(Q)) \cap f(\mathbf{S_2})$ and $D \notin f(l_\psi(Q))$, $D \notin f(l_\theta(Q))$. Then there exist $r, s \in \mathbb{R}$, $r \neq 0$, $s \neq 0$ such that

$$D = f(Q) + rU(\varphi) + sU(\bar{\varphi}).$$

First we will suppose that the set $K_0(f^{-1}(D)) \cap l_\psi(Q)$ is nonempty. Then there exists $P \in \mathbf{S_2}$ such that $P \in K_0(f^{-1}(D)) \cap l_\psi(Q)$ and further there exist $\omega, a, d \in \mathbb{R}$ such that

$$P = Q + aU(\psi) = f^{-1}(D) + dU(\omega)$$

and from this

$$f^{-1}(D) = Q + aU(\psi) - dU(\omega).$$

We show that $f(l_\omega(P)) \subseteq l_{\bar{\varphi}}(f(P))$. By Lemma 3.6, there exists $\alpha \in [0, 2\pi)$ such that $f(l_\omega(P)) \subseteq l_\alpha(f(P))$. Since $D \in f(l_\omega(P))$, we get that $D \in l_\alpha(f(P))$. Since $P \in l_\psi(Q)$, we get that $f(P) \in l_\varphi(f(Q))$. This together implies that there exist $u, v \in \mathbb{R}$ such that

$$D = f(P) + vU(\alpha) = f(Q) + uU(\varphi) + vU(\alpha).$$

We also have that

$$D = f(Q) + rU(\varphi) + sU(\bar{\varphi}),$$

with $r \neq 0, s \neq 0$, hence we have by Lemma 2.11 that $\alpha = \bar{\varphi}$.

We show that $a \neq 0$. If $a = 0$, then $f(P) = f(Q)$ and $D = f(Q) + vU(\bar{\varphi})$, hence $D \in l_{\bar{\varphi}}(f(Q)) \cap f(\mathbf{S_2}) = f(l_\theta(Q))$, but we supposed that $D \notin f(l_\theta(Q))$.

Now we show that $\omega = \theta$. We will suppose that $\omega \neq \theta$ and get a contradiction. Let $b \in \mathbb{R}$ be such that $b \neq 0$, $b \neq a$. We take

$$A = Q + aU(\psi) - bU(\omega) \in l_\omega(P),$$
$$B = Q + (a - b)U(\psi) \in l_\psi(Q),$$
$$C = Q + (a - b)U(\theta) \in l_\theta(Q).$$

We have supposed that $\omega \neq \theta$, and we also have that $\psi \neq \theta$ and further also $\omega \neq \psi$, because $D \notin f(l_\psi(Q))$, hence by Lemma 2.17, there exist $V, W \in \mathbf{S_2}$ such that $A, B, C \in K_0^+(V)$ and $A, B, C \in K_0^-(W)$ and $V < W$. Then Lemma 3.4 yields that $f(A), f(B), f(C) \in K_0^+(f(V)) \cap K_0^-(f(W))$. Since $A \in l_\omega(P)$, $B \in l_\psi(Q), C \in l_\theta(Q)$, we have that

$$
\begin{aligned}
f(A) &\in l_{\bar\varphi}(f(P)) \subseteq p_{\varphi,\bar\varphi}(f(Q)) = p_{\varphi,\bar\varphi}(f(A)),\\
f(B) &\in l_{\varphi}(f(Q)) \subseteq p_{\varphi,\bar\varphi}(f(Q)) = p_{\varphi,\bar\varphi}(f(A)),\\
f(C) &\in l_{\bar\varphi}(f(Q)) \subseteq p_{\varphi,\bar\varphi}(f(Q)) = p_{\varphi,\bar\varphi}(f(A)).
\end{aligned}
$$

Together it implies that $f(A), f(B), f(C) \in p_{\varphi,\bar\varphi}(f(A)) \cap K_0^+(f(V)) \cap K_0^-(f(W))$ and by Lemma 2.18 either $f(A) = f(B)$ or $f(A) = f(C)$ or $f(B) = f(C)$, hence either $A = B$ or $A = C$ or $B = C$. But this is not possible, since we have

$$
\begin{aligned}
&A - B = b[U(\psi) - U(\omega)], \text{ and } b \neq 0, \omega \neq \psi,\\
&B - C = (a - b)[U(\psi) - U(\theta)], \text{ and } a - b \neq 0, \theta \neq \psi,\\
&C - A = (a - b)U(\theta) - aU(\psi) + bU(\omega), \text{ and } a \neq 0, b \neq 0, \omega \neq \psi,
\end{aligned}
$$

and Lemma 2.11 implies that $A \neq B \neq C \neq A$. Thus, we got a contradiction, and we have proven that $\omega = \theta$ and

$$f^{-1}(D) = Q + aU(\psi) - dU(\theta), \quad f^{-1}(D) \in p_{\psi,\theta}(Q), \quad D \in f(p_{\psi,\theta}(Q)).$$

It remains to prove this for such D when the set $K_0(f^{-1}(D)) \cap l_\psi(Q)$ is empty. Let $\delta_1, \delta_2 \in [0, 2\pi)$ be such that

$$l_\varphi(D) \cap f(\mathbf{S_2}) = f(l_{\delta_1}(f^{-1}(D))) \quad \text{and} \quad l_{\bar\varphi}(D) \cap f(\mathbf{S_2}) = f(l_{\delta_2}(f^{-1}(D))).$$

Such δ_1, δ_2 exist by Lemma 3.6, and we have that $\delta_1 \neq \delta_2$ because $\varphi \neq \bar\varphi$. Hence, at least one of them is different from ψ. Denote it δ.

Now, by Lemma 2.15, we further have that for all $t \neq 0$ the set $l_\psi(Q) \cap K_0(f^{-1}(D) + tU(\delta))$ is nonempty. Denote $D_t = f(f^{-1}(D) + tU(\delta)))$. We have that

$$D_t \in p_{\varphi,\bar\varphi}(D) \cap f(\mathbf{S_2}) = p_{\varphi,\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}).$$

Since $D \notin f(l_\psi(Q))$, $D \notin f(l_\theta(Q))$, there exists t_0 such that for all $|t| < t_0$, $D_t \notin f(l_\psi(Q))$, $D_t \notin f(l_\theta(Q))$.

Hence, we can again show in the same way as before, that for all $t \neq 0$, $|t| < t_0$,

$$D_t = f(f^{-1}(D) + tU(\delta))) \in f(p_{\psi,\theta}(Q)).$$

Since f is continuous, then also $D \in f(p_{\psi,\theta}(Q))$. □

Lemma 3.8. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $Q \in \mathbf{S_2}$ and $\varphi, \bar\varphi, \psi, \bar\psi \in [0, 2\pi)$, $\varphi \neq \bar\varphi$ be such that*

$$l_\varphi(f(Q)) \cap f(\mathbf{S_2}) = f(l_\psi(Q)), \quad l_{\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}) = f(l_{\bar\psi}(Q)), \tag{3.4}$$

$$p_{\varphi,\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}) \subseteq f(p_{\psi,\bar\psi}(Q)). \tag{3.5}$$

Let $P \in \mathbf{S_2}$ be such that $P \in p_{\psi,\bar\psi}(Q)$. Then $f(P) \in p_{\varphi,\bar\varphi}(f(Q))$ and

$$l_\varphi(f(P)) \cap f(\mathbf{S_2}) = f(l_\psi(P)), \quad l_{\bar\varphi}(f(P)) \cap f(\mathbf{S_2}) = f(l_{\bar\psi}(P)). \tag{3.6}$$

Proof. First suppose that $P \in l_\psi(Q)$. Then $l_\psi(P) = l_\psi(Q)$ and we obtain from (3.4) that $f(P) \in l_\varphi(f(Q)$, which implies that also $l_\varphi(f(P)) = l_\varphi(f(Q))$, hence $l_\varphi(f(P)) \cap f(\mathbf{S_2}) = f(l_\psi(P))$.

Further we will suppose that $P \notin l_\psi(Q)$. Since $P \in p_{\psi,\bar\psi}(Q)$, then $l_\psi(P) \subseteq p_{\psi,\bar\psi}(Q)$. Let $C \in p_{\psi,\bar\psi}(Q)$ be such that $C \in l_\psi(P) \cap l_{\bar\psi}(Q)$. Since $P \notin l_\psi(Q)$, then $C \neq Q$. Further, $l_\psi(C) = l_\psi(P)$ and we obtain from (3.4) that $f(C) \in l_{\bar\varphi}(f(Q))$, hence $f(C) \in p_{\varphi,\bar\varphi}(f(Q))$ and also $l_\varphi(f(C)) \subseteq p_{\varphi,\bar\varphi}(f(Q))$. Thus, we obtain from (3.5) that

$$l_\varphi(f(C)) \cap f(\mathbf{S_2}) \subseteq p_{\varphi,\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}) \subseteq f(p_{\psi,\bar\psi}(Q)). \tag{3.7}$$

By Lemma 3.6, there exists $\gamma \in [0, 2\pi)$ such that

$$l_\varphi(f(C)) \cap f(\mathbf{S_2}) = f(l_\gamma(C)). \tag{3.8}$$

Then, by (3.7), $l_\gamma(C) \subseteq p_{\psi,\bar\psi}(Q)$ and either $\gamma = \psi$ or $\gamma = \bar\psi$. Since $f(Q) \in l_{\bar\varphi}(f(C))$ and $C \neq Q$, then $f(Q) \notin l_\varphi(f(C))$, hence, by (3.8), $Q \notin l_\gamma(C)$. This together with $C \in l_{\bar\psi}(Q)$ implies that $\gamma \neq \bar\psi$ and hence $\gamma = \psi$. Then, by (3.8),

$$l_\varphi(f(C)) \cap f(\mathbf{S_2}) = f(l_\psi(C)) = f(l_\psi(P)).$$

Finally,

$$f(P) \in f(l_\psi(P)) = l_\varphi(f(C)) \cap f(\mathbf{S_2}) \subseteq p_{\varphi,\bar\varphi}(f(Q)) \cap f(\mathbf{S_2})$$

and from this further $l_\varphi(f(P)) = l_\varphi(f(C))$, hence the first equality in (3.6) holds. The proof of the second equality in (3.6) is analogical. □

Lemma 3.9. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \bar\varphi \in [0, 2\pi)$, $\varphi \neq \bar\varphi$ and $Q \in \mathbf{S_2}$. Then there exist $\psi, \bar\psi, \theta, \bar\theta \in [0, 2\pi)$ such that*

$$\begin{aligned}
&l_\varphi(f(Q)) \cap f(\mathbf{S_2}) = f(l_\psi(Q)), \quad l_{\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}) = f(l_{\bar\psi}(Q)),\\
&p_{\varphi,\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}) = f(p_{\psi,\bar\psi}(Q)),\\
&l_\theta(f(Q)) \cap f(\mathbf{S_2}) = f(l_\varphi(Q)), \quad l_{\bar\theta}(f(Q)) \cap f(\mathbf{S_2}) = f(l_{\bar\varphi}(Q)),\\
&p_{\theta,\bar\theta}(f(Q)) \cap f(\mathbf{S_2}) = f(p_{\varphi,\bar\varphi}(Q)).
\end{aligned}$$

Proof. We have by Lemma 3.7 that there exist $\psi, \bar\psi \in [0, 2\pi)$ such that

$$\begin{aligned}
&l_\varphi(f(Q)) \cap f(\mathbf{S_2}) = f(l_\psi(Q)), \quad l_{\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}) = f(l_{\bar\psi}(Q)),\\
&p_{\varphi,\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}) \subseteq f(p_{\psi,\bar\psi}(Q)).
\end{aligned}$$

Then also

$$f(p_{\psi,\bar\psi}(Q)) \subseteq p_{\varphi,\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}),$$

which follows from Lemma 3.8. Now we have by Lemma 3.6 that there exist $\theta, \bar\theta \in [0, 2\pi)$ such that

$$l_\theta(f(Q)) \cap f(\mathbf{S_2}) = f(l_\varphi(Q)), \quad l_{\bar\theta}(f(Q)) \cap f(\mathbf{S_2}) = f(l_{\bar\varphi}(Q)).$$

From the first part of Lemma 3.9 we have that for these $\theta, \bar\theta$ there exist $\omega, \bar\omega$ such that

$$\begin{aligned}
&l_\theta(f(Q)) \cap f(\mathbf{S_2}) = f(l_\omega(Q)),\ l_{\bar\theta}(f(Q)) \cap f(\mathbf{S_2}) = f(l_{\bar\omega}(Q)),\\
&p_{\theta,\bar\theta}(f(Q)) \cap f(\mathbf{S_2}) = f(p_{\omega,\bar\omega}(Q)).
\end{aligned}$$

Thus, we get that $\omega = \varphi$, $\bar\omega = \bar\varphi$ and $p_{\theta,\bar\theta}(f(Q)) \cap f(\mathbf{S_2}) = f(p_{\varphi,\bar\varphi}(Q))$. □

3.2. PARAMETERS φ, ψ, f_1, f_2

Now we introduce four real (nonnegative) parameters that are assigned to each continuous function f with property (1.1) on $\mathbf{S_2}$. These we denote as φ, ψ and f_1, f_2. The geometric meaning of these parameters is as follows. The parameter φ is the azimuthal angle of the point $f(I) - f(0)$. The parameters f_1 and f_2 are the first two coordinates of the point $f(I)$ in the coordinate system with the center $f(0)$ and axes $U(\varphi)$, $U(\varphi+\pi)$ and $U(\varphi+\frac{\pi}{2}) - I$. The parameter ψ is the azimuthal angle of the preimage of $l_\varphi(f(0)) \cap f(\mathbf{S_2})$.

In this subsection we show, that the image of any vertical plane or a $\frac{\pi}{4}$-line with azimuthal angle ψ is a subset of a vertical plane or a $\frac{\pi}{4}$-line with azimuthal angle φ. Further, the image of any vertical line is parallel with $f(I) - f(0)$. We use this to show that if a point is shifted by the vector with coordinates $(x, y, 0)$ in the coordinate system with the center 0 and axes $U(\psi)$, $U(\psi+\pi)$ and $U\left(\psi+\frac{\pi}{2}\right) - I$, then its image is shifted by the vector with coordinates $(2xf_1, 2yf_2, 0)$ in the coordinate system with the center 0 and axes $U(\varphi)$, $U(\varphi+\pi)$ and $U(\varphi+\frac{\pi}{2}) - I$. And further, if a point is shifted by the vector $(0,0,z)$, then its image is shifted by the vector $\pm(0,0,2z\sqrt{f_1 f_2})$ in the same coordinate systems. This together then gives us the desired form of f.

Lemma 3.10. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Then there exist $\varphi, \psi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$, $f_1 > 0$, $f_2 > 0$ such that*

$$\begin{aligned} &f(I) - f(0) = f_1 U(\varphi) + f_2 U(\varphi+\pi), \\ &l_\varphi(f(0)) \cap f(\mathbf{S_2}) = f(l_\psi(0)), \quad l_{\varphi+\pi}(f(0)) \cap f(\mathbf{S_2}) = f(l_{\psi+\pi}(0)), \\ &p_{\varphi,\varphi+\pi}(f(0)) \cap f(\mathbf{S_2}) = f(p_{\psi,\psi+\pi}(0)). \end{aligned} \tag{3.9}$$

Proof. We have $0 < I$, hence $f(0) < f(I)$. By Lemma 2.9, there exist $\varphi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$, $f_1 > 0, f_2 > 0$ such that the first equation from 3.9 holds.

By Lemma 3.9, there exist $\psi, \theta \in [0, 2\pi)$ such that

$$\begin{aligned} &l_\varphi(f(0)) \cap f(\mathbf{S_2}) = f(l_\psi(0)),\ l_{\varphi+\pi}(f(0)) \cap f(\mathbf{S_2}) = f(l_\theta(0)), \\ &p_{\varphi,\varphi+\pi}(f(0)) \cap f(\mathbf{S_2}) = f(p_{\psi,\theta}(0)). \end{aligned}$$

The first equation from (3.9) implies that $f(I) \in p_{\varphi,\varphi+\pi}(f(0))$, hence $I \in p_{\psi,\theta}(0)$ and there exist s, r such that $I = 0 + sU(\psi) + rU(\theta)$. We get, by Lemma 2.12, that $s = r$ and either $\theta = \psi + \pi$ or $\theta = \psi - \pi$. Hence, $p_{\psi,\theta}(0) = p_{\psi,\psi+\pi}(0)$, $l_\theta(0) = l_{\psi+\pi}(0)$ and all relations in (3.9) hold. $\square$

Lemma 3.11. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \psi \in [0, 2\pi)$ be such that relations* (3.9) *from Lemma 3.10 hold and $Q \in \mathbf{S_2}$. Then*

$$\begin{aligned} &f(l_\psi(Q)) = l_\varphi(f(Q)) \cap f(\mathbf{S_2}), \quad f(l_{\psi+\pi}(Q)) = l_{\varphi+\pi}(f(Q)) \cap f(\mathbf{S_2}), \\ &f(p_{\psi,\psi+\pi}(Q)) = p_{\varphi,\varphi+\pi}(f(Q)) \cap f(\mathbf{S_2}). \end{aligned} \tag{3.10}$$

Proof. First suppose that $Q \in p_{\psi,\psi+\pi}(0)$. Then $p_{\psi,\psi+\pi}(Q) = p_{\psi,\psi+\pi}(0)$, and by (3.9) and Lemma 3.8, $f(Q) \in p_{\varphi,\varphi+\pi}(f(0))$, $p_{\varphi,\varphi+\pi}(f(Q)) = p_{\varphi,\varphi+\pi}(f(0))$, and relations (3.9) imply relations (3.10).

Now let $Q \notin p_{\psi,\psi+\pi}(0)$. By Lemma 3.9, there exist $\omega, \theta \in [0, 2\pi)$ such that

$$\begin{aligned} &f(l_\omega(Q)) = l_\varphi(f(Q) \cap f(\mathbf{S_2}), \quad f(l_\theta(Q)) = l_{\varphi+\pi}(f(Q)) \cap f(\mathbf{S_2}), \\ &f(p_{\omega,\theta}(Q)) = p_{\varphi,\varphi+\pi}(f(Q)) \cap f(\mathbf{S_2}). \end{aligned} \tag{3.11}$$

If $Q \notin p_{\psi,\psi+\pi}(0)$, then $f(Q) \notin p_{\varphi,\varphi+\pi}(f(0))$ by (3.9), and

$$p_{\varphi,\varphi+\pi}(f(0)) \cap p_{\varphi,\varphi+\pi}(f(Q)) = \emptyset.$$

This implies that also $p_{\psi,\psi+\pi}(0) \cap p_{\omega,\theta}(Q) = \emptyset$, hence $p_{\omega,\theta}(Q) = p_{\psi,\psi+\pi}(Q)$ and this further implies that either $\omega = \psi$, $\theta = \psi + \pi$ or $\omega = \psi + \pi$, $\theta = \psi$. That the first case is true follows from (3.9) and the continuity of the function f. Hence, relations (3.11) are equivalent with relations (3.10). □

Lemma 3.12. *Let* $f\colon \mathbf{S_2} \to \mathbf{S_2}$ *be continuous and have property* (1.1) *on* $\mathbf{S_2}$. *Let* $P, Q \in \mathbf{S_2}$, $P \neq Q$ *and denote*

$$l_1 = \{Q + z(P - Q) : z \in \mathbb{R}\}, \quad l_2 = \{f(Q) + z(f(P) - f(Q)) : z \in \mathbb{R}\}.$$

Then $f(l_1) = l_2 \cap f(\mathbf{S_2})$.

Proof. From Lemma 2.16, we have that there exist $\varphi, \bar\varphi, \psi, \bar\psi \in [0, 2\pi)$, all different, and $a, b, c, d \in \mathbb{R}$ such that

$$f(P) - f(Q) = aU(\varphi) + bU(\bar\varphi) = cU(\psi) + dU(\bar\psi),$$

hence

$$f(P) \in p_{\varphi,\bar\varphi}(f(Q)), \quad f(P) \in p_{\psi,\bar\psi}(f(Q)).$$

Further, by Lemma 3.9, there exist $\theta, \bar\theta, \omega, \bar\omega \in [0, 2\pi)$ such that

$$\begin{aligned} p_{\varphi,\bar\varphi}(f(Q)) \cap f(\mathbf{S_2}) &= f(p_{\theta,\bar\theta}(Q)), \\ p_{\psi,\bar\psi}(f(Q)) \cap f(\mathbf{S_2}) &= f(p_{\omega,\bar\omega}(Q)), \end{aligned}$$

and $\theta, \bar\theta, \omega, \bar\omega$ are all different. Then, we have that for all $z \in \mathbb{R}$ holds

$$\begin{aligned} P &\in p_{\theta,\bar\theta}(Q) \cap p_{\omega,\bar\omega}(Q), \\ Q + z(P - Q) &\in p_{\theta,\bar\theta}(Q) \cap p_{\omega,\bar\omega}(Q), \\ f(Q + z(P - Q)) &\in p_{\varphi,\bar\varphi}(f(Q)) \cap p_{\psi,\bar\psi}(f(Q)) \cap f(\mathbf{S_2}). \end{aligned}$$

We have that $l_1 \subseteq p_{\theta,\bar\theta}(Q) \cap p_{\omega,\bar\omega}(Q)$, and now we show the opposite inclusion. Let $A \in \mathbf{S_2}$ be such that $A \in p_{\theta,\bar\theta}(Q) \cap p_{\omega,\bar\omega}(Q)$. Since also $P \in p_{\theta,\bar\theta}(Q) \cap p_{\omega,\bar\omega}(Q)$, there exist r_1, s_1, v_1, w_1 and r_2, s_2, v_2, w_2, such that

$$\begin{aligned} A &= Q + r_1U(\theta) + s_1U(\bar\theta) = Q + v_1U(\omega) + w_1U(\bar\omega), \\ P &= Q + r_2U(\theta) + s_2U(\bar\theta) = Q + v_2U(\omega) + w_2U(\bar\omega), \end{aligned}$$

and we get by Lemma 2.21 that there exists a t such that $r_1 = ts_1$, $r_2 = ts_2$. Since $P \neq Q$, we have $s_2 \neq 0$, and we get from the above equations that

$$\begin{aligned} s_2(A - Q) &= s_1 s_2[tU(\theta) + (\bar\theta)] = s_1(P - Q), \\ A &= Q + \frac{s_1}{s_2}(P - Q), \end{aligned}$$

hence $A \in l_1$ and $l_1 = p_{\theta,\bar\theta}(Q) \cap p_{\omega,\bar\omega}(Q)$.

Since

$$l_2 \subseteq p_{\varphi,\bar\varphi}(f(Q)) \cap p_{\psi,\bar\psi}(f(Q)),$$

then

$$l_2 \cap f(\mathbf{S_2}) \subseteq f(p_{\theta,\bar\theta}(Q) \cap p_{\omega,\bar\omega}(Q)),$$

which is equivalent with $l_2 \cap f(\mathbf{S_2}) \subseteq f(l_1)$.

Now we show the opposite inclusion. Since

$$f(Q + z(P - Q)) \in p_{\varphi,\bar\varphi}(f(Q)) \cap p_{\psi,\bar\psi}(f(Q)),$$

there exist real continuous functions r, s, v, w such that

$$f(Q + z(P - Q)) = f(Q) + r(z)U(\varphi) + s(z)U(\bar\varphi) = f(Q) + v(z)U(\psi) + w(z)U(\bar\psi).$$

Further, we get by Lemma 2.21 that there exists a t such that $r(z) = ts(z)$ for all $z \in \mathbb{R}$ and since $P \neq Q$ and f is injective, we have $s(1) \neq 0$, hence

$$f(Q + z(P - Q)) = f(Q) + s(z)(tU(\varphi) + U(\bar\varphi) = f(Q) + \frac{s(z)}{s(1)}(f(P) - f(Q)).$$

This implies the inclusion $f(l_1) \subseteq l_2 \cap f(\mathbf{S_2})$. □

Lemma 3.13. *Let $f \colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \psi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$ be such that relations* (3.9) *from Lemma 3.10 hold. Let $Q \in \mathbf{S_2}$. Then for all $z \in \mathbb{R}$ there exists $s \in \mathbb{R}$ such that*

$$f(Q + zI) = f(Q) + s[f(I) - f(0)]. \tag{3.12}$$

Proof. We have from continuity of f, that there exists $s_0 \in \mathbb{R}$, $s_0 \neq 0$ such that $f(Q) + s_0[f(I) - f(0)] \in f(\mathbf{S_2})$. Let denote

$$\begin{aligned} l_1 &= \{sI : s \in \mathbb{R}\}, \\ l_2 &= \{f(0) + s[f(I) - f(0)] : s \in \mathbb{R}\}, \\ k_1 &= \{Q + s[f^{-1}(f(Q) + s_0[f(I) - f(0)]) - Q] : s \in \mathbb{R}\}, \\ k_2 &= \{f(Q) + s[f(I) - f(0)] : s \in \mathbb{R}\}. \end{aligned}$$

From Lemma 3.12, we have that $f(l_1) = l_2 \cap f(\mathbf{S_2})$ and $f(k_1) = k_2 \cap f(\mathbf{S_2})$.

If the intersection $l_2 \cap k_2$ is nonempty, then there exist $A \in \mathbf{S_2}$, $s_1, s_2 \in \mathbb{R}$ such that

$$A = f(0) + s_1[f(I) - f(0)] = f(Q) + s_2[f(I) - f(0)],$$

hence

$$f(Q) = f(0) + (s_1 - s_2)[f(I) - f(0)] \in l_2$$

and thus $Q \in l_1$. For such Q, relation (3.12) holds by Lemma 3.12.

Now suppose that $l_2 \cap k_2 = \emptyset$. Then also $f^{-1}(l_2 \cap k_2 \cap f(\mathbf{S_2})) = \emptyset$, and we get that $l_1 \cap k_1 = \emptyset$.

By Lemma 2.23, there exist α, β such that $f(Q), f(I) \in p_{\alpha,\beta}(f(0))$. Then we have that $k_2 \subseteq p_{\alpha,\beta}(f(0))$. Further we have by Lemma 3.9 that there exist γ, δ such that $p_{\alpha,\beta}(f(0)) \cap f(\mathbf{S_2}) = f(p_{\gamma,\delta}(0))$. Since $Q, I \in p_{\gamma,\delta}(0)$, we also have that $l_1, k_1 \subseteq p_{\gamma,\delta}(0)$. This together with $l_1 \cap k_1 = \emptyset$ implies that $k_1 = \{Q + sI, s \in \mathbb{R}\}$, and since $f(k_1) \subseteq k_2$, the statement of the lemma is proven. □

Lemma 3.14. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \psi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$ be such that relations* (3.9) *from Lemma 3.10 hold. Then for all $Q \in \mathbf{S_2}$ there exists $s_Q \in \mathbb{R}$ such that*

$$f(Q + xU(\psi) + yU(\psi + \pi)) = f(Q) + 2xs_Q f_1 U(\varphi) + 2ys_Q f_2 U(\varphi + \pi) \tag{3.13}$$

holds for any $x, y \in \mathbb{R}$.

Proof. Let $Q \in \mathbf{S_2}$. Let $u, v \in \mathbb{R}$. We take $A, B, C, D, E \in \mathbf{S_2}$ as

$$\begin{aligned}
A &= Q + vI = Q + \tfrac{v}{2}\left[U(\psi) + U(\psi + \pi)\right],\\
B &= Q + (u-1)vI = Q + \tfrac{(u-1)v}{2}\left[U(\psi) + U(\psi + \pi)\right],\\
C &= Q + uvI = Q + \tfrac{uv}{2}\left[U(\psi) + U(\psi + \pi)\right],\\
D &= Q + \tfrac{(u-1)v}{2}U(\psi) = B - \tfrac{(u-1)v}{2}U(\psi + \pi),\\
E &= A + \tfrac{(u-1)v}{2}U(\psi) = C - \tfrac{(u-1)v}{2}U(\psi + \pi) = D + vI\\
&= Q + \tfrac{uv}{2}U(\psi) + \tfrac{v}{2}U(\psi + \pi).
\end{aligned} \tag{3.14}$$

By Lemma 3.11, Lemma 3.13 and (3.9), there exist $t_1, t_2, \ldots, t_8 \in \mathbb{R}$ such that

$$f(A) = f(Q) + t_1[f(I) - f(0)] = f(Q) + t_1(f_1U(\varphi) + f_2U(\varphi + \pi)), \tag{3.15}$$

$$f(B) = f(Q) + t_2[f(I) - f(0)] = f(Q) + t_2(f_1U(\varphi) + f_2U(\varphi + \pi)),$$

$$f(C) = f(Q) + t_3[f(I) - f(0)] = f(Q) + t_3(f_1U(\varphi) + f_2U(\varphi + \pi)), \tag{3.16}$$

$$f(D) = f(Q) + t_4U(\varphi), \tag{3.17}$$

$$\begin{aligned} f(D) &= f(B) + t_5U(\varphi + \pi)\\ &= f(Q) + t_5U(\varphi + \pi) + t_2(f_1U(\varphi) + f_2U(\varphi + \pi)), \end{aligned} \tag{3.18}$$

$$f(E) = f(A) + t_6U(\varphi) = f(Q) + t_6U(\varphi) + t_1(f_1U(\varphi) + f_2U(\varphi + \pi)), \tag{3.19}$$

$$\begin{aligned} f(E) &= f(C) + t_7U(\varphi + \pi)\\ &= f(Q) + t_7U(\varphi + \pi) + t_3(f_1U(\varphi) + f_2U(\varphi + \pi)), \end{aligned} \tag{3.20}$$

$$\begin{aligned} f(E) &= f(D) + t_8[f(I) - f(0)]\\ &= f(Q) + t_4U(\varphi) + t_8(f_1U(\varphi) + f_2U(\varphi + \pi)). \end{aligned} \tag{3.21}$$

Equations (3.17), (3.18) imply $t_4 = f_1 t_2$, equations (3.21), (3.20) imply $t_4 + f_1 t_8 = f_1 t_3$, and equations (3.21), (3.19) imply $f_2 t_8 = f_2 t_1$. Since $f_1 > 0, f_2 > 0$, this further implies $t_1 + t_2 = t_3$, hence

$$\begin{aligned} f(C) - f(Q) &= [f(A) - f(Q)] + [f(B) - f(Q)]\,, \\ f(Q + uvI) - f(Q) &= [f(Q + vI) - f(Q)] + [f(Q + (u-1)vI) - f(Q)]\,. \end{aligned}$$

This holds for all $u, v \in \mathbb{R}$. Now we can prove by induction that

$$\begin{aligned} f(Q + uvI) - f(Q) &= u\,[f(Q + vI) - f(Q)]\,, \quad u = 0, 1, 2, \ldots, v \in \mathbb{R}, \\ f(Q - uvI) - f(Q) &= -u\,[f(Q + vI) - f(Q)]\,, \quad u = 0, 1, 2, \ldots, v \in \mathbb{R}. \end{aligned}$$

Together we have

$$f(Q + uvI) - f(Q) = u\,[f(Q + vI) - f(Q)]\,, \quad u \in \mathbb{Z}, v \in \mathbb{R}. \tag{3.22}$$

When we put $v = \frac{1}{u}$, we get

$$f(Q + \tfrac{1}{u} I) - f(Q) = \tfrac{1}{u}\,[f(Q + I) - f(Q)]\,, \quad u \in \mathbb{Z} \setminus \{0\}. \tag{3.23}$$

From (3.22), (3.23) we further get that if $z = \frac{p}{q}$, $p, q \in \mathbb{Z}$, then

$$f(Q + zI) - f(Q) = p\left[f(Q + \tfrac{1}{q} I) - f(Q)\right] = z\,[f(Q + I) - f(Q)]\,.$$

And we get from the continuity of f that

$$f(Q + zI) - f(Q) = z\,[f(Q + I) - f(Q)] = z s_Q\,[f(I) - f(0)]\,, \quad z \in \mathbb{R}, \tag{3.24}$$

where $s_Q \in \mathbb{R}$ is the parameter s from Lemma 3.13 with $z = 1$.

Now let again $u, v \in \mathbb{R}$ and take A, B, C, D, E as in (3.14) and let $x = \frac{uv}{2}, y = \frac{v}{2}$. From (3.15), (3.16), (3.24) we have that

$$\begin{aligned} f(Q + 2xI) &= f(Q) + t_3[f(I) - f(0)] = f(Q) + 2x s_Q\,[f(I) - f(0)]\,, \\ f(Q + 2yI) &= f(Q) + t_1[f(I) - f(0)] = f(Q) + 2y s_Q\,[f(I) - f(0)]\,, \end{aligned}$$

and we get that $t_1 = 2y s_Q, t_3 = 2x s_Q$.

Now from (3.20), (3.19), we have that

$$f(E) = f(Q) + f_1 t_3 U(\varphi) + f_2 t_1 U(\varphi + \pi)),$$

and from (3.14) is $E = Q + xU(\psi) + yU(\psi + \pi)$, hence we get

$$f(Q + xU(\psi) + yU(\psi + \pi)) = f(Q) + 2 f_1 x s_Q U(\varphi) + 2 f_2 y s_Q U(\varphi + \pi)). \tag{3.25}$$

This relation holds for all $x, y \in \mathbb{R}$ such that $x = \frac{uv}{2}, y = \frac{v}{2}$ for some u, v. These are all $x, y \in \mathbb{R}$ except of $x \neq 0, y = 0$. If we take A, B, C, D, E as in (3.14), but we exchange $U(\psi)$ and $U(\psi + \pi)$, we get that relation (3.25) holds for all $x, y \in \mathbb{R}$ except of $y \neq 0, x = 0$. Together we get that it holds for all $x, y \in \mathbb{R}$. □

Lemma 3.15. *Let $f : \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \psi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$ be such that relations* (3.9) *from Lemma 3.10 hold. Let $Q \in \mathbf{S_2}$ and $x, y \in \mathbb{R}$. Then*

$$f(Q + xU(\psi) + yU(\psi + \pi)) = f(Q) + 2xf_1U(\varphi) + 2yf_2U(\varphi + \pi). \tag{3.26}$$

Proof. We show that for all $Q \in \mathbf{S_2}$, relation (3.13) from Lemma 3.13 holds with $s_Q = 1$. Since $f(I) = f(0) + [f(I) - f(0)]$, we have that $s_0 = 1$. Now let $Q \in \mathbf{S_2}$. By Lemma 2.9, there exist $\alpha \in [0, 2\pi)$, $a, b \in \mathbb{R}$ such that

$$\begin{aligned} Q &= aU(\alpha) + bU(\alpha + \pi) = (a - b)U(\alpha) + 2bI \in l_\alpha(2bI) \subseteq p_{\alpha,\alpha+\pi}(I), \\ Q + 2I &= (a - b)U(\alpha) + 2(b + 1)I \in l_\alpha(2(b + 1)I) \subseteq p_{\alpha,\alpha+\pi}(I). \end{aligned} \tag{3.27}$$

By Lemma 3.9 and Lemma 3.8, there exist $\beta, \gamma \in [0, 2\pi)$ such that

$$f(p_{\alpha,\alpha+\pi}(I)) \subseteq p_{\beta,\gamma}(f(I)) \quad \text{and} \quad f(l_\alpha(A)) \subseteq l_\beta(f(A))$$

for all $A \in p_{\alpha,\alpha+\pi}(I)$. Then we get from (3.27) that there exist $c, d \in \mathbb{R}$ such that

$$\begin{aligned} f(Q) \in l_\beta(f(2bI)), &\quad f(Q) = f(2bI) + cU(\beta), \\ f(Q + 2I) \in l_\beta(f(2(b + 1)I)), &\quad f(Q + 2I) = f(2(b + 1)I) + dU(\beta). \end{aligned}$$

Now we use (3.13) and get from these equations that

$$\begin{aligned} f(Q) &= f(0) + 2bf_1U(\varphi) + 2bf_2U(\varphi + \pi) + cU(\beta), \\ f(Q + 2I) &= f(0) + 2(b + 1)f_1U(\varphi) + 2(b + 1)f_2U(\varphi + \pi) + dU(\beta). \end{aligned}$$

and also

$$f(Q + 2I) = f(Q) + 2s_Qf_1U(\varphi) + 2s_Qf_2U(\varphi + \pi).$$

Together we get that

$$\begin{aligned} f(Q + 2I) - f(Q) &= 2f_1U(\varphi) + 2f_2U(\varphi + \pi) + (d - c)U(\beta) \\ &= 2s_Qf_1U(\varphi) + 2s_Qf_2U(\varphi + \pi), \end{aligned}$$

$$2(1 - s_Q)\,[f_1U(\varphi) + f_2U(\varphi + \pi)] = (c - d)U(\beta),$$

and since $f_1 > 0, f_2 > 0$, we get by Lemma 2.11 that $1 - s_Q = 0$. □

Lemma 3.16. *Let $f \colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \psi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$ be such that relations* (3.9) *from Lemma 3.10 hold. Let $Q \in \mathbf{S_2}$. Then*

$$f(p_{\psi,\psi+\pi}(Q)) = p_{\varphi,\varphi+\pi}(f(Q)). \tag{3.28}$$

Proof. By Lemma 3.11, $f(p_{\psi,\psi+\pi}(Q)) = p_{\varphi,\varphi+\pi}(f(Q)) \cap f(\mathbf{S_2})$, hence we have that $f(p_{\psi,\psi+\pi}(Q)) \subseteq p_{\varphi,\varphi+\pi}(f(Q))$. It remains to show the opposite inclusion. Let $A \in p_{\varphi,\varphi+\pi}(f(Q))$. Then there exist $a, b \in \mathbb{R}$ such that

$$A = f(Q) + aU(\varphi) + bU(\varphi + \pi).$$

Since $f_1 f_2 > 0$, there exist $x, y \in \mathbb{R}$ such that $x = \frac{a}{2f_1}$, $y = \frac{b}{2f_1}$. Then

$$A = f(Q) + 2xf_1 U(\varphi) + 2yf_2 U(\varphi + \pi).$$

And, by Lemma 3.15, we further get that $f(Q + xU(\psi) + yU(\psi + \pi)) = A$, hence $A \in f(p_{\psi,\psi+\pi}(Q))$. □

Lemma 3.17. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \psi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$ be such that relations* (3.9) *from Lemma 3.10 hold. Let $Q \in \mathbf{S_2}$ and $z \in \mathbb{R}$. Then*

$$f(Q + z\left(U\left(\psi + \tfrac{\pi}{2}\right) - I\right)) = f(Q) + 2\sigma z\sqrt{f_1 f_2}\left[U(\varphi + \tfrac{\pi}{2}) - I\right],$$

where $\sigma = 1$ or $\sigma = -1$.

Proof. Denote

$$P = Q + z\left(U\left(\psi + \tfrac{\pi}{2}\right) - I\right).$$

We may assume that $z > 0$. Let $V, W \in \mathbf{S_2}$ be such that $V = Q - zI$, $W = Q + zI$. Then

$$W = V + zU(\psi) + zU(\psi + \pi),$$

and further, we have by Lemma 2.20 that

$$p_{\psi,\psi+\pi}(P) \cap K_0(V) \cap K_0(W) = \{P\}.$$

By Lemmas 3.4 and 3.16, the images of these sets are $f(K_0(V)) = K_0(f(V)) \cap f(\mathbf{S_2})$, $f(K_0(W)) = K_0(f(W)) \cap f(\mathbf{S_2})$, $f(p_{\psi,\psi+\pi}(P)) = p_{\varphi,\varphi+\pi}(f(P))$, hence

$$\begin{aligned}\{f(P)\} &= f(p_{\psi,\psi+\pi}(P)) \cap f(K_0(V)) \cap f(K_0(W))\\ &= p_{\varphi,\varphi+\pi}(f(P)) \cap K_0(f(V)) \cap K_0(f(W)).\end{aligned}$$

Further, by Lemma 3.15, we have that

$$\begin{aligned}f(V) &= f(Q - zI) = f\left(Q - \tfrac{z}{2}U(\psi) - \tfrac{z}{2}U(\psi + \pi)\right)\\ &= f(Q) - zf_1 U(\varphi) - zf_2 U(\varphi + \pi),\\ f(W) &= f(Q + zI) = f(Q) + zf_1 U(\varphi) + zf_2 U(\varphi + \pi)\\ &= f(V) + 2zf_1 U(\varphi) + 2zf_2 U(\varphi + \pi).\end{aligned}$$

Now, by Lemma 2.19, we get that $f(P)$ is of the form

$$\begin{aligned}f(P) &= \tfrac{f(V)+f(W)}{2} \pm 2z\sqrt{f_1 f_2}\left[U(\varphi + \tfrac{\pi}{2}) - I\right]\\ &= f(Q) \pm 2z\sqrt{f_1 f_2}\left[U(\varphi + \tfrac{\pi}{2}) - I\right].\end{aligned}$$

This completes the proof. □

Now, by Lemma 3.15 and Lemma 3.17, we finally get the relation for the image of any $Q \in \mathbf{S_2}$.

Lemma 3.18. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \psi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$ be such that relations* (3.9) *from Lemma 3.10 hold. Let $Q \in \mathbf{S_2}$ and $x, y, z \in \mathbb{R}$ be such that*

$$Q = xU(\psi) + yU(\psi + \pi) + z\left(U\left(\psi + \tfrac{\pi}{2}\right) - I\right).$$

Then

$$f(Q) = f(0) + 2xf_1U(\varphi) + 2yf_2U(\varphi + \pi) + 2\sigma z\sqrt{f_1f_2}\left(U(\varphi + \tfrac{\pi}{2}) - I\right),$$

where $\sigma = 1$ or $\sigma = -1$.

Proof. The relation follows from Lemma 3.15 and Lemma 3.17. □

Now we write the relation from the previous lemma in the matrix form.

Lemma 3.19. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be continuous and have property* (1.1) *on $\mathbf{S_2}$. Let $\varphi, \psi \in [0, 2\pi)$ and $f_1, f_2 \in \mathbb{R}$ be such that relations* (3.9) *from Lemma 3.10 hold. Let $Q \in \mathbf{S_2}$. Then $f(Q) = f(0) + LQL^T$, where*

$$L = \left[U\left(\tfrac{\varphi}{2}\right) - I\right]\left[\sqrt{2f_1} \quad 00 \quad \sigma\sqrt{2f_2}\right]\left[U\left(\tfrac{\psi}{2}\right) - I\right], \quad \sigma = 1 \text{ or } \sigma = -1.$$

Proof. Let $Q \in \mathbf{S_2}$. By Lemma 2.8, there exist $x, y, z \in \mathbb{R}$ such that

$$Q = xU(\psi) + yU(\psi + \pi) + z\left(U\left(\psi + \tfrac{\pi}{2}\right) - I\right).$$

By identities (2.6), it holds that

$$\left[U\left(\tfrac{\psi}{2}\right) - I\right] Q \left[U\left(\tfrac{\psi}{2}\right) - I\right] = \begin{bmatrix} 2x & -z \\ -z & 2y \end{bmatrix}.$$

Further, it follows from Lemma 3.18 and identities (2.6) that

$$\begin{aligned}
&\left[U\left(\tfrac{\varphi}{2}\right) - I\right] f(Q) \left[U\left(\tfrac{\varphi}{2}\right) - I\right] = \begin{bmatrix} 4xf_1 & 2\sigma z\sqrt{f_1f_2} \\ 2\sigma z\sqrt{f_1f_2} & 4yf_2 \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2f_1} & 0 \\ 0 & \sigma\sqrt{2f_2} \end{bmatrix} \begin{bmatrix} 2x & -z \\ -z & 2y \end{bmatrix} \begin{bmatrix} \sqrt{2f_1} & 0 \\ 0 & \sigma\sqrt{2f_2} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2f_1} & 0 \\ 0 & \sigma\sqrt{2f_2} \end{bmatrix} \left[U\left(\tfrac{\psi}{2}\right) - I\right] Q \left[U\left(\tfrac{\psi}{2}\right) - I\right] \begin{bmatrix} \sqrt{2f_1} & 0 \\ 0 & \sigma\sqrt{2f_2} \end{bmatrix},
\end{aligned}$$

and identity (2.3) implies the statement of the lemma. □

Hypothesis 1.4 with $n = 2$ is a direct consequence of Lemma 3.19.

Theorem 3.20. *Let $f\colon \mathbf{S_2} \to \mathbf{S_2}$ be a continuous function such that f has property* (1.1) *on the whole $\mathbf{S_2}$. Then f is of the form $f(Q) = K + L^TQL$, where K is a symmetric 2×2 matrix and L is an invertible 2×2 matrix.*

Proof. It follows from Lemma 3.19. □

4. CONCLUSION AND FURTHER QUESTIONS

We were able to answer our question for the special case and proved that the only continuous symmetric 2×2 matrix functions with the order preserving property (1.1) on the whole $\mathbf{S}$ are the functions of the form $f(Q) = K + L^T QL$. It is still not resolved, whether this also holds for $n > 2$, and for the functions that have property (1.1) only on a subset of $\mathbf{S}$.

An example of such functions is the type we get when we put $\mathcal{A} = 0$. If $\mathcal{A} = 0$ then $M^+ = \{Q \in \mathbf{S} \colon Q > 0\}$, $M^- = \{Q \in \mathbf{S} \colon Q < 0\}$ and further if $\mathcal{C}, \mathcal{D}$ are such that $S = \left[\begin{smallmatrix} 0 & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is symplectic, then

$$\mathcal{B}^T \mathcal{D} = \mathcal{D}^T \mathcal{B}, \quad \mathcal{C}^T \mathcal{B} = -I$$

and

$$(\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} = (-\mathcal{B}^{T-1} + \mathcal{D}Q)(\mathcal{B}Q)^{-1} = -\mathcal{B}^{T-1} Q^{-1} \mathcal{B}^{-1} + \mathcal{D}\mathcal{B}^{-1}.$$

The matrix $\mathcal{D}\mathcal{B}^{-1}$ is symmetric and the matrix $\mathcal{B}^{-1}$ is invertible. Proposition 1.2 becomes the following.

Proposition 4.1. *Let $f \colon \mathbf{S} \to \mathbf{S}$ be defined as $f(Q) = K - L^T Q^{-1} L$, where K is a symmetric $n \times n$ matrix and L is an invertible $n \times n$ matrix. Then f has property* (1.1) *on the set $M^+ = \{Q \in \mathbf{S} \colon Q > 0\}$ and on the set $M^- = \{Q \in \mathbf{S} \colon Q < 0\}$.*

The proof of this proposition is again simple. However, we do not yet know, how it is with the converse of this proposition, or even how exactly it should be formulated.

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