Opuscula Math. 44, no. 4 (2024), 565–585
https://doi.org/10.7494/OpMath.2024.44.4.565 Opuscula Mathematica

GEOMETRIC PROPERTIES OF THE LATTICE OF POLYNOMIALS WITH INTEGER COEFFICIENTS

Artur Lipnicki and Marek J. Śmietański

Communicated by Palle E.T. Jorgensen

Abstract. This paper is related to the classic but still being examined issue of approximation of functions by polynomials with integer coefficients. Let r, n be positive integers with $n \geq 6r$. Let $\boldsymbol{P}_n \cap \boldsymbol{M}_r$ be the space of polynomials of degree at most n on $[0,1]$ with integer coefficients such that $P^{(k)}(0)/k!$ and $P^{(k)}(1)/k!$ are integers for $k = 0, \ldots, r-1$ and let $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ be the additive group of polynomials with integer coefficients. We explore the problem of estimating the minimal distance of elements of $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ from $\boldsymbol{P}_n \cap \boldsymbol{M}_r$ in $L_2(0,1)$. We give rather precise quantitative estimations for successive minima of $\boldsymbol{P}_n^{\mathbb{Z}}$ in certain specific cases. At the end, we study properties of the shortest polynomials in some hyperplane in $\boldsymbol{P}_n \cap \boldsymbol{M}_r$.

Keywords: approximation by polynomials with integer coefficients, lattice, covering radius, roots of polynomial.

Mathematics Subject Classification: 41A10, 52C07, 26C10, 65H04.

1. INTRODUCTION

The paper presents the idea of examining approximation problems by polynomials with integer coefficients in the context of lattices. Its importance results from, among other things, connections with such areas of the number theory as Diophantine approximations. There are considered the elements of algebraic integer rings instead of the coefficients taken from $\mathbb{Z}$.

Throughout the work, by an integer polynomial, we mean a polynomial with integer coefficients.

Results regarding the approximation of functions by integer polynomials were initiated by Pál in 1914 [8]. Pál proved that a continuous real-valued function on the interval $[-\alpha, \alpha]$ with $\alpha \in (0,1)$ can be uniformly approximated by polynomials with integer coefficients if and only if $f(0)$ is integer number.

In 1923 Okada in [7] expanded the Pál's assertion to the case $\alpha \in [1,2)$ demonstrating that if there exists a polynomial $X \in \mathbb{Z}[x]$ satisfying the condition $\|X\|_{[-\alpha,\alpha]} < 1$ then the function $f \in C[-\alpha, \alpha]$ can be uniformly approximated by polynomials with integer coefficients if and only if $f(x_j) = 0$, for $x_j \in X^{-1}(\{0\}) \cap [\alpha, \alpha]$. Okada also proved the existence of such polynomials P. In the same year, Fekete independently received a more precise result. His proof was later simplified by Aparicio in 1955. It turned out that the smaller the degree of the polynomial X is then the function f must meet fewer conditions so that it can be approximated by integer polynomials.

In 1925 Chlodovski in a fairly simple way showed that if $[a, b] \subset (0, 1)$ and $f \in C[a, b]$ then $\mathrm{dist}_{[a,b]}(f, \mathbb{Z}[x]) = 0$. The border case, when the Chlodovski's result cannot be applied, is interval $[0, 1]$.

In 1931 the following result of L. Kantorovich [5] appeared: a continuous function $f : [0, 1] \to \mathbb{R}$ can be uniformly approximated by polynomials with integer coefficients if and only if $f(0), f(1) \in \mathbb{Z}$. The necessity of the condition is obvious. The sufficiency is a consequence of Okada's Theorem.

We expand the function f to an even function $\tilde{f} \in C[-1, 1]$. We subtract from f the polynomial $p(x) = (f(1) - f(0))x^2 + f(0) \in \mathbb{Z}$ and next we apply the Okada's Theorem to function $\tilde{f} - p$. Conversely, assuming that $f \in C[-1, 1]$ satisfies $f(-1), f(0), f(1) \in \mathbb{Z}$ and $f(1) \equiv f(-1) \mod 2$ then

$$f(x) = f_0(x) + f_1(x), \quad f_i(x) = \frac{f(x) + (-1)^i f(-x)}{2}, \quad f_1(x) = x\tilde{f}_1(x),$$

where $f_0(x)$ and $\tilde{f}_1(x)$ are even and $f_0(\sqrt{x}), \tilde{f}_1(\sqrt{x}) \in C[0, 1]$ take integer values on the ends of the interval. These functions are approximated by integer polynomials and further, we construct the approximations of function f. Kantorovich showed somewhat more, namely, using the assumption $f(0) = f(1) = 0$ he obtained

$$E_n^{\mathbb{Z}}(f) \le 2E_n(f) + 1/n,$$

where $E_n(f)$ and $E_n^{\mathbb{Z}}(f)$ are distances f from the space consisting of polynomials with real (resp. integer) coefficients of degree at least n. The proof lies in the fact that polynomial P_n of the best approximation was replaced by the polynomial $\tilde{P}_n = \sum_{k=1}^{n-1} a_k x^k (1 - x)^k$ and the coefficients a_k were replaced by $[a_k]$. So one can see that the key role is played by the distance of the polynomial $P \in P_n$ $(n \ge 2)$ satisfying $P(0) = P(1) = 0$ of polynomial P_n of similar properties. The result of Okada was adapted by Aparicio to the case of polynomial approximation of the function from $L_p[a, b]$: if $b - a < 4$ for any function of $L_p[a, b]$ we can approximate (in the norm L_p) by integer polynomials.

Since polynomials are dense in $L_p[a, b]$ it is enough to prove that every polynomial can be approximated in $L_p[a, b]$. This leads to the following question: let P be a polynomial of degree at most n; how well P can be approximated by integer polynomials of degree at most n?

In the sequel, we shall restrict our considerations to the special case $[a, b] = [0, 1]$. We can give quite precise and less complicated estimations in this case. Most applications of the polynomial approximations concern the $[0, 1]$ interval, which is still under investigation. For example, they are considered applied in searching minimal polynomials, Chebyshev's constant the optimum.

We assume that m, n, r are non-negative integers. We denote by $\boldsymbol{P}_n$ the space of polynomials of degree n on the interval $[0, 1]$ with the established norm (in this paper we consider uniform or Euclidean norm). Let $\boldsymbol{P}_n^{\mathbb{Z}}$ be the additive subgroup of the space $\boldsymbol{P}_n$ consisting of polynomials with integer coefficients. Let $\boldsymbol{M}_r$ be the space of all polynomials divisible by the polynomial $x^r(1-x)^r$. When $n \le 2r-1$, then $\boldsymbol{P}_n \cap \boldsymbol{M}_r = \{0\}$, therefore, we will still to assume that $n \ge 2r$. Let us denote

$$\gamma_{r,n} := \max_{P \in \boldsymbol{P}_n \cap \boldsymbol{H}_r} d(P, \boldsymbol{P}_n^{\mathbb{Z}}).$$

In other words, $\gamma_{r,n}$ is covering radius of the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ with norm $L_p(0,1)$. Our question concerning the accuracy of the approximation level can be stated as follows: what are the values of $\gamma_{r,n}$? Responding to our question Kantorovich [5] obtained the inequality$\gamma_{r,n} \le 1/2n$ for the uniform norm. Trigub [9] observed, that $\gamma_{r,n} \asymp 1/n^2$, but in his work he did not provide the proof. In the first section we compiled some basic facts about estimations of $\gamma_{r,n}$ with the uniform and Euclidean norm. We get in this case good estimations for covering radius. This leads to the following question: what is the geometric lattice structure, if there is a good base?

The geometry of the base of our lattice can be quite complex in terms of the length of the generating elements. It is therefore reasonable to ask how can one choose a good basis consisting of almost orthogonal vectors, that have short lengths. We will ask for further estimations of the generating elements. The objective of our paper is to show estimations of the norms of subsequent generators of the selected lattice. We will give estimates of subsequent norms (lattice minima) for even and odd numbers.

Results of Section 2 will be used to estimate some metric values in the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$. By $\boldsymbol{E}$ (resp. by $\boldsymbol{F}$) we will denote the subspace of $\boldsymbol{P}$ consisting of polynomials P such that $P(1-x) = P(x)$ (resp. $P(1-x) = -P(x)$). For all r the space $\boldsymbol{P}_n \cap \boldsymbol{M}_r$ can be written in the form of orthogonal direct sum

$$(\boldsymbol{P}_n \cap \boldsymbol{M}_r \cap \boldsymbol{E}) \oplus (\boldsymbol{P}_n \cap \boldsymbol{M}_r \cap \boldsymbol{F}).$$

Suppose that n is an even number, $n = 2m$. Polynomials $x^i(1-x)^i$ $(r \le i \le m)$ $(r = 0, 1, \ldots, m)$ are the basis of the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r \cap \boldsymbol{E}$. Similarly, for $r = 0, 1, \ldots, m-1$ polynomials $(2x-1)x^i(1-x)^i$, $r \le i \le m-1$ are the basis of the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r \cap \boldsymbol{F}$. But they aren't good bases, because the norms of projections on the approximate coordinates can have big values. Note that

$$(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r \cap \boldsymbol{E}) \oplus (\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r \cap \boldsymbol{F}) \subsetneq \boldsymbol{P}_n^{\boldsymbol{Z}} \cap \boldsymbol{M}_r.$$

On the other hand

$$\boldsymbol{P}_n^{\boldsymbol{Z}} \cap \boldsymbol{M}_r \subsetneq \frac{1}{2}\left[(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r \cap \boldsymbol{E}) \oplus (\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r \cap \boldsymbol{F})\right].$$

It appears that in lattice $\boldsymbol{P}_n^{\boldsymbol{Z}} \cap \boldsymbol{E}$ we can find the base $S_1, \ldots, S_{m+1}$ such that,

$$\operatorname{span}\{S_{r+1}, \ldots, S_{m+1}\} = \boldsymbol{P}_n \cap \boldsymbol{M}_r \cap \boldsymbol{E}, \quad 0 \le r \le m.$$

The polynomial S_r is almost orthogonal to the subspace $\boldsymbol{P}_n \cap \boldsymbol{M}_r \cap \boldsymbol{E}$ for $r \lesssim m/3$. Similarly, for $\boldsymbol{P}_n^{\boldsymbol{Z}} \cap \boldsymbol{F}$ we can find the base $T_1, \ldots, T_m$ such that

$$\operatorname{span}\{T_{r+1}, \ldots, T_m\} = \boldsymbol{P}_n \cap \boldsymbol{M}_r \cap \boldsymbol{F}, \quad 0 \le r \le m-1,$$

wherein the polynomial T_r is almost orthogonal to the subspace $\boldsymbol{P}_n \cap \boldsymbol{M}_r \cap \boldsymbol{F}$ for $r \lesssim m/3$.

Let us note here that there is a polynomial $S_1 \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{E}$ orthogonal to the subspace $\boldsymbol{P}_n \cap \boldsymbol{M}_1 \cap \boldsymbol{E}$. Similarly, there is a polynomial $T_1 \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{F}$ orthogonal to the subspace $\boldsymbol{P}_n \cap \boldsymbol{M}_1 \cap \boldsymbol{F}$. We will search these polynomials in Section 4.

Let us consider now

$$\Phi_{n,r} : \boldsymbol{P}_n \cap \boldsymbol{E} \mapsto X_{n,r} := (\boldsymbol{P}_n \cap \boldsymbol{E})/(\boldsymbol{P}_n \cap \boldsymbol{M}_r \cap \boldsymbol{E}).$$

The set $L_{n,r} := \Phi(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{E})$ is the lattice in Euclidean space $X_{n,r}$. Our considerations will imply (being imprecise), that when r is a small company to n, the images of polynomials $x^i(1-x)^i$ $(0 \le i \le r-1)$ are the basis of the lattice $L_{n,r}$ composed of almost orthogonal vectors. The lattice $L_{n,r}$ can be represented as an almost orthogonal direct sum of one-dimensional subgroups.

We can therefore study the structure of the above lattice, considering the estimation of subsequent lattice minima. We then get some idea of the geometry of the lattice.

This paper is organized as follows. In Section 2 we discuss the basic properties and definitions related to the concept of a lattice. In this section, we also present specific examples of lattices together with a determination of the values of the covering radius and subsequent lattice minima. Section 3 contains estimations of the covering radius in the case of a lattice of polynomials with integer coefficients. In Section 4 we study the properties of the lattice of polynomials in the L_2 norm. We also determine the explicit form of the polynomial constituting the last generator of the lattice. The main result of this paper is Theorem 4.9 with a double estimation for the norms of subsequent generators of lattice along with the value of the covering radius. Finally, we consider some polynomials and the properties of their roots, presenting results of numerical computations and some conclusions in Section 5.

2. LATTICE

We shall treat X as an n-dimensional space with the norm $\|\cdot\|_p$, as the usual norm in L_p, $1 \le p \le \infty$. By $d_p(f, P)$ we denote the corresponding distance of a function $f \in L_p$ from a subset $P \subset L_p$. The closed unit ball in X will be denoted by B_X. By a *lattice* L in X we mean a non-zero finite dimensional discrete additive subgroup of X. Given a lattice L, by $\mu(L; X)$ we denote its *covering radius*:

$$\mu(L; X) := \max_{x \in \operatorname{span} L} d(x, L).$$

In other words, the covering radius of a lattice is the minimal r such that any point in space is within distance at most r from the lattice. Let $Y = \operatorname{span} L$ and $\dim Y = m$. The quantities

$$\lambda_i\,(L; X) := \min\{s > 0 : (L \cap sB_Y) \ge i\}, \quad i = 1, \dots, m,$$

are called the *successive minima* of L. By the definition of $\lambda_i\,(L; X)$, all lattice points inside the open ball $B(0; \lambda_i\,(L; X))$ are contained in some $(i-1)$-dimensional hyperplane. The first minimum lattice $\lambda_1\,(L; X)$ is the length of the shortest non-zero vector of lattice L. To simplify the notation, we shall write $\mu(L)$ and $\lambda_i\,(L)$ instead of $\mu(L;X)$ and $\lambda_i\,(L;X)$, respectively.

We can consider how to easily estimate the radius covering a given lattice and to what extent we can expect an estimation for subsequent lattice minima.

Lemma 2.1. *Let L be the lattice in X ($dimL = n$) with the norm $\|\cdot\|$, $M = \operatorname{span} L$ and $u \in X \setminus M$. Then*

$$\tilde{L} = L + \mathbb{Z}u \equiv \{v + ku : v \in L \text{ and } k \in \mathbb{Z}\}$$

is the lattice in X ($dim\tilde{L} = n+1$). Let us denote $h = d(u, M)$. Then

$$\tfrac{1}{2}h \le \mu(\tilde{L}) \le \tfrac{1}{2}h + \mu(L) \tag{2.1}$$

and

$$h \le \lambda_{n+1}(\tilde{L}) \le \max\left\{h + \mu(L), \lambda_n(L)\right\}. \tag{2.2}$$

If X is Euclidean space we can write the inequalities (2.1) *and* (2.2) *as*

$$\tfrac{1}{2}h \le \mu(\tilde{L}) \le \sqrt{\tfrac{1}{4}h^2 + [\mu(L)]^2} \tag{2.3}$$

and

$$h \le \lambda_{n+1}(\tilde{L}) \le \max\left\{\sqrt{h^2 + [\mu(L)]^2}, \lambda_n(L)\right\}. \tag{2.4}$$

The proof is based of the standard arguments. By a *lattice* in $C[0,1]$ we mean an additive subgroup generated by a finite number of linearly independent vectors. It is not to hard to see if $n \ge 2r$, then $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ is a lattice generated by the Bernstein polynomials

$$x^k(1-x)^r, \quad k = r, \dots, n-r,$$

hence it follows that $\boldsymbol{P}_n \cap \boldsymbol{M}_r = \operatorname{span}(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r)$.

For simplicity of notation we write $\mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty)$, $\lambda_i(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty)$ for the uniform norm and we denote by $\mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_2)$, $\lambda_i(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_2)$ the covering radius for the $L_2(0,1)$. When we approximate polynomial in $\boldsymbol{P}_n \cap \boldsymbol{M}_r$ by elements of $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ then $\mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty)$, $\mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_2)$ are maximal errors.

In the following two examples, we can see what the values of the covering radius and subsequent lattice minima look like (in a specific case).

Example 2.2. Let $X = C[0,1]$ and $V_1(x) = 1$, $V_2(x) = x$. It is not hard to see that V_1, V_2 is a basis of the lattice $\boldsymbol{P}_1^{\mathbb{Z}}$. In this case

$$\lambda_1\left(\boldsymbol{P}_1^{\mathbb{Z}}; L_2\right) = \lambda_2\left(\boldsymbol{P}_1^{\mathbb{Z}}; L_2\right) = \frac{\sqrt{3}}{3} \approx 0.58, \quad \mu\left(\boldsymbol{P}_1^{\mathbb{Z}}; L_2\right) = \frac{1}{3}.$$

The shortest non-zero elements of the lattice (the first minimum) are the vertices of a regular hexagon. The first and second minimum are achieved at least among for polynomials

$$2V_2(x) - V_1(x) = W_0(x) = (2x-1), \quad V_2(x) = x.$$

Example 2.3. Let $X = C[0,1]$ and $U_1(x) = x(1-x)$, $V(x) = x^2(1-x)$ be a basis of the lattice $\boldsymbol{P}_3^{\mathbb{Z}} \cap \boldsymbol{M}_1$. The shortest non-zero elements of the lattice (the first minimum) are the vertices regular hexagon (see Example 2.2). In this case

$$\lambda_1 \left(\boldsymbol{P}_3^{\mathbb{Z}} \cap \boldsymbol{M}_1; L_2 \right) = \frac{1}{\sqrt{210}} \approx 0.069,$$

for polynomials $\pm W_1(x)$ and

$$\lambda_2 \left(\boldsymbol{P}_3^{\mathbb{Z}} \cap \boldsymbol{M}_1; L_2 \right) = \frac{1}{\sqrt{105}} \approx 0.097,$$

for polynomial $V(x) = x^2(1-x)$. In case of covering radius we obtain

$$\mu \left(\boldsymbol{P}_3^{\mathbb{Z}} \cap \boldsymbol{M}_1; L_2 \right) = \frac{\sqrt{30}}{105} \approx 0.052.$$

3. APPROXIMATION BY POLYNOMIALS WITH INTEGER COEFFICIENTS IN $L_\infty(0,1)$

Let r, n ($n \geq 2r+3$) be non-negative integers. Then

$$\gamma_{r,n} \leq \mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty). \tag{3.1}$$

The detailed proof can be found in [6]. It is easy to check that

$$\mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty) \leq \frac{1}{2}\binom{n}{r}^{-1}. \tag{3.2}$$

The proof of the above inequality given by Kantorovich [5] used the fact that the polynomials $x^k(1-x)^{n-k}$, where $1 \leq k \leq n-1$, form a basic of the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1$, and was based on an estimation which may be written in the form

$$\mu(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty) < \frac{1}{2} \max_{x \in [0,1]} \sum_{k=r}^{n-r} x^k (1-x)^{n-k} < \frac{1}{2}\binom{n}{r}^{-1}.$$

For the proofs we refer the reader to [6]. The estimation obtained in the inequality (3.2) cannot be improved by this method. It is easy to check that

$$\max_{x \in [0,1]} \sum_{k=r}^{n-r} x^k (1-x)^{n-k} \geq n^{-r}(1-1/n)^{n-r}.$$

Lemma 3.1. *Let $r \in \mathbb{N}$ and let $n \geq 2r+3$. Then*

$$\mu \left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty \right) \leq \tfrac{1}{2} d \left(U_r, \boldsymbol{P}_{n-1} \cap \boldsymbol{M}_{r+1} \right) + \mu \left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{r+1}; L_\infty \right),$$

where $U_r(x) = x^r(1-x)^r$, $r = 1, 2, \ldots$

For the proofs we refer the reader again to [6]. Let $n \geq 6r$. According to Lemma 3.1 we can prove that

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty\right) < C_1 \cdot C_2^r \cdot \frac{r^{2r+1/2}}{n^{2r}}, \tag{3.3}$$

where C_1, C_2 are some numeric constants. One may take $C_1 = 2\sqrt{\pi}+1, C_2 = 16$. On the other hand, let f be the linear functional on $(\boldsymbol{P}_n, \|\cdot\|_\infty)$ given by $f(P) = P^{(r)}(0)/r!$. For any $P \in \boldsymbol{P}_n^{\mathbb{Z}}$ we can write

$$\frac{1}{2} \leq \left| f\left(\frac{1}{2}U_r\right) - f(P)\right| \leq \|f\| \cdot \|\frac{1}{2}U_r - P\|_\infty.$$

So we get

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty\right) \geq d_\infty\left(\frac{1}{2}U_r, \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r\right) \geq \|f\|^{-1}.$$

If $P \in \boldsymbol{P}_n$, then, by the Markov inequality,

$$|P^{(r)}(0)| \leq 2^r \cdot \frac{n^2(n^2-1^2)\dots(n^2-(r-1)^2)}{1\cdot 3\cdot 5\dots(2r-1)}\|P\|_\infty.$$

It is easy to check that (according to Stirling's formula)

$$\|f\| < \frac{1}{2\pi^{1/2}} \cdot \frac{e^2 r}{r^{2r+1/2}}.$$

Finally, we obtain

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r; L_\infty\right) > c_1 \cdot c_2^r \cdot \frac{r^{2r+1/2}}{n^{2r}}, \tag{3.4}$$

taking e.g. $c_1 = \sqrt{\pi}$, $c_2 = e^{-2}$. The detailed proof is in [6]. Fix $r \in \mathbb{N}$. From the above one gets $\gamma_{r,n} \asymp n^{-2r}$ as $n \to \infty$. From Lemma 3 in [9] it follows that $\gamma_{r,n} = O(n^{-r})$. The results of [9] allow to obtain the bound $\gamma_{r,n} = O(n^{-2r})$, but don't give more precise estimations of the form (3.3) and (3.4).

4. APPROXIMATION BY POLYNOMIALS WITH INTEGER COEFFICIENTS IN $L_2(0,1)$

Let r, n ($n \geq 2r+4$) be non-negative integers and $a, b \in \mathbb{R}$. In this section, we will consider the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_r$ with the Euclidean norm. Banaszczyk and Lipnicki proved in [1]) the following inequalities

$$\frac{\sqrt{2}}{4}\frac{C_r}{n^{2r+1}}\left(1+O(n^{-1})\right) \leq \mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{2r}; L_2\right) \leq \frac{\sqrt{2}}{2}\frac{C_r}{n^{2r+1}}\left(1+O(n^{-1})\right),$$

as $n \to \infty$. We consider in more detail the geometry of the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{2r} \cap \boldsymbol{E}$ (resp. $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_{2r} \cap \boldsymbol{F}$). Let us denote

$$H(a, b; n, r) := \{P \in \boldsymbol{P}_n \cap \boldsymbol{M}_r \, : \, P^{(r)}(0) = a,\ P^{(r)}(1) = b\}.$$

Then by $R(a, b; n, r)$ we denote the shortest polynomial in the hyperplane $H(a, b; n, r)$.

Lemma 4.1. *Let $s_1, s_2, \ldots, s_k, a \in \mathbb{R}$ and $u_1, u_2, \ldots, u_k$ be a sequence of the orthogonal system in unit space and let $K = \sum_{i=1}^{k} \|u_i\|_2^{-2}$. Then*

$$\min_{s_1+\ldots+s_k=a} \|s_1u_1 + \ldots + s_ku_k\|_2^2 = a^2K^{-1},$$

for

$$s_i = \frac{a}{\|u_i\|_2^2} \cdot K^{-1}, \quad i = 1, \ldots, k.$$

Corollary 4.2. *Let $a, b, s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_l \in \mathbb{R}$ and $u_1, u_2, \ldots, u_k, v_1, \ldots, v_l$ be a sequence of orthogonal system in unit space. Then*

$$\min_{s_1+\ldots+s_k=a, t_1+\ldots+t_l=b} \|s_1u_1 + \ldots + s_ku_k + t_1v_1 + \ldots + t_lv_l\|_2^2 = a^2K^{-1} + b^2L^{-1},$$

for

$$s_i = \frac{a}{\|u_i\|_2^2} \cdot K^{-1}, \quad t_j = \frac{b}{\|v_j\|_2^2} \cdot L^{-1}, \quad i = 1, \ldots, k, j = 1, \ldots, l,$$

where

$$K = \sum_{i=1}^{k} \frac{1}{\|u_i\|_2^2}, \quad L = \sum_{j=1}^{l} \frac{1}{\|v_j\|_2^2}.$$

Let $n \in \mathbb{N}_0$. By P_n we denote the following Legendre polynomials on interval $[0, 1]$:

$$P_n(x) = \frac{1}{n!} \cdot \frac{d^n\left(x^n(x-1)^n\right)}{dx^n}. \tag{4.1}$$

Each Legendre polynomial $P_n(x)$ is an nth-degree polynomial.

Lemma 4.3. *Let $n \in \mathbb{N}$ be an even number. Then*

$$\begin{aligned} R(a, b; n, 0) = \frac{a+b}{2} \cdot \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n/2} (4i+1)P_{2i} \\ + \frac{b-a}{2} \cdot \frac{2}{(n+1)(n+2)} \sum_{i=1}^{n/2} (4i-1)P_{2i-1} \end{aligned}$$

and

$$\|R(a, b; n, 0)\|_2^2 = \left(\frac{a+b}{2}\right)^2 \cdot \frac{2}{(n+1)(n+2)} + \left(\frac{b-a}{2}\right)^2 \cdot \frac{2}{n(n+1)}.$$

Proof. Let us take arbitrary $P \in \boldsymbol{P}_n$. We may write

$$P = t_0P_0 + t_1P_1 + \ldots + t_nP_n,$$

for some $t_0, \ldots, t_n \in \mathbb{R}$. By assumption $P_i(0) = (-1)^i, P_i(1) = 1$ $(i = 0, 1, \ldots, n,)$. Therefore $P \in H(a, b; n, 0)$ if and only if

$$t_0 + t_2 + t_4 + \ldots + t_n = \frac{a+b}{2}, \tag{4.2}$$

$$t_1 + t_3 + \ldots + t_{n-1} = \frac{b-a}{2}. \tag{4.3}$$

In this way, our problem is reduced to minimize the expression

$$\|t_0P_0 + t_1P_1 + \ldots + t_nP_n\|_2^2,$$

if equality holds in (4.2)–(4.3). By Corollary 4.2, we get

$$t_{2i} = \frac{a+b}{2} \cdot \frac{1}{\|P_{2i}\|_2^2} \left(\sum_{j=0}^{n/2} \frac{1}{\|P_{2j}\|_2^2} \right)^{-1}, \quad i = 0, 1, \ldots, n/2,$$

$$t_{2i-1} = \frac{b-a}{2} \cdot \frac{1}{\|P_{2i-1}\|_2^2} \left(\sum_{j=1}^{n/2} \frac{1}{\|P_{2j-1}\|_2^2} \right)^{-1}, \quad i = 1, \ldots, n/2.$$

This gives

$$\begin{aligned}\|t_0P_0 + t_1P_1 + \ldots + t_nP_n\|_2^2 = &\left(\frac{a+b}{2}\right)^2 \cdot \left(\sum_{j=0}^{n/2} \frac{1}{\|P_{2j}\|_2^2} \right)^{-1} \\ &+ \left(\frac{b-a}{2}\right)^2 \cdot \left(\sum_{j=1}^{n/2} \frac{1}{\|P_{2j-1}\|_2^2} \right)^{-1}.\end{aligned}$$

From the equality

$$\|P_i\|_2^2 = \frac{1}{2i+1}, \quad i \in \mathbb{N}_0,$$

we can see that

$$\sum_{j=0}^{n/2} \frac{1}{\|P_{2j}\|_2^2} = \sum_{j=0}^{n/2} (4j+1) = \frac{1}{2}(n+1)(n+2).$$

By a similar argument, we have

$$\sum_{j=0}^{n/2} \frac{1}{\|P_{2j-1}\|_2^2} = \sum_{j=1}^{n/2} (4j-1) = \frac{1}{2}n(n+1).$$

We conclude from the above that

$$t_{2i} = \frac{a+b}{2} \cdot (4i+1) \cdot \frac{2}{(n+1)(n+2)}, \quad i = 0, 1, \ldots, n/2,$$

$$t_{2i-1} = \frac{b-a}{2} \cdot (4i-1) \cdot \frac{2}{n(n+1)}, \quad i = 1, \ldots, n/2.$$

Finally, from the above we obtain

$$R(a,b;n,0) = \frac{a+b}{2} \cdot \frac{2}{(n+1)(n+2)} \cdot \sum_{i=0}^{n/2} (4i+1) P_{2i}$$
$$+ \frac{b-a}{2} \cdot \frac{2}{(n+1)(n+2)} \sum_{i=1}^{n/2} (4i-1) P_{2i-1}$$

and

$$\|R(a,b;n,0)\|_2^2 = \left(\frac{a+b}{2}\right)^2 \cdot \frac{2}{(n+1)(n+2)} + \left(\frac{b-a}{2}\right)^2 \cdot \frac{2}{n(n+1)}. \qquad \square$$

In a special case for $a = \pm 1$ and $b = 1$ we get simpler form

$$R(a,b;n,0) = \frac{a+b}{2} R(1,1;n,0) + \frac{b-a}{2} R(-1,1;n,0),$$

where

$$R(1,1;n,0) = \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n/2} (4i+1) P_{2i}, \tag{4.4}$$

$$R(-1,1;n,0) = \frac{2}{n(n+1)} \sum_{i=0}^{n/2} (4i-1) P_{2i-1}.$$

According to Lemma 4.3 we obtain

$$\|R(1,1;n,0)\|_2^2 = \frac{2}{(n+1)(n+2)}, \tag{4.5}$$

$$\|R(-1,1;n,0)\|_2^2 = \|R(1,-1;n,0)\|_2^2 = \frac{2}{n(n+1)}. \tag{4.6}$$

Lemma 4.4. *Let $n \in \mathbb{N}$ be an odd number. Then*

$$R(a,b;n,0) = \frac{a+b}{2} \cdot \frac{2}{n(n+1)} \sum_{i=0}^{(n-1)/2} (4i+1) P_{2i}$$
$$+ \frac{b-a}{2} \cdot \frac{2}{(n+1)(n+2)} \sum_{i=1}^{(n+1)/2} (4i-1) P_{2i-1}$$

and

$$\|R(a,b;n,0)\|_2^2 = \left(\frac{a+b}{2}\right)^2 \cdot \frac{2}{n(n+2)} + \left(\frac{b-a}{2}\right)^2 \cdot \frac{2}{(n+1)(n+2)}.$$

The proof is analogous to that of Lemma 4.3. In a special case, we get

$$R(a,b;n,0) = \frac{a+b}{2} R(1,1;n,0) + \frac{b-a}{2} R(-1,1;n,0),$$

where

$$R(1,1;n,0) = \frac{2}{n(n+1)} \sum_{i=0}^{(n-1)/2} (4i+1)P_{2i}, \tag{4.7}$$

$$R(-1,1;n,0) = \frac{2}{(n+1)(n+2)} \sum_{i=1}^{(n+1)/2} (4i-1)P_{2i-1}. \tag{4.8}$$

According to Lemma 4.4 we obtain

$$\|R(1,1;n,0)\|_2^2 = \frac{2}{n(n+1)}, \tag{4.9}$$

$$\|R(-1,1;n,0)\|_2^2 = \|R(1,-1;n,0)\|_2^2 = \frac{2}{(n+1)(n+2)}. \tag{4.10}$$

It is clear that the determination of the form of polynomials by this method is quite time-consuming. We find the form of the polynomial R in special cases by applying some different method.

Using the previous lemmas, we will find the exact form of the minimal polynomial for a given lattice. In the case of Theorems 4.6 and 4.7, we will prove that the indicated polynomial form satisfies the assumptions of the last minimum for the selected lattice. We will divide the considerations into two cases – for even and odd numbers.

Theorem 4.5. *Let m, r be non-negative integers and $m \geq r+2$. Then*

$$R(r!, r!; 2m, r) = (2m-2r+1)!(2r+1)! \cdot \sum_{k=1}^{r+1} \frac{P_r^{(k-1)}(0)}{(2m+2k+1)!} \cdot P_{2m+k+1}^{(k)}.$$

Proof. Let us denote

$$S_{r,m} = R_E(r!; 2m, r) = r!R_E(1,1;2m,r) = R(r!, r!; 2m, r) = r!R(1,1;2m,r),$$

where $0 \leq r \leq m$. By the definition,

$$\|S_{r,m}\| = \alpha(2m, r) = d_2(U_r, \boldsymbol{P}_{2m} \cap \boldsymbol{M}_{r+1}).$$

Next, we can see that

$$S_{0,m} = \frac{1}{2m(2m+1)} \frac{d}{dx} P_{2m+1}(x)$$

and

$$S_{r,m} = s_{r,m,1}P'_{2m+1} + s_{r,m,2}P''_{2m+2} + \ldots + s_{r,m,r+1}P^{(r+1)}_{2m+r+1} = \sum_{k=1}^{r+1} s_{r,m,k}P^{(k)}_{2m+k}.$$

And now

$$s_{r,m,k} = \frac{a_{r,k}}{(2m-2r+1)\cdot \ldots \cdot (2m+2k)} = \frac{(2m-2r)!}{(2m+2k)!} \cdot a_{r,k}, \quad k = 1, \ldots, r+1.$$

We determined $a_{r,k}$ from the system of equations

$$\begin{aligned} a_{r,1} + \frac{a_{r,2}}{2!} + \frac{a_{r,3}}{3!} + \ldots + \frac{a_{r,r+1}}{(r+1)!} &= 0, \\ \frac{a_{r,1}}{2!} + \frac{a_{r,2}}{3!} + \frac{a_{r,3}}{4!} + \ldots + \frac{a_{r,r+1}}{(r+2)!} &= 0, \\ \frac{a_{r,1}}{3!} + \frac{a_{r,2}}{4!} + \frac{a_{r,3}}{5!} + \ldots + \frac{a_{r,r+1}}{(r+3)!} &= 0, \\ \frac{a_{r,1}}{r!} + \frac{a_{r,2}}{(r+1)!} + \frac{a_{r,3}}{(r+2)!} + \ldots + \frac{a_{r,r+1}}{(2r+1)!} &= (-1)^r r!. \end{aligned}$$

It is easy to see that

$$\Delta_r = (-1)^{r(r+1)/2} \cdot \frac{2!3! \cdot \ldots \cdot r!}{(r+1)!(r+2)! \cdot \ldots (2r+1)!}.$$

It follows that

$$a_{r,k} = (-1)^{r-k+1} \cdot \frac{(r+k-1)!}{(r-k+1)!} \cdot \frac{(2r+1)!}{(k-1)!}.$$

Obviously

$$S_{r,m} = (-1)^{r+1}(2m-2r)!(2r+1)! \cdot \sum_{k=1}^{r+1} (-1)^k \cdot \frac{(r+k-1)!}{(r-k+1)!} \cdot \frac{P_{2m+k}^{(k)}}{(2m+k)!(k-1)!}$$

or

$$S_{r,m} = (2m-2r+1)!(2r+1)! \cdot \sum_{k=1}^{r+1} \frac{P_r^{(k-1)}(0)}{(2m+2k+1)!} \cdot P_{2m+k+1}^{(k)}. \qquad \square$$

It is easy to check that for $r \geq 1$ the polynomial $R(1,1;n,r)$ does not need to have integer coefficients.

However, in our situation, we will see that our polynomial will have integer coefficients. For this reason, we will find an estimation for the value of the last minimum in the lattice. There hold the following two theorems (Theorems 4.6 and 4.7).

Theorem 4.6. *Let $n \in \mathbb{N}$ be an even number. Then $R(1,1;n,0) \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{E}$.*

Proof. Let $n \in \mathbb{N}$ be an even number. The main idea of the proof is to take $R(1,1;n,0)^{(k)}(0)/k! \in \mathbb{Z}$, for $k = 0, 1, \ldots, n$. It is not hard to see that

$$P_i^{(k)}(0) = (-1)^{i+k} \cdot \frac{(i+k)!}{k!(i-k)!}, \quad k = 0, 1, \ldots, i,$$

for $i = 0, 1, \ldots, n$. According to formula (4.4) let us write

$$R(1,1;n,0) = \frac{2}{(n+1)(n+2)} \sum_{i=0}^{n/2} (4i+1) P_{2i}.$$

This gives

$$R(1,1;n,0)^{(k)}(0) = \frac{2}{(n+1)(n+2)} \sum_{i=[k/2]}^{n/2} (4i+1)(-1)^{k+2i} \cdot \frac{(k+2i)!}{k!(2i-k)!}.$$

Then

$$\begin{aligned} &R(1,1;n,0)^{(k)}(0) \\ &= \frac{2}{(n+1)(n+2)} \cdot \frac{(-1)^k}{k!} \sum_{i=[k/2]}^{n/2} (4i+1)(2i-k+1)(2i-k+2)\cdot \ldots \cdot (2i+k). \end{aligned} \tag{4.11}$$

Next, we have

$$\begin{aligned} &\sum_{i=[k/2]}^{n/2} (4i+1)(2i-k+1)(2i-k+2)\cdot \ldots \cdot (2i+k) \\ &= \frac{1}{2(k+1)}(n-k+1)(n-k+2)\cdot \ldots \cdot (n+k+2), \end{aligned}$$

which is a standard calculation. We can write the equation (4.11) as

$$R(1,1;n,0)^{(k)}(0) = (-1)^k \cdot k! \cdot \frac{1}{k+1} \cdot \binom{n}{k} \cdot \binom{n+k+2}{k}.$$

This proves that

$$\frac{R(1,1;n,0)^{(k)}(0)}{k!} = (-1)^k \cdot \frac{1}{k+1} \cdot \binom{n}{k} \cdot \binom{n+k+2}{k}.$$

It is easy to check that

$$\frac{1}{k+1}\binom{n}{k}\binom{n+k+2}{k} \in \mathbb{Z}.$$

□

We obtain a similar situation in the case of odd values of index n. In this situation, our polynomial $R(-1,1;n,0)$ will also be the best choice for the last minimum of the lattice (the last lattice generator).

Theorem 4.7. *Let $n \geq 3$ be an odd number. Then $R(-1,1;n,0) \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{F}$.*

Proof. The proof is analogous to that of Theorem 4.6. □

The situation described in Theorems 4.6 and 4.7 is quite unique. It is easy to check that for $r \geq 1$ the polynomial $R(1,1;n,r)$ doesn't need to have integer coefficients. The last theorems show a part of the natural geometry of the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{E}$ (resp. $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{F}$). In the case of the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{E}$ we have the polynomial $R(1,1;n,0)$ (the minimal polynomial), which is the first orthogonal one to the subspace $\boldsymbol{P}_n \cap \boldsymbol{M}_1 \cap \boldsymbol{E}$.

Proposition 4.8. *Let $n \geq 6$ be an even number. Then the value of $\|R(1,1;n,0)\|_2$ is the last minimum of the lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{E}$.*

Proof. Now we have to know that the value of the covering radius of the lattice lattice $\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1 \cap \boldsymbol{E}$ is smaller than the value of $\|R(1,1,n,0)\|_2$. For every even number $n \geq 6$ there is the following inequality

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1 \cap \boldsymbol{E}; L_2\right) < \frac{2}{n^2}. \tag{4.12}$$

From the Theorem 4.6 we have $R(1,1;n,0) \in \boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{E}$. Next, from (4.6) we get

$$\|R(1,1;n,0)\|_2 = \sqrt{\frac{2}{(n+1)(n+2)}}. \tag{4.13}$$

The polynomial $R(1,1;n,0)$ is orthogonal to $\boldsymbol{P}_n \cap \boldsymbol{M}_1$. According to (4.13) we have

$$\mu\left(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1 \cap \boldsymbol{E}; L_2\right) < \frac{2}{n^2}.$$

According to Lemma 2.1 we can write

$$\lambda_{n/2}(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1 \cap \boldsymbol{E}; L_2) < \frac{4}{n^2}.$$

From the above and from (4.13) we get

$$\lambda_{n/2}(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{M}_1 \cap \boldsymbol{E}; L_2) < \|R(1,1;n,0)\|_2.$$

It follows that

$$\lambda_{n/2+1}(\boldsymbol{P}_n^{\mathbb{Z}} \cap \boldsymbol{E}; L_2) = \|R(1,1;n,0)\|_2 = \sqrt{\frac{2}{(n+1)(n+2)}}. \tag{4.14}$$

□

Let us denote

$$\alpha(2m,r) := d_2(U_r, \boldsymbol{P}_{2m} \cap \boldsymbol{M}_{r+1}),$$
$$\beta(2m-1,r) := d_2(W_r, \boldsymbol{P}_{2m-1} \cap \boldsymbol{M}_{r+1}).$$

Based on the above considerations, we can estimate the norms of subsequent generators of our lattice along with the value of the covering radius. We obtain the following results (for even and odd numbers, respectively):

Theorem 4.9. *Let $m \geq 1$. Then*

$$\lambda_{m+1}\left(\boldsymbol{P}_{2m}^{\mathbb{Z}} \cap \boldsymbol{E}; L_2\right) = \alpha(2m,0) \tag{4.15}$$

and

$$\alpha^2(2m, m-i+1) \leq \lambda_i^2\left(\boldsymbol{P}_{2m}^{\mathbb{Z}} \cap \boldsymbol{E}; L_2\right) \leq \sum_{k=m-i+1}^{m} \alpha^2(2m,k), \tag{4.16}$$

for $i = \lfloor m/3 \rfloor, \ldots, m$.

Proof. Our proof starts with the observation that from (4.14) we get (4.15). From Lemma 2.1 we have

$$\alpha^2\left(2m, m-i+1\right) \leq \lambda_i^2\left(\boldsymbol{P}_{2m}^{\mathbb{Z}} \cap \boldsymbol{E}; L_2\right)$$

and

$$\lambda_i^2\left(\boldsymbol{P}_{2m}^{\mathbb{Z}} \cap \boldsymbol{E}; L_2\right) \leq \sum_{k=m-i+1}^{m} \alpha^2(2m, k).$$

This gives (4.16). □

Theorem 4.10. *Let $m \geq 1$. Then*

$$\lambda_{m+1}\left(\boldsymbol{P}_{2m-1}^{\mathbb{Z}} \cap \boldsymbol{F}; L_2\right) = \beta(2m-1, 0),$$

and

$$\beta^2\left(2m-1, m-i\right) \leq \lambda_i^2\left(\boldsymbol{P}_{2m-1}^{\mathbb{Z}} \cap \boldsymbol{F}; L_2\right) \leq \sum_{k=m-i}^{m-1} \beta^2(2m-1, k),$$

for $i = \lfloor m/3 \rfloor, \ldots, m-1$.

Proof. The proof is analogous to that of Theorem 4.9. □

5. EXAMPLES

In this section, we study the properties of shortest polynomials $R(-1,1;n,0)$ and $R(1,1;n,0)$ in the context of their roots. We only took into account the polynomials for odd $n \in \mathbb{N}$, as in the even case properties are analogous.

First, we present the polynomials $R(-1,1;n,0)$ for $n \in \{1,3,5,7,9\}$, based on the recursive form (charts of all these polynomials are presented in Figure 1):

$$\begin{aligned}
R(-1,1;1,0) &= P_1 = 2x - 1,\\
R(-1,1;3,0) &= \frac{1}{10}(3R(-1,1;1,0) + 7P_3)\\
&= 14x^3 - 21x^2 + 9x - 1,\\
R(-1,1;5,0) &= \frac{1}{21}(10R(-1,1;3,0) + 11P_5)\\
&= 132x^5 - 330x^4 + 300x^3 - 120x^2 + 20x - 1,\\
R(-1,1;7,0) &= \frac{1}{36}(21R(-1,1;5,0) + 15P_7)\\
&= 1430x^7 - 5005x^6 + 7007x^5 - 5005x^4 + 1925x^3 - 385x^2\\
&\quad + 35x - 1,\\
R(-1,1;9,0) &= \frac{1}{55}(36R(-1,1;7,0) + 19P_9)\\
&= 16796x^9 - 75582x^8 + 143208x^7 - 148512x^6 + 91728x^5\\
&\quad - 34398x^4 + 7644x^3 - 936x^2 + 54x - 1.
\end{aligned}$$

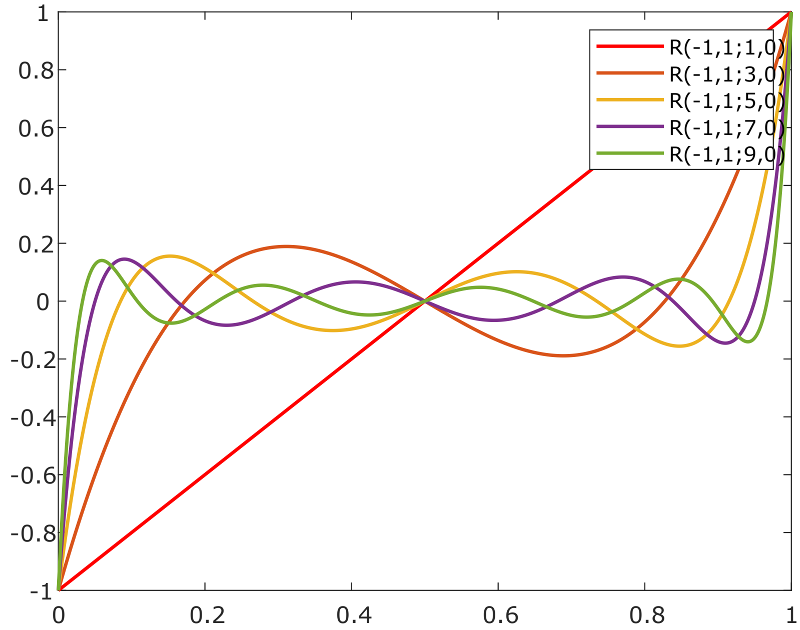

Fig. 1. Graphs of polynomials $R(-1,1;n,0)$

Next, we present the first 5 polynomials $R(1,1;n,0)$ that have roots (obviously, polynomial $R(1,1;1,0)=P_0=1$ has no roots), based on the recursive form (charts of all these polynomials are presented in Figure 2):

$$\begin{aligned}
R(1,1;3,0) &= \frac{1}{6}(R(1,1;1,0)+5P_2) = 5x^2-5x+1,\\
R(1,1;5,0) &= \frac{1}{15}(6R(1,1;3,0)+9P_4)\\
&= 42x^4-84x^3+56x^2-14x+1,\\
R(1,1;7,0) &= \frac{1}{28}(15R(1,1;5,0)+13P_6)\\
&= 429x^6-1287x^5+1485x^4-825x^3+225x^2-27x+1,\\
R(1,1;9,0) &= \frac{1}{45}(28R(1,1;7,0)+17P_8) = 4862x^8-19448x^7+32032x^6\\
&\quad -28028x^5+14014x^4-4004x^3+616x^2-44x+1,\\
R(1,1;11,0) &= \frac{1}{66}(45R(1,1;9,0)+21P_{10})\\
&= 184756x^{10}-923780x^9+1969110x^8-2333760x^7+1681680x^6\\
&\quad -756756x^5+210210x^4-34320x^3+2970x^2-110x+1.
\end{aligned}$$

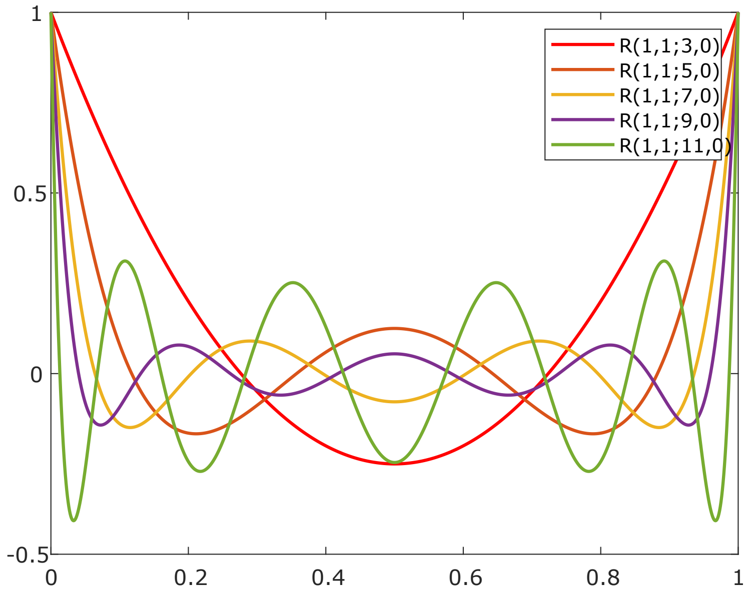

Fig. 2. Graphs of polynomials $R(1,1;n,0)$

Now, we briefly describe the numerical tests performed using the classical Newton–Raphson method, which has the following iterative form:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots$$

where f denotes a given polynomial (i.e. $R(-1,1;n,0)$ or $R(1,1;n,0)$) and f' is the usual derivative of this polynomial.

To find roots of polynomials $R(-1,1;n,0)$ and $R(1,1;n,0)$ we used the code written in C++ with double precision. We performed computations with the equidistant starting points $x_0 = i \cdot 0.1$ for $i = 0, \ldots, 10$ to find all the roots located inside the interval $[0,1]$. For $R(1,1;11,0)$ we had to perform two additional calculations to achieve really all its roots. At the time we used starting points 0.05 and 0.95. We didn't perform obviously the calculations for $R(-1,1;1,0)$ as it is linear and has only one root 0.5.

All results are presented in Tables 1 and 2, where N_{it} denotes the number of performed iterations to satisfy the stopping criterion

$$|x_{k+1} - x_k| < 10^{-12}.$$

Moreover, "$-$" denotes that the calculations were not performed, and "$\times$" – a failure that occurs when the starting point was a point of local extremum.

Table 1
Approximations of the roots of polynomials $R(-1,1;n,0)$
and results of numerical computations

x_0	N_{it}	R(-1,1;3,0)	N_{it}	R(-1,1;5,0)	N_{it}	R(-1,1;7,0)	N_{it}	R(-1,1;9,0)
0.0	7	0.17267316465	7	0.084888051861	7	0.050121002294	7	0.032999284796
0.1	6	0.17267316465	5	0.084888051861	13	0.949878997706	5	0.107758263168
0.2	5	0.17267316465	5	0.265575603265	6	0.161406860245	4	0.217382336502
0.3	10	0.17267316465	5	0.265575603265	4	0.318441268087	6	0.352120932207
0.4	5	0.5	13	0.915111948139	11	0.050121002294	6	0.352120932207
0.5	1	0.5	1	0.5	1	0.5	1	0.5
0.6	5	0.5	13	0.084888051861	11	0.949878997706	6	0.647879067793
0.7	10	0.82732683535	5	0.734424396735	4	0.681558731913	9	0.647879067793
0.8	5	0.82732683535	5	0.734424396735	6	0.838593139755	8	0.782617663498
0.9	6	0.82732683535	5	0.915111948139	13	0.050121002294	14	0.892241736831
1.0	7	0.82732683535	7	0.915111948139	7	0.949878997706	12	0.967000715204

Table 2
Approximations of the roots of polynomials $R(1,1;n,0)$
and results of numerical computations

x_0	N_{it}	R(1,1;3,0)	N_{it}	R(1,1;5,0)	N_{it}	R(1,1;7,0)	N_{it}	R(1,1;9,0)	N_{it}	R(1,1;11,0)
0.0	6	0.27639320225	7	0.11747233804	7	0.064129925745	7	0.040233045917	6	0.013046735741
0.05		—		—		—		—	5	0.067468316656
0.1	6	0.27639320225	5	0.11747233804	8	0.064129925745	5	0.130613067447	7	0.013046735741
0.2	5	0.27639320225	10	0.11747233804	4	0.204149909283	18	0.040233045917	7	0.160295215850
0.3	5	0.27639320225	5	0.35738424176	7	0.604649608951	5	0.261037525095	5	0.283302302935
0.4	6	0.27639320225	5	0.35738424176	4	0.395350391049	4	0.417360521167	5	0.425562830509
0.5		×		×		×		×		×
0.6	6	0.72360679775	5	0.64261575824	4	0.395350391049	4	0.582639478834	6	0.574437169489
0.7	5	0.72360679775	5	0.64261575824	7	0.395350391049	5	0.738962474905	5	0.716697697064
0.8	5	0.72360679775	10	0.88252766197	4	0.795850090717	18	0.959766954083	9	0.839704784147
0.9	6	0.72360679775	5	0.88252766197	8	0.935870074255	5	0.869386932552	17	0.986953264264
0.95		—		—		—		—	9	0.932531683332
1.0	6	0.72360679775	7	0.88252766197	7	0.935870074255	7	0.959766954083	25	0.986953264264

Insights into computational experiments:

– there are no problems with the numerical approximating of all roots of the tested polynomials, if we use appropriate starting points, i.e. if the starting points are selected densely enough;
– the convergence of the Newton–Raphson method is slower than in the general case, as a more detailed study indicates that it is slightly slower than quadratic (we omit the detailed estimations);
– the Newton–Raphson method doesn't work if starting point is a point of local extremum (see Theorem 2.6 of [2] - local convergence theorem) and it works noticeably slower if the starting point is close to a point of local extremum, first of all, $x_0 = 0.3, 0.7$ for $R(-1, 1; 3, 0)$, $x_0 = 0.4, 0.6$ for $R(-1, 1; 5, 0)$ and $x_0 = 0.1, 0.9$ for $R(-1, 1; 7, 0)$ in Table 1 or $x_0 = 0.2, 0.8$ for $R(1, 1; 5, 0)$ and for $R(1, 1; 9, 0)$ in Table 2.

Insights into properties of the roots of considered polynomials $R(-1, 1; n, 0)$ and $R(1, 1; n, 0)$:

— all roots are real numbers, i.e. these polynomials don't have complex roots;
— the roots are located symmetrically with respect to the center 0.5 (obviously, inside the interval $[0, 1]$), wherein they have a wider spacing near the center, and are closer together near the endpoints of the interval $[0, 1]$;
— the higher degree of the polynomial, the more densely the roots are located near the endpoints of $[0, 1]$;
— the sum of all roots of a given polynomial is always equal to half of the degree of this polynomial;
— the product of all roots of a given polynomial is always equal to the inverse of its leading coefficient.

REFERENCES

[1] W. Banaszczyk, A. Lipnicki, *On the lattice of polynomials with integer coefficients: the covering radius in $L_p(0, 1)$*, Ann. Polon. Math. **121** (2015), 123–144.

[2] R.L. Burden, J.D. Faires, A.M. Burden, *Numerical Analysis, 10th Edition*, Cengage Learning, 2015.

[3] L.B.O. Ferguson, *Approximation by Polynomials with Integer Coefficients*, Amer. Math. Society, Providence, R.I., 1980.

[4] L.B.O. Ferguson, *What can be approximated by polynomials with integer coefficients*, Amer. Math. Monthly **113** (2006), 403–414.

[5] L.V. Kantorowič, *Quelques observations sur l'approximation de fonctions au moyen de polynômes à coefficients entiers*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1931), 1163–1168.

[6] A. Lipnicki, *Uniform approximation by polynomials with integer coefficients*, Opuscula Math. **36** (2016), no. 4, 489–498.

[7] Y. Okada, *On approximate polynomials with integral coefficients only*, Tohoku Math. J. **23** (1923), 26–35.

[8] J. Pál, *Zwei kleine Bemerkungen*, Tohoku Math. J. **6** (1914), 42–43.

[9] R.M. Trigub, *Approximation of functions by polynomials with integer coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **26:2** (1962), 261–280.

Artur Lipnicki
artur.lipnicki@wmii.uni.lodz.pl
https://orcid.org/0000-0001-9848-6157

University of Lodz
Faculty of Mathematics and Computer Science
Banacha 22, 90–238 Łódź, Poland

Marek J. Śmietański (corresponding author)
marek.smietanski@wmii.uni.lodz.pl
https://orcid.org/0000-0002-6557-6436

University of Lodz
Faculty of Mathematics and Computer Science
Banacha 22, 90-238 Łódź, Poland

Received: December 18, 2023.
Revised: January 26, 2024.
Accepted: January 27, 2024.