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ISOPERIMETRIC INEQUALITIES IN NONLOCAL DIFFUSION PROBLEMS WITH INTEGRABLE KERNEL

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Abstract. We deduce isoperimetric estimates for solutions of linear stationary and evolution problems. Our main result establishes the comparison in norm between the solution of a problem and its symmetric version when nonlocal diffusion defined through integrable kernels is replacing the usual local diffusion defined by a second order differential operator. Since an appropriate kernel rescaling allows to define a sequence of solutions of the nonlocal diffusion problems converging to their local diffusion counterparts, we also find the corresponding isoperimetric inequalities for the latter, i.e. we prove the classical Talenti's theorem. The novelty of our approach is that we replace the measure geometric tools employed in Talenti's proof, such as the geometric isoperimetric inequality or the coarea formula, by the Riesz's rearrangement inequality. Thus, in addition to providing a proof for the nonlocal diffusion case, our technique also introduces an alternative proof to Talenti's theorem.

Keywords: nonlocal diffusion, Schwarz's symmetrization, Talenti's theorem, Riesz's inequality.

Mathematics Subject Classification: 35B45, 35R09, 35J25, 35K20.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set, and let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radially non-increasing function with $\|\rho\|_{L^1(\mathbb{R}^N)} = 1$.

Let $\Omega^\star \subset \mathbb{R}^N$ be the symmetric rearrangement of Ω, this is, the ball of $\mathbb{R}^N$ centered at zero with the same volume as Ω, and let $g^\star$ be the Schwarz's symmetrization of the nonnegative measurable function vanishing at infinity $g : \mathbb{R}^N \to \mathbb{R}$, see Section 2.1 for definitions. For a function $g : \Omega \to \mathbb{R}$ we also denote by $g^\star$ to the Schwarz's symmetrization of the extension by zero of g to $\mathbb{R}^N$.

1.1. THE STATIONARY PROBLEM

Let $f \in L^p(\Omega)$, with $p \geq 1$, be a nonnegative function. Let $u : \mathbb{R}^N \to \mathbb{R}$ be the solution of the integral problem

$$u(x) = \alpha \int_{\mathbb{R}^N} \rho(x-y)u(y)dy + \gamma f(x) \quad \text{for } x \in \Omega, \tag{1.1}$$

$$u(x) = 0 \quad \text{for } x \in \mathbb{R}^N \backslash \Omega, \tag{1.2}$$

where α and γ are positive constants, and let $v : \mathbb{R}^N \to \mathbb{R}$ be the solution to the corresponding *symmetrized problem* corresponding to (1.1)–(1.2), this is

$$v(x) = \alpha \int_{\mathbb{R}^N} \rho(x-y)v(y)dy + \gamma f^\star(x) \quad \text{for } x \in \Omega^\star, \tag{1.3}$$

$$v(x) = 0 \quad \text{for } x \in \mathbb{R}^N \backslash \Omega^\star. \tag{1.4}$$

In this article, we prove the isoperimetric inequality

$$\|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\star)}. \tag{1.5}$$

As a first, straightforward corollary of this result, we deduce a similar inequality for the linear stationary nonlocal diffusion problem

$$-\int_{\mathbb{R}^N} \rho(x-y)(u(y)-u(x))dy + cu(x) = f(x) \quad \text{for } x \in \Omega, \tag{1.6}$$

$$u(x) = 0 \quad \text{for } x \in \mathbb{R}^N \backslash \Omega. \tag{1.7}$$

Indeed, this problem may be expressed in the form of (1.1)–(1.2) just by setting $\alpha = \gamma = 1/(1+c)$.

A second, more interesting corollary is that the isoperimetric property carries out to the local diffusion problem

$$-\Delta U + cU = f \quad \text{in } \Omega, \tag{1.8}$$

$$U = 0 \quad \text{on } \partial\Omega, \tag{1.9}$$

through the approximation of U by the sequence u_ε defined by the solutions of (1.6)–(1.7) corresponding to the rescaled kernel $\rho_\varepsilon(x) = C_1 \varepsilon^{-(N+2)} \rho(x/\varepsilon)$, for $\varepsilon > 0$ and C_1 given in (2.2). More concretely, under appropriate assumptions it holds

$$\|u_\varepsilon - U\|_{L^p(\Omega)} \to 0 \quad \text{and} \quad \|v_\varepsilon - V\|_{L^p(\Omega^*)} \to 0, \tag{1.10}$$

where v_ε and V are the solutions of the symmetric versions of (1.6)–(1.7) (with kernel ρ_ε) and (1.8)–(1.9), respectively. Then, using the isoperimetric inequality (1.5) we deduce $\|U\|_{L^p(\Omega)} \leq \|V\|_{L^p(\Omega^\star)}$, as detailed in Remark 2.7.

The isoperimetric inequality for the local diffusion problem (1.8)–(1.9) was proven by Talenti in his celebrated article [45]. Since then, the technique of proof for the linear local diffusion problem with homogeneous Dirichlet boundary conditions has been extended to many other problems, see Section 1.3.

Two fundamental ingredients in Talenti's proof, used in most of the subsequent works, are the classical (geometric) isoperimetric inequality and the Fleming and Rishel's formula, whose proofs rely on measure geometric techniques.

In this article we introduce a new proof of the isoperimetric inequality for the nonlocal and local diffusion stationary problems (1.6)–(1.7) and (1.8)–(1.9), and for their corresponding evolution counterparts, which do not use neither the classical isoperimetric inequality nor the Fleming and Rishel's or any other coarea formula.

The main ingredient of our proof is the Riesz's rearrangement inequality [43]

$$\int_{\mathbb{R}^N} f_1(x)\Big(\int_{\mathbb{R}^N} f_2(x-y)f_3(y)dy\Big)dx \leq \int_{\mathbb{R}^N} f_1^\star(x)\Big(\int_{\mathbb{R}^N} f_2^\star(x-y)f_3^\star(y)dy\Big)dx, \tag{1.11}$$

where $f_1, f_2, f_3 : \mathbb{R}^N \to \mathbb{R}$ are measurable nonnegative functions vanishing at infinity. Testing (1.1) with a nonnegative function φ, and (1.3) with $\varphi^\star$, Riesz's inequality allows to establish an integral comparison between the solutions of the original and the symmetrized problems leading to the isoperimetric inequality for the nonlocal diffusion problem. In addition, the approximation result (1.10) establishes the inequality for the local diffusion case too.

1.2. THE EVOLUTION PROBLEM

We follow a similar approach to establish the isoperimetric inequality for the evolution nonlocal and local diffusion problems, see Section 2.2 for their definitions. In this case, we start proving the property for each step of an explicit time discretization of the evolution problem corresponding to (1.6)–(1.7), written in the form

$$\begin{aligned} u_{n+1}(x) &= \alpha \int_{\mathbb{R}^N} \rho(x-y)u_n(y)dy + \beta u_n(x) + \gamma g_n(x) \quad &&\text{for } x\in\Omega,\\ u_{n+1}(x) &= 0 &&\text{for } x \in \mathbb{R}^N\backslash\Omega, \end{aligned}$$

for $n = 0, 1, \ldots, M$, where α, β and γ are positive constants, $g_n(x) = g(t_n, x)$, and u_0 and $g : (0, T) \times \Omega$ are nonnegative functions. Here, $\{t_n\}_{n=0}^M$ denotes a uniform mesh of the interval $(0, T)$. Like in the stationary problem, we deduce $\|u_{n+1}\|_{L^p(\Omega)} \leq \|v_{n+1}\|_{L^p(\Omega^\star)}$, where

$$\begin{aligned} v_{n+1}(x) &= \alpha \int_{\mathbb{R}^N} \rho(x-y)v_n(y)dy + \beta v_n(x) + \gamma f_n^\star(x) \quad &&\text{for } x\in\Omega^\star,\\ v_{n+1}(x) &= 0 &&\text{for } x \in \mathbb{R}^N\backslash\Omega^\star, \end{aligned}$$

with $v_0 = u_0^\star$. A piecewise constant interpolation of both sequences provides the isoperimetric inequality for the evolution nonlocal diffusion problem, and a subsequent kernel rescaling argument leads to the corresponding isoperimetric inequality for the local diffusion problem.

1.3. EXTENSIONS AND COMMENTS

After the seminal article by Talenti [45], dealing with the linear local diffusion problem with homogeneous Dirichlet boundary conditions, a big effort has been devoted to its generalization. The literature on the subject is vast and deals with a considerable variety of problems, among which: semilinear and nonlinear elliptic equations [4, 5, 16, 18, 21, 28], their evolution counterparts [1, 2, 9, 17, 21, 25, 41, 47], systems of equations [22, 23], eigenvalue problems [3, 11, 13, 19], free boundary problems [12, 20, 48], Neumann and Robin boundary conditions [6, 7, 26, 27, 36], nonlocal fractional-type equations [14, 29, 30, 44, 49–51], to mention a few of them. The review on the topic [46] written by Talenti is a fundamental starting point to approach to this variety.

In what respects to our work, as far as we know, our proof seems to be the first proof of the isoperimetric inequality for stationary or evolution problems with nonlocal diffusion with integrable kernel. As already noticed, our method of proof, based on Riesz's inequality, differs considerably from the usual approach to the problem. However, Lieb [37] uses a similar approach (replacing the use of the classical isoperimetric inequality and the coarea formula by Riesz's inequality) to prove the Pólya–Szegö's theorem (with power 2). See [35, Remark 2.3.2].

The rest of the article is organized as follows. In Section 2, we introduce some common tools related to Schwarz's symmetrization and state our assumptions and main results. In Sections 3 and 4, we prove the results related to the stationary and the evolution problems, respectively.

2. MAIN RESULTS

2.1. SCHWARZ'S SYMMETRIZATION

Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure, and let $\chi_E : \mathbb{R}^N \to \mathbb{R}$ be its characteristic function, i.e., defined by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise.

The *symmetric rearrangement of* E is the ball $E^\star \subset \mathbb{R}^N$ centered at zero with $|E^\star| = |E|$, i.e., with radius $(|E|/\omega_N)^{1/N}$, where ω_N denotes the volume of the $N-$dimensional unit ball.

For a nonnegative measurable function $f : \mathbb{R}^N \to \mathbb{R}$ vanishing at infinity, the *Schwarz's symmetrization of* f is

$$f^\star(x) = \int_0^\infty \chi_{\{f>s\}^\star}(x) ds,$$

where, by definition, $(\chi_E)^\star = \chi_{E^\star}$. Thus, the level sets of $f^\star$ are the rearrangements of the level sets of f, implying the equi-measurability property

$$|\{x : f^\star(x) > s\}| = |\{x : f(x) > s\}|. \tag{2.1}$$

The Schwarz's symmetrization of a function fulfils some optimization properties with respect to integration. In this article we shall make a recurrent use of the Hardy–Littlewood's inequality

$$\int_{\mathbb{R}^N} f_1(x)f_2(x)dx \leq \int_{\mathbb{R}^N} f_1^\star(x)f_2^\star(x)dx,$$

of the Riesz's inequality given by (1.11) and of the following generalized Riesz's inequality deduced in [15, 38]: Let $f_1,\dots,f_m$ be nonnegative functions defined on $\mathbb{R}^N$ and vanishing at infinity. Let $k \leq m$ and let $B = (b_{ij})$ be a $k \times m$ matrix (with $1 \leq i \leq k$, $1 \leq j \leq m$). Define

$$I(f_1,\dots,f_m) = \int_{\mathbb{R}^N} \cdots \int_{\mathbb{R}^N} \prod_{j=1}^{m} f_j\Big(\sum_{i=1}^{k} b_{ij}x_i\Big)dx_1\cdots dx_k.$$

Then $I(f_1,\dots,f_m) \leq I(f_1^\star,\dots,f_m^*)$.

Remark 2.1. This generalization of Riesz's inequality allows to localize the original inequality (1.11) in bounded measurable sets $\Omega \subset \mathbb{R}^N$. Indeed, assuming $f_i \in L^1(\Omega)$ for $i = 1,2,3$, defining $f_4 = f_5 = \chi_\Omega$ and

$$B = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \end{pmatrix}$$

we obtain

$$\int_{\Omega} f_1(x)\Big(\int_{\Omega} f_2(x-y)f_3(y)dy\Big)dx \leq \int_{\Omega^\star} f_1^\star(x)\Big(\int_{\Omega^\star} f_2^\star(x-y)f_3^\star(y)dy\Big)dx.$$

There is an extensive literature on Schwarz's symmetrization, as well as on other types of symmetrizations and functional rearrangements. We refer the reader to the books [10, 34, 35, 39, 40, 42] for a detailed description of these tools, and to the review article [46] as an abundant source of applications and references.

2.2. THE PROBLEMS

Since the different problems we deal with have a common structure, we introduce some notation to refer to them with brevity.

Let $E \subset \mathbb{R}^N$ be an open and bounded set with smooth boundary, let $\varphi : E \to \mathbb{R}$, $\psi : (0,T) \times E \to \mathbb{R}$, and $\zeta : E \to \mathbb{R}$ be nonnegative measurable functions. Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radially non-increasing function with $\|J\|_{L^1(\mathbb{R}^N)} = 1$, and $J_\varepsilon(x) = C_1\varepsilon^{-(N+2)}J(x/\varepsilon)$, for $\varepsilon > 0$ and C_1 given in (2.2).

(i) *Stationary nonlocal diffusion problem*, $\mathrm{SN}(E,h,\varepsilon)$:

$$-\int_{\mathbb{R}^N} J_\varepsilon(x-y)(w(y)-w(x))dy + cw(x) = h(x) \quad \text{for } x \in E,$$
$$w(x) = 0 \quad \text{for } x \in \mathbb{R}^N\backslash E.$$

(ii) *Stationary local diffusion problem*, SL(E,h):

$$\begin{aligned} & -\Delta W + cW = h && \text{in } E, \\ & W = 0 && \text{on } \partial E. \end{aligned}$$

(iii) *Evolution nonlocal diffusion problem*, EN$(E,\psi,\zeta,\varepsilon)$:

$$\partial_t w(t,x) - \int\limits_{\mathbb{R}^N} J_\varepsilon(x-y)(w(t,y)-w(t,x))dy + cw(t,x) = \psi(t,x)$$
$$\text{for } (t,x) \in (0,T) \times E,$$

$$\begin{aligned} & w(t,x) = 0 && \text{for } (t,x) \in (0,T) \times \mathbb{R}^N \backslash E, \\ & w(0,x) = \zeta(x) && \text{for } x \in E. \end{aligned}$$

(iv) *Evolution local diffusion problem*, EL(E,ψ,ζ):

$$\begin{aligned} & \partial_t W - \Delta W + cW = \psi && \text{in } (0,T) \times E, \\ & W = 0 && \text{on } (0,T) \times \partial E, \\ & W(0,\cdot) = \zeta && \text{in } E. \end{aligned}$$

2.3. ASSUMPTIONS

The minimal hypothesis for proving the well-posedness of the nonlocal diffusion and the local diffusion problems are different. Proving the convergence of the solutions of the nonlocal diffusion rescaled problems to their local diffusion versions requires additional assumptions.

Since in this article we are interested in the comparison results between solutions of a problem and of its symmetrized version, we give here just the assumptions needed to prove them.

The additional hypotheses needed to ensure the existence, uniqueness and regularity of solutions, as well as the convergence of solutions of nonlocal diffusion rescaled problems to their local diffusion counterpart, are assumed to hold. The reader is referred to [8, 24] for details on these questions.

Assumptions (H):

1. $J \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, for some $1 < r \leq 2$, is a non-negative function which is *radially non-increasing*, this is, such that (i) $J(x) = J(y)$ if $|x| = |y|$, and (ii) $J(x) \leq J(y)$ if $|x| \geq |y|$. Besides, we assume $\|J\|_{L^1(\mathbb{R}^N)} = 1$ and define the constant

$$C_1 = 2/\sigma^2 \quad \text{with} \quad \sigma^2 = \int\limits_{\mathbb{R}^N} |x|^2 J(x)dx < \infty. \tag{2.2}$$

2. The data, $\Omega \subset \mathbb{R}^N$, $F : \Omega \to \mathbb{R}$, $G : (0,T) \times \Omega \to \mathbb{R}$, and $v_0 : \Omega \to \mathbb{R}$ are regular enough to provide unique solutions with L^p regularity, for some $1 \leq p \leq \infty$, to problems $\mathrm{SN}(\Omega, F, \varepsilon)$, $\mathrm{SL}(\Omega, F)$, $\mathrm{EN}(\Omega, G, v_0, \varepsilon)$, $\mathrm{EL}(\Omega, G, v_0)$, and their corresponding symmetrized versions. We always assume that, at least, $F \in L^p(\Omega)$ and $G \in L^p((0,T) \times \Omega)$ for some $1 \leq p \leq \infty$.
3. In addition to $(\mathrm{H})_2$, we assume that the the regularity of the data is enough to provide the convergence of solutions of nonlocal diffusion rescaled problems to their local diffusion counterpart. Explicitly, if $u_\varepsilon, u_\varepsilon \in L^p(\mathbb{R}^N)$ are the solutions of $\mathrm{SN}(\Omega, F, \varepsilon)$ and $\mathrm{SN}(\Omega^\star, F^\star, \varepsilon)$, and if $U \in L^p(\Omega)$ and $V \in L^p(\Omega^\star)$ are the solutions of $\mathrm{SL}(\Omega, F, \varepsilon)$ and $\mathrm{SL}(\Omega^\star, F^\star, \varepsilon)$ then we have

$$\|u_\varepsilon - U\|_{L^p(\Omega)} \to 0 \quad \text{and} \quad \|v_\varepsilon - V\|_{L^p(\Omega^\star)} \to 0.$$

   A similar property is assumed for the solutions of the evolution problems.
4. This assumption is related to the convergence of time discrete approximations schemes to their corresponding time continuous versions. See (4.1)–(4.2) for details.

Remark 2.2. The condition $J \in L^r(\mathbb{R}^N)$ for $1 < r \leq 2$ is unusual, although not specially limiting. The use of the central limit theorem in the proof of Theorem 2.3 requires it.

2.4. MAIN RESULTS

We obtain similar comparison results for the stationary and the evolution problems. For the stationary problems, we have:

Theorem 2.3. *Assume* (H) *and set* $\varepsilon > 0$. *Let* $u_\varepsilon, v_\varepsilon \in L^p(\mathbb{R}^N)$ *be the solutions of* $\mathrm{SN}(\Omega, F, \varepsilon)$ *and* $\mathrm{SN}(\Omega^\star, F^\star, \varepsilon)$, *respectively. Then* v_ε *is radially non-increasing and*

$$\|u_\varepsilon\|_{L^p(\Omega)} \leq \|v_\varepsilon\|_{L^p(\Omega^\star)}. \tag{2.3}$$

Corollary 2.4. *Let* U, V *be the solutions of* $\mathrm{SN}(\Omega, F)$ *and* $\mathrm{SN}(\Omega^\star, F^\star)$, *respectively. Then* V *is radially non-increasing and*

$$\|U\|_{L^p(\Omega)} \leq \|V\|_{L^p(\Omega^\star)}.$$

For the evolution problems, we have similar results.

Theorem 2.5. *Assume* (H) *and set* $\varepsilon > 0$. *Let* $u_\varepsilon, v_\varepsilon \in L^p((0,T) \times \mathbb{R}^N)$ *be the solutions of* $\mathrm{EN}(\Omega, G, u_0, \varepsilon)$ *and* $\mathrm{EN}(\Omega^\star, G^\star, u_0^\star, \varepsilon)$, *respectively. Then* $v_\varepsilon(t, \cdot)$ *is radially non increasing for a.e.* $t \in (0,T)$ *and*

$$\|u_\varepsilon\|_{L^p((0,T)\times\Omega)} \leq \|v_\varepsilon\|_{L^p((0,T)\times\Omega^\star)}. \tag{2.4}$$

Corollary 2.6. *Let* U, V *be the solutions of* $\mathrm{EL}(\Omega, G, v_0)$ *and* $\mathrm{SL}(\Omega^\star, G^\star, u_0^\star)$, *respectively. Then* $V(t, \cdot)$ *is radially non increasing for a.e.* $t \in (0,T)$ *and*

$$\|U\|_{L^p((0,T)\times\Omega)} \leq \|V\|_{L^p((0,T)\times\Omega^\star)}.$$

Remark 2.7. Under assumption $(\mathrm{H})_3$, the corollaries are straightforward to prove. For instance, in the stationary case, using the triangle inequality and Theorem 2.3 we obtain

$$\begin{aligned}\|U\|_{L^p(\Omega)} &\leq \|U-u_\varepsilon\|_{L^p(\Omega)} + \|u_\varepsilon\|_{L^p(\Omega)} \leq \|U-u_\varepsilon\|_{L^p(\Omega)} + \|v_\varepsilon\|_{L^p(\Omega^\star)} \\ &\leq \|U-u_\varepsilon\|_{L^p(\Omega)} + \|v_\varepsilon - V\|_{L^p(\Omega^\star)} + \|V\|_{L^p(\Omega^\star)},\end{aligned}$$

and letting $\varepsilon \to 0$, we deduce the assertion. The evolution case is treated similarly.

Remark 2.8. The result of Talenti [45] for the stationary problem is the pointwise estimate $u_\varepsilon^\star \leq v_\varepsilon$ a.e. in $\Omega^\star$, and thus is stronger than the estimate in norm provided by Corollary 2.4. Similarly, for a nonlocal diffusion problem given in terms of fractional diffusion, Vázquez and Volzone [49] prove the mass concentration property

$$\int\limits_{B_R(0)} u_\varepsilon^\star(x)dx \leq \int\limits_{B_R(0)} v_\varepsilon(x)dx$$

for all $R > 0$, which is also stronger than our estimate in norm.

Although we expect that similar (stronger) results may be obtained for the problems involving nonlocal diffusion with integrable kernel when the measure geometric tools are employed, the most general estimates we may find with the techniques employed in this article are of the form

$$\int\limits_{\Omega} u(x)\varphi(x)dx \leq \int\limits_{\Omega^\star} v(x)\varphi^\star(x)dx,$$

for any $\varphi \in L^{p'}(\Omega)$, which, in the theorems, we have expressed in terms of L^p norms for the sake of presentation. In particular, choosing $\varphi = \chi_{\{u>s\}}$ and using the equi-measurability property (2.1), we deduce

$$\int\limits_{u^\star>s} u^\star(x)dx \leq \int\limits_{u^\star>s} v(x)dx,$$

which gives a mass comparison on the superlevels of $u^\star$.

2.5. NOTATION

For a measurable set $E \subset \mathbb{R}^N$ and a function $\varphi : E \to \mathbb{R}$, we define the *extension by zero of* φ *to* $\mathbb{R}^N$ as

$$\overline{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in E, \\ 0 & \text{if } x \in \mathbb{R}^N \backslash E. \end{cases}$$

For $\varphi, \psi \in L^1(\mathbb{R}^N)$, we denote the convolution of φ with ψ in $\mathbb{R}^N$ by the usual symbol $*$, this is,

$$\varphi * \psi(x) = \int\limits_{\mathbb{R}^N} \varphi(x-y)\psi(y)dy.$$

We recall that the convolution in $\mathbb{R}^N$ is commutative. For the convolution of φ with ψ in E, we use the symbol $\circledast_E$:

$$\varphi \circledast_E \psi(x) = \int_E \varphi(x-y)\psi(y)dy.$$

If the context is clear, we just write $\circledast$. Observe that if $\varphi, \psi \in L^1(\mathbb{R}^N)$ with $\psi = 0$ in $\mathbb{R}^N \backslash E$ then $\varphi * \psi = \varphi \circledast_E \psi$.

We write $(J*)^1 = J$, and denote by $(J*)^k$, for $k = 1, 2, \ldots$, to the recurrent convolution in $\mathbb{R}^N$ of k kernels, J. This is, $(J*)^2 = J * J$, $(J*)^3 = J * J * J$, and so on. Observe that $\|(J*)^k\|_{L^1(\mathbb{R}^N)} = \|J\|^k_{L^1(\mathbb{R}^N)}$. Correspondingly, we write $(J \circledast_E)^k$ to denote the convolution in E of k kernels, J. In this case, $\|(J \circledast_E)^k\|_{L^1(E)} \leq \|J\|^k_{L^1(E)}$.

3. PROOF OF THEOREM 2.3

Problem $\mathrm{SN}(E, h, \varepsilon)$ may be reformulated as

$$w(x) = \alpha \int_{\mathbb{R}^N} \rho_\epsilon(x-y)w(y)dy + \xi(x) \quad \text{for } x \in E, \tag{3.1}$$

$$w(x) = 0 \quad \text{for } x \in \mathbb{R}^N \backslash E, \tag{3.2}$$

where $\alpha = C_1/(C_1 + c\varepsilon^2)$, $\xi = \varepsilon^2 h/(C_1 + c\varepsilon^2)$, and $\rho_\varepsilon = (\varepsilon^2/C_1)J_\varepsilon$. Observe that $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|J\|_{L^1(\mathbb{R}^N)} = 1$, for all $\varepsilon > 0$ and that $\alpha \leq 1$.

The parameter ε is kept fixed in the proofs of Theorems 2.3 and 2.5. For the sake of clarity we shall remove it from the notation.

We refer to problem (3.1)–(3.2) as to $\mathrm{AUX}(E, \xi)$. Let u and v be the solutions of $\mathrm{AUX}(\Omega, f)$ and $\mathrm{AUX}(\Omega^\star, f^\star)$, respectively, for $f = \varepsilon^2 F/(C_1 + c\varepsilon^2)$. Then u and v are the solutions of $\mathrm{SN}(\Omega, F, \varepsilon)$ and $\mathrm{SN}(\Omega^\star, F^\star, \varepsilon)$, respectively.

The proof of the theorem uses two lemmas. We first prove the main result stating the lemmas when required and, afterwards, we prove the lemmas. We start with the comparison property stated in the theorem.

Let $\varphi_0 \in L^{p'}(\Omega)$ be a non-negative function. Multiplying the first equation of $\mathrm{AUX}(\Omega, f)$ by φ_0, integrating in Ω and using that $u = 0$ in $\mathbb{R}^N \backslash \Omega$, and the symmetry of ρ to interchange the convolved functions, we obtain

$$\begin{aligned} \int_\Omega u(x)\varphi_0(x)dx &= \alpha \int_\Omega u(y) \int_\Omega \rho(x-y)\varphi_0(x)dxdy + \int_\Omega f(x)\varphi_0(x)dx \\ &= \int_\Omega u(x)\varphi_1(x)dx + \int_\Omega f(x)\varphi_0(x)dx, \end{aligned} \tag{3.3}$$

with $\varphi_1 = \alpha\rho \circledast_\Omega \varphi_0$. Multiplying now the first equation of AUX(Ω, f) by φ_1 we similarly get

$$\int_\Omega u(x)\varphi_1(x)dx = \int_\Omega u(x)\varphi_2(x)dx + \int_\Omega f(x)\varphi_1(x)dx,$$

with $\varphi_2 = \alpha^2(\rho\circledast_\Omega)^2 \circledast_\Omega \varphi_0$. Thus, replacing in (3.3), we obtain

$$\int_\Omega u(x)\varphi_0(x)dx = \int_\Omega u(x)\varphi_2(x)dx + \int_\Omega f(x)(\varphi_0(x) + \varphi_1(x))dx.$$

Therefore, the following identity holds for $k = 1, 2, \ldots$:

$$\int_\Omega u(x)\varphi_0(x)dx = \int_\Omega u(x)\varphi_k(x)dx + \sum_{i=0}^{k-1}\int_\Omega f(x)\varphi_i(x)dx, \tag{3.4}$$

with $\varphi_k = \alpha^k(\rho\circledast_\Omega)^k \circledast_\Omega \varphi_0$.

In a similar way, testing the first equation of AUX$(\Omega^\star, f^\star)$ with $\varphi_0^\star$, we obtain

$$\int_{\Omega^\star} v(x)\varphi_0^\star(x)dx = \int_{\Omega^\star} v(x)\psi_k(x)dx + \sum_{i=0}^{k-1}\int_{\Omega^\star} f^\star(x)\psi_i(x)dx, \tag{3.5}$$

with $\psi_k = \alpha^k(\rho\circledast_{\Omega^\star})^k \circledast_{\Omega^\star} \varphi_0^\star$. The next lemma ensures that ψ_k is radially non-increasing.

Lemma 3.1. *Let $\xi \in L^1(\Omega^\star)$ be a radially non-increasing function. Then $(\rho\circledast_{\Omega^\star})^k\circledast_{\Omega^\star}\xi$ is radially non-increasing.*

Using Riesz's inequality, see Remark 2.1, we deduce

$$\begin{aligned}\int_\Omega f(x)\varphi_i(x)dx &= \alpha^i\int_\Omega f(x)\int_\Omega (\rho\circledast_\Omega)^i(x-y)\varphi_0(y)dydx \\ &\le \alpha^i\int_{\Omega^\star} f^\star(x)\int_{\Omega^\star}(\rho\circledast_{\Omega^\star})^i(x-y)\varphi_0^\star(y)dydx = \int_{\Omega^\star} f^\star(x)\psi_i(x)dx.\end{aligned}$$

This estimate in combination with (3.4) and (3.5) yields

$$\int_\Omega u(x)\varphi_0(x)dx \le \int_{\Omega^\star} v(x)\varphi_0^\star(x)dx + \int_\Omega u(x)\varphi_k(x)dx - \int_{\Omega^\star} v(x)\psi_k(x)dx. \tag{3.6}$$

The last two terms of the right hand side of this expression tend to cero as $k \to \infty$. This is the assertion of our second lemma.

Lemma 3.2. *Let $E \subset \mathbb{R}^N$ be a bounded measurable set, $w \in L^p(E)$ and $\xi \in L^{p'}(E)$, for $1 \leq p \leq \infty$. Then*

$$\lim_{k\to\infty} \int_E w(x)(\rho *)^k * \overline{\xi}(x)dx = 0. \tag{3.7}$$

The same convergence is attained when replacing $*$ by $\circledast_E$.

Therefore, taking the limit $k \to \infty$ in (3.6), using the lemma to estimate the last two integrals of the right hand side, and recalling that $\alpha \leq 1$ we obtain

$$\int_\Omega u(x)\varphi_0(x)dx \leq \int_{\Omega^\star} v(x)\varphi_0^\star(x)dx.$$

For $p = 1$, we choose $\varphi_0 = 1$ to obtain $\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\star)}$. For $p > 1$, recalling that $(\Phi(u))^\star = \Phi(u^\star)$ for any non-decreasing function Φ we deduce, for $\Phi(s) = s^{p-1}$,

$$\int_{\Omega^\star} |u(x)|^p dx \leq \int_{\Omega^\star} v(x)|u^\star(x)|^{p-1}dx. \tag{3.8}$$

Finally, (2.3) follows from Hölder's inequality and the equi-measurability property $\|u\|_{L^p(\Omega)} = \|u^\star\|_{L^p(\Omega^\star)}$.

We continue the proof of the theorem showing that the solution of the symmetrized problem $\mathrm{AUX}(\Omega^\star, f^\star)$, and thus of the stationary problem $\mathrm{SN}(\Omega^\star, F^\star, \varepsilon)$, is radially non-increasing.

Let $v_0 \in L^1(\Omega^\star)$ be a radially non-increasing function, and consider the sequence $v_k : \Omega^\star \to \mathbb{R}$ given by, for $k = 1, 2, \ldots$, and $x \in \Omega^\star$,

$$v_k(x) = \alpha\rho \circledast_{\Omega^\star} v_{k-1}(x) + f^\star(x). \tag{3.9}$$

By Lemma 3.1, v_k is radially non-increasing. Solving the recursion (3.9), we obtain

$$v_k(x) = \alpha^k(\rho\circledast_{\Omega^\star})^k \circledast_{\Omega^\star} v_0(x) + \Big(f^\star(x) + \sum_{i=1}^{k-1} \alpha^i(\rho\circledast_{\Omega^\star})^i * f^\star(x)\Big), \quad \text{for } k = 1, 2, \ldots \tag{3.10}$$

For $x \in \Omega^\star$, the solution v of $\mathrm{AUX}(\Omega^\star, f^\star)$ satisfies

$$v(x) = \alpha\rho \circledast_{\Omega^\star} v(x) + f^\star(x), \tag{3.11}$$

where we used that $v = 0$ in $\mathbb{R}^\mathbb{N} \backslash \Omega^\star$. Therefore, for $k = 2, 3, \ldots$ and $x \in \Omega^\star$, we deduce the implicit formula

$$v(x) = \alpha^k(\rho\circledast_{\Omega^\star})^k \circledast_{\Omega^\star} v(x) + \Big(f^\star(x) + \sum_{i=1}^{k-1} \alpha^i(\rho\circledast_{\Omega^\star})^i \circledast_{\Omega^\star} f^\star(x)\Big). \tag{3.12}$$

From (3.10) and (3.12), we obtain, for $x \in \Omega^\star$ and $k = 2, 3, \ldots$,

$$v_k(x) = v(x) + \alpha^k (\rho \circledast_{\Omega^\star})^k \circledast_{\Omega^\star} (v_0 - v)(x).$$

Defining $v_0 = f^\star$ and using (3.11) we obtain, for $x \in \Omega^\star$ and $k = 2, 3, \ldots$

$$v_k(x) = v(x) - \alpha^{k+1} (\rho \circledast_{\Omega^\star})^{k+1} \circledast_{\Omega^\star} v(x),$$

implying

$$\|v_k - v\|_{L^1(\Omega^*)} = \alpha^{k+1} \int_{\Omega^*} (\rho \circledast_{\Omega^\star})^{k+1} \circledast_{\Omega^\star} v(x) dx. \tag{3.13}$$

Using Lemma 3.2 with $E = \Omega^\star$, $w = 1$ and $\xi = v|_{\Omega^\star}$ in (3.13) we deduce that $v_k \to v|_{\Omega^\star}$ strongly in $L^1(\Omega^\star)$. At least for a subsequence v_{k_j}, we have that $v_{k_j} \to v|_{\Omega^\star}$ a.e. in $\Omega^\star$. Since v_{k_j} are radially non-increasing, it follows that the solution v of AUX$(\Omega^\star, f^\star)$ is also radially non-increasing. This finishes the proof of Theorem 2.3. □

Proof of Lemma 3.1. We first prove the result for a radially non-increasing function ξ defined in $\mathbb{R}^N$. We start by checking that $A(x) = \rho * \xi(x)$ is radial.

Let $x_1, x_2 \in \mathbb{R}^N$ be such that $|x_1| = |x_2|$. Then, there exists a orthogonal rotation matrix, G, such that $x_2 = Gx_1$ and $|\det G| = 1$. Introducing the change of integration variable $y = Gz$ and taking into account that $|Gz| = |z|$, we obtain

$$A(x_2) = \int_{\mathbb{R}^N} \rho(G(x_1 - z))\xi(Gz)dz = \int_{\mathbb{R}^N} \rho(x_1 - z)\xi(z)dz = A(x_1),$$

so that A is radial.

We now check that A is radially non-increasing. For a radial function $\zeta : \mathbb{R}^N \to \mathbb{R}$ we introduce the notation $\zeta_R : \mathbb{R} \to \mathbb{R}$ to denote the real function such that $\zeta(x) = \zeta_R(|x|)$. We start assuming that $\xi \in W^{1,1}(\mathbb{R}^N)$. Then

$$\partial_{x_j} A(x) = \int_{\mathbb{R}^N} \rho_R(|y|)\xi'_R(|x - y|)\frac{x_j - y_j}{|x - y|}dy = \int_{\mathbb{R}^N} \rho_R(|x - z|)\xi'_R(|z|)\frac{z_j}{|z|}dz,$$

and thus

$$\begin{aligned}
\nabla A(x) \cdot x &= \int_{\mathbb{R}^N} \rho_R(|x - z|)\xi'_R(|z|)\frac{z \cdot x}{|z|}dz \\
&= \int_{z \cdot x > 0} \rho_R(|x - z|)\xi'_R(|z|)\frac{z \cdot x}{|z|}dz + \int_{z \cdot x < 0} \rho_R(|x - z|)\xi'_R(|z|)\frac{z \cdot x}{|z|}dz \\
&= \int_{z \cdot x > 0} \rho_R(|x - z|)\xi'_R(|z|)\frac{z \cdot x}{|z|}dz - \int_{z \cdot x > 0} \rho_R(|x + z|)\xi'_R(|z|)\frac{z \cdot x}{|z|}dz \\
&= \int_{z \cdot x > 0} \big(\rho_R(|x - z|) - \rho_R(|x + z|)\big)\xi'_R(|z|)\frac{z \cdot x}{|z|}dz.
\end{aligned}$$

For $z \in \mathbb{R}^N$ we have the equivalence $|x-z| < |x+z| \iff z \cdot x > 0$. Since ρ_R and ξ_R are non-increasing, we deduce that $\nabla A(x) \cdot x \leq 0$, this is, A is radially non-increasing.

The general case $\xi \in L^1(\mathbb{R}^N)$ is then proven by approximating ξ by a sequence $\xi_n \in W^{1,1}(\mathbb{R}^N)$, which satisfies the property $A_n(x) \leq A_n(y)$ if $|x| \geq |y|$. Then, the pointwise convergence of a subsequence $\xi_{n_j} \to \xi$ implies that this property is kept by the limit, A.

Finally, if ξ is radially non-increasing in $\Omega^\star$ we deduce that, for $x \in \Omega^\star$,

$$\int_{\Omega^\star} \rho(x-y)\xi(y)dy = \int_{\mathbb{R}^N} \rho(x-y)\bar{\xi}(y)dy, \tag{3.14}$$

and since $\bar{\xi}$ is radially non-increasing in $\mathbb{R}^N$, we deduce that $\rho \circledast_{\Omega^\star} \xi$ is radially non-increasing in $\Omega^\star$. For the concatenation of convolutions we follow a similar argument interchanging ξ by $(\rho \circledast_{\Omega^\star})^k \circledast_{\Omega^\star} \xi$, for $k = 1, 2, \ldots$ in (3.14). □

Proof of Lemma 3.2. We have

$$\left| \int_E w(x)(\rho *)^k * \bar{\xi}(x)dx \right| \leq \|w\|_{L^p(E)} \|(\rho *)^k * \bar{\xi}\|_{L^{p'}(E)}.$$

Since E is bounded, there exists a ball B_R such that $E \subset B_R$ and $x - y \in B_R$ for all $x, y \in E$. Like in the case $E = \mathbb{R}^N$, we have

$$\|(\rho *)^k * \bar{\xi}\|_{L^{p'}(E)} \leq \|(\rho *)^k\|_{L^1(B_R)} \|\xi\|_{L^{p'}(E)}.$$

Indeed,

$$\begin{aligned}
\int_E \left| \int_{\mathbb{R}^N} (\rho *)^k (x-y)\bar{\xi}(y)dy \right|^{p'} dx &= \int_E \left| \int_E (\rho *)^k (x-y)\xi(y)dy \right|^{p'} dx \\
&= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} (\rho *)^k (x-y)\bar{\xi}(y)\chi_E(y)\chi_{B_R}(x-y)dy \right|^{p'} \chi_E(x)dx \\
&= \|\big((\rho *)^k \chi_{B_R}\big) * (\xi \chi_E)\|^{p'}_{L^{p'}(\mathbb{R}^N)} \\
&\leq \|(\rho *)^k \chi_{B_R}\|^{p'}_{L^1(\mathbb{R}^N)} \|\xi \chi_E\|^{p'}_{L^{p'}(\mathbb{R}^N)}.
\end{aligned}$$

According to the central limit theorem [33, §46, Th. 1], we have

$$\sigma\sqrt{k}(\rho *)^k(\sigma\sqrt{k}x) \to \Phi(x) \quad \text{as } k \to \infty \quad \text{uniformly for } x \text{ in } \mathbb{R}^N, \tag{3.15}$$

where Φ is the Gaussian of zero mean and identity covariance matrix. Let, for $x \in \mathbb{R}^N$,

$$U_k(x) = \sigma\sqrt{k}(\rho *)^k(\sigma\sqrt{k}x)\chi_{B_{r_k}}(x)dx,$$

with $r_k = R/(\sigma\sqrt{k})$, so that $\int_{B_R}(\rho*)^k = \int_{\mathbb{R}^N} U_k$. The uniform convergence (3.15) implies that for all $\delta > 0$ there exists $K \in \mathbb{N}$ such that $|\sigma\sqrt{k}(\rho*)^k(0) - \Phi(0)| < \delta$ for $k > K$. Since $(\rho*)^k$ is radially non-increasing, we deduce

$$U_k(x) \leq (\delta + \Phi(0))\chi_{B_{r_k}}(x) \leq (\delta + \Phi(0))\chi_{B_{r_1}}(x).$$

The first inequality shows that $U_k(x) \to 0$ for a.e. $x \in \mathbb{R}^N$, while the second implies that U_k is dominated by $(\delta + \Phi(0))\chi_{B_{r_1}} \in L^1(\mathbb{R}^N)$. Therefore, the theorem of dominated convergence ensures that $\|(\rho*)^k\|_{L^1(B_R)} \to 0$ as $k \to \infty$, and (3.7) follows.

Finally, on noting that

$$\begin{aligned}\|(\rho\circledast_E)^k \circledast_E \bar{\xi}\|^{p'}_{L^{p'}(E)} &= \int_E \left| \int_E (\rho\circledast_E)^k(x-y)\xi(y)dy \right|^{p'} dx \\ &\leq \|(\rho\circledast_E)^k\|^{p'}_{L^1(B_R)}\|\xi\|^{p'}_{L^{p'}(E)} \leq \|(\rho*)^k\|^{p'}_{L^1(B_R)}\|\xi\|^{p'}_{L^{p'}(E)},\end{aligned}$$

we see that the same arguments may be employed when replacing $*$ by $\circledast_E$. □

4. PROOF OF THEOREM 2.5

Consider the partition of the interval $[0, T]$ given by $t_n = n\tau$, for $n = 0, \ldots, N$, with $\tau = T/N$. We introduce the following explicit time discretization of the problem $\mathrm{EN}(E, \psi, \zeta, \varepsilon)$. We set $w_0 = \zeta$ in E and, for $n = 0, \ldots, N-2$,

$$w_{n+1}(x) = (1 - \tau c)w_n(x) + \tau \int_{\mathbb{R}^N} J_\varepsilon(x-y)(w_n(y) - w_n(x))dy + \tau\psi_n(x)$$

for $x \in E$, and $w_{n+1}(x) = 0$ for $x \in \mathbb{R}^N \backslash E$, where $\psi_n(x) = \psi(t_n, x)$. We also consider the piecewise constant interpolator

$$w^{(\tau)}(t,x) = w_n(x) \quad \text{for} \quad (t,x) \in (t_n, t_{n+1}] \times E \quad \text{and} \quad n = 0, \ldots, N-1.$$

Let u and v be the solutions of $\mathrm{EN}(\Omega, G, u_0, \varepsilon)$ and $\mathrm{EN}(\Omega^\star, G^\star, u_0^\star, \varepsilon)$, and let $u^{(\tau)}$ and $v^{(\tau)}$ be their corresponding approximations according to the discrete scheme. Here, $G^\star$ denotes the Schwarz symmetrization of G with respect to the space variable, i.e. $(G(t,\cdot))^\star$.

Assumption $(\mathrm{H})_4$*: The data,* $\Omega \subset \mathbb{R}^N$*,* $G : (0,T) \times \Omega \to \mathbb{R}$*, and* $u_0 : \Omega \to \mathbb{R}$ *are regular enough to provide the convergence, as* $\tau \to 0$*,*

$$u^{(\tau)} \to u \quad \textit{strongly in } L^p((0,T) \times \Omega), \tag{4.1}$$

$$v^{(\tau)} \to v \quad \textit{strongly in } L^p((0,T) \times \Omega^\star). \tag{4.2}$$

Remark 4.1. The convergence property assumed in $(\mathrm{H})_4$ has been proven for a general class of nonlinear nonlocal diffusion evolution problems. See [31, 32] for details.

The semi-discrete problem may be reformulated as: $w_0 = \zeta$ in E and, for $n = 0, \dots, N-2$,

$$w_{n+1}(x) = \alpha \int_{\mathbb{R}^N} \rho_\varepsilon(x-y) w_n(y) dy + \beta w_n(x) + \tau \psi_n(x) \quad \text{for} \quad x \in E, \tag{4.3}$$

and $w_{n+1} = 0$ for $x \in \mathbb{R}^N \backslash E$, where

$$\alpha = C_1 \frac{\tau}{\varepsilon^2}, \quad \rho_\varepsilon = \frac{\varepsilon^2}{C_1} J_\varepsilon, \quad \beta = \Big(1 - (C_1 + c\varepsilon^2)\frac{\tau}{\varepsilon^2}\Big),$$

yielding $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} = 1$. We choose the time step $\tau < \varepsilon^2/(C_1 + c\varepsilon^2)$, implying $\beta > 0$. Observe that this is the usual stability condition associated to explicit time discretization schemes.

Like in the proof of Theorem 2.3, we remove from the notation the reference to the parameter ε and refer to the previous scheme as to $\mathrm{DEN}(E, \psi, \zeta)$.

Let u_n and v_n be the sequences defined by the schemes $\mathrm{DEN}(\Omega, G, u_0)$ and $\mathrm{DEN}(\Omega^\star, G^\star, u_0^\star)$, respectively. From (4.3), it is clear that, being an addition of nonnegative radially non-increasing functions, v_n is radially non-increasing.

Multiplying the identity (4.3) corresponding to $\mathrm{DEN}(\Omega, G, u_0)$ by any non-negative $\varphi \in L^{p'}(\Omega)$, integrating in Ω, and using the symmetry of ρ to interchange the convolution of ρ with u_n for the convolution of ρ with φ, and taking into account that $u_n = 0$ in $\mathbb{R}^N \backslash \Omega$, we get

$$\int_\Omega u_{n+1}\varphi = \alpha \int_\Omega u_n(x) \int_\Omega \rho(x-y)\varphi(y) dy dx + \int_\Omega (\beta u_n + \tau G_n)\varphi. \tag{4.4}$$

Proceeding similarly for $\mathrm{DNS}(\Omega^\star, G^\star, u_0^\star)$, we obtain

$$\int_{\Omega^\star} v_{n+1}\varphi^* = \alpha \int_{\Omega^\star} v_n(x) \int_{\Omega^\star} \rho(x-y)\varphi^\star(y) dy dx + \int_{\Omega^\star} (\beta v_n + \tau G_n^\star)\varphi^\star. \tag{4.5}$$

For $n = 0$, we have

$$\int_\Omega u_1\varphi = \alpha \int_\Omega u_0(x) \int_\Omega \rho(x-y)\varphi(y) dy dx + \int_\Omega (\beta u_0 + \tau G_0)\varphi, \tag{4.6}$$

$$\int_{\Omega^\star} v_1\varphi^* = \alpha \int_{\Omega^\star} v_0(x) \int_{\Omega^\star} \rho(x-y)\varphi^\star(y) dy dx + \int_{\Omega^\star} (\beta v_0 + \tau G_0^\star)\varphi^\star. \tag{4.7}$$

Using Riesz's inequality and $v_0 = u_0^\star$ in the first term of the right hand side of (4.6) yields

$$\int_\Omega u_0(x) \int_\Omega \rho(x-y)\varphi(y) dy dx \leq \int_{\Omega^\star} v_0(x) \int_{\Omega^\star} \rho(x-y)\varphi^\star(y) dy dx.$$

Hardy–Littlewood's inequality gives

$$\int_{\Omega} u_0\varphi \le \int_{\Omega^\star} v_0\varphi^\star \quad \text{and} \quad \int_{\Omega} G_0\varphi \le \int_{\Omega^\star} G_0^\star\varphi^\star.$$

Therefore, (4.6) may be estimated as

$$\begin{aligned}\int_{\Omega} u_1\varphi &\le \alpha \int_{\Omega^\star} v_0(x) \int_{\Omega^\star} \rho(x-y)\varphi^\star(y)dydx + \int_{\Omega^\star} (\beta v_0 + \tau G_0^\star)\varphi^\star \\ &= \int_{\Omega^\star} v_1\varphi^\star,\end{aligned}$$

where the last equality is deduced from (4.7).

Now we proceed by induction. Assume that for some $1 < k < N-2$ we have

$$\int_{\Omega} u_k\varphi \le \int_{\Omega^\star} v_k\varphi^\star. \tag{4.8}$$

Introducing the notation

$$\begin{aligned} I_k^j &= \int_{\Omega} u_k(x)(\rho \circledast_{\Omega})^j \circledast_{\Omega} \varphi(x)dx, \quad J_k^j = \int_{\Omega} G_k(x)(\rho \circledast_{\Omega})^j \circledast_{\Omega} \varphi(x)dx, \\ I_k^{\star j} &= \int_{\Omega^\star} v_k(x)(\rho \circledast_{\Omega^\star})^j \circledast_{\Omega^\star} \varphi^\star(x)dx, \quad J_k^{\star j} = \int_{\Omega^\star} G_k^\star(x)(\rho \circledast_{\Omega^\star})^j \circledast_{\Omega^\star} \varphi^\star(x)dx, \end{aligned}$$

we may write (4.4) and (4.5) for $n = k$ as

$$I_{k+1}^0 = \alpha I_k^1 + \beta I_k^0 + \tau J_k^0, \qquad I_{k+1}^{\star 0} = \alpha I_k^{\star 1} + \beta I_k^{\star 0} + \tau J_k^{\star 0}. \tag{4.9}$$

The induction assumption (4.8) implies $I_k^0 \le I_k^{\star 0}$ and Hardy–Littlewood's inequality yields $J_k^0 \le J_k^{\star 0}$. Estimating the term I_k^1 is more complicated. Replacing the test function φ by $\rho \circledast_{\Omega} \varphi$ and the index $n+1$ by k in (4.4), we obtain

$$I_k^1 = \alpha I_{k-1}^2 + \beta I_{k-1}^1 + \tau J_{k-1}^1. \tag{4.10}$$

Similar replacements of the test function and indices give

$$I_{k-1}^1 = \alpha I_{k-2}^2 + \beta I_{k-2}^1 + \tau J_{k-2}^1, \quad I_{k-1}^2 = \alpha I_{k-2}^3 + \beta I_{k-2}^2 + \tau J_{k-2}^2,$$

and from (4.10) we deduce

$$I_k^1 = \beta^2 I_{k-2}^1 + 2\alpha\beta I_{k-2}^2 + \alpha^2 I_{k-2}^3 + \tau(J_{k-1}^1 + \beta J_{k-2}^1 + \alpha J_{k-2}^2).$$

Performing a finite amount of this kind of test function replacements we reach the initial time index $n = 0$, obtaining the expression

$$I_k^1 = \sum_{j=0}^{k} \binom{k}{j} \beta^{k-j}\alpha^j I_0^{1+j} + \tau \sum_{j=0}^{k-1} \sum_{m=0}^{j} \binom{j}{m} \beta^{j-m}\alpha^m J_{k-j-1}^{1+m}.$$

A similar expression is obtained for the symmetrized problem, this is

$$I_k^{\star 1} = \sum_{j=0}^{k} \binom{k}{j} \beta^{k-j} \alpha^j I_0^{\star 1+j} + \tau \sum_{j=0}^{k-1} \sum_{m=0}^{j} \binom{j}{m} \beta^{j-m} \alpha^m J_{k-j-1}^{\star 1+m}.$$

The generalized Riesz's inequality provides the estimates

$$I_0^{1+j} \leq I_0^{\star 1+j} \quad \text{and} \quad J_{k-j-1}^{1+m} \leq J_{k-j-1}^{\star 1+m},$$

with which we finally deduce $I_k^1 \leq I_k^{\star 1}$. Returning to (4.9), we obtain

$$I_{k+1}^0 = \alpha I_k^1 + \beta I_k^0 + \tau J_k^0 \leq \alpha I_k^{\star 1} + \beta I_k^{\star 0} + \tau J_k^{\star 0} = I_{k+1}^{\star 0},$$

this is,

$$\int_\Omega u_{k+1} \varphi \leq \int_{\Omega^\star} v_{k+1} \varphi^\star.$$

This finishes the inductive argument, which proves that

$$\int_\Omega u_n \varphi \leq \int_{\Omega^\star} v_n \varphi^\star, \quad \text{for all} \quad n = 0, 1, \ldots, N-2.$$

By choosing $\varphi = 1$, for $p = 1$ and $\varphi = u_n^{p-1}$, for $p > 1$, we deduce like in (3.8) that $\|u_n\|_{L^p(\Omega)} \leq \|v_n\|_{L^p(\Omega^\star)}$, for all $n = 0, 1, \ldots, N-2$, implying that

$$\|u^{(\tau)}\|_{L^p((0,T)\times\Omega)} \leq \|v^{(\tau)}\|_{L^p((0,T)\times\Omega^\star)}.$$

We now recover the ε-notation and pass to the limit $\tau \to 0$ to prove the estimate (2.4) of Theorem 2.5. We have

$$\begin{aligned} \|u_\varepsilon\|_{L^p((0,T)\times\Omega)} &\leq \|u_\varepsilon - u_\varepsilon^{(\tau)}\|_{L^p(0,T)\times\Omega)} + \|u_\varepsilon^{(\tau)}\|_{L^p(0,T)\times\Omega)} \\ &\leq \|u_\varepsilon - u_\varepsilon^{(\tau)}\|_{L^p(0,T)\times\Omega)} + \|v_\varepsilon^{(\tau)}\|_{L^p(0,T)\times\Omega^\star)} \\ &\leq \|u_\varepsilon - u_\varepsilon^{(\tau)}\|_{L^p(0,T)\times\Omega)} + \|v_\varepsilon - v_\varepsilon^{(\tau)}\|_{L^p(0,T)\times\Omega^\star)} + \|v_\varepsilon\|_{L^p(0,T)\times\Omega^\star)}, \end{aligned}$$

and taking $\tau \to 0$ and using the assumption $(\mathrm{H})_4$ we deduce (2.4). Finally, the same kind of argument combined with assumption $(\mathrm{H})_3$ allows to pass to the limit $\varepsilon \to 0$ to prove Corollary 2.6.

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