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POSITIVE SOLUTIONS OF NONPOSITONE SUBLINEAR ELLIPTIC PROBLEMS

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Abstract. Consider the problem $-\Delta u = \lambda f(\cdot, u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, where Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary when $n \geq 2$, $\lambda > 0$, and where $f \in C\left(\overline{\Omega} \times [0, \infty)\right)$ satisfies $\lim_{s\to\infty} s^{-p} f(\cdot, s) = \gamma$ for some $p \in (0, 1)$ and some $\gamma \in C(\overline{\Omega})$ such that $\gamma \not\equiv 0$ *a.e.* in Ω and, for some positive constants c and c', $\gamma^- \leq c d_\Omega^\beta$ for some $\beta \in \left(\frac{n-1}{n}, \infty\right)$ and $(-\Delta)^{-1}\gamma \geq c' d_\Omega$, where $d_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$ and $\gamma^- := -\min(0, \gamma)$. Under these assumptions we show that for λ large enough, the above problem has a positive weak solution $u \in C^1(\overline{\Omega})$ such that, for some constant $c'' > 0$, $u \geq c'' d_\Omega$ in Ω.

Keywords: elliptic sublinear problems, nonpositone problems, positive solutions, Leray–Schauder degree.

Mathematics Subject Classification: 35J25, 35A01, 35B09, 35J15.

1. INTRODUCTION

Let Ω be a regular enough bounded domain in $\mathbb{R}^n$, let $\lambda \in (0, \infty)$, and let $f \in C\left(\overline{\Omega} \times [0, \infty)\right)$. The existence of positive solutions of nonpositone elliptic problems of the form

$$\begin{cases} -\Delta u = \lambda f(\cdot, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

(i.e., in the case when f changes sign in $\overline{\Omega} \times [0, \infty)$) have received considerable interest in the literature and was studied in several works. Let us mention some of them.

Castro and Shivaji studied in [5] existence and uniqueness of classical solutions of problem (1.4) in the one-dimensional case when $\Omega = (0, 1)$, $m \equiv 1$ in Ω, $l \in (0, \infty)$, and with h such that $h(0) = 0$; and the same authors addressed in [6] the analogous n dimensional radial problem in the case when Ω is an open ball in $\mathbb{R}^n$ centered at the origin.

Castro, Garner, and Shivaji studied in [7] sublinear problems of the form

$$\begin{cases} -\Delta u = \lambda h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where Ω is a regular enough bounded domain in $\mathbb{R}^n$, $\lambda \in (0,\infty)$, and $h : [0,\infty) \to \mathbb{R}$ is a smooth function such that h' is bounded from below, and satisfies $h(0) < 0$, and $h(\sigma) > 0$ for some $\sigma > 0$. By using the sub-supersolutions method, and under some additional assumptions on h, they proved the existence of positive solutions of (1.2) for λ positive and large enough, and that the obtained solution $u = u_\lambda$ satisfies $\lim_{\lambda\to\infty} \sup_{x\in\Omega} |u_\lambda(x)| = \infty$.

Concerning superlinear nonpositone problems let us mention the following works. Ambrosetti, Arcoya, and Buffoni studied in [1], superlinear subcritical problems of the form (1.1) in the case when Ω is a smooth bounded domain in $\mathbb{R}^n$, $\lambda > 0$, and with $f \in C\left(\overline{\Omega} \times [0,\infty)\right)$ satisfying that $\max_{\overline{\Omega}} f(\cdot,0) < 0$ and that $\lim_{s\to\infty} s^{-p} f(\cdot,s) = \gamma$ uniformly on $\overline{\Omega}$ with $\gamma \in C(\overline{\Omega})$ such that $\min_{\overline{\Omega}} \gamma > 0$ and $p \in (1, 2^* - 1)$ (where $2^* := \frac{2n}{n-2}$ if $n \geq 3$ and $2^* := \infty$ if $n = 1$ or $n = 2$). Under these assumptions, by using bifurcation results and degree theoretic arguments, they proved, in Theorem 5 of [1], that there exists $\lambda^* > 0$ such that (1.1) has a positive solution for any $\lambda \in (0, \lambda^*)$, and that $\lambda = 0$ is a bifurcation from infinity for (1.1).

Berestycki, Capuzzo Dolcetta, and Nirenberg investigated in [3] the existence of positive solutions of superlinear subcritical problems of the form

$$\begin{cases} -\Delta u = \gamma h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded and smooth enough domain, with $\gamma \in C^2(\overline{\Omega})$ such that:

(B1)

$$\Omega^+ := \{x \in \Omega : \gamma(x) > 0\} \neq \varnothing, \quad \Omega^- := \{x \in \Omega : \gamma(x) < 0\} \neq \varnothing,$$
$$\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega \quad \text{and} \quad \nabla\gamma(x) \neq 0 \text{ for any } x \in \Gamma,$$

and with $h \in C^1([0,\infty))$ such that:

(B2) there exists $s_1 > 0$ with the property that $h(s) > 0$ for any $s > s_1$,
(B3) $h(0) = h'(0)$,
(B4) $\lim_{s\to\infty} s^{-p} h(s) = l$ for some $p \in (1, \frac{n+2}{n-1})$ and $l \in (0,\infty)$.

Under these assumptions [3] proved some Liouville type theorems which give, via a blow-up method inspired in [14], a priori estimates of the solutions of (1.3). From these estimates, and by using Leray–Schauder degree arguments, [3, Theorem 1] proves that (1.3) has at least a positive solution.

It is worth to mention that [3] covers also more general second order linear operators, and with more general boundary conditions allowed.

Costa, Ramos Quoirin, and Therani, addressed in [9] problems of the form

$$\begin{cases} -\Delta u = \lambda m(h(u) - l) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $m : \Omega \to \mathbb{R}$, $l \in (0, \infty)$, $h : [0, \infty) \to \mathbb{R}$, and $\lambda \in (0, \infty)$, with h superlinear, subcritical, and regularly varying at infinity, and with m allowed to change sign in Ω, and vanish on an open set in Ω. By using variational methods they proved the existence of positive solutions for λ positive and small enough.

Garcia-Melian, Iturriaga, and Ramos Quoirin studied in [13] problem (1.3) in the slightly superlinear case when h is regularly varying at infinity with index one at infinity, and $\gamma \in C(\overline{\Omega})$ is such that $\Omega^+ := \{x \in \Omega : \gamma(x) > 0\} \neq \varnothing$ and satisfies one of the following assumptions:

(G1) $\gamma > 0$ in $\overline{\Omega}$,
(G2) $\overline{\Omega^+} \subset \Omega$ and $\gamma > 0$ in a neighborhood of $\partial\Omega$,
(G3) $\partial\Omega^+$ is an n dimensional manifold of dimension $n - 1$ and, for some positive constants c_1 and c_2, $c_1 \leq \frac{d_G(x)}{\gamma(x)} \leq c_2$ on a neighborhood of $\partial\Omega$.

Under these hypotheses they proved, in [13, Theorem 1.1], that (1.3) has at least one positive solution.

Ma, Zhang, and Zhu studied in [22] problem (1.4) in the case when Ω is a regular bounded domain of $\mathbb{R}^N$ with $\mathbb{N} \geq 3$, λ and $l \in (0, \infty)$, $m \in C\left(\overline{\Omega}\right)$, $h : [0, \infty) \to [0, \infty)$ is superlinear at infinity and has regular variation at infinity of index p for some $p \in (1, \frac{N+2}{N-2})$. They assumed also that $h(0) = 0$, $h(s) > 0$ for any $s > 0$, and that $\lim_{s\to\infty} s^{-p} h(s) = \gamma$ for some $\gamma \in (0, \infty)$. In addition, there it was assumed also that m satisfies one of the following three conditions:

(M1) $m > 0$ in $\overline{\Omega}$,
(M2) $\overline{\Omega^+} \subset \Omega$, where $\Omega^+ := \{x \in \Omega : m(x) > 0\}$, and $m \geq 0$ in a neighborhood of $\partial\Omega^+$,
(M3) $\partial\Omega^+$ is a $n - 1$ dimensional C^1 manifold, $\partial\Omega^+ \subset \Omega$ and there exist positive constants q, c_1, and c_2, such that $c_1 \leq \left|d_\Omega^{-q} m\right| \leq c_2$ in a neighborhood of $\partial\Omega^+$.

Under these hypotheses, by using Leray–Schauder degree arguments and a priori estimates for the solutions of (1.4), they proved in [22, Theorem 1.2] that there exists $\lambda_0 > 0$ such that (1.4) has a positive solution for any $\lambda \in (0, \lambda_0)$.

Concerning sublinear nonpositone problems, Dancer and Shi studied in [11] problem (1.3) in the case when $\gamma = 1$, $\lambda > 0$, and Ω is a bounded and C^2 domain in $\mathbb{R}^n$ which satisfies an uniform inner ball condition. They proved, in Theorem 1.1 of [11], that if h satisfies:

(D1) $h \in C^{1,\alpha}([0, \infty))$ for some $\alpha \in (0, 1)$ and $h(0) < 0$,
(D2) for some $\beta > 0$ it holds that $h(s) > 0$ for any $s > \beta$ and $F(s) < 0$ whenever $s \in (0, \beta)$, where $F(s) := \int_0^s f$,
(D3) $\lim_{s\to\infty} s^{-1} h(s) = 0$,

(D4) $\liminf_{s\to\infty}(h(s)-sh'(s))>0$,
(D5) there exists $s_0>0$ such that $h'(s)>0$ for any $s>s_0$,

then (1.3) has a unique weak positive solution for λ positive and large enough. They also proved that if (D3) is replaced by

(D3') $\limsup_{s\to\infty}s^{-1}h(s)>0$,

(i.e., in the superlinear case of (1.3)) then (1.3) has no positive solutions for λ large enough.

Costa, Tehrani, and Yang considered in [10], via variational methods, problem (1.3) in the case when $h\in C([0,\infty),\mathbb{R})$ and $h(0)<0$. In the sublinear case (when $\lim_{s\to\infty}s^{-1}h(s)=0$) they proved the existence of positive solutions for λ large enough. They studied also some superlinear subcritical cases of (1.3), namely those where $\lim_{s\to\infty}s^{-p}h(s)=\infty$ and $|h(s)|\le C\left(1+s^{p-2}\right)$ for some $p\in(0,2^*-2)$ (with $2^*=\frac{2n}{n-2}$ when $n>2$ and $2^*=\infty$ when $n=1,2$), and for these superlinear problems they obtained positive solutions for λ positive and small enough.

Kaufmann and Ramos Quoirin addressed in [18] sublinear cases of problem (1.4). They assumed that $h\in C(\mathbb{R})$ satisfies $h(0)=0$, $\lim_{s\to\infty}s^{-p}h(s)=1$ for some $p\in(0,1)$, and that $m\in C\left(\overline{\Omega}\right)$ is such that m^- is sufficiently small with respect to m^+ in the sense that: There exists $\delta>0$ such that the problem

$$\begin{cases}-\Delta w=(1-\delta)m^+w^p-m^-w^p & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{1.5}$$

has a solution $w\in P^{\circ}_{C^1_{\mathcal{B}}(\overline{\Omega})}$, where

$$C^1_{\mathcal{B}}(\overline{\Omega}):=\left\{u\in C^1\overline{\Omega}:u=0\text{ on }\partial\Omega\right\}$$

and $P^{\circ}_{C^1_{\mathcal{B}}(\overline{\Omega})}$ denotes the interior of the positive cone in $C^1_{\mathcal{B}}(\overline{\Omega})$ and with, as usual, $m^+:=\max(m,0)$ and $m^-:=-\min(m,0)$. Under these assumptions, by using the sub-supersolutions method, they proved, in Theorem 1.2 of [18], that there exists $\lambda_0>0$ such that for any $\lambda\ge\lambda_0$ problem (1.4) has a solution $u=u_\lambda\in P^{\circ}_{C^1_{\mathcal{B}}(\overline{\Omega})}$, which, in addition, satisfies $\lim_{\lambda\to\infty}u_\lambda(x)=\infty$ any $x\in\Omega$.

On the other hand, Bachar, Mâagli, and Eltayeb studied in [2] the existence of nonnegative solutions of semipositone problems of the form

$$\begin{cases}-\Delta u=a+\lambda f(\cdot,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{1.6}$$

where Ω is a $C^{1,1}$ bounded domain in $\mathbb{R}^n$, $n\ge 2$, $a:\Omega\to\mathbb{R}$ is a nonnegative nonidentically zero function in a suitable Kato class $K(\Omega)$, $\lambda\in(0,\infty)$, and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is measurable and such that, for some nonnegative and nontrivial $q\in K(\Omega)$, it holds that

$$|f(x,s)-f(x,t)|\le q(x)|s-t|\text{ for a.e. }(x,s)\in\Omega\times(0,\infty),\ (x,t)\in\Omega\times(0,\infty).$$

Under these assumptions it was proved in Theorem 3.3 of [2] that there exists $\lambda^* > 0$ such that (1.6) has a nontrivial and nonnegative solution (in the distributional sense) for any $\lambda \in (0, \lambda^*)$. Estimates from above and from below for the found solution were also given there.

Additional references on semipositone problems and some comments on the state of the art for these problems, can be found in the excellent surveys [8, 21, 25], and [28].

Let us present our result. Our aim in this paper is to prove the following theorem:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary when $n \geq 2$, let $\lambda \in (0, \infty)$, and let $f : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$. Assume the following conditions:*

(H1) $f \in C\left(\overline{\Omega} \times [0, \infty)\right)$,

(H2) $\lim_{s\to\infty} s^{-p} f(\cdot, s) = \gamma$ *uniformly on Ω, for some $p \in (0, 1)$ and some $\gamma \in C\left(\overline{\Omega}\right)$ such that*

$$(-\Delta)^{-1}\gamma \geq c d_\Omega \text{ in } \Omega$$

for some positive constant c, and such that

$$\gamma^- \leq c' d_\Omega^\beta \tag{1.7}$$

with c' and β positive constants such that $\beta > \frac{n-1}{n}$.

Then exists $\lambda_0 > 0$ such that, for any $\lambda > \lambda_0$ the problem

$$\begin{cases} -\Delta u = \lambda f(\cdot, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.8}$$

has a weak solution $u \in C^1_{\mathcal{B}}(\overline{\Omega})$. Moreover, it satisfies $u \geq c d_\Omega$ in Ω, with c a positive constant c, and where $d_\Omega : \overline{\Omega} \to \mathbb{R}$ is the distance to the boundary function defined by

$$d_\Omega(x) := \operatorname{dist}(x, \partial\Omega). \tag{1.9}$$

We assume, from now on, that Ω and f, satisfy the assumptions of Theorem 1.1, and that $\lambda > 0$.

We stress that our nonlinearity f is rather general, in the sense that no conditions on the sign of f are imposed, except for those implied by (H2). Notice also that (1.7) is automatically fulfilled if γ is nonnegative in a neighborhood of $\partial\Omega$.

The paper is organized as follows. In Section 2 we introduce some notations and general facts we use.

In Section 3 we introduce the operator $T : C(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ defined by

$$T(u) := (-\Delta)^{-1}\left(\gamma\left(u^+\right)^p\right).$$

Lemma 3.8 shows that T is continuous and compact, and Lemma 3.9 says that $T : D \to C^1_{\mathcal{B}}(\overline{\Omega})$ is continuously Fréchet differentiable, where D is the open set in $C^1_{\mathcal{B}}\left(\overline{\Omega}\right)$ given by

$$D := \left\{u \in C^1_{\mathcal{B}}(\overline{\Omega}) : \inf_{\overline{\Omega}}\left(u d_\Omega^{-1}\right) > 0\right\},$$

and Remark 3.6 says that T has a unique fixed point $\boldsymbol{u}$ in $C^1_{\mathcal{B}}(\overline{\Omega})$ and that it belongs to D. By defining $\mathcal{O}$ as the open ball in $C^1_{\mathcal{B}}(\overline{\Omega})$ centered at $\boldsymbol{u}$ and with radius ρ, for a suitable carefully chosen $\rho > 0$, Lemma 3.18 shows that the Leray–Schauder degree $\deg_{LS}(I - T, \mathcal{O}, 0)$ is well defined and that $\deg_{LS}(I - T, \mathcal{O}, 0) = \pm 1$.

In Section 4 we prove Theorem 1.1. To do it we follow an approach inspired in [22], used there to study some semipositone superlinear subcritical problems. Following [22], for $\alpha \in (0, \infty)$ and $v \in \overline{\mathcal{O}}$, we define $T_\alpha(v)$ by

$$T_\alpha(v) := (-\Delta)^{-1}\left(\alpha^p f\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p\right),$$

and we show that there exists $\alpha_0 > 0$ such that, for any $\alpha \in (0, \alpha_0)$, the Leray–Schauder degree $\deg_{LS}(I - (T_\alpha + T), \mathcal{O}, 0)$ is well defined and satisfies

$$\deg_{LS}(I - (T_\alpha + T), \mathcal{O}, 0) = \deg_{LS}(I - T, \mathcal{O}, 0) = \pm 1,$$

and then, for such α, the problem $-\Delta v = \alpha^p f(\cdot, \frac{v}{\alpha})$ in Ω, $v = 0$ on $\partial\Omega$ has a solution in $\mathcal{O}$. Therefore, by defining $u = \frac{v}{\alpha}$, we see that, for any $\lambda > \lambda_0 := \frac{1}{\alpha_0}$, u is a weak solution of problem (1.1) which belongs to $\frac{1}{\alpha}D$,and so it satisfies $u \geq c d_\Omega$ for some $c > 0$.

2. PRELIMINARIES

Let us introduce some notations and assumptions we use along our work. Along the whole paper d_Ω will denote the distance function defined by (1.9) and we assume, also from now on, the conditions (H1) and (H2) stated in Theorem 1.1.

If h is a real function defined on Ω (respectively defined *a.e.* in Ω), we will write h^+ and h^- for the functions defined on Ω (resp. defined a.e. in Ω) by

$$h^+(x) := \max(0, h(x)) \quad \text{and} \quad h^-(x) := -\min(0, h(x)).$$

$C^1_{\mathcal{B}}(\overline{\Omega})$ will denote the Banach space

$$C^1_{\mathcal{B}}(\overline{\Omega}) := \left\{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\right\},$$

endowed with the norm

$$\|u\|_{C^1_{\mathcal{B}}(\overline{\Omega})} := \|u\|_{L^\infty(\Omega)} + \||\nabla u|\|_{L^\infty(\Omega)}.$$

For $u \in C^1_{\mathcal{B}}(\overline{\Omega})$ and $r > 0$, we will write $B_r(u)$ (respectively $\overline{B}_r(u)$) to denote the open (resp. closed) ball in $C^1_{\mathcal{B}}(\overline{\Omega})$ centered at u and with radius r.

Also, $P_{C^1_{\mathcal{B}}(\overline{\Omega})}$ will denote the positive cone in $C^1_{\mathcal{B}}(\overline{\Omega})$, defined by

$$P_{C^1_{\mathcal{B}}(\overline{\Omega})} := \left\{u \in C^1_{\mathcal{B}}(\overline{\Omega}) : u \geq 0 \text{ in } \Omega\right\},$$

and we will write $P^\circ_{C^1_{\mathcal{B}}(\overline{\Omega})}$ for the interior of $P_{C^1_{\mathcal{B}}(\overline{\Omega})}$ in $C^1_{\mathcal{B}}(\overline{\Omega})$, which can be described as

$$P^\circ_{C^1_{\mathcal{B}}(\overline{\Omega})} := \left\{u \in C^1_{\mathcal{B}}(\overline{\Omega}) : u > 0 \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ in } \partial\Omega\right\},$$

where ν is the outward unit normal at $\partial\Omega$, or, alternatively, by

$$P^{\circ}_{C^1_{\mathcal{B}}(\overline{\Omega})} = \left\{ u \in C^1_{\mathcal{B}}(\overline{\Omega}) : \inf_{\Omega} u d_{\Omega}^{-1} > 0 \right\}.$$

As usual, $(-\Delta)^{-1} : L^{\infty}(\Omega) \to \bigcap_{1 \le q < \infty} W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ will denote the solution operator of the problem

$$\begin{cases} -\Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

i.e., the operator defined, for $h \in L^{\infty}(\Omega)$, by $(-\Delta)^{-1}h =: u$, where u is the unique solution of (2.1). We recall that, for such a h, by the embedding theorems for Sobolev spaces, $(-\Delta)^{-1}h \in C^1_{\mathcal{B}}(\overline{\Omega})$ and that, for $r > n$, $(-\Delta)^{-1} : L^r(\Omega) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is a continuous and compact operator.

If g and h are real functions defined on Ω (respectively defined a.e. in Ω), we will write $g \lessapprox h$ to mean that, for some positive constant c, $g \le ch$ in Ω (resp. $g \le ch$ a.e. in Ω. Also, $g \gtrapprox h$ will mean that $h \lessapprox g$, and $g \approx h$ will mean that $g \lessapprox h$ and $h \lessapprox g$.

If $h \in C(\Omega)$ we will write $h > 0$ in Ω to mean that $h(x) > 0$ for any $x \in \Omega$; and if h is defined a.e. in Ω (for instance when $h \in L^q(\Omega)$ for some $q \ge 1$), we will write $h > 0$ in Ω to mean that $h(x) > 0$ for a.e. $x \in \Omega$,

If $b \in L^r(\Omega)$ for some $r > \frac{n}{2}$ and $b^+ \not\equiv 0$, we will denote by $\lambda_1(-\Delta, b)$ the unique positive principal eigenvalue of $-\Delta$ on Ω with weight function b, that is, $\lambda_1(-\Delta, b)$ is the unique positive number ρ such that the problem

$$\begin{cases} -\Delta\varphi = \rho b\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \\ \varphi > 0 & \text{in } \partial\Omega \end{cases}$$

has a weak solution (called a positive principal eigenfunction). If $r > n$ then, elliptic regularity, any such a φ belongs to $C^1_{\mathcal{B}}\left(\overline{\Omega}\right)$. We recall also that $\lambda_1\left(-\Delta, b\right)$ is a simple eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda b\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

and that, if φ is a positive principal eigenfunction, then $\varphi \approx d_{\Omega}$ in Ω. Moreover, $\lambda_1\left(-\Delta, b\right)$ is given by the Rayleigh variational formula

$$\lambda_1(-\Delta, b) = \inf_{u \in C^1(\overline{\Omega}) : \int_{\Omega} bu^2 > 0} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} bu^2}. \tag{2.2}$$

For more details about principal eigenvalues with weight and their associated eigenfunctions, see, e.g. [12].

3. SOME FACTS ABOUT AN AUXILIARY PROBLEM

Lemma 3.1. *Let $\gamma \in C(\overline{\Omega}) \to \mathbb{R}$ be such that $\gamma^+ \not\equiv 0$ and $\gamma^- \leq c d_\Omega$ for some positive constant c, and satisfying that $(-\Delta)^{-1}\gamma \in \left(P_{C^1_{\mathcal{B}}(\overline{\Omega})}\right)^\circ$. If $u \in C^1_{\mathcal{B}}(\overline{\Omega})$ is a non-identically zero weak solution of*

$$\begin{cases} -\Delta u = \gamma\,(u^+)^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

then:

(i) $u \geq 0$ *in* Ω,
(ii) $\{x \in \Omega : u(x) = 0\} \subset \{x \in \Omega : \gamma\,(x) \leq 0\}$.

Proof. We first prove (i). Let $U := \{x \in \Omega : u(x) < 0\}$. Then U is open in Ω. We claim that $U = \varnothing$. Indeed, suppose that $U \neq \varnothing$, then $u \leq 0$ on ∂U. Let us see that $u = 0$ on ∂U. By the way of contradiction suppose that $x \in \partial U$ and that $u(x) < 0$. Then either $x \in \Omega$ or $x \in \partial\Omega$, but $x \in \partial\Omega$ is impossible because $u = 0$ on $\partial\Omega$. Thus, $x \in \Omega$, and so, since u is continuous in Ω, there exists a neighborhood N_x of x such that $u\,(x) < 0$ in N_x, which is impossible because, since $x \in \partial U$,

$$N_x \cap (\Omega \setminus U) \neq \varnothing.$$

Thus, $u = 0$ on ∂U. Now, in the sense of distributions,

$$\begin{cases} -\Delta u = \gamma\,(u^+)^p = 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Then, by the weak maximum principle, as stated, e.g. in Theorem 8.1 of [16], $u = 0$ in U, which contradicts the definition of U. Thus, $U = \varnothing$ and so $u \geq 0$ in Ω. Therefore, (i) holds.

To prove (ii), consider the sets Ω_+ and $\Omega_{0,u}$ given by

$$\Omega_+ := \{x \in \Omega : \gamma(x) > 0\}, \quad \Omega_{0,u} := \{x \in \Omega : u(x) = 0\},$$

and let C_{Ω^+} be an arbitrary connected component of Ω^+. We claim that $\Omega_{0,u} \cap C_{\Omega^+} = \varnothing$. Indeed, suppose that $\Omega_{0,u} \cap C_{\Omega^+} \neq \varnothing$ and let $x \in \Omega_{0,u} \cap C_{\Omega^+}$. Then $\gamma(x) > 0$ and $u(x) = 0$. Thus, for r positive and small enough, $\overline{B}_r(x) \subset \Omega$ and $\gamma > 0$ in $B_r(x)$, where $B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$. Now, in the distributional sense,

$$\begin{cases} -\Delta u = \gamma\,(u^+)^p \geq 0 & \text{in } B_r\,(x)\,, \\ u \geq 0 & \text{on } \partial B_r(x), \end{cases}$$

$u \in C^1\left(\overline{B}_r(x)\right)$ and so clearly $\min_{B_r(x)} u$ is attained at x. Since $\gamma\,(u^+)^p \in L^q(\Omega)$ for any $q \in (1, \infty)$, the generalized strong maximum principle, as stated, e.g. in [16, Theorem 9.6], implies that $u = 0$ in $B_r(x)$. Thus, $B_r(x) \subset \Omega_{0,u} \cap C_{\Omega^+}$ and so $\Omega_{0,u} \cap C_{\Omega^+}$ is open in C_{Ω^+}. On the other hand, $u \in C(\overline{\Omega})$ and so $\Omega_{0,u} \cap C_{\Omega^+}$ is closed in C_{Ω^+}. Then, since C_{Ω^+} is connected, $\Omega_{0,u} \cap C_{\Omega^+} = C_{\Omega^+}$ and so $C_{\Omega^+} \subset \Omega_{0,u}$. Since this holds for each connected component of Ω^+, the lemma follows. $\square$

Lemma 3.2. *Let γ be as in Lemma 3.1. If $u \in C^1_{\mathcal{B}}(\overline{\Omega})$ is a non-identically zero weak solution of problem* (3.1) *then there exists a positive constant c such that $u \geq c d_{\Omega}^{\frac{1}{1-p}}$ in Ω.*

Proof. Notice that $\gamma\,(u^+)^p \in C^1_{\mathcal{B}}(\overline{\Omega})$ and then $u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ for any $q \in [1,\infty)$, and so u is a strong solution of (3.1). By Lemma 3.1, $u \geq 0$ in Ω. Let $\varepsilon > 0$. Then $x \mapsto \frac{1}{1-p}(u+\varepsilon)^{1-p}(x)$ belongs to $C^1_{\mathcal{B}}(\overline{\Omega})$ and a direct computation shows that

$$\begin{aligned} -\Delta\left((u+\varepsilon)^{1-p} - \varepsilon^{1-p}\right) &= -(1-p)(u+\varepsilon)^{-p}\Delta u + p\,(1-p)\,(u+\varepsilon)^{-1-p}\,|\nabla u|^2 \\ &= (1-p)\gamma\frac{u^p}{(u+\varepsilon)^p} + p(1-p)(u+\varepsilon)^{-1-p}\,|\nabla u|^2 \\ &\geq (1-p)\gamma\frac{u^p}{(u+\varepsilon)^p} \text{ in } \Omega, \end{aligned}$$

and since $(u+\varepsilon)^{1-p} - \varepsilon^{1-p} = 0$ on $\partial\Omega$, the weak maximum principle gives

$$(u+\varepsilon)^{1-p} - \varepsilon^{1-p} \geq (1-p)(-\Delta)^{-1}\left(\frac{\gamma u^p}{(u+\varepsilon)^p}\right) \text{ in } \Omega. \tag{3.2}$$

Let $r \in (n,\infty)$, and let $\Omega_{0,u}$ be defined as in the proof of Lemma 3.1. Thus,

$$\lim_{\varepsilon\to 0^+} \frac{\gamma u^p}{(u+\varepsilon)^p} = \gamma \text{ in } \Omega \setminus \Omega_{0,u},$$

and we have also, for any $\varepsilon > 0$, $\frac{\gamma u^p}{(u+\varepsilon)^p} = 0$ in $\Omega_{0,u}$. Since

$$\left|\frac{\gamma u^p}{(u+\varepsilon)^p}\right|^r \leq \|\gamma\|_\infty^r \in L^1(\Omega),$$

then, by the Lebesgue dominated convergence theorem, $\lim_{\varepsilon\to 0^+}\frac{\gamma u^p}{(u+\varepsilon)^p} = \gamma\chi_{\Omega\setminus\Omega_{0,u}}$ with convergence in $L^r(\Omega)$ and so

$$\lim_{\varepsilon\to 0^+}(-\Delta)^{-1}\left(\frac{\gamma u^p}{(u+\varepsilon)^p}\right) = (-\Delta)^{-1}\left(\gamma\chi_{\Omega\setminus\Omega_{0,u}}\right)$$

with convergence in $C^1_{\mathcal{B}}(\overline{\Omega})$ and so, from (3.2),

$$u^{1-p} \geq (1-p)(-\Delta)^{-1}\left(\gamma\chi_{\Omega\setminus\Omega_{0,u}}\right) \text{ in } \Omega.$$

By Lemma 3.1, $\gamma \leq 0$ in $\Omega_{0,u}$ and then $\gamma\chi_{\Omega\setminus\Omega_{0,u}} \geq \gamma$, Thus $u^{1-p} \geq (-\Delta)^{-1}(\gamma) \geq c d_\Omega$ in Ω, with c a positive constant, and the lemma follows. □

The next lemma improves Lemma 3.2.

Lemma 3.3. *If $u \in C^1_{\mathcal{B}}(\overline{\Omega})$ is a non-identically zero solution, in the sense of distributions, of problem* (3.1)*, then there exists a positive constant c such that $u \geq c d_\Omega$ in Ω.*

Proof. By Lemma 3.1 $u \geq 0$ in Ω, and, as observed within the proof of Lemma 3.2, $u \in \bigcap_{1\leq q<\infty} W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ for any $q \in (1,\infty)$ and u is a strong solution of (3.1). By Lemma 3.2 and (H2) we have $u \geq c d_\Omega^{\frac{1}{1-p}}$ and, for some $\beta \in \left(\frac{n-1}{n}, \infty\right)$, $\gamma^- \leq c' d_\Omega^\beta$ in Ω with c and c' positive constants. Then,

$$\begin{cases} -\Delta u + \frac{c' d_\Omega^\beta}{c^{1-p} d_\Omega} u \geq -\Delta u + \frac{\gamma^-}{u^{1-p}} u = -\Delta u + \gamma^- u^p = \gamma^+ u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Notice that, since $\beta > \frac{n-1}{n}$, we have that $0 \leq \frac{c' d_\Omega^\beta}{c^{1-p} d_\Omega} \in L^r(\Omega)$ for some $r > n$.

We study first the case $n \geq 2$. For $\varepsilon > 0$, consider the mapping $\Phi : \partial\Omega \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ defined by

$$\Phi(y,t) = y + t\nu(y),$$

where, for $y \in \partial\Omega$, $\nu(y)$ denotes the unit outward normal to Ω at y. Since Ω is a C^2 bounded domain, we have that $\Phi \in C^1(\partial\Omega \times (-\varepsilon,\varepsilon))$ Then, for ε positive and small enough, $N_\varepsilon := \Phi(\partial\Omega \times (-\varepsilon,\varepsilon))$ is a tubular open neighborhood of $\partial\Omega$ (see, e.g. [17, Chapter 4, Section 5], see also [21, Theorem 6.17]) and [4, Chapter 6, Section 6.2]), and Φ is a C^1 diffeomorphism from $\partial\Omega \times (-\varepsilon,\varepsilon)$ onto N_ε. Moreover, if N_ε^- is defined by $N_\varepsilon^- := \Phi(\partial\Omega \times (-\varepsilon, 0])$ then, taking into account that ν is the outward unit normal we have, again for ε positive and small enough, $N_\varepsilon^- = A_\varepsilon$, where $A_\varepsilon := \left\{x \in \overline{\Omega} : d_\Omega(x) < \varepsilon\right\}$. We fix such a ε with the above properties.

From the Hopf boundary lemma, as stated in [27, Theorem 1.1], and taking into account (3.3), we have that $\frac{\partial u}{\partial \nu}(y) < 0$ for any $y \in \partial\Omega$, i.e.,

$$\frac{\partial(u \circ \Phi)(\cdot, t)}{\partial t}_{|t=0} = \langle (\nabla u)(y), \nu(y) \rangle < 0 \text{ for any } y \in \partial\Omega.$$

Therefore, since $u \in C^1(\overline{\Omega})$ and since $\nu \in C^1(\partial\Omega)$ (because Ω is a C^2 domain), we have that, by diminishing ε if necessary, there exists a positive constant c_1 such that

$$\frac{\partial(u \circ \Phi)(y,t)}{\partial t} \leq -c_1 \text{ in } \partial\Omega \text{ for any } y \in \partial\Omega \text{ and } t \in (-\varepsilon, 0].$$

For $x \in A_\varepsilon$, we have $x = y + t\nu(y)$ for some $y \in \partial\Omega$ and $t \in (-\varepsilon, 0]$ (which, in particular implies that $d_\Omega(x) \leq |t|$) and so

$$\begin{aligned} u(x) &= u(y + t\nu(y)) - u(y) = (u \circ \Phi)(x,t) - (u \circ \Phi)(x,0) \\ &= \int_0^t \frac{\partial(u \circ \Phi)(y,t)}{\partial s} ds \\ &= -\int_{-|t|}^0 \frac{\partial(u \circ \Phi)(y,t)}{\partial s} ds \geq c_1 |t| \geq c_1 d_\Omega(x) \text{ for any } x \in A_\varepsilon. \end{aligned} \tag{3.4}$$

Let $m := \min_{\Omega\setminus A_\varepsilon} u$. From (3.3) and the strong maximum principle (as stated, e.g. in Theorem 9.6 of [16]) we have that $m > 0$. Now, $d_\Omega \leq \varepsilon = \frac{e}{m} m \leq \frac{\varepsilon}{m} u$ in A_ε and so

$$u \geq \frac{m}{\varepsilon} d_\Omega \text{ in } A_\varepsilon, \tag{3.5}$$

and thus, from (3.4) and (3.5),

$$u \geq \max\left\{c_1, \frac{m}{\varepsilon}\right\} d_\Omega \text{ in } A_\varepsilon$$

which ends the proof of the lemma when $n \geq 2$.

Consider now the case $n = 1$, i.e., the case when $\Omega := (a, b)$ with $-\infty < a < b < \infty$. In this case $\frac{\partial u}{\partial \nu}(a) := -u'(a)$, $\frac{\partial u}{\partial \nu}(b) := u'(b)$. Notice that the assumption on the smoothness of Ω was used in Theorem 1 of [27] only to guarantee that Ω satiisfies the inner ball condition which, in the one-dimensional case, clearly holds. In fact, for any $\rho < b-a$, the interval $(a, a+\rho)$ (respectively $(b-\rho, b)$) is an inner ball whose closure intersects $\partial\Omega$ only at a (resp. at b). Then, by Theorem 1 of [27] we have that $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$. Therefore, since $u \in C^1(\overline{\Omega})$, there exist $\delta > 0$ and $\eta > 0$ such that $u'(x) > \eta$ for any $x \in (a, a+\delta)$ and $u'(x) < -\eta$ for any $x \in (b, b-\delta)$. Then, since $u(a) = u(b) = 0$, the mean value theorem gives that, for some positive constant c,

$$u \geq c d_\Omega \text{ in } A_\delta,$$

Also, by the strong maximum principle, $u > 0$ in $\Omega \setminus A_\delta$, and now the proof ends as in the case $n \geq 2$. □

Lemma 3.4. $\left\|u d_\Omega^{-1}\right\|_{L^\infty(\Omega)} \leq \|u\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$ *for any* $u \in C^1_{\mathcal{B}}(\overline{\Omega})$.

Proof. For $x \in \Omega$, let $y_x \in \partial\Omega$ be such that $d_\Omega(x) = |x - y_x|$ and let

$$I_x := \{\theta x + (1-\theta) y_x : 0 < \theta < 1\}.$$

Then $I_x \subset \Omega$. Indeed, if not, then for some $\theta \in (0,1)$, $\theta x + (1-\theta)y_x \notin \Omega$ and so there exists $\widetilde{\theta} \in [\theta, 1)$ such that $z := \widetilde{\theta} x + \left(1 - \widetilde{\theta}\right) y_x \in \partial\Omega$, but this contradicts the definition of d_Ω because $|x - z| < |x - y_x| = d_\Omega(x)$. Thus, $I_x \subset \Omega$. Now, from the mean value theorem,

$$\frac{|u(x)|}{d_\Omega(x)} = \frac{\left|u(x) - u(y_\{x\})\right|}{|x - y_x|} \leq \sup_{z \in I_x} |\nabla u(z)| \leq \|u\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$$

and the lemma follows. □

In what follows, we put

$$D := \left\{u \in C^1_{\mathcal{B}}(\overline{\Omega}) : \inf_\Omega \left(u d_\Omega^{-1}\right) > 0\right\}. \tag{3.6}$$

Remark 3.5. Notice that $D \neq \varnothing$ (for instance, if φ_1 is a positive principal eigenfunction for $-\Delta$ in Ω, with weight function $\mathbf{1}$ and homogeneous Dirichlet boundary condition, then $\varphi_1 \in D$). Moreover, D is open in $C^1_{\mathcal{B}}(\overline{\Omega})$. Indeed, for $u \in D$, let $\rho := \inf_\Omega \left(ud_\Omega^{-1}\right)$. Then, for any $v \in C^1_{\mathcal{B}}(\overline{\Omega})$ such that $\|v\|_{C^1_{\mathcal{B}}(\overline{\Omega})} < \frac{1}{2}\rho$, we have $u+v \in C^1_{\mathcal{B}}(\overline{\Omega})$ and

$$(u+v)d_\Omega^{-1} \geq \rho + vd_\Omega^{-1} \geq \rho - \left\|vd_\Omega^{-1}\right\|_\infty \geq \rho - \|v\|_{C^1_{\mathcal{B}}(\overline{\Omega})} > \frac{\rho}{2} \text{ in } \Omega.$$

Thus $u+v \in D$, i.e. $B_{\frac{1}{2}\rho}(u) \subset D$.

Remark 3.6. If $u \in C^1_{\mathcal{B}}(\overline{\Omega})$ is a non-identically zero weak solution of problem (3.1) then, by Lemma 3.3, $u > 0$ in Ω and $u \in D$. On the other hand, by [19, Theorem 1.1], (3.1) there exists $v \in H^1_0(\Omega)$ such that v is the unique positive weak solution of (3.1). Then it follows that (3.1) has a unique non-identically zero weak solution $u \in H^1_0(\Omega)$ and that such a u belongs to $C^1_{\mathcal{B}}(\overline{\Omega}) \cap D$.

Definition 3.7. Let $T : C(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ be defined by

$$T(u) := (-\Delta)^{-1}\left(\gamma\left(u^+\right)^p\right).$$

It is clear that $\gamma\left(u^+\right)^p \in C\left(\overline{\Omega}\right)$ for any $u \in C(\overline{\Omega})$. Then $T(u)$ is well defined and $T(u) \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ for any $q \in [1,\infty)$. In particular, by taking $q > n$, $T(u) \in C^1_{\mathcal{B}}(\overline{\Omega})$.

Lemma 3.8.

(i) $T : C\left(\overline{\Omega}\right) \to C^1_{\mathcal{B}}(\overline{\Omega})$ *is continuous.*
(ii) $T : C^1_{\mathcal{B}}\left(\overline{\Omega}\right) \to C^1_{\mathcal{B}}(\overline{\Omega})$ *is continuous.*
(iii) $T : C^1_{\mathcal{B}}\left(\overline{\Omega}\right) \to C^1_{\mathcal{B}}(\overline{\Omega})$ *is a compact operator.*

Proof. To see (i), observe that if $a \geq 0$, $b \geq 0$, and $0 < p < 1$, then $|a^p - b^p| \leq |a-b|^p$. Let $u \in C^1_{\mathcal{B}}(\overline{\Omega})$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $C^1_{\mathcal{B}}(\overline{\Omega})$ that converges to u in $C^1_{\mathcal{B}}(\overline{\Omega})$. Then

$$\left|\gamma\left(u_j^+\right)^p - \gamma\left(u^+\right)^p\right| \leq |\gamma|\left|u_j^+ - u^+\right|^p \text{ in } \Omega$$

and thus $\left\{\gamma\left(u_j^+\right)^p\right\}_{j\in\mathbb{N}}$ converges to $\gamma\left(u^+\right)^p$ in $C(\overline{\Omega})$ (i.e. $v \mapsto \gamma v^p$ is continuous from $C^1_{\mathcal{B}}(\overline{\Omega})$ into $C(\overline{\Omega})$). Then $\left\{(-\Delta)^{-1}\left(\gamma\left(u_j^+\right)^p\right)\right\}_{j\in\mathbb{N}}$ converges to $(-\Delta)^{-1}\left(\gamma\left(u^+\right)^p\right)$ in $W^{2,q}(\Omega)$ for any $q \in [1,\infty)$ and so, by taking $q > n$,

$$\lim_{j\to\infty}\left\|(-\Delta)^{-1}\left(\gamma\left(u_j^+\right)^p - \gamma\left(u^+\right)^p\right)\right\|_{C^1_{\mathcal{B}}(\overline{\Omega})} = 0 \tag{3.7}$$

which gives (i).

Since the inclusion $C^1_{\mathcal{B}}\left(\overline{\Omega}\right) \hookrightarrow C(\overline{\Omega})$ is continuous, (ii) follows immediately from (i).

To see (iii) notice that, as observed above, $v \mapsto \gamma\left(v^+\right)^p$ is continuous from $C^1_{\mathcal{B}}(\overline{\Omega})$ into $C(\overline{\Omega})$, Also, $(-\Delta)^{-1} : C(\overline{\Omega}) \to W^{2,q}(\Omega)$ is continuous for any $q \in [1,\infty)$ and recalling that, for any $q > n$, the embedding $W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \hookrightarrow C^1_{\mathcal{B}}(\overline{\Omega})$ is compact (with $W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ provided with the topology inherit from $W^{2,q}(\Omega)$), we get that $T : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is compact. $\square$

Lemma 3.9. *Let D and T be as given by* (3.6) *and Definition 3.7, respectively. Then $T : D \to C^1_{\mathcal{B}}(\overline{\Omega})$ is continuously Fréchet differentiable and its Fréchet differential $D_uT : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ at $u \in D$ is given by*

$$(D_uT)(\psi) = p(-\Delta)^{-1}\left(\gamma u^{p-1}\psi\right) \text{ for any } \psi \in C^1_{\mathcal{B}}(\overline{\Omega}). \tag{3.8}$$

Proof. Let $u \in D$, let $\rho := \inf_\Omega \left(ud_\Omega^{-1}\right)$ and let $\varphi \in C^1_{\mathcal{B}}(\overline{\Omega})$ be such that $\|\varphi\|_{C^1_{\mathcal{B}}(\overline{\Omega})} < \frac{\rho}{4}$. Then $u+\varphi \in D$ and the mean value theorem gives

$$T(u+\varphi) - T(u) = (-\Delta)^{-1}(\gamma((u+\varphi)^p - u^p)) = (-\Delta)^{-1}\left(\gamma p\left((u+\theta\varphi)^{p-1}\varphi\right)\right)$$

for some $\theta : \Omega \to \mathbb{R}$ such that $0 < \theta < 1$ in Ω. The mean value theorem gives

$$\begin{aligned}&(-\Delta)^{-1}\left(\gamma p\left((u+\theta\varphi)^{p-1}\varphi\right)\right)\\&= p(-\Delta)^{-1}\left(\gamma u^{p-1}\varphi\right) + p(p-1)(-\Delta)^{-1}\left(\gamma(u+\theta\eta\varphi)^{p-2}(\theta-1)\varphi^2\right)\end{aligned}$$

for some $\eta : \Omega \to \mathbb{R}$ such that $0 < \eta < 1$ in Ω. Then

$$T(u+\varphi) = T(u) + (D_uT)(\varphi) + p(p-1)(-\Delta)^{-1}\left(\gamma(u+\theta\eta\varphi)^{p-2}(\theta-1)\varphi^2\right) \tag{3.9}$$

with D_uT defined by (3.8). Notice that D_uT is well defined on $C^1_{\mathcal{B}}(\overline{\Omega})$ and that it is continuous from $C^1_{\mathcal{B}}(\overline{\Omega})$ into $L^\infty(\Omega)$. Indeed, for $\psi \in C^1_{\mathcal{B}}(\overline{\Omega})$ and $u \in D$, we have $u \geq cd_\Omega$ in Ω for some positive constant c (with c depending on u). Then

$$\left|\gamma u^{p-1}\psi\right| = |\gamma| d_\Omega^p \left(d_\Omega^{-1}u\right)^{p-1}\left|\psi d_\Omega^{-1}\right| \leq c^{1-p}\|\gamma\|_\infty \|d_\Omega\|_\infty^p \left|\psi d_\Omega^{-1}\right|$$

and so, by taking $q > n$, $\left\|\gamma u^{p-1}\psi\right\|_q \leq c''\|\psi\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$, where c' is a positive constant independent of ψ. Thus,

$$\begin{aligned}\|D_uT(\psi)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} &= \left\|(-\Delta)^{-1}\left(\gamma u^{p-1}\psi\right)\right\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\\&\leq c\left\|\gamma u^{p-1}\psi\right\|_q \leq c'\left\|\psi d_\Omega^{-1}\right\|_\infty \leq c''\|\psi\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\end{aligned}$$

with c, c' and c'' positive constants independent of ψt. Then $D_uT : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is continuous.

On the other hand, for $\varphi \in C^1_{\mathcal{B}}(\overline{\Omega})$, we have

$$\begin{aligned}\|T(u+\varphi) - T(u) - (D_uT)(\varphi)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} &= c\left\|(-\Delta)^{-1}\left(\gamma(u+\theta\eta\varphi)^{p-2}(\theta-1)\varphi^2\right)\right\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\\&\leq c'\left\|(u+\theta\eta\varphi)^{p-2}\varphi^2\right\|_q \leq c''\left\|\left(\varphi d_\Omega^{-1}\right)^2\right\|_\infty\\&\leq c'''\|\varphi\|^2_{C^1_{\mathcal{B}}(\overline{\Omega})},\end{aligned}$$

and so

$$\lim_{\|\varphi\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\to 0} \frac{\|T(u+\varphi) - T(u) - (D_uT)(\varphi)\|_{C^1_{\mathcal{B}}(\overline{\Omega})}}{\|\varphi\|_{C^1_{\mathcal{B}}(\overline{\Omega})}} = 0.$$

Then T is Fréchet differentiable at u and its Fréchet derivative (at u) is D_uT.

Finally, for $u \in D$, $v \in D$ and $\varphi \in C^1_{\mathcal{B}}(\overline{\Omega})$, from (3.8), by the mean value theorem we have

$$\begin{aligned}\|(D_uT - D_vT)(\varphi)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} &= p\left\|(-\Delta)^{-1}\left(\gamma\left(u^{p-1} - v^{p-1}\right)\varphi\right)\right\|_{C^1_{\mathcal{B}}(\overline{\Omega})} \\ &= p(1-p)\left\|(-\Delta)^{-1}\left(\gamma\xi^{p-2}_{u,v}(u-v)\varphi\right)\right\|_{C^1_{\mathcal{B}}(\overline{\Omega})},\end{aligned} \tag{3.10}$$

where $\xi_{u,v} : \Omega \to \mathbb{R}$ is a function such that, for any $x \in \Omega$, $\xi_{u,v}(x)$ belongs to the open segment with endpoints $u(x)$ and $v(x)$. Since $u \in D$ there exists a positive constant c_u such that $u \geq c_u d_\Omega$ in Ω and, by Lemma 3.4, $|u - v| \leq d_\Omega \|u - v\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$ in Ω. Thus, for $v \in B_{\frac{c_u}{2}}(u)$, we have

$$v = u - (u - v) \geq c_u d_\Omega - d_\Omega \|u - v\|_{C^1_{\mathcal{B}}(\overline{\Omega})} \geq \left(c_u - \frac{c_u}{2}\right) d_\Omega = \frac{c_u}{2} d_\Omega \text{ in } \Omega. \tag{3.11}$$

Therefore, since also $u \geq \frac{c_u}{2} d_\Omega$ in Ω, it follows that $\xi_{u,v} \geq \frac{c_u}{2} d_\Omega$ in Ω. Again by Lemma 3.4, $|\varphi| \leq \|\varphi\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$. Thus, for $v \in B_{\frac{c_u}{2c_\Omega}}(u)$,

$$\left|\gamma\xi^{p-2}_{u,v}(u-v)\varphi\right| \leq \|\gamma\|_\infty \left(\frac{c_u}{2}\right)^{p-2} \frac{1}{d_\Omega^{2-p}} d_\Omega^2 \|u - v\|_{C^1_{\mathcal{B}}(\overline{\Omega})} \|\varphi\|_{C^1_{\mathcal{B}}(\overline{\Omega})} \text{ in } \Omega,$$

and so

$$\left\|\gamma\xi^{p-2}_{u,v}(u-v)\varphi\right\|_\infty \leq c\|u - v\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\|\varphi\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$$

with c a positive constant independent of v and φ (for $v \in B_{\frac{c_u}{2}}(u)$). Then, from (3.10), for such v,

$$\|(D_uT - D_vT)(\varphi)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} \leq c'\|u - v\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\|\varphi\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$$

with c' independent of v and φ, which concludes the proof that T is continuously differentiable. □

Definition 3.10. From now on we will denote by $\boldsymbol{u}$ the (unique) non-identically zero fixed point of T in $C^1_{\mathcal{B}}(\overline{\Omega})$.

Remark 3.11. There exist positive constants $c_{\boldsymbol{u}}$ and $c'_{\boldsymbol{u}}$ such that $c_{\boldsymbol{u}} d_\Omega \leq \boldsymbol{u} \leq c'_{\boldsymbol{u}} d_\Omega$. Indeed, since $\boldsymbol{u} \in D$ there exists a constant $c_{\boldsymbol{u}} > 0$ such that $\boldsymbol{u} \geq c_{\boldsymbol{u}} d_\Omega$ in Ω and, by Lemma 3.4 we have also, for some constant $c'_{\boldsymbol{u}}$, $\boldsymbol{u} \leq c'_{\boldsymbol{u}} d_\Omega$ in Ω.

Remark 3.12. Since $I - T : D \to C^1_{\mathcal{B}}(\overline{\Omega})$ is continuous, and since $\boldsymbol{u} \neq 0$ and $T\boldsymbol{u} = \boldsymbol{u}$ it follows that there exists $\rho_0 > 0$ such that $\|T(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} \geq \frac{1}{2}\|\boldsymbol{u}\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$ for any $v \in \overline{B}_{\rho_0}(\boldsymbol{u})$.

Lemma 3.13. *Let $\boldsymbol{u}$ be as given by Definition 3.10. Then $D_{\boldsymbol{u}}T : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is a compact operator.*

Proof. As seen in the proof of Lemma 3.9, $D_{\boldsymbol{u}}T : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is continuous. To see the compactness of $D_{\boldsymbol{u}}T$ consider an arbitrary bounded sequence in $C^1_{\mathcal{B}}(\overline{\Omega})$ and, for $j \in \mathbb{N}$, let $v_j := D_{\boldsymbol{u}}T(\psi_j)$. Then

$$\begin{cases} -\Delta v_j = p\gamma \boldsymbol{u}^{p-1}\psi_j & \text{in } \Omega, \\ v_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

For $x \in \Omega$ let $y_x \in \partial\Omega$ be such that $d_\Omega(x) = |x - y_x|$. Then, as seen in the proof of Lemma 3.4,

$$I_x := \{\theta x + (1-\theta)y_x : 0 < \theta < 1\} \subset \Omega.$$

Thus, for $x \in \Omega$ and $j \in \mathbb{N}$, the mean value theorem gives

$$|\psi_j(x)| = \left|\psi_j(x) - \psi_j(y_{\{}x\})\right| \le \||\nabla\psi_j|\|_\infty |x - y_x| \le \|\psi_j\|_{C^1_{\mathcal{B}}(\overline{\Omega})} d_\Omega(x),$$

and so $\psi_j(x) \le c d_\Omega(x)$ with c a positive constant independent of x and j. On the other hand, as observed in Remark 3.11, we have $\boldsymbol{u}(x) \ge c_{\boldsymbol{u}} d_\Omega(x)$ with $\boldsymbol{c_u}$ a positive constant independent of x, and so, since $p < 1$ and $\gamma \in C\left(\overline{\Omega}\right)$, we get that

$$\left|p\gamma(x)\boldsymbol{u}^{p-1}(x)\psi_j(x)\right| \le c c_{\boldsymbol{u}}^{p-1} p \left\|\gamma d_\Omega^p\right\|_\infty.$$

Then $\left\{p\gamma\boldsymbol{u}^{p-1}\psi_j\right\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$, and so, by elliptic regularity, $\{v_j\}_{j\in\mathbb{N}}$ has a subsequence that converges in $C^1_{\mathcal{B}}(\overline{\Omega})$. $\square$

Definition 3.14. We define $\mathcal{O} := B_\rho(\boldsymbol{u})$ where $\rho > 0$ is such that:

(1) $\overline{B}_{2\rho}(\boldsymbol{u}) \subset D$, with D as given by (3.6),
(2) $\rho < \frac{1}{2}c_{\boldsymbol{u}}$ with $c_{\boldsymbol{u}}$ as given by Remark 3.11,
(3) $\rho \le \rho_0$ with ρ_0 as given by Remark 3.12.

Notice that $\mathcal{O}$ is an open and bounded set in $C^1_{\mathcal{B}}(\overline{\Omega})$. We will write $\overline{\mathcal{O}}$ for the closure of $\mathcal{O}$ in $C^1_{\mathcal{B}}(\overline{\Omega})$.

Remark 3.15. Notice that there exist positive constants $c_{\mathcal{O}}$ and $c'_{\mathcal{O}}$ such that $c_{\mathcal{O}} d_\Omega \le v \le c'_{\mathcal{O}} d_\Omega$ in Ω for any $v \in \overline{\mathcal{O}}$. Indeed, let $c_{\boldsymbol{u}}$ be as given by Remark 3.11 and let ρ be as given by Definition 3.14. Then, for any $v \in \overline{\mathcal{O}}$,

$$v = \boldsymbol{u} - (\boldsymbol{u} - v) \ge c_u d_\Omega - \|\boldsymbol{u} - v\|_{C^1_{\mathcal{B}}(\overline{\Omega})} d_\Omega \ge c_u d_\Omega - \rho d_\Omega \ge \frac{1}{2} c_u d_\Omega \text{ in } \Omega.$$

Also,

$$v = (v - \boldsymbol{u}) + \boldsymbol{u} \le \|v - \boldsymbol{u}\|_{C^1_{\mathcal{B}}(\overline{\Omega})} d_\Omega + \|\boldsymbol{u}\|_{C^1_{\mathcal{B}}(\overline{\Omega})} d_\Omega \le \left(\rho + \|\boldsymbol{u}\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\right) d_\Omega$$

and thus, by taking $c'_{\mathcal{O}} := \left(\rho + \|\boldsymbol{u}\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\right)$ we have $v \le c'_{\mathcal{O}} d_\Omega$ in Ω for any $v \in \overline{\mathcal{O}}$.

The proof of the following lemma is an adaptation of part of the proof of [15, Theorem 3.1].

Lemma 3.16. *Let $D_{\boldsymbol{u}}T$ be the Fréchet differential of $T : D \to C^1_{\mathcal{B}}(\overline{\Omega})$ at $\boldsymbol{u}$. Then $D_{\boldsymbol{u}}T$ has no eigenvalues in (p, ∞).*

Proof. Let $\sigma \in \mathbb{R}$ be an eigenvalue of $D_{\boldsymbol{u}}T$, and let $z \in C^1_{\mathcal{B}}(\overline{\Omega})$ be an associated eigenfunction. Thus,

$$\begin{cases} -\Delta z = \frac{1}{\sigma} p\gamma \boldsymbol{u}^{p-1} z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

and thus, by taking z as a test function in (3.13),

$$\sigma^{-1} = \frac{\int_\Omega |\nabla z|^2}{\int_\Omega p\gamma \boldsymbol{u}^{p-1} z^2}. \tag{3.14}$$

Also,

$$\begin{cases} -\Delta \boldsymbol{u} = \gamma \boldsymbol{u}^p & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

For $\varepsilon > 0$, we take $\frac{z^2}{u+\varepsilon}$ as a test function in (3.15) to obtain

$$\int\limits_\Omega \left\langle \nabla \boldsymbol{u}, \nabla \left(\frac{z^2}{\boldsymbol{u}+\varepsilon} \right) \right\rangle = \int\limits_\Omega \gamma \boldsymbol{u}^p \frac{z^2}{\boldsymbol{u}+\varepsilon}.$$

i.e.

$$\int\limits_\Omega \left\langle \nabla \boldsymbol{u}, \frac{2z\nabla z}{\boldsymbol{u}+\varepsilon} - \frac{z^2 \nabla \boldsymbol{u}}{(\boldsymbol{u}+\varepsilon)^2} \right\rangle = \int\limits_\Omega \gamma \boldsymbol{u}^p \frac{z^2}{\boldsymbol{u}+\varepsilon},$$

that is,

$$\int\limits_\Omega \left(-\frac{z^2 |\nabla \boldsymbol{u}|^2}{(\boldsymbol{u}+\varepsilon)^2} + \left\langle \frac{2z}{\boldsymbol{u}+\varepsilon} \nabla \boldsymbol{u}, \nabla z \right\rangle \right) = \int\limits_\Omega \gamma \boldsymbol{u}^p \frac{z^2}{\boldsymbol{u}+\varepsilon},$$

Now,

$$-\frac{z^2 |\nabla \boldsymbol{u}|^2}{(\boldsymbol{u}+\varepsilon)^2} + \left\langle \frac{2z}{\boldsymbol{u}+\varepsilon} \nabla \boldsymbol{u}, \nabla z \right\rangle \le |\nabla z|^2,$$

and so

$$\int\limits_\Omega |\nabla z|^2 \ge \int\limits_\Omega \gamma \boldsymbol{u}^p \frac{z^2}{\boldsymbol{u}+\varepsilon}$$

which, by taking the limit as $\varepsilon \to 0^+$, gives

$$\int\limits_\Omega |\nabla z|^2 \ge \int\limits_\Omega \gamma \boldsymbol{u}^{p-1} z^2$$

and thus, by (3.14), $\sigma \le p$. $\square$

Remark 3.17. We recall that if X is a Banach space, if $\mathcal{U}$ is a open bounded set in X, and if $F : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ is a continuous and compact operator such that $0 \notin (I - F)(\partial\mathcal{U})$, then the Leray–Schauder degree $\deg_{LS}(I - F, \mathcal{U}, 0)$ is a well defined integer which has the following properties (see, e.g. [29, Chapter 13, Section 13.6], see also [24, Chapter 3, Section 3.2]).

(i) $\deg_{LS}(I, \mathcal{U}, 0) = 1$.
(ii) If $\deg_{LS}(I - F, \mathcal{U}, 0) \neq 0$ then F has at least a fixed point $v \in \mathcal{U}$.
(iii) If $F : \overline{\mathcal{U}} \to X$ and $\widetilde{F} : \overline{\mathcal{U}} \to X$ are two continuous and compact operators such that $0 \notin (I - F)(\partial\mathcal{U})$ and $0 \notin \left(I - \widetilde{F}\right)(\partial\mathcal{U})$ and if

$$\sup_{v \in \mathcal{U}} \left\| F(v) - \widetilde{F}(v) \right\|_X < \inf_{v \in \partial U} \|F(v)\|_X$$

then

$$\deg_{LS}(I - F, \mathcal{U}, 0) = \deg_{LS}\left(I - \widetilde{F}, \mathcal{U}, 0\right).$$

(iv) (homotopic invariance) If $H \in C\left([0,1] \times \overline{\mathcal{U}}, X\right)$ is a compact operator such that $0 \notin (I - H(t,\cdot))(\partial\mathcal{U})$ for any $t \in [0,1]$ then $t \mapsto \deg_{LS}(H(t,\cdot), \mathcal{U}, 0)$ is constant on $[0,1]$. In particular,

$$\deg_{LS}(I - H(0,\cdot), \mathcal{U}, 0) = \deg_{LS}(I - H(1,\cdot), \mathcal{U}, 0).$$

(v) If $\mathcal{U}_0$ is an open subset of $\mathcal{U}$ and if $F : \overline{\mathcal{U}} \to X$ is a continuous and compact operator such that $0 \notin (I - F)\left(\overline{\mathcal{U}} \setminus \mathcal{U}_0\right)$ then

$$\deg_{LS}(I - F, \mathcal{U}, 0) = \deg_{LS}(I - F, \mathcal{U}_0, 0).$$

Lemma 3.18. *Let $\mathcal{O}$ be as given by Definition 3.14. Then*

$$\deg_{LS}(I - T, \mathcal{O}, 0) = \pm 1.$$

Proof. By Lemma 3.8, $T : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is a continuous and compact operator, and, since T has a unique fixed point in $C^1_{\mathcal{B}}(\overline{\Omega})$, which, by the definition of $\mathcal{O}$, belongs to $\mathcal{O}$, then T has no fixed points in $\theta\mathcal{O}$, and so $\deg_{LS}(I - T, \mathcal{O}, 0)$ is well defined.

By Lemma 3.9, $T : D \to C^1_{\mathcal{B}}(\overline{\Omega})$ is continuously Fréchet differentiable and its Fréchet differential $D_u T$ at an arbitrary $u \in D$ is given by

$$(D_u T)(\psi) = p(-\Delta)^{-1}\left(\gamma u^{p-1}\psi\right) \tag{3.16}$$

for any $\psi \in C^1_{\mathcal{B}}(\overline{\Omega})$. We recall that, by Remark 3.6 and by the definition of $\boldsymbol{u}$, the function $\boldsymbol{u}$ is the unique fixed point of T in $C^1_{\mathcal{B}}(\overline{\Omega})$ and so

$$\left\{v \in C^1_{\mathcal{B}}(\overline{\Omega}) : v \in (I - T)^{-1}(0)\right\} = \{\boldsymbol{u}\}. \tag{3.17}$$

We claim that $I - D_{\boldsymbol{u}}T : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is invertible with continuous inverse. Indeed, if $\psi \in C^1_{\mathcal{B}}(\overline{\Omega}) \setminus \{0\}$ and $\psi - (D_{\boldsymbol{u}}T)(\psi) = 0$, then 1 would be an eigenvalue of $D_{\boldsymbol{u}}T$, in contradiction with Lemma 3.16. Thus, $I - D_{\boldsymbol{u}}T$ is injective.

On the other hand, by Lemma 3.13, $D_{\boldsymbol{u}}T$ is compact, and since $\boldsymbol{u}$ is not an eigenvalue of $D_{\boldsymbol{u}}T$, 1 does not belong to the spectrum of T and so $D_{\boldsymbol{u}}T : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is surjective, and then the open mapping theorem (as stated, e.g. in [26]) gives that $I - D_u T : C^1_{\mathcal{B}}\left(\overline{\Omega}\right) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is invertible with continuous inverse. Now (see, e.g. [23, p. 197]),

$$\deg_{LS}\left(I - T, \mathcal{O}, 0\right) = \sum_{v \in (I-T)^{-1}(0)} (-1)^{\sigma(v)}, \tag{3.18}$$

where $\sigma(v)$ denotes the sum of the algebraic multiplicities of the eigenvalues of $D_v T$ contained in $(1, \infty)$ (with the convention that $\sigma(v) = 0$ if no such a eigenvalue exists). By (3.17), the sum in (3.18) reduces to only one term (namely for $v = \boldsymbol{u}$) and since $\sigma\left(\boldsymbol{u}\right) = 0$ (because, by Lemma 3.16, $D_{\boldsymbol{u}}T$ has no eigenvalues in $(1, \infty)$) we conclude that $\deg_{LS}\left(I - T, \mathcal{O}, 0\right) = \pm 1$. □

4. PROOF OF THE MAIN RESULT

We recall our assumptions, stated at the introduction, on f: Assume the following conditions:

(H1) $f \in C\left(\overline{\Omega} \times [0, \infty)\right)$.

(H2) $\lim_{s \to \infty} s^{-p} f(\cdot, s) = \gamma$ uniformly on Ω, for some $p \in (0, 1)$ and some $\gamma \in C\left(\overline{\Omega}\right)$ such that for some positive constants c, c' and β, with $\beta > \frac{n-1}{n}$,

$$(-\Delta)^{-1}\gamma \geq c d_\Omega \quad \text{and} \quad \gamma^- \leq c' d_\Omega^\beta \quad \text{in } \Omega.$$

We recall also that we will write $\boldsymbol{u}$ to denote the unique non identically zero fixed point of T in $C^1_{\mathcal{B}}\left(\overline{\Omega}\right)$.

Definition 4.1. Let $f^* : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ be defined by $f^*(\cdot, s) := s^{-p} f(\cdot, s)$.

Lemma 4.2. *Let $\mathcal{O}$ be as given by Definition 3.14. Then:*

(i) *for any $v \in \overline{\mathcal{O}}$,*

$$\lim_{\alpha \to 0^+} \alpha^p f\left(\cdot, \frac{v}{\alpha}\right) = \gamma v^p \ \textit{in } \Omega,$$

(ii) *for any $\alpha \in (0, 1)$, $v \in \overline{\mathcal{O}}$ and $r \in [1, \infty)$, the function $\alpha^p f\left(\cdot, \frac{v}{\alpha}\right)$ belongs to $L^r(\Omega)$, and the map $v \mapsto \alpha^p f\left(\cdot, \frac{v}{\alpha}\right)$ is continuous from $\overline{\mathcal{O}}$ into $L^r(\Omega)$.*

(iii) *for any $v \in \overline{\mathcal{O}}$, $\lim_{\alpha \to 0^+} \alpha^p f\left(\cdot, \frac{v}{\alpha}\right) = \gamma v^p$ with convergence in $L^r(\Omega)$ for any $r \in [1, \infty)$.*

Proof. (i) follows immediately from (H2).

To prove (ii), consider an arbitrary function $v \in \overline{\mathcal{O}}$. By Remarks 3.11 and 3.15, there exist positive constants $c_{\boldsymbol{u}}$ and $c'_{\boldsymbol{u}}$ such that $c_{\boldsymbol{u}} d_\Omega \leq \boldsymbol{u} \leq c'_{\boldsymbol{u}} d_\Omega$ in Ω, and there exist positive constants $c_{\mathcal{O}}$ and $c'_{\mathcal{O}}$ such that $c_{\mathcal{O}} d_\Omega \leq v \leq c'_{\mathcal{O}} d_\Omega$ in Ω for any $v \in \overline{\mathcal{O}}$. Then there exist positive constants $c''_{\mathcal{O}}$ and $c'''_{\mathcal{O}}$ such that

$$c''_{\mathcal{O}} \leq \frac{v}{\boldsymbol{u}} \leq c'''_{\mathcal{O}} \text{ in } \Omega \text{ for any } v \in \overline{\mathcal{O}}. \tag{4.1}$$

Therefore, by (H1) and (H2), it follows that, for any $\alpha > 0$, $\alpha f\left(\cdot, \frac{v}{\alpha}\right) \in L^\infty(\Omega)$. Then $\alpha^p f\left(\cdot, \frac{v}{\alpha}\right)$ belongs to $L^r(\Omega)$ for any $r \in [1, \infty)$, and so the first assertion of (ii) holds.

To see the second assertion of (ii), consider, for an arbitrary $v \in \overline{\mathcal{O}}$, a sequence $\{v_j\}_{j\in\mathbb{N}} \subset \overline{\mathcal{O}}$ such that $\{v_j\}_{j\in\mathbb{N}}$ converges to v in $C^1_{\mathcal{B}}(\overline{\Omega})$. For $\alpha \in (0, \infty)$ and $r \in [1, \infty)$ we have, by (H1),

$$\lim_{j\to\infty} \left|\alpha^p f\left(\cdot, \frac{v_j}{\alpha}\right) - \alpha^p f\left(\cdot, \frac{v}{\alpha}\right)\right|^r = 0 \text{ a.e. in } x \in \Omega. \tag{4.2}$$

Also, by (4.1), we have $\frac{v}{\alpha} \in L^\infty(\Omega)$ and that $\left\{\frac{v_j}{\alpha}\right\}_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Then, by (H1), there exists $\widetilde{M} > 0$ such that, for all $j \in \mathbb{N}$.

$$\left|\alpha^p f\left(\cdot, \frac{v_j}{\alpha}\right) - \alpha^p f\left(\cdot, \frac{v}{\alpha}\right)\right|^r \leq \widetilde{M} \text{ a.e. in } \Omega. \tag{4.3}$$

Therefore, since $\widetilde{M}\boldsymbol{u}^{rp} \in L^1(\Omega)$, by (4.2), (4.3), and the Lebesgue dominated convergence theorem we get that

$$\lim_{j\to\infty} \left\|\alpha^p f\left(\cdot, \frac{v_j}{\alpha}\right) - \alpha^p f\left(\cdot, \frac{v}{\alpha}\right)\right\|_r = 0,$$

and so the second assertion of (ii) holds.

To prove (iii), consider $\varepsilon > 0$ and an arbitrary function $v \in \overline{\mathcal{O}}$. By (H2), there exists $t_0 = t_0(\varepsilon, v) > 0$ such that $|f^*(\cdot, t) - \gamma| \leq \frac{\varepsilon}{2\|v\|^p_\infty}$ in Ω whenever $t \geq t_0$. For $\alpha > 0$, let

$$E_{v,\alpha.\varepsilon} := \{x \in \Omega : v(x) \geq \alpha t_0(\varepsilon)\}, \tag{4.4}$$

$$F_{v,\alpha,\varepsilon} := \{x \in \Omega : v(x) < \alpha t_0(\varepsilon)\}. \tag{4.5}$$

Let f^* be as given by Definition 4.1. Then

$$\begin{aligned}\alpha^p f\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p &= \alpha^p \left(\frac{v}{\alpha}\right)^p f^*\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p \\ &= v^p\left(f^*\left(\cdot, \frac{v}{\alpha}\right) - \gamma\right),\end{aligned}$$

and thus, for each $\alpha \in (0, 1)$,

$$\left|\alpha^p f\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p\right| \leq \frac{\varepsilon}{2} \text{ in } E_{v,\alpha,\varepsilon}. \tag{4.6}$$

and so

$$\int\limits_{E_{v,\alpha,\varepsilon}} \left|\alpha^p f\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p\right|^r \leq \left(\frac{\varepsilon}{2}\right)^r |\Omega|. \tag{4.7}$$

On the other hand, by Remark 3.15, there exists a positive constant c such that

$$\frac{v(x)}{\alpha(x)} > c \text{ in } \Omega \text{ for any } v \in \overline{\mathcal{O}} \text{ and } \alpha \in (0, 1), \tag{4.8}$$

and, by (H1) and (H2), f^* is bounded in $\overline{\Omega} \times [c, \infty)$. Let $M^{**} > 0$ be such that

$$|f^*| \le M^{**} \text{ in } \overline{\Omega} \times (c, \infty). \tag{4.9}$$

Thus, taking into account the definition of f^*, and (4.8), and (4.9), we get that, for any $v \in \overline{\mathcal{O}}$ and $\alpha \in (0,1)$,

$$\begin{aligned} \left| \alpha^p f\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p \right| &= \left| v^p \left(f^*\left(\cdot, \frac{v}{\alpha}\right) - \gamma \right) \right| \le v^p \left(M^{**} + \|\gamma\|_\infty \right) \\ &\le \alpha^p t_0^p(\varepsilon, v) \left(M^{**} + \|\gamma\|_\infty \right) \text{ in } F_{v,\alpha,\varepsilon}. \end{aligned} \tag{4.10}$$

Let $\alpha_0 := a_0(\varepsilon, v)$ be defined by

$$\alpha_0(\varepsilon, v) := \left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}} (t_0(\varepsilon, v))^{-1} \left[\left(M^{**} + \|\gamma\|_\infty\right)^{-r} |\Omega|^{-1} \right]^{\frac{1}{pr}}. \tag{4.11}$$

Then, for $r \in [1, \infty)$, by (4.10),

$$\int\limits_{F_{v,\alpha,\varepsilon}} \left| \alpha^p f\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p \right|^r \le \left(\frac{\varepsilon}{2}\right)^r \text{ whenever } 0 < \alpha \le \alpha_0, \tag{4.12}$$

and so (iii) follows from (4.7) and (4.12). □

Definition 4.3. For $\alpha \in (0, \infty)$ and $v \in \overline{\mathcal{O}}$ let $T_\alpha(v)$ be defined by

$$T_\alpha(v) := (-\Delta)^{-1} \left(\alpha^p f\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p \right).$$

Lemma 4.4. *For $\alpha \in (0, \infty)$ and $v \in \overline{\mathcal{O}}$, $T_\alpha(v)$ is well defined, belongs to $C^1_{\mathcal{B}}(\overline{\Omega})$, and $T_\alpha : \overline{\mathcal{O}} \to C^1_{\mathcal{B}}(\overline{\Omega})$ is a continuous and compact operator.*

Proof. From Lemma 4.2 the mapping $v \mapsto \alpha^p f\left(\cdot, \frac{v}{\alpha}\right)$ is continuous from $\overline{\mathcal{O}}$ into $L^q(\Omega)$ for any $q \in [1, \infty)$. Then, since $(-\Delta)^{-1} : L^q(\Omega) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is a continuous and compact operator, and since, as seen in the previous section, $v \mapsto T(v) := (-\Delta)^{-1}(\gamma v^p)$ is a continuous and compact operator from $\overline{\mathcal{O}}$ into $C^1_{\mathcal{B}}(\overline{\Omega})$, the lemma follows. □

Lemma 4.5. $\lim_{\alpha \to 0^+} \sup_{v \in \overline{\mathcal{O}}} \|T_\alpha(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} = 0$.

Proof. Let f^* be as given by Definition 4.1. Let $v \in \overline{\mathcal{O}}$, $\alpha \in (0,1)$. Then

$$\alpha^p f\left(\cdot, \frac{v}{\alpha}\right) - \gamma v^p = v^p \left(f^*\left(\cdot, \frac{v}{\alpha}\right) - \gamma \right), \tag{4.13}$$

Also, by (H2), $\lim_{t \to \infty} f^*(\cdot, t) = \gamma$ uniformly on $\overline{\Omega}$. Let $\varepsilon > 0$. Then there exists $t_0 = t_0(\varepsilon) > 0$ such that $|f^*(x,t) - \gamma(x)| \le \varepsilon$ for all $x \in \Omega$ whenever $t \ge t_0(\varepsilon)$. Let $E_{\alpha,v,\varepsilon}$ and $F_{\alpha,v,\varepsilon}$ be defined by (4.4) and (4.5), respectively, and let $r > n$. Then $\left| f^*\left(\cdot, \frac{v}{\alpha}\right) - \gamma \right| \le \varepsilon$ in $E_{\alpha,v,\varepsilon}$. Let T be as given by Definition 3.7. Since $v \in \overline{\mathcal{O}} := \overline{B}_\rho(u)$ and taking into account the conditions imposed on ρ (cf. Definition 3.14), we have

$$|v| \le \|v\|_{C^1_{\mathcal{B}}(\overline{\Omega})} \le \|v - \boldsymbol{u}\|_{C^1_{\mathcal{B}}(\overline{\Omega})} + \|\boldsymbol{u}\|_{C^1_{\mathcal{B}}(\overline{\Omega})} \le \|\boldsymbol{u}\|_{C^1_{\mathcal{B}}(\overline{\Omega})} + \frac{1}{2} c_{\boldsymbol{u}}$$

with c_u (which is independent of v) as given by Remark 3.11. Thus, by (4.13),

$$\begin{aligned}\int\limits_{E_{\alpha,v,\varepsilon}} \left|\alpha^p f\left(\cdot,\frac{v}{\alpha}\right)-\gamma v^p\right|^r &= \int\limits_{E_{\alpha,v,\varepsilon}} \left|v^p\left(f^*\left(\cdot,\frac{v}{\alpha}\right)-\gamma\right)\right|^r \\ &\leq \left(\|u\|_{C^1_{\mathcal{B}}(\overline{\Omega})}+\frac{1}{2}c_{\boldsymbol{u}}\right)^{pr}|\Omega|\varepsilon^r.\end{aligned} \tag{4.14}$$

On the other hand, by Remark 3.15 there exists a positive constant c such that $\frac{v}{\alpha}\geq c$ for any $v\in\overline{\mathcal{O}}$ and $\alpha\in(0,1)$. Since $v<\alpha t_0$ in $F_{\alpha,v,\varepsilon}$, from (H1) it follows that there exists $M^{**}>0$ such that $\left|f\left(\cdot,\frac{v}{\alpha}\right)\right|\leq M^{**}$ in $F_{\alpha,v,\varepsilon}$ for any $v\in\overline{\mathcal{O}}$ and $\alpha\in(0,1)$. Therefore, since $v<\alpha t_0$ in $F_{\alpha,v,\varepsilon}$ we get

$$\begin{aligned}\int\limits_{F_{\alpha,v,\varepsilon}} \left|\alpha^p f\left(\cdot,\frac{v}{\alpha}\right)-\gamma v^p\right|^r &= \left\|\alpha^p f\left(\cdot,\frac{v}{\alpha}\right)-\gamma v^p\right\|^r_{L^r(F_{\alpha,v,\varepsilon})} \\ &\leq \left(\left\|\alpha^p f\left(\cdot,\frac{v}{\alpha}\right)\right\|_{L^r(F_{\alpha,v,\varepsilon})}+\|\gamma v^p\|_{L^r(F_{\alpha,v,\varepsilon})}\right)^r \\ &\leq \left(\alpha^p M^{**}|\Omega|^{\frac{1}{r}}+\|\gamma\|_\infty\|v^p\|_{L^r(F_{\alpha,v,\varepsilon})}\right)^r \\ &\leq \left(\alpha^p M^{**}|\Omega|^{\frac{1}{r}}+\|\gamma\|_\infty\alpha^p t_0^p(\varepsilon)\,|\Omega|^{\frac{1}{r}}\right)^r \\ &\leq \alpha^{pr}\left(M^{**}+\|\gamma\|_\infty t_0^p(\varepsilon)\right)^r|\Omega|\,.\end{aligned}$$

Let

$$\widetilde{\alpha}_0(\varepsilon):=\left(\frac{1}{2}\right)^{\frac{1}{pr}}\varepsilon^{\frac{1}{p}}\left(M^{**}+\|\gamma\|_\infty t_0^p(\varepsilon)\right)^{-\frac{1}{p}}|\Omega|^{-\frac{1}{pr}}.$$

Then, for any $v\in\overline{\mathcal{O}}$,

$$\int\limits_{F_{\alpha,v,\varepsilon}} \left|\alpha^p f\left(\cdot,\frac{v}{\alpha}\right)-\gamma v^p\right|^r \leq \frac{1}{2}\varepsilon^r \text{ whenever } 0<\alpha\leq\widetilde{\alpha}_0(\varepsilon)\,. \tag{4.15}$$

Then, by (4.14) and (4.15) we have, for any $v\in\overline{\mathcal{O}}$,

$$\left\|\alpha^p f\left(\cdot,\frac{v}{\alpha}\right)-\gamma v^p\right\|_r\leq c\varepsilon \text{ if } \alpha\geq\widetilde{\alpha}_0(\varepsilon)$$

where c is a positive constant independent of α and v. Thus, for another positive constant c' independent of α and v, we have,

$$\sup_{v\in\overline{\mathcal{O}}}\|T_\alpha(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})}=\sup_{v\in\overline{\mathcal{O}}}\left\|(-\Delta)^{-1}\left(\alpha^p f\left(\cdot,\frac{v}{\alpha}\right)-\gamma v^p\right)\right\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\leq c'\varepsilon \quad \text{if } \alpha\geq\alpha_0(\varepsilon)$$

which gives the lemma. □

Lemma 4.6. $\inf_{v\in\partial\mathcal{O}} \|v - T(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} > 0$.

Proof. We proceed by the way of contradiction. Suppose that there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset \partial\mathcal{O}$ such that

$$\lim_{j\to\infty} \left\|v_j - T(v_{\{}j\})\right\|_{C^1_{\mathcal{B}}(\overline{\Omega})} = 0. \tag{4.16}$$

Now, $\{v_j\}_{j\in\mathbb{N}}$ is bounded in $C^1_B(\overline{\Omega})$ and $T : C^1_B(\overline{\Omega}) \to C^1_B(\overline{\Omega})$ is compact then, after pass to a subsequence if necessary (still denoted $\{v_j\}_{j\in\mathbb{N}}$) we can assume that $\left\{T(v_{\{}j\})\right\}_{j\in\mathbb{N}}$ converges, in $C^1_B(\overline{\Omega})$, to some $v \in C^1_B(\overline{\Omega})$. Thus, from (4.16) it follows that $\{v_j\}_{j\in\mathbb{N}}$ converges to v in $C^1_B(\overline{\Omega})$. Since $T : C^1_B(\overline{\Omega}) \to C^1_B(\overline{\Omega})$ is continuous, we conclude that $T(v) = v$. But $v \in \partial\mathcal{O}$, which contradicts the fact that T_λ has a unique non-identically zero fixed point in $C^1_B(\overline{\Omega})$ (namely $\boldsymbol{u}$, which belongs to $\mathcal{O}$). □

Lemma 4.7. *For α positive and small enough,*

(i) $\sup_{v\in\mathcal{O}} \|T_\alpha(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} < \inf_{v\in\partial\mathcal{O}} \|v - T(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$,
(ii) *$T + T_\alpha$ has no fixed points in $\partial\mathcal{O}$.*

Proof. (i) follows immediately from Lemmas 4.6 and 4.5.

To see (ii) we proceed by the way of contradiction. Suppose that $v \in \partial\mathcal{O}$ and $(T + T_\alpha)(v) = v$. Then $v - T(v) = T_\alpha(v)$. Since T_α is continuous, we have

$$\sup_{w\in\overline{\mathcal{O}}} \|T_\alpha(w)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} = \sup_{v\in\mathcal{O}} \|T_\alpha(w)\|_{C^1_{\mathcal{B}}(\overline{\Omega})}.$$

Thus,

$$\begin{aligned}\sup_{w\in\overline{\mathcal{O}}} \|T_\alpha(w)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} &\geq \|T_\alpha(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} = \|v - T(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})}\\ &\geq \inf_{w\in\partial\mathcal{O}} \|w - T(w)\|_{C^1_{\mathcal{B}}(\overline{\Omega})},\end{aligned}$$

which contradicts (i). □

Lemma 4.8. $\inf_{v\in\partial\mathcal{O}} \|T(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} > 0$.

Proof. The lemma follows immediately from Remark 3.12 and Definition 3.14 of $\mathcal{O}$. □

Proof of Theorem 1.1. To prove the theorem we follow some ideas of [22] used there to deal with superlinear problems. By Lemma 3.8, $T : \overline{\mathcal{O}} \to C^1_{\mathcal{B}}(\overline{\Omega})$ is a continuous and compact operator, and by Lemma 3.18, T has no fixed pints in $\partial\mathcal{O}$ and $\deg_{LS}(I - T, \mathcal{O}, 0) = \pm 1$. On the other hand, taking into account Lemma 4.4, Lemma 3.8 gives that for $\alpha > 0$, $T + T_\alpha : \overline{\mathcal{O}} \to C^1_{\mathcal{B}}(\overline{\Omega})$ is a continuous and compact operator, and Lemma 4.7 says that, for α positive and small enough, $T + T_\alpha$ has no fixed pints in $\partial\mathcal{O}$ (and thus $\deg_{LS}(I - (T + T_\alpha), \mathcal{O}, 0)$ is well defined). By Lemma 4.8, $\inf_{v\in\partial\mathcal{O}} \|T(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} > 0$, and by Lemma 4.6,

$$\lim_{\alpha\to 0^+} \sup_{v\in\overline{\mathcal{O}}} \|T_\alpha(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} = 0.$$

Then, for α positive and small enough,

$$\sup_{v\in\overline{\mathcal{O}}}\|T_\alpha(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})} < \inf_{v\in\partial\mathcal{O}}\|-T(v)\|_{C^1_{\mathcal{B}}(\overline{\Omega})}$$

and thus, by Remark 3.17(iii), for such α we have

$$\deg_{LS}\left(I-(T_\alpha+T),\mathcal{O},0\right)=\deg_{LS}\left(I-T,\mathcal{O},0\right)=\pm1.$$

Then there exists $\alpha_0>0$ such that, for $0<\alpha<\alpha_0$, the problem

$$\begin{cases} -\Delta v=\alpha^p f\left(\cdot,\frac{v}{\alpha}\right) & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega,\\ v>0 & \text{in } \Omega \end{cases}$$

has a weak solution v in $\mathcal{O}$. By writing $u=\frac{v}{\alpha}$ it follows that, for $0<\alpha<\alpha_0$, the problem

$$\begin{cases} -\Delta u=\alpha^{p-1} f(\cdot,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\\ u>0 & \text{in } \Omega \end{cases}$$

has a weak solution $u\in D$. Then, for $\lambda>\alpha_0^{p-1}$ problem (1.8) has a weak solution $u\in D$, which, by the definition of D, satisfies, for some constant $c>0$, $u\geq cd_\Omega$ in Ω. □

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