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ON A NONLOCAL $p(x)$-LAPLACIAN DIRICHLET PROBLEM INVOLVING SEVERAL CRITICAL SOBOLEV–HARDY EXPONENTS

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Abstract. The aim of this work is to present a result of multiplicity of solutions, in generalized Sobolev spaces, for a non-local elliptic problem with $p(x)$-Laplace operator containing k distinct critical Sobolev–Hardy exponents combined with singularity points

$$\begin{cases} M(\psi(u))(-\Delta_{p(x)}u + |u|^{p(x)-2}u) = \sum_{i=1}^{k} h_i(x)\dfrac{|u|^{p^*_{s_i}(x)-2}u}{|x|^{s_i(x)}} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $0 \in \Omega$ and $1 < p^- \leq p(x) \leq p^+ < N$. The real function M is bounded in $[0, +\infty)$ and the functions h_i $(i = 1, \ldots, k)$ and f are functions whose properties will be given later. To obtain the result we use the Lions' concentration-compactness principle for critical Sobolev–Hardy exponent in the space $W_0^{1,p(x)}(\Omega)$ due to Yu, Fu and Li, and the Fountain Theorem.

Keywords: generalized Lebesgue–Sobolev spaces, $p(x)$-Laplacian nonlocal operator, Sobolev–Hardy critical exponents, concentration-compactness principle for critical Sobolev–Hardy exponent, fountain theorem.

Mathematics Subject Classification: 35J60, 58E05.

1. INTRODUCTION

In this paper we study the following class of $p(x)$-Laplacian equations coupled with the homogeneous Dirichlet boundary conditions given by

$$\begin{cases} M(\psi(u))(-\Delta_{p(x)}u + |u|^{p(x)-2}u) = \sum_{i=1}^{k} h_i(x)\dfrac{|u|^{p^*_{s_i}(x)-2}u}{|x|^{s_i(x)}} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with $0 \in \Omega$. Furthermore, the variable exponent $p(x)$ is Lipschitz continuous and radially symmetric on $\overline{\Omega}$ with the growth condition

$$1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p(x) \leq \max_{x\in\overline{\Omega}} p(x) =: p^+ < N,$$

while $s_1(x), \ldots, s_k(x)$ are Lipschitz continuous, radially symmetric on $\overline{\Omega}$ such that $0 \leq s_i(x) \ll p(x)$ for all $i \in \{1, \ldots, k\}$, where $s_i(x) \ll p(x)$ denotes the fact that $\inf(p(x) - s_i(x)) > 0$, with

$$|\{x \in \Omega : s_i(x) = s_j(x) \text{ for all } i \neq j\}| = 0,$$

where $|\cdot|$ denotes the Lebesgue measure. The critical Sobolev–Hardy exponent for each $i = 1, \ldots, k$ is defined by

$$p^*_{s_i}(x) = \frac{p(x) \cdot (N - s_i(x))}{N - p(x)}.$$

Note that

$$p^*_0(x) = \frac{Np(x)}{N - p(x)} = p^*(x)$$

is the critical Sobolev exponent. The $p(x)$-Laplace operator $\Delta_{p(x)}$ given by

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_j}\right)$$

is a natural extension of the p-Laplace operator. We define

$$\psi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\, dx$$

and $M : \mathbb{R}^+ \to \mathbb{R}^+$ where $\mathbb{R}^+ = [0, +\infty)$ is a class of continuous functions satisfying the following growth condition:

(M_0) $m_0 \leq M(\tau) \leq m_1$ for all $\tau \geq 0$,

where m_0 and m_1 are positive constants.

Every real function h_i is continuous on $\overline{\Omega}$ and satisfy:

$$h_i(x) = h_i(|x|) > 0,\ \forall x \in \overline{\Omega} - \{0\} \quad \text{and} \quad h_i(0) = 0, \tag{1.2}$$

and

$$\lim_{x\to 0} h_i(x) \cdot \frac{1}{|x|^{s_i(x)}} = +\infty, \quad \forall i \in \{1, \ldots, k\}. \tag{1.3}$$

A typical example for functions h_i's can be given by $\phi(x) = \frac{1}{|\ln|x||}$ if $x \neq 0$, and $\phi(x) = 0$ if $x = 0$.

We assume the following hypotheses for the function $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$:

(f_1) f satisfies the Carathéodory condition;
(f_2) There are constants c_1, c_2 such that $|f(x,t)| \leq c_1 + c_2|t|^{q(x)-1}$, where $q : \overline{\Omega} \longrightarrow \mathbb{R}$ is a measurable Lebesgue function such that $p(x) \ll q(x) \ll p^*_{s_i}(x)$ for all $i \in \{1, \ldots, k\}$ and for all $x \in \overline{\Omega}$,
(f_3) $f(x,t) = f(|x|, t)$ for all $(x,t) \in \Omega \times \mathbb{R}$,
(f_4) $f(x,t) = -f(x,-t)$ for all $(x,t) \in \Omega \times \mathbb{R}$.

This problem is an extension of the application, of the presented result by Yu, Fu and Li in [30], where they presented a version of the Lions' concentration-compactness principle for the critical Sobolev–Hardy exponent in $W_0^{1,p(x)}(\Omega)$, in order to solve the $p(x)$-Laplacian problem with only one Sobolev–Hardy critical term

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = h(x)\dfrac{|u|^{p^*_s(x)-2}u}{|x|^{s(x)}} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is bounded in $\mathbb{R}^N$ such that $0 \in \Omega$, $p(x), s(x)$ are Lipschitz continuous and radially symmetric on $\overline{\Omega}$ with $1 < p^- \leq p(x) \leq p^+ < N$, and $0 \leq s(x) \ll p(x)$, h and f satisfy the same conditions as this paper.

Singularity problems with critical exponents of the Sobolev–Hardy type have been studied frequently in recent years, starting when $p(x)$ is a constant function p. This is the case, for $p = 2$, of the singular critical problem with the usual Laplace operator given by

$$\begin{cases} -\Delta u - \mu\dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s)-2}u}{|x|^s} + \lambda|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ bounded and smooth; $0 \in \Omega$, $0 \leq s < 2$, $2^*(s) := \frac{2(N-s)}{N-2}$, $2 \leq r < 2^*$, $\lambda > 0$ and $0 \leq \mu < \widetilde{\mu} := \left(\frac{N-2}{2}\right)^2$. The works of Jannelli [18], Ferrero and Gazzola [14] and Cao and Peng [7] show the existence of solutions for problem (1.4) when $s = 0$ and $r = 2$ using local compactness arguments and min-max principles.

If s varies in the interval $[0,2)$ many interesting results for (1.4) due to Kang and Peng have been obtained. For example, in [21] they conclude that (1.4) has a positive solution in $H_0^1(\Omega)$ under certain assumptions for r, μ and λ, applying the Mountain Pass Theorem. In [22], with $\lambda_1(\mu)$ being the first eigenvalue of the operator $-\Delta - \mu/|x|^2$ in $H_0^1(\Omega)$ and $N \geq 7$, they guarantee the existence of at least one pair of sign-changing solutions, as $r = 2$, $0 \leq \mu < \overline{\mu} - 4$ and $0 < \lambda < \lambda_1(\mu)$. On the other hand, in [20] they prove the existence of sign-changing solutions in the range $2 < r < 2^*$ and $\lambda > 0$, employing the technique used in [17] and [29].

A particular case of equations (1.4) when $r = 2$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ is an open bounded domain and $0 \leq \mu < \widetilde{\mu} - \left(\frac{N+2}{N}\right)^2$, is investigated by Cao and Han in [6] and again by Kang and Peng in [23], where they prove at least one non-trivial solution in $H_0^1(\Omega)$ for a certain range of energy level and with the critical Sobolev–Hardy growth.

The authors Li and Lin through [26] studied the Laplacian problem generated by two terms with critical Sobolev–Hardy exponents, that is, the problem

$$\begin{cases} \Delta u + \lambda \dfrac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \dfrac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ bounded and smooth and $0 \in \partial\Omega$, $0 \leq s_2 < s_1 \leq 2$, $0 \neq \lambda \in \mathbb{R}$. In this paper they first show the existence of a least-energy solution, and then prove the existence of positive integer solutions under $\Omega = \mathbb{R}^N_+$ with certain conditions for s_1, s_2 and λ.

In [19], Kang presents the p-Laplacian version of (1.4) with arbitrary p such that $1 < p < N$ $(N \geq 3)$ in its singular quasilinear form

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = \dfrac{u^{p^*(s)-1}}{|x|^s} + \lambda a(x) u^{r-1} & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \quad u \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

considering $0 \leq s < p$, $1 < r < p^*(s)$, $\lambda > 0$ and $0 \leq \mu < \widetilde{\mu} := \left(\frac{N-p}{p}\right)^p$ with $a(x) \in C(\mathbb{R}^N) \cap L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$ and $a(0) > 0$. He obtains the existence of non-trivial solutions to (1.5) via mountain pass arguments and analysis techniques.

In addition to mathematical motivations, the interest in elliptic equations of the $p(x)$-Laplacian nature has motivations in the context of physical applications, such as, for example, in the field of nonlinear elastic mechanics and in dynamic models of electrorheological fluids that allow changing the mechanical properties of these fluids when exposed to electromagnetic fields external (see [1, 2, 27] and the references therein). In [28], Růžička presents another physical implication for this type of equations through image processing.

Motivated by the problems mentioned above, and inspired by [30], we study the problem (1.1) establishing the following theorem as the main result of this paper:

Theorem 1.1. *Assume that* (M_0), (1.2), (1.3) *and* (f_1)–(f_4) *hold. Moreover, assume* $\frac{m_1 p^+}{m_0} < p^{*\,-}_{s_i}$ *for all* $i \in \{1, \ldots, k\}$. *Then, the problem* (1.1) *has a sequence* $(u_n) \subset W_0^{1,p(x)}(\Omega)$ *of solutions such that, for its energy functional* $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$, *we have* $J(u_n) \to +\infty$, *as* $n \to +\infty$.

Our result shows that there are infinitely many solutions, for example, for the following problem, naturally with the hypotheses of Theorem 1.1, given by

$$\begin{cases} (2 + \sin(\psi(u)))(-\Delta_{p(x)} u + |u|^{p(x)-2} u) = \sum\limits_{i=1}^{k} h_i(x) \frac{|u|^{p^*_{s_i}(x)-2} u}{|x|^{s_i(x)}} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Problems in the form (1.1) are associated with the energy functional

$$\begin{aligned} J(u) = \widehat{M}(\psi(u)) - \int_\Omega \frac{h_1(x)}{p^*_{s_1}(x)} \frac{|u|^{p^*_{s_1}(x)}}{|x|^{s_1(x)}}\, dx - \ldots - \int_\Omega \frac{h_k(x)}{p^*_{s_k}(x)} \frac{|u|^{p^*_{s_k}(x)}}{|x|^{s_k(x)}}\, dx \\ - \int_\Omega F(x,u)\, dx, \end{aligned} \tag{1.6}$$

for all $u \in W_0^{1,p(x)}(\Omega)$, where

$$F(x,t) = \int_0^t f(x,s)ds \quad \text{and} \quad \widehat{M}(t) := \int_0^t M(s)ds.$$

This functional is differentiable and its Fréchet derivative is given by

$$\begin{aligned} J'(u)v = M(\psi(u)) \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv \right) dx \\ - \int_\Omega h_1(x) \frac{|u|^{p^*_{s_1}(x)-2} uv}{|x|^{s_1(x)}}\, dx - \ldots - \int_\Omega h_k(x) \frac{|u|^{p^*_{s_k}(x)-2} uv}{|x|^{s_k(x)}}\, dx \\ - \int_\Omega f(x,u) v\, dx, \end{aligned}$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1.1) if and only if u is a critical point of J.

This paper is organized as follows: in Section 2 we introduce a summary on Sobolev–Lebesgue spaces of variable exponents, and in Section 3 we prove Theorem 1.1.

2. PRELIMINARIES ON VARIABLE EXPONENT SPACES

In this paper we consider

$$\mathcal{C}^+(\overline{\Omega}) := \left\{ h \in C(\overline{\Omega}) : h(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}$$

and for each $h \in \mathcal{C}^+(\overline{\Omega})$ we define

$$h^+ := \max_{\overline{\Omega}} h(x) \quad \text{and} \quad h^- := \min_{\overline{\Omega}} h(x).$$

We denote by $\mathcal{M}(\Omega)$ the set of real measurable functions defined on Ω.

Definition 2.1. Let $p(x) \in \mathcal{C}^+(\overline{\Omega})$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\}. \tag{2.1}$$

$L^{p(x)}(\Omega)$ is a Banach space equipped with the Luxemburg norm defined as

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

For all $p(x) \in \mathcal{C}^+(\overline{\Omega})$, we define $L^{p'(x)}(\Omega)$, the dual space of $L^{p(x)}(\Omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

for all $x \in \overline{\Omega}$.

The proofs of the following propositions and theorems can be found in Kováčik and Rákosník [24], Fan, Shen and Zhao [13], Fan and Zhao [12], Fu [15], Fan and Zhang [11], Diening [9], Bonder and Silva [5], Corrêa and Costa [8] and Bonder, Saintier and Silva [3, 4].

Proposition 2.2. *Let*

$$\rho_p(u) := \int_\Omega |u(x)|^{p(x)} \, dx.$$

For all $u, u_n \in L^{p(x)}(\Omega)$, we have:

(i) *for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho_p\left(\frac{u}{\lambda}\right) = 1$,*
(ii) *$|u|_{p(x)} < 1 (= 1; > 1)$ if and only if $\rho_p(u) < 1 (= 1; > 1)$,*
(iii) *if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \leq \rho_p(u) \leq |u|_{p(x)}^{p^+}$,*
(iv) *if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \leq \rho_p(u) \leq |u|_{p(x)}^{p^-}$,*
(v) *$\lim_{n\to+\infty} |u_n|_{p(x)} = 0$ if and only if $\lim_{n\to+\infty} \rho_p(u_n) = 0$,*
(vi) *$\lim_{n\to+\infty} |u_n|_{p(x)} = +\infty$ if and only if $\lim_{n\to+\infty} \rho_p(u_n) = +\infty$.*

Proposition 2.3. *If $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, then*

$$\left| \int_\Omega u(x) \cdot v(x) \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} \cdot |v|_{p'(x)}.$$

Proposition 2.4. *If $|\Omega| < +\infty$ and $p, q \in \mathcal{C}^+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for all $x \in \Omega$, then we have continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.*

Definition 2.5. Let $\Omega \subset \mathbb{R}^N$ be an open set, $m \in \mathbb{Z}_+^*$, $\alpha \in \mathbb{N}^N$ and $p(x) \in \mathcal{C}^+(\overline{\Omega})$. The generalized Lebesgue–Sobolev space $W^{m,p(x)}(\Omega)$ is defined by

$$W^{m,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), \text{ where } |\alpha| \leq m \right\}.$$

$W^{m,p(x)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{m,p(x)} = \sum_{|\alpha| \leq m} |D^\alpha u|_{p(x)}.$$

We define $W_0^{1,p(x)}(\Omega)$ as being the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(x)}$. The space $W_0^{1,p(x)}(\Omega)$ is Banach.

The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive and separable Banach spaces.

In the space $W^{1,p(x)}(\Omega)$ there is an equivalent norm for $\|\cdot\|_{1,p(x)}$:

$$\|u\|_{W^{1,p(x)}} := \inf\left\{\lambda > 0 : \int\limits_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}$$

for all $u \in W^{1,p(x)}(\Omega)$. In fact, we have

$$\frac{1}{2}\left(|\nabla u|_{p(x)} + |u|_{p(x)}\right) \le \|u\|_{W^{1,p(x)}} \le 2\left(|\nabla u|_{p(x)} + |u|_{p(x)}\right).$$

Proposition 2.6 (Poincaré inequality). *If $u \in W_0^{1,p(x)}(\Omega)$, then*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)},$$

where C is a constant that does not depend on u.

Note that, by the Poincaré inequality, the norms $\|\cdot\|_{1,p(x)}$ and $\|u\| = |\nabla u|_{p(x)}$ are equivalent in $W_0^{1,p(x)}(\Omega)$. From now on we will work in $W_0^{1,p(x)}(\Omega)$ with the norm $\|u\| = |\nabla u|_{p(x)}$.

Proposition 2.7. *Consider*

$$\rho_{1,p(x)}(u) := \int\limits_\Omega \left(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}\right) dx, \quad u \in W^{1,p(x)}(\Omega).$$

For all $u, u_n \in W^{1,p(x)}(\Omega)$, we have:

(i) $\|u\|_{1,p(x)} < 1(=1; > 1)$ *if and only if* $\rho_{1,p(x)}(u) < 1(=1; > 1)$,
(ii) *if* $\|u\|_{1,p(x)} > 1$, *then* $\|u\|_{1,p(x)}^{p^-} \le \rho_{1,p(x)}(u) \le \|u\|_{1,p(x)}^{p^+}$,
(iii) *if* $\|u\|_{1,p(x)} < 1$, *then* $\|u\|_{1,p(x)}^{p^+} \le \rho_{1,p(x)}(u) \le \|u\|_{1,p(x)}^{p^-}$,
(iv) $\lim_{n\to+\infty} \|u_n\|_{1,p(x)} = 0$ *if and only if* $\lim_{n\to+\infty} \rho_{1,p(x)}(u_n) = 0$,
(v) $\lim_{n\to+\infty} \|u_n\|_{1,p(x)} = +\infty$ *if and only if* $\lim_{n\to+\infty} \rho_{1,p(x)}(u_n) = +\infty$.

Theorem 2.8. *Let Ω be a bounded in $\mathbb{R}^N$, $p \in C(\overline{\Omega})$ with $1 < p^- \le p^+ < +\infty$. Then for any measurable function $q(x)$ with $1 \le q(x) \ll p^*(x)$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Theorem 2.9. *If* $p : \overline{\Omega} \longrightarrow \mathbb{R}$ *is Lipschitz continuous and* $1 < p^- \le p^+ < +\infty$*, then for any measurable function* $q(x)$ *with* $p(x) \le q(x) \le p^*(x)$*, there is a continuous embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$*.*

Proposition 2.10 (Fan and Zhang [11])**.** *Let* $L_{p(x)} : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))'$ *be such that*

$$L_{p(x)}(u)(v) = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \; dx, \quad u, v \in W_0^{1,p(x)}(\Omega).$$

Then:

(i) $L_{p(x)} : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))'$ *is a continuous, bounded and strictly monotone operator,*
(ii) $L_{p(x)}$ *is a mapping of type* S_+*, i.e. if* $u_n \rightharpoonup u$ *in* $W_0^{1,p(x)}(\Omega)$ *and*

$$\limsup (L_{p(x)}(u_n) - L_{p(x)}(u), u_n - u) \le 0,$$

then $u_n \to u$ *in* $W_0^{1,p(x)}(\Omega)$*,*
(iii) $L_{p(x)} : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))'$ *is a homeomorphism.*

Theorem 2.11. *Let* $\Omega \subset \mathbb{R}^N$ *be a measurable subset. Suppose that* $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a Carathéodory function and satisfies*

$$|f(x,t)| \le \alpha(x) + \beta |t|^{\frac{p_1(x)}{p_2(x)}}, \quad \forall x \in \Omega, t \in \mathbb{R},$$

where $p_1(x), p_2(x) \ge 1$ *for all* $x \in \Omega$*,* $\alpha(x) \in L^{p_2(x)}(\Omega)$ *such that* $\alpha(x) \ge 0, x \in \Omega$ *and* $\beta \ge 0$ *is a constant. Then the Nemytskii operator from* $L^{p_1(x)}(\Omega)$ *to* $L^{p_2(x)}(\Omega)$ *defined by* $(N_f u)(x) = f(x, u(x))$ *is a continuous and bounded operator.*

Definition 2.12 (Yu, Fu and Li [30])**.** Let $\Omega \subset \mathbb{R}^N$ be an open set, $p(x) \in \mathcal{C}^+(\overline{\Omega})$ and $a(x)$ a real measurable function with $a(x) > 0$ for all $x \in \Omega$. We define the space

$$L^{p(x)}_{a(x)}(\Omega) := \left\{ u \in \mathcal{M}(\Omega) : \int_\Omega a(x) |u(x)|^{p(x)} \; dx < +\infty \right\}. \tag{2.2}$$

$L^{p(x)}_{a(x)}(\Omega)$ is a Banach space with the norm

$$|u|_{p(x),a(x)} = \inf \left\{ \lambda > 0 : \int_\Omega a(x) \left| \frac{u(x)}{\lambda} \right|^{p(x)} \; dx \le 1 \right\}. \tag{2.3}$$

Theorem 2.13. *Let*

$$\rho_{p,a}(u) = \int\limits_{\Omega} a(x)|u(x)|^{p(x)}\,dx.$$

If $u, u_n \in L^{p(x)}_{a(x)}(\Omega)$, then:

(i) *for $u \neq 0$, $|u|_{p(x),a(x)} = \lambda$ if and only if $\rho_{p,a}\left(\frac{u}{\lambda}\right) = 1$,*
(ii) *$|u|_{p(x),a(x)} < 1(= 1; > 1)$ if and only if $\rho_{p,a}(u) < 1(= 1; > 1)$,*
(iii) *if $|u|_{p(x),a(x)} > 1$ then $|u|^{p^-}_{p(x),a(x)} \leq \rho_{p,a}(u) \leq |u|^{p^+}_{p(x),a(x)}$,*
(iv) *if $|u|_{p(x),a(x)} < 1$ then $|u|^{p^+}_{p(x),a(x)} \leq \rho_{p,a}(u) \leq |u|^{p^-}_{p(x),a(x)}$,*
(v) *$\lim_{n\to+\infty} |u_n|_{p(x),a(x)} = 0$ if and only if $\lim_{n\to+\infty} \rho_{p,a}(u_n) = 0$,*
(vi) *$\lim_{n\to+\infty} |u_n|_{p(x),a(x)} = +\infty$ if and only if $\lim_{n\to+\infty} \rho_{p,a}(u_n) = +\infty$.*

Theorem 2.14. *Assume that $0 \in \overline{\Omega}$ and the boundary of Ω possesses the cone property. Suppose that $p(x), s(x), q(x) \in C(\overline{\Omega})$, $0 \leq s(x) < N$ for $x \in \overline{\Omega}$. If $q(x)$ satisfies $1 \leq q(x) < p^*_s(x)$ for $x \in \overline{\Omega}$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{|x|^{-s(x)}}(\Omega)$.*

Theorem 2.15 (Yu, Fu and Li [30])**.** *Assume that $0 \in \overline{\Omega}$ and the boundary of Ω possesses the cone property. Suppose that $p(x), s(x), q(x) \in C(\overline{\Omega})$, $0 \leq s(x) \ll p(x)$ for $x \in \overline{\Omega}$. There is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p^*_s(x)}_{|x|^{-s(x)}}(\Omega)$.*

For the lemma below, consider $p(x)$ Lipschitz continuous with

$$1 < p^- \leq p(x) \leq p^+ < +\infty$$

and $s(x)$ continuous on $\overline{\Omega}$.

Lemma 2.16 (Fu [16])**.** *Let $(u_n) \subset L^{p(x)}_{|x|^{-s(x)}}(\Omega)$ be bounded, and $u_n \to u \in L^{p(x)}_{|x|^{-s(x)}}(\Omega)$, a.e. on Ω, then*

$$\lim_{n\to\infty} \int\limits_{\Omega} \left(\frac{|u_n|^{p(x)}}{|x|^{s(x)}} - \frac{|u_n - u|^{p(x)}}{|x|^{s(x)}} \right) dx = \int\limits_{\Omega} \frac{|u|^{p(x)}}{|x|^{s(x)}}\,dx.$$

Let $\mathscr{M}(\overline{\Omega})$ be the class of nonnegative Borel measures with finite total mass on $\overline{\Omega}$. Let $(\mu_n) \in \mathscr{M}(\overline{\Omega})$, we say that $\mu_n \to \mu$ weakly-$*$ in $\mathscr{M}(\overline{\Omega})$ when

$$(\mu_n, u) = \int\limits_{\overline{\Omega}} u d\mu_n \longrightarrow \int\limits_{\overline{\Omega}} u d\mu = (\mu, u)$$

for every function $u \in C(\overline{\Omega}) \cap C^{\infty}_0(\Omega)$.

Now we reproduce the concentration-compactness principle of Lions for critical Sobolev–Hardy exponent $p^*_s(x)$ extended by Yu, Fu and Li [30], in the space $W^{1,p(x)}_0(\Omega)$.

Theorem 2.17. *Let (u_n) be a sequence in $W_0^{1,p(x)}(\Omega)$ with norm $\|u_n\|_{1,p(x)} \leq 1$ such that*

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega),$$
$$|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \to \mu \text{ weakly-}* \text{ in } \mathscr{M}(\overline{\Omega}),$$
$$\frac{|u_n|^{p_s^*(x)}}{|x|^{s(x)}} \to \nu \text{ weakly-}* \text{ in } \mathscr{M}(\overline{\Omega}),$$

as $n \to +\infty$. Then, the limit measures are of the form

$$\mu = |\nabla u|^{p(x)} + |u|^{p(x)} + \sum_{j\in\mathscr{J}} \mu_j \delta_{x_j} + \mu_0\delta_0 + \widetilde{\mu}, \quad \mu(\overline{\Omega}) \leq 1,$$
$$\nu = \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}} + \sum_{j\in\mathscr{J}} \nu_j \delta_{x_j} + \nu_0\delta_0, \quad \nu(\overline{\Omega}) \leq C^*,$$

where $\mathscr{J}$ is a countable set, $\{\mu_j\} \subset [0,+\infty)$, $\{\nu_j\} \subset [0,+\infty)$, $\mu_0 \geq 0$, $\nu_0 \geq 0$, $\{x_j\} \in \overline{\Omega}$, $\widetilde{\mu} \in \mathscr{M}(\overline{\Omega})$ is a nonatomic positive measure, δ_{x_j} and δ_0 are atomic measures which concentrate on x_j and 0, respectively, and

$$C^* = \sup\left\{ \int\limits_\Omega \frac{|u|^{p_s^*(x)}}{|x|^{s(x)}}\, dx : u \in W_0^{1,p(x)}(\Omega), \|u\|_{1,p(x)} \leq 1 \right\}.$$

The atoms and the regular part satisfy the generalized Sobolev inequalities:

$$\nu(\overline{\Omega}) \leq C^* \max\left\{ \mu(\overline{\Omega})^{\frac{p_s^{*+}}{p^-}}, \mu(\overline{\Omega})^{\frac{p_s^{*-}}{p^+}} \right\},$$
$$\nu_j \leq C^* \max\left\{ \mu_j^{\frac{p_s^{*+}}{p^-}}, \mu_j^{\frac{p_s^{*-}}{p^+}} \right\},$$
$$\nu_0 \leq C^* \max\left\{ \mu_0^{\frac{p_s^{*+}}{p^-}}, \mu_0^{\frac{p_s^{*-}}{p^+}} \right\}.$$

In the next section, we will apply this principle distinctly for each $p_{s_i}^*(x)$ $(i = 1, \ldots, k)$ of our problem in order to ensure important convergences.

3. PROOF OF THEOREM 1.1

Here we will proceed in a similar way to the proof given in [30], with important adaptations. We denote by $O(N)$ the group of orthogonal linear transformations in $\mathbb{R}^N$. Let G be a subgroup of $O(N)$, an open subset Ω of $\mathbb{R}^N$ is G-invariant if $\tau\Omega = \Omega$, for every $\tau \in G$. For $x \neq 0$, taking $|G_x|_\sharp$ as the cardinality of $G_x = \{\tau x : \tau \in G\}$, we define the cardinality of G by

$$|G|_\sharp = \inf_{x\in\mathbb{R}^N, x\neq 0} |G_x|_\sharp.$$

Considering G a subgroup of $O(N)$ and Ω a G-invariant open subset of $\mathbb{R}^N$, the action of G on $W_0^{1,p(x)}(\Omega)$ is defined by $\tau u(x) = u(\tau^{-1}x)$ for any $u \in W_0^{1,p(x)}(\Omega)$.

The subspace of invariant functions is defined by

$$W_{0,O(N)}^{1,p(x)}(\Omega) := \left\{ u \in W_0^{1,p(x)}(\Omega) : \tau u = u \text{ for all } \tau \in G \right\}. \tag{3.1}$$

We have that the functional J in (1.6) is G-invariant, because $(J \circ \tau)(u) = J(u)$ for all $\tau \in G$.

So, let us establish the functional

$$\widetilde{J} = J|_{W_{0,O(N)}^{1,p(x)}(\Omega)}.$$

Based on the principle of symmetric criticality due to Krawcewicz and Marzantowicz [25], we claim that any critical point of J is critical point of $\widetilde{J}$. Thus, it is sufficient and necessary to find critical points of $\widetilde{J}$.

The first step is to show that $\widetilde{J}$ satisfies the condition (PS). To verify this, we need to prove that (PS) sequences are bounded.

Lemma 3.1. *If $(u_n) \subset W_{0,O(N)}^{1,p(x)}(\Omega)$ is a $(PS)_c$ sequence for $\widetilde{J}$, then (u_n) is bounded in $W_{0,O(N)}^{1,p(x)}(\Omega)$.*

Proof. Let (u_n) be a sequence $(PS)_c$ for $\widetilde{J}$, that is,

$$\widetilde{J}(u_n) \to c \, (c \in \mathbb{R}) \text{ and } \widetilde{J}'(u_n) \to 0 \text{ in } \left(W_{0,O(N)}^{1,p(x)}(\Omega) \right)', \text{ as } n \to +\infty. \tag{3.2}$$

Considering $\theta \in \mathbb{R}$ such that

$$\frac{m_1 p^+}{m_0} < \theta < \min\{{p_{s_1}^*}^-, \cdots, {p_{s_k}^*}^-\},$$

where ${p_{s_i}^*}^- = \min_{\overline{\Omega}} p_{s_i}^*(x)$ for all $i \in \{1, \dots, k\}$, and C is such that

$$C \geq \widetilde{J}(u_n), \quad \forall n \geq 1,$$

we write

$$
\begin{aligned}
C+\|u_n\| &\geq \widetilde{J}(u_n)-\frac{1}{\theta}\widetilde{J}'(u_n)u_n\\
&=\widehat{M}\left(\int\limits_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)dx\right)\\
&-\int\limits_\Omega \frac{h_1(x)}{p_{s_1}^*(x)}\frac{|u_n|^{p_{s_1}^*(x)}}{|x|^{s_1(x)}}dx-\ldots-\int\limits_\Omega \frac{h_k(x)}{p_{s_k}^*(x)}\frac{|u_n|^{p_{s_k}^*(x)}}{|x|^{s_k(x)}}dx-\int\limits_\Omega F(x,u_n)\,dx\\
&-\frac{1}{\theta}M\left(\int\limits_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)dx\right)\int\limits_\Omega\left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)dx\\
&+\frac{1}{\theta}\int\limits_\Omega h_1(x)\frac{|u_n|^{p_{s_1}^*(x)}}{|x|^{s_1(x)}}dx+\ldots+\frac{1}{\theta}\int\limits_\Omega h_k(x)\frac{|u_n|^{p_{s_k}^*(x)}}{|x|^{s_k(x)}}dx+\int\limits_\Omega \frac{1}{\theta}f(x,u_n)u_n dx,
\end{aligned}
$$

which implies

$$
\begin{aligned}
C+\|u_n\| \geq \left(\frac{m_0}{p^+}-\frac{m_1}{\theta}\right)&\int\limits_\Omega\left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)\,dx\\
&+\left(\frac{1}{\theta}-\frac{1}{{p_{s_1}^*}^-}\right)\int\limits_\Omega h_1(x)\frac{|u_n|^{p_{s_1}^*(x)}}{|x|^{s_1(x)}}\,dx+\\
&+\ldots+\left(\frac{1}{\theta}-\frac{1}{{p_{s_k}^*}^-}\right)\int\limits_\Omega h_k(x)\frac{|u_n|^{p_{s_k}^*(x)}}{|x|^{s_k(x)}}\,dx\\
&+\int\limits_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n-F(x,u_n)\right)\,dx.
\end{aligned}
$$

Suppose that (u_n) is not bounded in $W_{0,O(N)}^{1,p(x)}(\Omega)$. Then, if necessary, passing to a subsequence again denoted by (u_n) we have for the terms of the sequence such that $\|u_n\|>1$, by Proposition 2.7,

$$
\begin{aligned}
C+\|u_n\| \geq \left(\frac{m_0}{p^+}-\frac{m_1}{\theta}\right)\|u_n\|^{p^-}&+\left(\frac{1}{\theta}-\frac{1}{{p_{s_1}^*}^-}\right)\int\limits_\Omega h_1(x)\frac{|u_n|^{p_{s_1}^*(x)}}{|x|^{s_1(x)}}dx\\
&+\ldots+\left(\frac{1}{\theta}-\frac{1}{{p_{s_k}^*}^-}\right)\int\limits_\Omega h_k(x)\frac{|u_n|^{p_{s_k}^*(x)}}{|x|^{s_k(x)}}\,dx\\
&+\int\limits_\Omega\left(\frac{1}{\theta}f(x,u_n)u_n-F(x,u_n)\right)dx.
\end{aligned}
\tag{3.3}
$$

From (f_2) we compute

$$\left|\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right| \le C_0\left(|u_n| + |u_n|^{q(x)}\right).$$

By Young's inequality, for $\epsilon \in (0,1)$, we obtain

$$\begin{aligned} |u_n| + |u_n|^{q(x)} &\le \epsilon |u_n|^{p^*_{s_1}(x)} + C(\epsilon) + \epsilon\left(|u_n|^{q(x)}\right)^{\frac{p^*_{s_1}(x)}{q(x)}} + C(\epsilon) \\ &\le \overline{\epsilon}|u_n|^{p^*_{s_1}(x)} + \overline{C(\epsilon)}, \end{aligned}$$

where $\overline{\epsilon} := 2\epsilon$ and $\overline{C(\epsilon)} := 2C(\epsilon)$ and consequently

$$\left|\int\limits_\Omega \left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right)dx\right| \le C_0\overline{\epsilon}\int\limits_\Omega |u_n|^{p^*_{s_1}(x)}dx + \overline{C}_0. \tag{3.4}$$

Now, by (1.3), there exists $\overline{H}_{i,1} > 0$ such that $h_i(x)/|x|^{s_i(x)} > \overline{H}_{i,1}$, as $x \to 0$ and, by (1.2), there exists $\overline{H}_{i,2} > 0$ such that $h_i(x)/|x|^{s_i(x)} > H_{i,2}$, as $x \in \overline{\Omega} - \{0\}$. Then there is a constant $\overline{H}_i > 0$ such that for all $x \in \overline{\Omega}$ we get $h_i(x)/|x|^{s_i(x)} > \overline{H}_i$ and consequently

$$\int\limits_\Omega h_i(x)\frac{|u_n|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}dx > \overline{H}_i\int\limits_\Omega |u_n|^{p^*_{s_i}(x)}dx, \quad \forall i \in \{1,\dots,k\}. \tag{3.5}$$

Then by (3.3), (3.4) and (3.5) we have

$$\begin{aligned} C + \|u_n\| \ge \left(\frac{m_0}{p^+} - \frac{m_1}{\theta}\right)\|u_n\|^{p^-} + \left[\left(\frac{1}{\theta} - \frac{1}{p^{*\,-}_{s_1}}\right)\overline{H}_1 - C_0\overline{\epsilon}\right]\int\limits_\Omega |u_n|^{p^*_{s_1}(x)}dx \\ + \ldots + \left(\frac{1}{\theta} - \frac{1}{p^{*\,-}_{s_k}}\right)\overline{H}_k\int\limits_\Omega |u_n|^{p^*_{s_k}(x)}dx - \overline{C}_0. \end{aligned}$$

Take

$$0 < \overline{\epsilon} < \frac{\overline{H}_1}{C_0}\left(\frac{1}{\theta} - \frac{1}{p^{*\,-}_{s_1}}\right)$$

to get

$$C + \|u_n\| \ge \left(\frac{m_0}{p^+} - \frac{m_1}{\theta}\right)\|u_n\|^{p^-} - \overline{C}_0,$$

which is a contradiction because $p^- > 1$. Hence, (u_n) is bounded in $W^{1,p(x)}_{0,O(N)}(\Omega)$. □

Before proving the condition (PS) for $\widetilde{J}$, let us present another technical lemma.

Lemma 3.2. *Let $(u_n) \subset W^{1,p(x)}_{0,O(N)}(\Omega)$ be a sequence $(PS)_c$ for $\widetilde{J}$. Then, $u_n \to u$ in $L^{p^*_{s_i}(x)}_{|x|^{-s_i(x)}}(\Omega)$ for all $i \in \{1,\dots,k\}$.*

Proof. Due to the previous lemma, if $(u_n) \subset W^{1,p(x)}_{0,O(N)}(\Omega)$ is a sequence $(PS)_c$ for $\widetilde{J}$, we claim that (u_n) is bounded, so there exists a subsequence, still denoted by (u_n), and $u \in W^{1,p(x)}_{0,O(N)}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } W^{1,p(x)}_{0,O(N)}(\Omega)\\ &u_n \to u \text{ in } \ L^{r(x)}(\Omega), \quad \forall r(x) \in [1, p^*(x)). \end{aligned}$$

According to Theorem 2.17 we have the following convergences:

$$\begin{aligned} |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \rightharpoonup \mu = |\nabla u|^{p(x)} + |u|^{p(x)} + \sum_{j\in\mathscr{J}} \mu_j \delta_{x_j} + \mu_0\delta_0 + \widetilde{\mu},\\ \frac{|u_n|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}} \rightharpoonup \nu^i = \frac{|u|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}} + \sum_{j\in\mathscr{J}} \nu^i_j \delta_{x_j} + \nu^i_0\delta_0 \quad (i = 1,\dots,k), \end{aligned} \tag{3.6}$$

weakly-$*$ in $\mathscr{M}(\overline{\Omega})$, as $n \to +\infty$ where $\mathscr{J}$ is a countable set and $\{x_j\} \in \overline{\Omega}$, $\mu_j, \nu^i_j \geq 0$, $\mu_0, \nu^i_0 \geq 0$, $\mu(\overline{\Omega}) \leq 1$, $\nu^i(\overline{\Omega}) \leq C^*$, with C^* defined in the theorem itself and $\widetilde{\mu} \in \mathscr{M}(\overline{\Omega})$ is a nonatomic positive measure. Furthermore, the following inequalities hold:

$$\begin{aligned} \nu^i_j &\leq C^* \max\left\{ \mu_j^{\frac{p^{*\,+}_{s_i}}{p^-}}, \mu_j^{\frac{p^{*\,-}_{s_i}}{p^+}} \right\},\\ \nu^i_0 &\leq C^* \max\left\{ \mu_0^{\frac{p^{*+}_{s_i}}{p^-}}, \mu_0^{\frac{p^{*\,-}_{s_i}}{p^+}} \right\},\\ \nu^i(\overline{\Omega}) &\leq C^* \max\left\{ \mu(\overline{\Omega})^{\frac{p^{*+}_{s_i}}{p^-}}, \mu(\overline{\Omega})^{\frac{p^{*-}_{s_i}}{p^+}} \right\}. \end{aligned} \tag{3.7}$$

The lemma will be proved if, for $i \in \{1,\dots,k\}$, we show that $\nu^i_0 = \nu^i_j = 0$ for all $j \in \mathscr{J}$, in (3.6), because, in this case, considering $\eta \in C^\infty_0(\mathbb{R}^N)$ such that $\eta = 1$ in Ω and compact support $\operatorname{supp}(\eta) \subset \Omega$, by (3.6) we obtain

$$\int_\Omega \frac{|u_n|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}} dx = \int_{\mathbb{R}^N} \frac{|u_n|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}} \eta dx \to \int_{\mathbb{R}^N} \eta d\nu^i = \int_\Omega \frac{|u|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}} dx \tag{3.8}$$

as $n \to +\infty$, for $i \in \{1,\dots,k\}$, and by Lemma 2.16 then $u_n \to u$ in $L^{p^*_{s_i}(x)}_{|x|^{-s_i(x)}}(\Omega)$.

Case 1. $\nu^i_0 = 0$ for all $i \in \{1,\dots,k\}$.

For convenience, we define $\phi \in C^\infty_0(\mathbb{R}^N)$ with $0 \leq \phi \leq 1$, $\phi(0) = 1$ and compact support in $B_1(0) \subset \mathbb{R}^N$. Consider $\epsilon > 0$, let $\phi_\epsilon(x) = \phi\left(\frac{x}{\epsilon}\right)$, for $x \in \mathbb{R}^N$ such that $|\nabla \phi_\epsilon| \leq 2/\epsilon$.

Since the sequence $(\phi_\epsilon u_n)$ is bounded and $\widetilde{J}'(u_n) \to 0$, we have

$$\lim_{n\to+\infty} \widetilde{J}'(u_n)\,(\phi_\epsilon u_n) = 0,$$

that is,

$$0=\lim_{n\to\infty}\Bigg[M\left(\psi(u_n)\right)\Bigg(\int\limits_{\Omega}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(\phi_\epsilon u_n)+|u_n|^{p(x)-2}u_n(\phi_\epsilon u_n)\Big)\,dx\Bigg)$$
$$-\int\limits_{\Omega}\left(h_1(x)\frac{|u_n|^{p^*_{s_1}(x)-2}u_n}{|x|^{s_1(x)}}(\phi_\epsilon u_n)\right)dx$$
$$-\ldots-\int\limits_{\Omega}\left(h_k(x)\frac{|u_n|^{p^*_{s_k}(x)-2}u_n}{|x|^{s_k(x)}}(\phi_\epsilon u_n)\right)dx$$
$$-\int\limits_{\Omega}f(x,u_n)(\phi_\epsilon u_n)dx\Bigg],$$

$$\begin{aligned}0=\lim_{n\to\infty}\Bigg[&M\left(\psi(u_n)\right)\Bigg(\int\limits_{B_\epsilon(0)}\Big(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\Big)\,\phi_\epsilon\ dx\Bigg)\\&+M(\psi(u_n))\Bigg(\int\limits_{B_\epsilon(0)}|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla\phi_\epsilon)u_n dx\Bigg)\\&-\int\limits_{B_\epsilon(0)}h_1(x)\frac{|u_n|^{p^*_{s_1}(x)}}{|x|^{s_1(x)}}\phi_\epsilon dx-\ldots-\int\limits_{B_\epsilon(0)}h_k(x)\frac{|u_n|^{p^*_{s_k}(x)}}{|x|^{s_k(x)}}\phi_\epsilon dx\\&-\int\limits_{B_\epsilon(0)}f(x,u_n)u_n\phi_\epsilon dx\Bigg].\end{aligned}\tag{3.9}$$

By (3.6), note that

$$\lim_{n\to+\infty}\int\limits_{B_\epsilon(0)}\Big(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\Big)\,\phi_\epsilon\ dx=\int\limits_{B_\epsilon(0)}\phi_\epsilon d\mu\to\mu_0\phi_\epsilon(0)=\mu_0,$$

$$\lim_{n\to+\infty}\int\limits_{B_\epsilon(0)}h_i(x)\frac{|u_n|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}\phi_\epsilon\ dx=\int\limits_{B_\epsilon(0)}h_i(x)\phi_\epsilon d\nu^i\to h_i(0)\nu_0^i\phi_\epsilon(0)=0,\tag{3.10}$$

as $\epsilon\to 0$.

By the same arguments as in [30], we can show that

$$\begin{aligned}\lim_{\epsilon\to0}\Big(\lim_{n\to+\infty}\textstyle\int_{B_\epsilon(0)}|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla\phi_\epsilon)u_n dx\Big)=0,\\ \lim_{\epsilon\to0}\Big(\lim_{n\to+\infty}\textstyle\int_{B_\epsilon(0)}f(x,u_n)u_n\phi_\epsilon dx\Big)=0.\end{aligned}\tag{3.11}$$

According to (3.9), using (M_0), (3.10) and (3.11), it results that

$$0 \le m_0\mu_0 \le \lim_{\epsilon\to 0}\lim_{n\to+\infty}(M(\psi(u_n)))\left(\int\limits_{B_\epsilon(0)}\left(|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\right)\phi_\epsilon dx\right)=0.$$

Hence, as $m_0>0$, it follows that $\mu_0=0$ and by (3.7)

$$0\le \nu_0^i \le C^*\max\left\{\mu_0^{p_{s_i}^{*}{}^{+}/p^-},\mu_0^{p_{s_i}^{*}{}^{-}/p^+}\right\}=0 \quad\iff\quad \nu_0^i=0 \quad (i=1,\dots,k).$$

Case 2. $\nu_j^i=0$ for all $j\in\mathscr{J}$ and for all $i\in\{1,\dots,k\}$.

Let us assume that for some $j_0\in\mathscr{J}$, $\nu_{j_0}^i>0$. Since we are working on the subspace $W_{0,O(N)}^{1,p(x)}(\Omega)$ in $O(N)$-invariant Ω domain, then

$$\nu^i(\tau x_{j_0})=\nu^i(x_{j_0})>0,\quad \forall\tau\in O(N).$$

However, we have $|O(N)|_\sharp=\infty$ and consequently $\nu^i\left(\{\tau x_{j_0}:\tau\in O(N)\}\right)=\infty$, which is a contradiction, since $\nu^i\in\mathscr{M}(\overline{\Omega})$ is a finite measure. So, for any $j\in\mathscr{J}$, we must have $\nu_j^i=0$.

So, for all cases we have $\nu_0^i=0$ and $\nu_j^i=0$, $j\in\mathscr{J}$ and $i\in\{1,\dots,k\}$ and by (3.8), we conclude $u_n\to u$ in $L_{|x|^{-s_i(x)}}^{p_{s_i}^*(x)}(\Omega)$ for all $i=1,\dots,k$. □

The next lemma deals with the Palais–Smale compactness condition for the functional $\widetilde{J}$.

Lemma 3.3. *Let* $(u_n)\subset W_{0,O(N)}^{1,p(x)}(\Omega)$ *be a sequence* $(PS)_c$ *for* $\widetilde{J}$. *Then,* (u_n) *has a convergent subsequence.*

Proof. Consider $(u_n)\subset W_{0,O(N)}^{1,p(x)}(\Omega)$ such that

$$\widetilde{J}(u_n)\to c,\quad \widetilde{J}'(u_n)\to 0,\ \text{as } n\to+\infty.$$

The result of Lemma 3.1 guarantees that (u_n) is bounded in $W_{0,O(N)}^{1,p(x)}(\Omega)$, and therefore, up to subsequence, there exists $u\in W_{0,O(N)}^{1,p(x)}(\Omega)$ such that

$$u_n\rightharpoonup u \text{ in } W_{0,O(N)}^{1,p(x)}(\Omega)$$

and

$$u_n\to u \text{ in } L^{r(x)}(\Omega),\quad \forall r(x)\in[1,p^*(x)).$$

From $\widetilde{J}'(u_n) \to 0$, we get

$$\begin{aligned}\widetilde{J}'(u_n)(u_n-u) = M(\psi(u_n))&\left(\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u)\,dx\right)\\&+M(\psi(u_n))\left(\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,dx\right)\\&-\int_\Omega\left[h_1(x)\frac{|u_n|^{p^*_{s_1}(x)-2}u_n}{|x|^{s_1(x)}}(u_n-u)\right]\,dx\\&-\ldots-\int_\Omega\left[h_k(x)\frac{|u_n|^{p^*_{s_k}(x)-2}u_n}{|x|^{s_k(x)}}(u_n-u)\right]\,dx\\&-\int_\Omega f(x,u_n)(u_n-u)\,dx\to 0.\end{aligned} \tag{3.12}$$

Moreover, as $u_n \to u$ in $L^{p(x)}(\Omega)$, we have $|u_n|^{p(x)-1} \to |u|^{p(x)-1}$ in $L^{\frac{p(x)}{p(x)-1}}(\Omega)$ and, by Lemma 3.2, $u_n \to u$ in $L^{p^*_{s_i}(x)}_{|x|^{-s_i(x)}}(\Omega)$, which implies that $|u_n|^{p^*_{s_i}(x)-1} \to |u|^{p^*_{s_i}(x)-1}$ in

$$L^{\frac{p^*_{s_i}(x)}{p^*_{s_i}(x)-1}}_{|x|^{-s_i(x)}}(\Omega) = L^{{p^*_{s_i}}'(x)}_{|x|^{-s_i(x)}}(\Omega)$$

for each $i \in \{1,\ldots,k\}$. Taking $h_i^+ = \max_{\overline{\Omega}} h_i(x)$ for each $i \in \{1,\ldots,k\}$, we use Hölder's inequality to obtain positive constants $\overline{C}_1$, $\overline{C}_2$ such that

$$\begin{aligned}\left|\int_\Omega |u_n|^{p(x)-2}u_n(u_n-u)\,dx\right| &\le \int_\Omega |u_n|^{p(x)-1}|u_n-u|\,dx\\&\le \overline{C}_1\left||u_n|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}|u_n-u|_{p(x)} \to 0\end{aligned}$$

and

$$\begin{aligned}&\left|\int_\Omega h_i(x)\frac{|u_n|^{p^*_{s_i}(x)-2}u_n}{|x|^{s_i(x)}}(u_n-u)\,dx\right|\\&\le \int_\Omega h_i(x)\frac{|u_n|^{p^*_{s_i}(x)-1}}{|x|^{s_i(x)}}|u_n-u|\,dx\\&\le \overline{C}_2\left|\frac{|u_n|^{p^*_{s_i}(x)-1}}{|x|^{s_i(x)/{p^*_{s_i}}'(x)}}\right|_{{p^*_{s_i}}'(x)}\left|\frac{|u_n-u|}{|x|^{s_i(x)/p^*_{s_i}(x)}}\right|_{p^*_{s_i}(x)} \to 0,\end{aligned}$$

as $n \to +\infty$.

Note that by (f_1) and (f_2) and Theorem 2.11, the Nemytskii operator $N_f\colon L^{q(x)}(\Omega) \to L^{q'(x)}(\Omega)$ is continuous and bounded, then $f(x,u_n) \in L^{q'(x)}(\Omega)$ is bounded. So, by Hölder's inequality,

$$\left| \int_\Omega f(x,u_n)(u_n - u)\, dx \right| \le \overline{C}_3 |f(x,u_n)|_{q'(x)} |u_n - u|_{q(x)} \to 0,$$

as $n \to +\infty$.

Remembering that

$$L_{p(x)}(u_n)(u_n - u) = \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla (u_n - u),$$

we obtain from (3.12) $L_{p(x)}(u_n)(u_n - u) \to 0$. We also have $L_{p(x)}(u)(u_n - u) \to 0$. So,

$$\big(L_{p(x)}(u_n) - L_{p(x)}(u), (u_n - u)\big) \to 0,$$

as $n \to +\infty$.

From Proposition 2.10, and the condition $(PS)c$ holds, we have that $u_n \to u$ in $W^{1,p(x)}_{0,O(N)}(\Omega)$ as $n \to \infty$. □

Since $W^{1,p(x)}_{0,O(N)}(\Omega) \subset W^{1,p(x)}_0(\Omega)$ is a reflexive and separable Banach space, there exist $(e_n) \subset W^{1,p(x)}_{0,O(N)}(\Omega)$ and $(e^*_m) \subset \Big(W^{1,p(x)}_{0,O(N)}(\Omega)\Big)'$ such that

$$\begin{aligned} W^{1,p(x)}_{0,O(N)}(\Omega) &= \overline{\operatorname{span}\{e_n : n = 1,2,\dots\}}, \\ \Big(W^{1,p(x)}_{0,O(N)}(\Omega)\Big)' &= \overline{\operatorname{span}\{e^*_m : m = 1,2,\dots\}} \end{aligned} \tag{3.13}$$

and

$$\langle e^*_m, e_n \rangle = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases} \tag{3.14}$$

We denote the subspaces

$$\begin{aligned} X_r &= \operatorname{span}\{e_1,\dots,e_r\}, \text{ for } r = 1,2,\dots, \\ Y_r &= \bigoplus_{l=1}^{r} X_l \quad \text{and} \quad Z_r = \overline{\bigoplus_{l=r}^{\infty} X_l}. \end{aligned} \tag{3.15}$$

Consider the following two results below.

Proposition 3.4 (Fan [10])**.** *Let X be a Banach space. Assume that $\Psi : X \longrightarrow \mathbb{R}$ is weakly-strongly continuous and $\Psi(0) = 0$. Let $\gamma > 0$ be given. Set*

$$\beta_r = \beta_r(\gamma) = \sup_{u \in Z_r, \|u\| \le \gamma} |\Psi(u)|.$$

Then $\beta_r \to 0$, as $r \to +\infty$.

Proposition 3.5 (Fan [10], Fountain Theorem)**.** *Assume*

(F1) *X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ is an even functional, the subspaces X_r, Y_r and Z_r are defined by* (3.15)*.*

If for each $r = 1, 2, \ldots$, there exists $\rho_r > \gamma_r > 0$ such that

(F2) $\inf_{u\in Z_r,\|u\|=\gamma_r} \varphi(u) \to +\infty$ *as* $r \to +\infty$,
(F3) $\max_{u\in Y_r,\|u\|=\rho_r} \varphi(u) \le 0$,
(F4) *φ satisfies $(PS)_c$ condition for every $c > 0$,*

then φ has a sequence of critical values tending to $+\infty$.

In order to apply Proposition 3.4, consider

$$\Psi(u) := \int\limits_{\Omega} F(x,u)\, dx.$$

Certainly $\Psi(u)$ is weakly-strongly continuous. In fact, since f satisfies (f_2), we can write

$$\begin{aligned}|F(x,u)| &\le \int\limits_0^u |f(x,s)|ds \le \int\limits_0^u (c_1 + c_2|s|^{q(x)-1})\, ds\\ &\le c_1|u| + \left(\frac{1}{q(x)}\right)\cdot c_2|u|^{q(x)},\\ |F(x,u)| &\le c_3\left(|u| + |u|^{q(x)}\right),\end{aligned}$$

where

$$c_3 := \max_{x\in\overline{\Omega}}\left\{c_1, \left(\frac{c_2}{q(x)}\right)\right\}.$$

Using Young's inequality for $\epsilon \in (0,1)$, we can have

$$\begin{aligned}|u| + |u|^{q(x)} &\le \epsilon|u|^{q(x)} + C(\epsilon) + |u|^{q(x)} = (\epsilon+1)|u|^{q(x)} + C(\epsilon)\\ &\le c_4\left(1 + |u|^{q(x)}\right),\end{aligned}$$

and consequently

$$|F(x,u)| \le c_5 + c_5|u|^{q(x)}. \tag{3.16}$$

Based on (f_1), (3.16), and Theorem 2.11, we conclude that the operator Nemytskii of the form $N_F : L^{q(x)}(\Omega) \longrightarrow L^1(\Omega)$ is continuous and bounded. Therefore, for the sequence $(u_n) \subset W^{1,p(x)}_{0,O(N)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p(x)}_{0,O(N)}(\Omega)$ by compact embedding we obtain the convergence $u_n \to u$ in $L^{q(x)}(\Omega)$ and through the continuity of N_F we have

$$u_n \to u \text{ in } L^{q(x)}(\Omega) \quad \Rightarrow \quad F(x,u_n) \to F(x,u) \text{ in } L^1(\Omega),$$

i.e. $\Psi(u_n) \to \Psi(u)$, showing that $\Psi(u)$ is weakly-strongly continuous on $W^{1,p(x)}_{0,O(N)}(\Omega)$.

Furthermore, $\Psi(0)=0$, so for $\gamma>0$ it follows from Proposition 3.4 that

$$\beta_r = \beta_r(\gamma) = \sup_{u\in Z_r, \|u\|\le\gamma} \left| \int_\Omega F(x,u)\,dx \right| \to 0, \quad \text{as } r\to+\infty. \tag{3.17}$$

Proceeding, given $\gamma>0$, for each $s_i(x)$ $(i=1,\dots,k)$ we define

$$\Phi_{s_i,r} = \Phi_{s_i,r}(\gamma) = \sup_{u\in Z_r, \|u\|\le\gamma} \int_\Omega \frac{|u|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}\,dx. \tag{3.18}$$

We prove the convergence

$$\Phi_{s_i,r} \to \sum_{j\in\mathscr{J}} \nu^i_j + \nu^i_0, \text{ as } r\to+\infty. \tag{3.19}$$

Indeed, from (3.18), for every integer $r>0$, there is $u_r\in Z_r$ such that $\|u_r\|\le\gamma$ and

$$0\le \Phi_{s_i,r} < \int_\Omega \frac{|u_r|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}\,dx + \frac{1}{r},$$

that is,

$$0\le \Phi_{s_i,r} - \int_\Omega \frac{|u_r|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}\,dx < \frac{1}{r}$$

which implies that

$$\lim_{r\to\infty} \Phi_{s_i,r} = \lim_{r\to\infty} \int_\Omega \frac{|u_r|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}\,dx. \tag{3.20}$$

Note that, for any r, $0\le\Phi_{s_i,r+1}\le\Phi_{s_i,r}$, then there exist $\Phi_{s_i}\ge0$ such that $\Phi_{s_i,r}\to\Phi_{s_i}$, as $r\to+\infty$. Taking a subsequence of the sequence (u_r), denoted again by (u_r), in the reflexive space $W^{1,p(x)}_{0,O(N)}(\Omega)$ such that $u_r\rightharpoonup u$ weakly in $W^{1,p(x)}_{0,O(N)}(\Omega)$, we deduce that $u=0$. In fact, choosing $m<r$, note that for $e^*_m\in\left(W^{1,p(x)}_{0,O(N)}(\Omega)\right)'$ we obtain

$$e^*_m(u_r) = t_r\langle e^*_m, e_r\rangle + t_{r+1}\langle e^*_m, e_{r+1}\rangle + t_{r+2}\langle e^*_m, e_{r+2}\rangle + \ldots = 0.$$

Therefore, in this case $e^*_m(u_r)\to0$, as $r\to+\infty$ and $m\in\mathbb{N}$. On the other hand,

$$u_r\rightharpoonup u \quad\Rightarrow\quad e^*_m(u_r)\to e^*_m(u), \text{ as } r\to+\infty \text{ and for all } e^*_m\in\left(W^{1,p(x)}_{0,O(N)}(\Omega)\right)'$$

thus, we conclude that $e^*_m(u)=0$ for all $e^*_m\in\left(W^{1,p(x)}_{0,O(N)}(\Omega)\right)'$, and therefore $u=0$.

So, since $u_r \rightharpoonup 0$ in $W^{1,p(x)}_{0,O(N)}(\Omega)$, according to Theorem 2.17 we have

$$\frac{|u_r|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}} \to \nu^i = \frac{|u|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}} + \sum_{j\in\mathscr{J}} \nu^i_j \delta_{x_j} + \nu^i_0\delta_0, \text{ weakly-}* \text{ in } \mathscr{M}(\overline{\Omega}),$$

where $\mathscr{J}$ is a countable set, $\{\nu^i_j\} \subset [0,+\infty)$, $\nu^i_0 \geq 0$, $\{x_j\} \in \overline{\Omega}$ such that δ_{x_j} and δ_0 are atomic measures which concentrate on x_j and 0.

Making $\eta \equiv 1$, we obtain

$$\int_\Omega \frac{|u_r|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}\, dx \to \int_\Omega \frac{|u|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}\, dx + \sum_{j\in\mathscr{J}} \nu^i_j + \nu^i_0 = \sum_{j\in\mathscr{J}} \nu^i_j + \nu^i_0, \text{ as } r \to +\infty$$

and, by (3.20), we have

$$\Phi_{s_i,r} = \sup_{u\in Z_r;\|u\|\leq\gamma} \int_\Omega \frac{|u|^{p^*_{s_i}(x)}}{|x|^{s_i(x)}}\, dx \to \sum_{j\in\mathscr{J}} \nu^i_j + \nu^i_0 \leq \mu(\overline{\Omega}) < \infty, \text{ as } r \to +\infty \tag{3.21}$$

which confirms (3.19).

Now, by (3.17) and (3.21), for each $\gamma = n \in \mathbb{N}$ there is an integer $r_n > 0$ such that for all $r \geq r_n$ we have

$$|\beta_r(n)| = \beta_r(n) < 1$$

and

$$\Phi_{s_i,r}(n) - \textstyle\sum_{j\in\mathscr{J}} \nu^i_j + \nu^i_0 < 1 \quad \Longleftrightarrow \quad \Phi_{s_i,r}(n) < \displaystyle\sum_{j\in\mathscr{J}} \nu^i_j + \nu^i_0 + 1. \tag{3.22}$$

Suppose further that $r_n < r_{n+1}$ for every $n \in \mathbb{N}$. Let us define the set $\{\gamma_r : r = 1, 2, \ldots\}$ such that

$$\gamma_r = \begin{cases} n, & r_n \leq r < r_n + 1, \\ 1, & 1 \leq r < r_1. \end{cases}$$

In this way, as $r \to +\infty$ we have $\gamma_r \to +\infty$.

Thus, for $u \in Z_r$ with $\|u\| = \gamma_r \geq 1$, we get

$$\widetilde{J}(u) \geq \frac{m_0}{p^+}\gamma_r^{p^-} - \frac{h_1^+}{{p^*_{s_1}}^-}\Phi_{s_1,r}(\gamma_r) - \ldots - \frac{h_k^+}{{p^*_{s_k}}^-}\Phi_{s_k,r}(\gamma_r) - \beta_r(\gamma_r)$$

since $h_i^+ := \max_{\overline{\Omega}} h_i(x)$. Using (3.22) we obtain

$$\begin{aligned} \widetilde{J}(u) \geq \frac{m_0}{p^+}\gamma_r^{p^-} &- \frac{h_1^+}{{p^*_{s_1}}^-}\left(\sum_{j\in\mathscr{J}} \nu^1_j + \nu^1_0 + 1\right) \\ &- \ldots - \frac{h_k^+}{{p^*_{s_k}}^-}\left(\sum_{j\in\mathscr{J}} \nu^k_j + \nu^k_0 + 1\right) - 1, \end{aligned}$$

that is,

$$\inf_{u\in Z_r,\|u\|=\gamma_r} \widetilde{J}(u) \geq \frac{m_0}{p^+}\gamma_r^{p^-} - \frac{h_1^+}{{p_{s_1}^*}^-}\left(\sum_{j\in\mathscr{J}} \nu_j^1 + \nu_0^1 + 1\right) - \ldots - \frac{h_k^+}{{p_{s_k}^*}^-}\left(\sum_{j\in\mathscr{J}} \nu_j^k + \nu_0^k + 1\right) - 1.$$

Making $r \to +\infty$ (i.e. $\gamma_r \to +\infty$) we have

$$\lim_{r\to+\infty}\left(\inf_{u\in Z_r,\|u\|=\gamma_r} \widetilde{J}(u)\right) \geq \lim_{\gamma_r\to+\infty}\left[\frac{m_0}{p^+}\gamma_r^{p^-} - \frac{h_1^+}{{p_{s_1}^*}^-}\left(\sum_{j\in\mathscr{J}} \nu_j^1 + \nu_0^1 + 1\right) - \ldots - \frac{h_k^+}{{p_{s_k}^*}^-}\left(\sum_{j\in\mathscr{J}} \nu_j^k + \nu_0^k + 1\right) - 1\right] \to +\infty$$

which shows that

$$\inf_{u\in Z_r,\|u\|=\gamma_r} \widetilde{J}(u) \to +\infty, \text{ as } r \to +\infty. \tag{3.23}$$

Next, again by (3.16) for $\epsilon \in (0,1)$ we have

$$\begin{aligned} |F(x,u)| &\leq c_5\left(1+|u|^{q(x)}\right) \\ &\leq c_5\left(1+\epsilon(|u|^{q(x)})^{\frac{p_{s_1}^*(x)}{q(x)}} + C(\epsilon)\right), \\ |F(x,u)| &\leq c_5\epsilon|u|^{p_{s_1}^*(x)} + c_6. \end{aligned}$$

Thus,

$$\left|\int_\Omega F(x,u)\,dx\right| \leq \int_\Omega |F(x,u)|\,dx \leq c_5\epsilon\int_\Omega |u|^{p_{s_1}^*(x)}\,dx + c_6|\Omega|.$$

Hence, it follows that

$$\widetilde{J}(u) \leq \frac{m_1}{p^-}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)\,dx - \frac{\overline{H}_1}{{p_{s_1}^*}^+}\int_\Omega |u|^{p_{s_1}^*(x)}\,dx - \ldots - \frac{\overline{H}_k}{{p_{s_k}^*}^+}\int_\Omega |u|^{p_{s_k}^*(x)}\,dx + c_5\epsilon\int_\Omega |u|^{p_{s_1}^*(x)}\,dx + c_6|\Omega|,$$

where $\overline{H}_i > 0$ is such that $\frac{h_i(x)}{|x|^{s_i(x)}} > \overline{H}_i$.

Taking ϵ small enough such that $c_5\epsilon \leq \frac{\overline{H}_1}{2p_{s_1}^{*}{}^{+}}$, we have

$$\begin{aligned}
\widetilde{J}(u) &\leq \frac{m_1}{p^-}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)\,dx - \frac{\overline{H}_1}{2p_{s_1}^{*}{}^{+}}\int_\Omega |u|^{p_{s_1}^*(x)}\,dx \\
&\quad - \ldots - \frac{\overline{H}_k}{p_{s_k}^{*}{}^{+}}\int_\Omega |u|^{p_{s_k}^*(x)}\,dx + c_6|\Omega|, \\
\widetilde{J}(u) &\leq \frac{m_1}{p^-}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)\,dx \\
&\quad -\overline{H}\int_\Omega \left(|u|^{p_{s_1}^*(x)} + \ldots + |u|^{p_{s_k}^*(x)}\right)dx + c_6|\Omega|,
\end{aligned}$$

where

$$\overline{H} := \min\left\{\frac{\overline{H}_1}{2p_{s_1}^{*+}}, \frac{\overline{H}_2}{p_{s_2}^{*}{}^{+}}, \ldots, \frac{\overline{H}_k}{p_{s_k}^{*+}}\right\}.$$

Therefore,

$$\widetilde{J}(u) \leq \frac{m_1}{p^-}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right)\,dx - \overline{H}\int_\Omega |u|^{p_{s_1}^*(x)}\,dx + c_6|\Omega|.$$

Considering $\psi \in Y_r$ with $\|\psi\| = 1$ and $t > 1$ note that $\|t\psi\| > 1$. So, by the previous expression, it follows that

$$\begin{aligned}
\widetilde{J}(t\psi) &\leq \frac{m_1}{p^-}\|t\psi\|^{p^+} - \overline{H}\int_\Omega |t\psi|^{p_{s_1}^*(x)}\,dx + c_6|\Omega| \\
&\leq \frac{m_1}{p^-}t^{p^+} - \overline{H}t^{p_{s_1}^{*}{}^{-}}\int_\Omega |\psi|^{p_{s_1}^*(x)}\,dx + c_6|\Omega|
\end{aligned}$$

since $p^+ < p_{s_1}^{*}{}^{-}$ for $t \to +\infty$ we conclude

$$\widetilde{J}(t\psi) \to -\infty.$$

Hence, there exists $t_* > \gamma_r > 1$ large enough such that $\widetilde{J}(t_*\psi) \leq 0$. Thus, just assume $\rho_r = \|t_*\psi\| = t_*$ to get

$$\max_{u\in Y_r, \|u\|=\rho_r} \widetilde{J}(u) \leq 0. \tag{3.24}$$

Since Lemma 3.3 proves that $\widetilde{J}$ satisfies condition $(PS)_c$, by results (3.23) and (3.24), we conclude via Fountain Theorem that $\widetilde{J}$ has an unbounded sequence of critical values. Thus, there is $(u_n) \subset W_0^{1,p(x)}(\Omega)$ critical points of J such that

$$J(u_n) \to +\infty, \text{ as } n \to +\infty$$

and (u_n) are weak solutions to (1.1).

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