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ASYMPTOTIC ANALYSIS FOR CONFLUENT HYPERGEOMETRIC FUNCTION IN TWO VARIABLES GIVEN BY DOUBLE INTEGRAL

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Abstract. We study an integrable connection with irregular singularities along a normally crossing divisor. The connection is obtained from an integrable connection of regular singular type by a confluence, and has irregular singularities along $x = \infty$ and $y = \infty$. Solutions are expressed by a double integral of Euler type with resonances among the exponents in the integrand. We specify twisted cycles that give main asymptotic behaviors in sectorial domains around (∞, ∞). Then we obtain linear relations among the twisted cycles, and get an explicit expression of the Stokes multiplier. The methods to derive the asymptotic behaviors for double integrals and to get linear relations among twisted cycles in resonant case, which we developed in this paper, seem to be new.

Keywords: strong asymptotic expansion, Stokes phenomenon, middle convolution, twisted homology.

Mathematics Subject Classification: 33C70, 34E05.

1. INTRODUCTION

Higher dimensional theory of integrable connections with irregular singularities was first studied in the pioneering work of Gérard–Sibuya [2]. Majima [9] generalized the notion of asymptotic expansion to several variables case in a natural way, and built a framework of the theory. Then Majima's theory was applied in various directions [3, 6, 8, 13, 14, 18, 22]. Now the general theory is completed by Mochizuki [10–12] and Sabbah [16, 17].

Together with the general theory, the study of explicit integrable connections is also important, because it will bring a new viewpoint to the general theory, and will reveal new relations to other research fields. There are several remarkable studies by Shimomura [19, 20], where the asymptotic behaviors of several confluent hypergeometric functions in two variables are clarified. We note that, in these studies, the confluent hypergeometric functions are expressed by single integrals, and asymptotic behaviors at a single singular locus are studied.

In the present paper, we study a confluent hypergeometric function in two variables obtained from Appell's hypergeometric function F_4 by a confluence. Appell's F_4 has an integral representation of double integral, and then our confluent F_4 is also expressed by a double integral. We derive an integrable connection for the confluent F_4, which has irregular singularities at $x = \infty$ and $y = \infty$. Then we study the asymptotic behavior of the integrable connection around $(x, y) = (\infty, \infty)$, and obtain the Stokes multipliers. Thus, we study a multiple integral, and derive the asymptotic behavior around an intersection of the singular locus, which seem to be new.

Moreover, in our double integral, resonances of exponents occur, which make the global analysis difficult. If we have an integral representation of solutions such that each twisted cycle gives a solution, the connection coefficients including the Stokes multipliers are obtained by solving linear relations among twisted cycles. A standard way to get linear relations among twisted cycles is to regard the integrals over a twisted cycles as iterated integrals of single variables and to apply Cauchy's theorem for the integrals of a single variable ([1]). This method works well for non-resonant case, however, for resonant case we do not have enough number of linear relations by this method. We overcame this difficulty by a natural idea as follows. By using the resonance, we may construct a higher dimensional twisted chain. Then the boundary of the twisted chain makes a new linear relation of twisted cycles. This *from the above* method seems to be new, and will become a good tool in the global analysis in resonant case.

This paper is organized as follows. In Section 2, we derive the integrable connection for confluent F_4 together with its integral representation. In Section 3, we calculate the formal reduction of the integrable connection, and obtain the main asymptotic behavior of the confluent F_4 at (∞, ∞) (Proposition 3.2). Section 4 is devoted to the local analysis. We find twisted cycles which give the main asymptotic behaviors at (∞, ∞), and also show that these cycles exhibit the full asymptotic behaviors in Majima's sense around (∞, ∞). In the last section, we obtain several linear relations among the twisted cycles, and then get the Stokes multipliers.

2. CONFLUENT F_4

The integrable connection which we are going to study in this paper is obtained by applying the higher dimensional Katz theory ([4]) and confluence. We start with a Pfaffian system of rank one

$$du = \left(\alpha_1 \frac{dx}{x} + \alpha_2 \frac{dx}{x-1} + \alpha_3 \frac{d(x-y)}{x-y} \right) u, \tag{2.1}$$

where $\alpha_1, \alpha_2, \alpha_3$ are complex parameters. As explained in [4, Example 14.2], we first operate the middle convolution in x-direction with parameter β, then operate the gauge transformation (addition) by $x^{-\alpha_1-\beta}(x-1)^{-\alpha_2-\beta}$, and finally operate the middle convolution in x-direction with parameter $\beta - \alpha_3$ to get the Pfaffian system of rank four

$$dU = \left(A_1 \frac{dx}{x} + A_2 \frac{dx}{x-1} + A_3 \frac{d(x-y)}{x-y} + A_4 \frac{dy}{y} + A_5 \frac{dy}{y-1} \right) U, \tag{2.2}$$

where A_j's are 4×4 constant matrices. Since the middle convolution is realized by an integral transformation, we get an integral representation of solutions of the last system. According to the manner explained in [5], we look at the integral representation in detail. After the first middle convolution, we have

$$\int_\Delta s_1^{\alpha_1}(s_1-1)^{\alpha_2}(s_1-y)^{\alpha_3}(s_1-x)^{\beta}\vec{\varphi}$$

as an integral representation of solutions, where $\vec{\varphi}$ is a vector of twisted cocycles given by

$$\vec{\varphi} = \left(\frac{ds_1}{s_1}, \frac{ds_1}{s_1-y}, \frac{ds_1}{s_1-1} \right)^T.$$

By operating the second middle convolution, we first get a system of rank nine, whose solution are expressed by the integral

$$\int_\Delta s_1^{\alpha_1}(s_1-1)^{\alpha_2}(s_1-y)^{\alpha_3}(s_1-s_2)^{\beta}s_2^{-\alpha_1-\beta}(1-s_2)^{-\alpha_2-\beta}(s_2-x)^{\beta-\alpha_3}\vec{\xi} \tag{2.3}$$

with the vector of cocycles

$$\vec{\xi} = \left(\vec{\varphi} \wedge \frac{ds_2}{s_2}, \vec{\varphi} \wedge \frac{ds_2}{s_2-y}, \vec{\varphi} \wedge \frac{ds_2}{s_2-1} \right)^T.$$

The Pfaffian system (2.2) is obtained by taking a quotient of the rank nine system on the space $\mathbb{C}^9/W$, where W is a five dimensional subspace. In order to get an integral representation of solutions for the quotient system, we take a basis $\{w_1, w_2, \dots, w_5\}$ of W, and extend it to a basis of $\mathbb{C}^9$ by adding vectors v_1, v_2, v_3, v_4. Set $P = (w_1, \dots, w_5, v_1, \dots, v_4)$. Then an integral representation of the system after the second middle convolution is given by the last four entries of $P^{-1}\xi$ as the vector of twisted cocycles. Temporarily we take

$$v_1 = e_2,\ v_2 = e_3,\ v_3 = e_7,\ v_4 = e_8,$$

where e_j denotes the j-th unit vector in $\mathbb{C}^9$. Then we have

$$\begin{aligned}
\zeta_1 &= \frac{ds_1 \wedge ds_2}{(s_1-y)s_2} + \frac{\beta}{\alpha_1+\alpha_2+\alpha_3+\beta} \cdot \frac{ds_1 \wedge ds_2}{(s_1-y)(s_1-s_2)}, \\
\zeta_2 &= \frac{ds_1 \wedge ds_2}{(s_1-1)s_2} + \frac{\beta}{\alpha_1+\alpha_2+\alpha_3+\beta} \cdot \frac{ds_1 \wedge ds_2}{(s_1-y)(s_1-s_2)}, \\
\zeta_3 &= \frac{ds_1 \wedge ds_2}{s_1(s_2-1)} + \frac{\beta}{\alpha_1+\alpha_2+\alpha_3+\beta} \cdot \frac{ds_1 \wedge ds_2}{(s_1-y)(s_1-s_2)}, \\
\zeta_4 &= \frac{ds_1 \wedge ds_2}{(s_1-y)(s_2-1)} + \frac{\beta}{\alpha_1+\alpha_2+\alpha_3+\beta} \cdot \frac{ds_1 \wedge ds_2}{(s_1-y)(s_1-s_2)},
\end{aligned}$$

and the integral representation is given by (2.3) with $\vec{\xi}$ replaced by $\vec{\zeta} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^T$. Note that the choice of $v_1, \dots, v_4$ determines the explicit forms of the residue matrices of the Pfaffian system after the middle convolution.

Now we consider the confluence. First we replace

$$(\alpha_1, \alpha_3, x, y) \mapsto (\alpha_1 - \gamma\epsilon^{-1}, \gamma\epsilon^{-1}, \epsilon x, \epsilon y), \tag{2.4}$$

and then take the limit $\epsilon \to 0$. By this operation, the integrand of (2.3) is formally changed as

$$\begin{aligned}
&s_1^{\alpha_1-\gamma\epsilon^{-1}}(s_1-1)^{\alpha_2}(s_1-\epsilon y)^{\gamma\epsilon^{-1}}(s_1-s_2)^{\beta}s_2^{-\alpha_1+\gamma\epsilon^{-1}-\beta}(s_2-1)^{-\alpha_2-\beta}(s_2-\epsilon x)^{\beta-\gamma\epsilon^{-1}}\\
&= s_1^{\alpha_1}(s_1-1)^{\alpha_2}\left(1-\epsilon\frac{y}{s_1}\right)^{\gamma\epsilon^{-1}}(s_1-s_2)^{\beta}s_2^{-\alpha_1}(s_2-1)^{-\alpha_2-\beta}\left(1-\epsilon\frac{x}{s_2}\right)^{\beta-\gamma\epsilon^{-1}}\\
&\to s_1^{\alpha_1}(s_1-1)^{\alpha_2}e^{-\gamma\frac{y}{s_1}}(s_1-s_2)^{\beta}s_2^{-\alpha_1}(s_2-1)^{-\alpha_2-\beta}e^{\gamma\frac{x}{s_2}}.
\end{aligned}$$

For the vector $\vec{\zeta}$ of cocycles, we see that ζ_3 and ζ_4 tends to the same limit, which breaks the independence. Therefore we need a gauge transformation before confluence. For example, the gauge transformation by the constant matrix

$$Q_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \frac{1}{\alpha_1} & 1 \\ & & & 1 \end{pmatrix}$$

changes the vector of cocycles to $Q_1^{-1}\vec{\zeta}$, and its result of the confluence becomes

$$\vec{\tilde{\zeta}} = (\tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta}_3, \tilde{\zeta}_4)^T$$

with

$$\begin{aligned}
\tilde{\zeta}_1 &= \frac{ds_1\wedge ds_2}{s_1 s_2} + \frac{\beta}{\alpha_1+\alpha_2+\beta}\cdot\frac{ds_1\wedge ds_2}{s_1(s_1-s_2)},\\
\tilde{\zeta}_2 &= \frac{ds_1\wedge ds_2}{(s_1-1)s_2} + \frac{\beta}{\alpha_1+\alpha_2+\beta}\cdot\frac{ds_1\wedge ds_2}{s_1(s_1-s_2)},\\
\tilde{\zeta}_3 &= \gamma y_1\frac{ds_1\wedge ds_2}{s_1^2(s_2-1)},\\
\tilde{\zeta}_4 &= \frac{ds_1\wedge ds_2}{s_1(s_2-1)} + \frac{\beta}{\alpha_1+\alpha_2+\beta}\cdot\frac{ds_1\wedge ds_2}{s_1(s_1-s_2)}.
\end{aligned}$$

Here we change the variables of integration to (t_1, t_2) given by

$$t_1 = \frac{1}{s_2}, \quad t_2 = \frac{1}{s_1}.$$

Then the integrand is changed to

$$t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1}(t_1-t_2)^{\beta}t_2^{-\alpha_1-\alpha_2-\beta}(1-t_2)^{\alpha_2}e^{-\gamma y t_2}.$$

Also by a simple calculation, we get

$$\begin{aligned}
\tilde{\zeta}_1 &= -\frac{dt_1 \wedge dt_2}{t_1 t_2} - \frac{\beta}{\alpha_1+\alpha_2+\beta} \cdot \frac{dt_1 \wedge dt_2}{t_1(t_1-t_2)}, \\
\tilde{\zeta}_2 &= -\frac{dt_1 \wedge dt_2}{t_1 t_2(1-t_2)} - \frac{\beta}{\alpha_1+\alpha_2+\beta} \cdot \frac{dt_1 \wedge dt_2}{t_1(t_1-t_2)}, \\
\tilde{\zeta}_3 &= -\gamma y \frac{dt_1 \wedge dt_2}{t_1(1-t_1)}, \\
\tilde{\zeta}_4 &= -\frac{dt_1 \wedge dt_2}{t_1 t_2(1-t_1)} - \frac{\beta}{\alpha_1+\alpha_2+\beta} \cdot \frac{dt_1 \wedge dt_2}{t_1(t_1-t_2)}.
\end{aligned}$$

Looking at these result, we notice that some linear combinations make these cocycles simple. For example, we find

$$\tilde{\zeta}_2 - \tilde{\zeta}_1 = \frac{dt_1 \wedge dt_2}{t_1(t_2-1)}, \quad \tilde{\zeta}_4 - \tilde{\zeta}_1 = \frac{dt_1 \wedge dt_2}{(t_1-1)t_2}.$$

Moreover, we can express these cocycles in simpler forms by using the help of twisted coboundaries. If we set

$$\begin{aligned}
\omega &= d \log \left(t_1^{\alpha_1+\alpha_2} (1-t_1)^{-\alpha_2-\beta} e^{\gamma x t_1} (t_1-t_2)^{\beta} t_2^{-\alpha_1-\alpha_2-\beta} (1-t_2)^{\alpha_2} e^{-\gamma y t_2} \right) \\
&= \left(\frac{\alpha_1+\alpha_2}{t_1} + \frac{-\alpha_2-\beta}{t_1-1} + \gamma x + \frac{\beta}{t_1-t_2} \right) dt_1 \\
&\quad + \left(\frac{-\alpha_1-\alpha_2-\beta}{t_2} + \frac{\alpha_2}{t_2-1} - \gamma y + \frac{\beta}{t_2-t_1} \right) dt_2,
\end{aligned}$$

the coboundary operator is given by

$$\nabla_\omega := d + \omega \wedge .$$

Then we have

$$\begin{aligned}
0 &= \nabla_\omega \left(\frac{dt_1}{t_1} \right) \\
&= (\alpha_1+\alpha_2+\beta) \left(\frac{dt_1 \wedge dt_2}{t_1 t_2} + \frac{\beta}{\alpha_1+\alpha_2+\beta} \cdot \frac{dt_1 \wedge dt_2}{t_1(t_1-t_2)} \right) \\
&\quad + \gamma y \frac{dt_1 \wedge dt_2}{t_1} - \alpha_2 \frac{dt_1 \wedge dt_2}{t_1(t_2-1)},
\end{aligned}$$

from which we derive

$$(\alpha_1+\beta)\tilde{\zeta}_1 + \alpha_2 \tilde{\zeta}_2 = \gamma y \frac{dt_1 \wedge dt_2}{t_1}.$$

In these ways, we find that the matrix Q_2 defined by

$$Q_2^{-1} = \begin{pmatrix} \alpha_1+\beta & \alpha_2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \alpha_1+\beta & \alpha_2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

sends $\vec{\tilde{\zeta}}$ to a simpler vector of cocycles

$$Q_2^{-1}\vec{\tilde{\zeta}} = \left(\gamma y \frac{dt_1 \wedge dt_2}{t_1}, \frac{dt_1 \wedge dt_2}{t_1(t_2-1)}, \gamma y \frac{dt_1 \wedge dt_2}{t_1 - 1}, \frac{dt_1 \wedge dt_2}{(t_1-1)t_2}\right)^T.$$

Although the twisted cocycles are simple enough, we operate further gauge transformation in order to make the residue matrices of the Pfaffian system simple. We take

$$Q_3 = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & -\frac{1}{\alpha_2} & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & \frac{1}{\alpha_1+\alpha_2+\beta} & 0 \end{pmatrix},$$

and then we have

$$Q_3^{-1}Q_2^{-1}\vec{\tilde{\zeta}} = \vec{\eta},$$

where $\vec{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4)^T$ is given by

$$\begin{aligned} \eta_1 &= \alpha_2 \frac{dt_1 \wedge dt_2}{(t_1-1)(t_2-1)} + \beta \frac{dt_1 \wedge dt_2}{(t_1-1)(t_2-t_1)}, \\ \eta_2 &= -\alpha_2 \frac{dt_1 \wedge dt_2}{t_1(t_2-1)}, \\ \eta_3 &= (\alpha_1+\alpha_2+\beta)\frac{dt_1 \wedge dt_2}{(t_1-1)t_2}, \\ \eta_4 &= (-\alpha_1-\alpha_2-\beta)\frac{dt_1 \wedge dt_2}{t_1 t_2} + \beta \frac{dt_1 \wedge dt_2}{t_1(t_2-t_1)}. \end{aligned} \tag{2.5}$$

The sequence of the above operations for the vector of cocycles including the confluence gives a sequence of operations for the Pfaffian systems. As the final result, we obtain the Pfaffian system

$$dU = \left(B_{x0}\frac{dx}{x} + B_{x\infty}\,dx + B_{xy}\frac{d(x-y)}{x-y} + B_{y0}\frac{dy}{y} + B_{y\infty}\,dy\right)U, \tag{2.6}$$

where

$$B_{x0} = \begin{pmatrix} \alpha_2 & \alpha_1+\alpha_2 & 0 & 0 \\ -\alpha_2 & -\alpha_1-\alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_2+\beta & \alpha_1+\alpha_2+\beta \\ 0 & 0 & -\alpha_2-\beta & -\alpha_1-\alpha_2-\beta \end{pmatrix},$$

$$B_{x\infty} = \begin{pmatrix} \gamma & & & \\ & 0 & & \\ & & \gamma & \\ & & & 0 \end{pmatrix}, \quad B_{xy} = \begin{pmatrix} \beta & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & \beta \end{pmatrix},$$

$$B_{y0} = \begin{pmatrix} -\alpha_2-\beta & 0 & \alpha_2+\beta & 0 \\ 0 & -\alpha_2 & 0 & \alpha_2 \\ -\alpha_1-\alpha_2-\beta & 0 & \alpha_1+\alpha_2+\beta & 0 \\ 0 & -\alpha_1-\alpha_2 & 0 & \alpha_1+\alpha_2 \end{pmatrix},$$

$$B_{y\infty} = \begin{pmatrix} -\gamma & & & \\ & -\gamma & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

By the above construction, we see that the system (2.6) has an integral representation of solutions given by

$$U(x,y) = \int_{\Delta} t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1}(t_1-t_2)^{\beta}t_2^{-\alpha_1-\alpha_2-\beta}(1-t_2)^{\alpha_2}e^{-\gamma y t_2}\vec{\eta}, \quad (2.7)$$

where the vector of cocycles $\vec{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4)^T$ is given by (2.5).

The Pfaffian system (2.6) is obtained from the Pfaffian system (2.2) by confluence. We know that the Pfaffian system (2.2) is also obtained as a pull-back from the Pfaffian system satisfied by Appell's hypergeometric function F_4 ([7]). Then the system (2.6) can be regarded as a Pfaffian system for the confluent F_4.

Now we look at the singular locus. The Pfaffian system (2.2) before confluence is regular holonomic. The singular locus of the Pfaffian system (2.2) is

$$\{x=0\} \cup \{x=1\} \cup \{x=\infty\} \cup \{y=0\} \cup \{y=1\} \cup \{y=\infty\} \cup \{x=y\}$$

in $\mathbb{P}^1 \times \mathbb{P}^1$, and is illustrated in Figure 1.

The process of confluence (2.4) sends $\{x=1\} \to \{x=\infty\}$ and $\{y=1\} \to \{y=\infty\}$ simultaneously. Then the singular locus of (2.6) after confluence becomes

$$\{x=0\} \cup \{y=0\} \cup \{x=y\} \cup \{x=\infty\} \cup \{y=\infty\},$$

where $\{x=\infty\}$ and $\{y=\infty\}$ have irregular singularities. We illustrate the singular locus in Figure 2.

We note that the confluence and unfolding of Pfaffian systems are studied in general by Oshima [15].

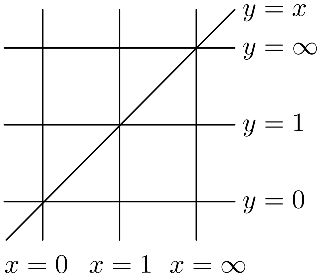

Fig. 1. Singular locus of (2.2); before confluence

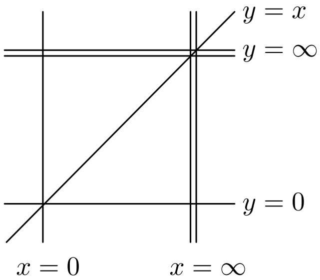

Fig. 2. Singular locus of (2.6); after confluence

3. FORMAL REDUCTION

The purpose of this paper is to study the behavior of the Pfaffian system (2.6) around the intersection of the divisors $\{x = \infty\}$ and $\{y = \infty\}$. In this section, we operate a formal reduction to (2.6) so as to obtain the main asymptotic behaviors.

As is seen in Figure 2, the intersection of $\{x = \infty\}$ and $\{y = \infty\}$ is not normal crossing. Then we first resolve it by a blowing-up. We set

$$y = zx,$$

and change the variables from (x, y) to (x, z). Then $\{x = \infty\}$ and $\{z = \infty\}$ intersects normally (see Figure 3). In these new variables, the Pfaffian system (2.6) can be written as

$$\begin{cases} \dfrac{\partial U}{\partial x} = \left[zB_{y\infty} + B_{x\infty} + \dfrac{B_{x0} + B_{xy} + B_{y0}}{x} \right] U, \\ \dfrac{\partial U}{\partial z} = \left[xB_{y\infty} + \dfrac{B_{y0}}{z} + \dfrac{B_{xy}}{z-1} \right] U. \end{cases} \tag{3.1}$$

Next we proceed to the simultaneous block diagonalization. The simultaneous block diagonalization in two variables case is the operation given in the following theorem.

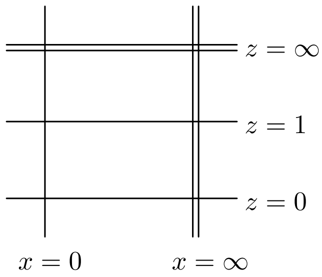

Fig. 3. Singular locus after blowing-up

Theorem 3.1. *Let $A(x,y), B(x,y)$ be $N \times N$-matrices of holomorphic functions at $(x,y) = (0,0)$, p,q positive integers and r,s integers. We consider a Pfaffian system*

$$du = \left(\frac{A(x,y)}{x^{p+1}y^r}\, dx + \frac{B(x,y)}{y^{q+1}x^s}\, dy \right) u, \tag{3.2}$$

and assume the integrability condition. We set

$$A(x,y) = \sum_{m,n=0}^{\infty} A_{mn} x^m y^n, \quad B(x,y) = \sum_{m,n=0}^{\infty} B_{mn} x^m y^n.$$

By the integrability condition, we have $[A_{00}, B_{00}] = O$, and hence may assume that A_{00} and B_{00} are simultaneously of block diagonal form

$$A_{00} = \left(\begin{array}{c|c} A_{00}^{11} & O \\ \hline O & A_{00}^{22} \end{array} \right), \quad B_{00} = \left(\begin{array}{c|c} B_{00}^{11} & O \\ \hline O & B_{00}^{22} \end{array} \right)$$

partitioned into the same size. We assume that A_{00}^{11} and A_{00}^{22} do not have any common eigenvalue. Then there exists a unique formal transformation $P(x,y)$ of the form

$$P(x,y) = I + \sum_{m+n \geq 1} P_{mn} x^m y^n, \quad P_{mn} = \left(\begin{array}{c|c} O & P_{mn}^{12} \\ \hline P_{mn}^{21} & O \end{array} \right)$$

such that the gauge transformation $u = P(x,y)v$ sends (3.2) to

$$dv = \left(\frac{\tilde{A}(x,y)}{x^{p+1}y^r}\, dx + \frac{\tilde{B}(x,y)}{y^{q+1}x^s}\, dy \right) v$$

with block diagonalized coefficients

$$\tilde{A}(x,y) = \left(\begin{array}{c|c} \tilde{A}(x,y)^{11} & O \\ \hline O & \tilde{A}(x,y)^{22} \end{array} \right), \ \tilde{B}(x,y) = \left(\begin{array}{c|c} \tilde{B}(x,y)^{11} & O \\ \hline O & \tilde{B}(x,y)^{22} \end{array} \right).$$

We can prove Theorem 3.1 in a similar manner as in one variable case ([21]) by the help of the integrability condition.

We apply Theorem 3.1 to the Pfaffian system (3.1) (we must rewrite the theorem to the variables (x^{-1}, y^{-1})). The leading coefficients of both equations in (3.1) are the same matrix

$$B_{y\infty} = \begin{pmatrix} -\gamma & & & \\ & -\gamma & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

which is already diagonal. We set

$$\begin{aligned} zB_{y\infty} + B_{x\infty} + \frac{B_{x0} + B_{xy} + B_{y0}}{x} &= z \sum_{m,n=0}^{\infty} C_{mn} x^{-m} z^{-n}, \\ xB_{y\infty} + \frac{B_{y0}}{z} + \frac{B_{xy}}{z-1} &= x \sum_{m,n=0}^{\infty} D_{mn} x^{-m} z^{-n}. \end{aligned}$$

Explicitly we have

$$C_{00} = B_{y\infty},\ C_{01} = B_{x\infty},\ C_{11} = B_{x0} + B_{xy} + B_{y0},\ C_{mn} = O \text{ (for the others)}$$

and

$$D_{00} = B_{y\infty},\ D_{11} = B_{y0} + B_{xy},\ D_{1n} = B_{xy}\ (n \geq 2),\ D_{mn} = O \text{ (for the others)}.$$

Thanks to Theorem 3.1, there exists a formal transformation

$$U = P(x,z)V, \quad P(x,z) = I + \sum_{m+n\geq 1} P_{mn} x^{-m} z^{-n}$$

that sends the system (3.1) to the $(2,2)\times(2,2)$-block diagonalized system

$$\begin{cases} \dfrac{\partial V}{\partial x} = \left(z \displaystyle\sum_{m,n=0}^{\infty} \tilde{C}_{mn} x^{-m} z^{-n} \right) V, \\ \dfrac{\partial V}{\partial z} = \left(x \displaystyle\sum_{m,n=0}^{\infty} \tilde{D}_{mn} x^{-m} z^{-n} \right) V. \end{cases} \tag{3.3}$$

In particular, we have

$$\begin{aligned}
\tilde{C}_{00} &= \tilde{D}_{00} = B_4 = \begin{pmatrix} -\gamma I_2 & \\ & I_2 \end{pmatrix}, \\
\tilde{C}_{01} &= B_{x\infty} = \left(\begin{array}{cc|cc} \gamma & & & \\ & 0 & & \\ \hline & & \gamma & \\ & & & 0 \end{array}\right), \quad \tilde{D}_{10} = O, \\
\tilde{C}_{11} &= \left(\begin{array}{cc|cc} 0 & \alpha_1+\alpha_2 & \multicolumn{2}{c}{\multirow{2}{*}{O}} \\ -\alpha_2 & -\alpha_1-2\alpha_2 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{O}} & \alpha_1+2\alpha_2+2\beta & \alpha_1+\alpha_2+\beta \\ & & -\alpha_2-\beta & 0 \end{array}\right), \\
\tilde{D}_{11} &= \left(\begin{array}{cc|cc} -\alpha_2 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{O}} \\ 0 & -\alpha_2 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{O}} & \alpha_1+\alpha_2+\beta & 0 \\ & & 0 & \alpha_1+\alpha_2+\beta \end{array}\right).
\end{aligned} \tag{3.4}$$

Moreover, it turns out that

$$\tilde{C}_{m0} = \tilde{D}_{0m} = O \quad (m \geq 1). \tag{3.5}$$

In order to get rid of the top terms in the right hand sides of (3.3), we operate the gauge transformation

$$V = \left(\begin{array}{c|c} e^{-\gamma xz} I_2 & \\ \hline & I_2 \end{array}\right) W \tag{3.6}$$

to the system (3.3). Then we get a $(2,2) \times (2,2)$-block diagonalized system

$$\begin{cases} \dfrac{\partial W}{\partial x} = E(x,z)W, \\ \dfrac{\partial W}{\partial z} = F(x,z)W \end{cases}$$

with

$$\begin{aligned}
E(x,z) &= \sum_{m,n=0}^{\infty} E_{mn} x^{-m} z^{-n}, \\
F(x,z) &= \sum_{m,n=0}^{\infty} F_{mn} x^{-m} z^{-n},
\end{aligned}$$

where

$$E_{mn} = \tilde{C}_{m,n+1}, \quad F_{mn} = \tilde{D}_{m+1,n}$$

and

$$E_{mn} = E_{mn}^{11} \oplus E_{mn}^{22}, \quad F_{mn} = F_{mn}^{11} \oplus F_{mn}^{22}.$$

Note that, thanks to (3.5), the terms of positive powers of x and z disappear. We partition W into $(2,2)$-sizes, and set $W=(W_1,W_2)^T$. Then we have two systems

$$\begin{cases} \dfrac{\partial W_1}{\partial x} = \left(\displaystyle\sum_{m,n=0}^{\infty} E^{11}_{mn} x^{-m} z^{-n}\right) W_1, \\ \dfrac{\partial W_1}{\partial z} = \left(\displaystyle\sum_{m,n=0}^{\infty} F^{11}_{mn} x^{-m} z^{-n}\right) W_1 \end{cases} \tag{3.7}$$

and

$$\begin{cases} \dfrac{\partial W_2}{\partial x} = \left(\displaystyle\sum_{m,n=0}^{\infty} E^{22}_{mn} x^{-m} z^{-n}\right) W_2, \\ \dfrac{\partial W_2}{\partial z} = \left(\displaystyle\sum_{m,n=0}^{\infty} F^{22}_{mn} x^{-m} z^{-n}\right) W_2. \end{cases} \tag{3.8}$$

Since E^{11}_{mn} (resp. F^{11}_{mn}) is the $(1,1)$-block of $E_{mn}=\tilde{C}_{m,n+1}$ (resp. $F_{mn}=\tilde{D}_{m+1,n}$), we have

$$E^{11}_{00} = \begin{pmatrix} \gamma & \\ & 0 \end{pmatrix}, \quad E^{11}_{10} = \begin{pmatrix} 0 & \alpha_1+\alpha_2 \\ -\alpha_2 & -\alpha_1-2\alpha_2 \end{pmatrix},$$
$$F^{11}_{00} = \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \quad F^{11}_{01} = \begin{pmatrix} -\alpha_2 & 0 \\ 0 & -\alpha_2 \end{pmatrix}.$$

We can apply Theorem 3.1 again to (3.7), and get a diagonalized system

$$\begin{cases} \dfrac{\partial \tilde{W}_1}{\partial x} = \left[\begin{pmatrix} \gamma & \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & \\ & -\alpha_1-2\alpha_2 \end{pmatrix} x^{-1} + \dots \right] \tilde{W}_1, \\ \dfrac{\partial \tilde{W}_1}{\partial z} = \left[\begin{pmatrix} 0 & \\ & 0 \end{pmatrix} + \begin{pmatrix} -\alpha_2 & 0 \\ 0 & -\alpha_2 \end{pmatrix} z^{-1} + \dots \right] \tilde{W}_1. \end{cases}$$

Then the systems split into sets of scalar differential equations. By solving the scalar equations, we get two asymptotic behaviors

$$e^{\gamma x} z^{-\alpha_2}, \; x^{-\alpha_1-2\alpha_2} z^{-\alpha_2}.$$

Similarly, by diagonalizing (3.8), we get the asymptotic behaviors

$$e^{\gamma x} x^{\alpha_1+2\alpha_2+2\beta} z^{\alpha_1+2\alpha_2+2\beta}, \; z^{\alpha_1+\alpha_2+\beta}.$$

Combining with (3.6), we obtain the following result.

Proposition 3.2. *The Pfaffian system* (3.1) *exhibits the asymptotic behaviors*

$$\begin{aligned} &e^{-\gamma xz} e^{\gamma x} z^{-\alpha_2}, \\ &e^{-\gamma xz} x^{-\alpha_1-2\alpha_2} z^{-\alpha_2}, \\ &e^{\gamma x} x^{\alpha_1+2\alpha_2+2\beta} z^{\alpha_1+\alpha_2+\beta}, \\ &z^{\alpha_1+\alpha_2+\beta} \end{aligned}$$

at $(x,z) = (\infty,\infty)$*. In terms of the variables* (x,y)*, these asymptotic behaviors are written as*

$$e^{\gamma x}e^{-\gamma y}x^{\alpha_2}y^{-\alpha_2},$$
$$e^{-\gamma y}x^{-\alpha_1-\alpha_2}y^{-\alpha_2},$$
$$e^{\gamma x}x^{\alpha_2+\beta}y^{\alpha_1+\alpha_2+\beta},$$
$$x^{-\alpha_1-\alpha_2-\beta}y^{\alpha_1+\alpha_2+\beta}.$$

4. LOCAL ANALYSIS

In this section, we study the asymptotic behaviors of the integrable Pfaffian system (2.6) around $(x,y) = (\infty,\infty)$. Before proceeding to the analysis, here we recall the definition of Majima's asymptotics in two variables case.

Let $S = S_1 \times S_2$ be a polysector centered at $(x,y) = (\infty,\infty) \in \mathbb{P}^1 \times \mathbb{P}^1$, and $f(x,y)$ a function holomorphic on S. We say that $f(x,y)$ is *asymptotically developable* on S, if there exist

$$\begin{cases} f_{mn} \in \mathbb{C} \quad (m,n = 0,1,\ldots), \\ f_{m*}(y) \in \mathcal{O}(S_2) \quad (m = 0,1,\ldots), \\ f_{*n}(x) \in \mathcal{O}(S_1) \quad (n = 0,1,\ldots) \end{cases}$$

such that, for any $M, N \in \mathbb{Z}_{\geq 0}$,

$$\left| f(x,y) - \sum_{m=0}^{M-1} f_{m*}(y)x^{-m} - \sum_{n=0}^{N-1} f_{*n}(x)y^{-n} + \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_{mn}x^{-m}y^{-n} \right| / (x^{-M}y^{-N})$$

is bounded on S. In this case, the following limits exist and are related to the above data as

$$\lim_{\substack{x\to\infty \\ x\in S_1}} f^{(m,0)}(x,y) = m! f_{m*}(y),$$
$$\lim_{\substack{y\to\infty \\ y\in S_2}} f^{(0,n)}(x,y) = n! f_{*n}(x),$$
$$\lim_{\substack{(x,y)\to(\infty,\infty) \\ (x,y)\in S}} f^{(m,n)}(x,y) = m!n! f_{mn},$$

where we use the notation

$$f^{(m,n)}(x,y) = \frac{\partial^{m+n}}{\partial x^m \partial y^n} f(x,y).$$

We use the integral representation (2.7) to obtain solutions exhibiting the asymptotic behaviors in Proposition 3.2. For simplicity, we remove the cocycle part of the integral, and consider the scalar function

$$u_\Delta = \int_\Delta t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1}t_2^{-\alpha_1-\alpha_2-\beta}(1-t_2)^{\alpha_2}e^{-\gamma y t_2}(t_1-t_2)^{\beta}\, dt_1 \wedge dt_2. \quad (4.1)$$

This integral looks like a coupling of Kummer's confluent hypergeometric integrals

$$\int_p^q t^a(1-t)^b e^{xt}\,dt$$

with the coupling term $(t_1-t_2)^\beta$. Note that there are two resonances in the exponents of the integral (4.1); the sum of the exponents of the factors

$$t_1^{\alpha_1+\alpha_2} t_2^{-\alpha_1-\alpha_2-\beta}(t_1-t_2)^\beta$$

is 0, and also the sum of the exponents of the factors

$$(1-t_1)^{-\alpha_2-\beta}(1-t_2)^{\alpha_2}(t_1-t_2)^\beta$$

is 0. If there is no resonance in the exponents, the rank of the Pfaffian system satisfied by the integral is greater than four. Then the resonances reduce the rank of the Pfaffian system for the integral (2.7) to four. In other words, the Pfaffian system (2.6) can be regarded as an irreducible component of a reducible Pfaffian system of rank greater than four. Therefore we should choose cycles Δ such that u_Δ exhibit the asymptotic behaviors in Proposition 3.2. Then the integrals $U(x,y)$ for these Δ become solutions of (2.6).

We use both variables (x,y) and (x,z) so that the descriptions become simple. We define two sectorial domains in (x,y) space. The first one S is defined by

$$\begin{cases} -\dfrac{\pi}{2} < \arg\gamma + \arg x < \dfrac{3}{2}\pi, \\ -\dfrac{3}{2}\pi < \arg\gamma + \arg y < \dfrac{\pi}{2}, \\ 0 < \arg x - \arg y < 2\pi, \end{cases} \tag{4.2}$$

and the second one S' is defined by

$$\begin{cases} \dfrac{\pi}{2} < \arg\gamma + \arg x < \dfrac{5}{2}\pi, \\ -\dfrac{\pi}{2} < \arg\gamma + \arg y < \dfrac{3}{2}\pi, \\ 0 < \arg x - \arg y < 2\pi. \end{cases} \tag{4.3}$$

The reason why we define these sectorial domains will be clarified in the following analysis. Here we note that, for any $(x,y) \in S$, there exists θ satisfying

$$\begin{aligned} \frac{\pi}{2} < \arg\gamma + \arg x + \theta < \frac{3}{2}\pi, \\ -\frac{\pi}{2} < \arg\gamma + \arg y + \theta < \frac{\pi}{2}, \\ 0 < \theta < \pi. \end{aligned} \tag{4.4}$$

In the following till the end of Section 4.5, we fix $(x,y) \in S$ and one such θ.

4.1. CYCLES

For the sake of the convergence of the integral (4.1), we assume

$$\begin{aligned}&\Re(\alpha_1+\alpha_2)>-1, \quad \Re(-\alpha_2-\beta)>-1, \quad \Re(-\alpha_1-\alpha_2-\beta)>-1,\\ &\Re(\alpha_2)>-1, \quad \Re(\beta)>-1.\end{aligned} \tag{4.5}$$

This assumption will be relaxed by the analytic continuation with respect to $(\alpha_1, \alpha_2, \beta)$.

For $a, b \in \mathbb{C} \cup \{\infty e^{\sqrt{-1}\varphi} \mid \varphi \in \mathbb{R}\}$ and $i = 1, 2$, we denote by $(t_i : a \to b)$ the open segment in the complex t_i-plane from a to b. We define four 2-chains Δ_{ij} $(i, j = 0, 1)$ by

$$\begin{aligned}\Delta_{01} &= (t_1 : 0 \to \infty e^{\sqrt{-1}\theta}) \times (t_2 : 1 \to \infty e^{\sqrt{-1}\theta}),\\ \Delta_{00} &= (t_1 : 0 \to \infty e^{\sqrt{-1}\theta}) \times (t_2 : 0 \to t_1),\\ \Delta_{10} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta}) \times (t_2 : 0 \to \infty e^{\sqrt{-1}\theta}),\\ \Delta_{11} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta}) \times (t_2 : 1 \to t_1).\end{aligned}$$

To each of these chains, we attach a branch of the integrand

$$t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1}t_2^{-\alpha_1-\alpha_2-\beta}(1-t_2)^{\alpha_2}e^{-\gamma y t_2}(t_1-t_2)^{\beta} \tag{4.6}$$

as follows:

$$\Delta_{01}:\begin{cases}\arg t_1 = \theta, & \\ \arg(1-t_1) = 0 & \text{at } t_1 = 0,\\ \arg t_2 = 0 & \text{at } t_2 = 1,\\ \arg(1-t_2) = \theta+\pi, & \\ \arg(t_1-t_2) = \pi & \text{at } (t_1,t_2) = (0,1),\end{cases}$$

$$\Delta_{00}:\begin{cases}\arg t_1 = \theta, & \\ \arg(1-t_1) = 0 & \text{at } t_1 = 0,\\ \arg t_2 = \theta, & \\ \arg(1-t_2) = 0 & \text{at } t_2 = 0,\\ \arg(t_1-t_2) = \theta, & \end{cases}$$

$$\Delta_{10}:\begin{cases}\arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta+\pi, & \\ \arg t_2 = \theta, & \\ \arg(1-t_2) = 0 & \text{at } t_2 = 0,\\ \arg(t_1-t_2) = 0 & \text{at } (t_1,t_2) = (1,0),\end{cases}$$

$$\Delta_{11}:\begin{cases}\arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta+\pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1,\\ \arg(1-t_2) = \theta+\pi, & \\ \arg(t_1-t_2) = \theta. & \end{cases}$$

In this way, the chains Δ_{ij} become twisted chains. By the choice of θ and the assumption (4.5), these chains becomes twisted cycles with respect to the integrand (4.6).

In the following, we often use Δ instead of u_Δ.

4.2. ASYMPTOTIC BEHAVIOR OF Δ_{01}

We study the asymptotic behavior of the integral Δ_{01}. By the change $t_2 = 1 - s_2$ of an integral variable, we have

$$\begin{aligned}\Delta_{01} &= \int_0^{\infty e^{\sqrt{-1}\theta}} dt_1 \int_1^{\infty e^{\sqrt{-1}\theta}} dt_2\, t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1} \\ &\quad \times t_2^{-\alpha_1-\alpha_2-\beta}(1-t_2)^{\alpha_2}e^{-\gamma y t_2}(t_1-t_2)^{\beta} \\ &= -\int_0^{\infty e^{\sqrt{-1}\theta}} dt_1 \int_0^{-\infty e^{\sqrt{-1}\theta}} ds_2\, t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1} \\ &\quad \times (1-s_2)^{-\alpha_1-\alpha_2-\beta}s_2^{\alpha_2}e^{-\gamma y(1-s_2)}(t_1-1+s_2)^{\beta} \\ &= e^{\pi\sqrt{-1}(\beta+1)}e^{-\gamma y}\int_0^{\infty e^{\sqrt{-1}\theta}} dt_1 \int_0^{-\infty e^{\sqrt{-1}\theta}} ds_2\, t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1} \\ &\quad \times (1-s_2)^{-\alpha_1-\alpha_2-\beta}s_2^{\alpha_2}e^{\gamma y s_2}(1-(t_1+s_2))^{\beta}.\end{aligned}$$

From the well-known formula of Taylor expansion

$$f(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}f^{(n)}(t)\,dt,$$

we obtain the following lemma.

Lemma 4.1. *For any holomorphic function $f(x,y)$ at $(x,y) = (0,0)$ and for any $M, N \in \mathbb{Z}_{>0}$, we have*

$$\begin{aligned}f(x,y) &= \sum_{m=0}^{M-1}\frac{f^{(m,0)}(0,y)}{m!}x^m + \sum_{n=0}^{N-1}\frac{f^{(0,n)}(x,0)}{n!}y^n - \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\frac{f^{(m,n)}(0,0)}{m!n!}x^m y^n \\ &\quad + \frac{x^M y^N}{(M-1)!(N-1)!}\int_0^1 ds \int_0^1 dt\,(1-s)^{M-1}(1-t)^{N-1}f^{(M,N)}(xs,yt).\end{aligned}$$

We will apply Lemma 4.1 to the function

$$g(t_1, s_2) = (1-t_1)^{-\alpha_2-\beta}(1-s_2)^{-\alpha_1-\alpha_2-\beta}(1-(t_1+s_2))^{\beta}$$

in (t_1, s_2). By rewriting $g(t_1, s_2)$ as

$$g(t_1, s_2) = (1-s_2)^{-\alpha_1-\alpha_2}(1-t_1)^{-\alpha_2-\beta}\left(1-\frac{t_1}{1-s_2}\right)^{\beta},$$

we get

$$\frac{g^{(m,0)}(0, s_2)}{m!} = (-1)^m \sum_{i=0}^{m} \binom{-\alpha_2-\beta}{m-i}\binom{\beta}{i}(1-s_2)^{\alpha_1-\alpha_2-i}.$$

In a similar way, we have

$$\frac{g^{(0,n)}(t_1, 0)}{n!} = (-1)^n \sum_{j=0}^{n} \binom{-\alpha_1-\alpha_2-\beta}{n-j}\binom{\beta}{j}(1-t_1)^{-\alpha_2-j}.$$

By expanding each factor of g, we also get

$$\begin{aligned}\frac{g^{(m,n)}(0,0)}{m!n!} &= (-1)^{m+n}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{-\alpha_2-\beta}{i}\binom{-\alpha_1-\alpha_2-\beta}{j}\binom{\beta}{m-i}\binom{\beta-n+i}{n-j} \\ &= (-1)^{m+n}\sum_{i=0}^{m}\binom{-\alpha_2-\beta}{i}\binom{\beta}{m-i}\binom{-\alpha_1-\alpha_2-m+i}{n} \\ &=: (-1)^{m+n}A_{mn}.\end{aligned}$$

Then we can express g as in Lemma 4.1. Putting the expression into the integral, we obtain

$$\begin{aligned}
&\frac{\Delta_{01}}{e^{\pi\sqrt{-1}(\beta+1)}e^{-\gamma y}} \\
&= \sum_{m=0}^{M-1}(-1)^m \sum_{i=0}^{m}\binom{-\alpha_2-\beta}{m-i}\binom{\beta}{i} \\
&\quad \times \int_0^{\infty e^{\sqrt{-1}\theta}} t_1^{\alpha_1+\alpha_2+m}e^{\gamma x t_1}\,dt_1 \int_0^{-\infty e^{\sqrt{-1}\theta}} s_2^{\alpha_2}(1-s_2)^{-\alpha_1-\alpha_2-i}e^{\gamma y s_2}\,ds_2 \\
&\quad + \sum_{n=0}^{N-1}(-1)^n \sum_{j=0}^{n}\binom{-\alpha_1-\alpha_2-\beta}{n-j}\binom{\beta}{j} \\
&\quad \times \int_0^{\infty e^{\sqrt{-1}\theta}} t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-j}e^{\gamma x t_1}\,dt_1 \int_0^{-\infty e^{\sqrt{-1}\theta}} s_2^{\alpha_2+n}e^{\gamma y s_2}\,ds_2 \\
&\quad - \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(-1)^{m+n}A_{mn} \int_0^{\infty e^{\sqrt{-1}\theta}} t_1^{\alpha_1+\alpha_2+m}e^{\gamma x t_1}\,dt_1 \int_0^{-\infty e^{\sqrt{-1}\theta}} s_2^{\alpha_2+n}e^{\gamma y s_2}\,ds_2 \\
&\quad + \int_0^{\infty e^{\sqrt{-1}\theta}} dt_1 \int_0^{-\infty e^{\sqrt{-1}\theta}} ds_2 \int_0^1 d\sigma \int_0^1 d\tau\,(-1)^{M+N}t_1^{\alpha_1+\alpha_2+M}e^{\gamma x t_1} \\
&\quad \times s_2^{\alpha_2+N}e^{\gamma y s_2}(1-\sigma)^{M-1}(1-\tau)^{N-1}\frac{g^{(M-1,N-1)}(t_1\sigma, s_2\tau)}{(M-1)!(N-1)!}.
\end{aligned}$$

The integrals in the above summands are evaluated as follows. By the change $\gamma x t_1 = -u_1$ of the integral variable, we have

$$\int_0^{\infty e^{\sqrt{-1}\theta}} t_1^{\alpha_1+\alpha_2+m}e^{\gamma x t_1}\,dt_1 = \left(-\frac{1}{\gamma x}\right)^{\alpha_1+\alpha_2+m+1}\Gamma(\alpha_1+\alpha_2+m+1).$$

By the change $\gamma y s_2 = -u_2$ of the integral variable, we have

$$\begin{aligned}
&\int_0^{-\infty e^{\sqrt{-1}\theta}} s_2^{\alpha_2}(1-s_2)^{-\alpha_1-\alpha_2-i}e^{\gamma y s_2}\,ds_2 \\
&= \left(-\frac{1}{\gamma y}\right)^{\alpha_1+1}\int_0^{+\infty} u_2^{\alpha_2}\left(1+\frac{u_2}{\gamma y}\right)^{-\alpha_1-\alpha_2-i} e^{-u_2}\,du_2.
\end{aligned}$$

Here we introduce the function

$$G(a,b;x) = \int_0^{+\infty} u^a \left(1+\frac{u}{x}\right)^b e^{-u}\,du.$$

Then we have

$$\int_0^{-\infty e^{\sqrt{-1}\theta}} s_2^{\alpha_2}(1-s_2)^{-\alpha_1-\alpha_2-i} e^{\gamma y s_2}\,ds_2 = \left(-\frac{1}{\gamma y}\right)^{\alpha_2+1} G(\alpha_2, -\alpha_1-\alpha_2-i;\gamma y).$$

The other integrals are evaluated in a similar way. We see that, in evaluating the integrals, the factor

$$\left(-\frac{1}{\gamma x}\right)^{\alpha_1+\alpha_2+1}\left(-\frac{1}{\gamma y}\right)^{\alpha_2+1}$$

appears in every term. Now we set

$$\Phi_{01} = e^{\pi\sqrt{-1}(\beta+1)} e^{-\gamma y}\left(-\frac{1}{\gamma x}\right)^{\alpha_1+\alpha_2+1}\left(-\frac{1}{\gamma y}\right)^{\alpha_2+1}.$$

Then, finally we get the asymptotic behavior

$$\begin{aligned}\frac{\Delta_{01}}{\Phi_{01}} &= \sum_{m=0}^{M-1}\Gamma(\alpha_1+\alpha_2+m+1)\sum_{i=0}^{m}\binom{-\alpha_2-\beta}{m-i}\binom{\beta}{i}G(\alpha_2,-\alpha_1-\alpha_2-i;\gamma y)(\gamma x)^{-m}\\ &+\sum_{n=0}^{N-1}\Gamma(\alpha_2+n+1)\sum_{j=0}^{n}\binom{-\alpha_1-\alpha_2-\beta}{m-j}\binom{\beta}{j}G(\alpha_1+\alpha_2,-\alpha_2-j;\gamma x)(\gamma y)^{-n}\\ &-\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}A_{mn}\Gamma(\alpha_1+\alpha_2+m+1)\Gamma(\alpha_2+n+1)(\gamma x)^{-m}(\gamma y)^{-n}\\ &+(\gamma x)^{-M}(\gamma y)^{-N}R^{01}_{MN},\end{aligned} \tag{4.7}$$

where R^{01}_{MN} is bounded as (x,y) tends to (∞,∞) in the sectorial domain S, since the derivative $g^{(M-1,N-1)}$ grows at most polynomially.

4.3. ASYMPTOTIC BEHAVIOR OF Δ_{00}

By the change $t_2 = t_1 s_2$ of an integral variable, we have

$$\begin{aligned}\Delta_{00} &= \int_0^{\infty e^{\sqrt{-1}\theta}} dt_1 \int_0^{t_1} dt_2\, t_1^{\alpha_1+\alpha_2}(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1} t_2^{-\alpha_1-\alpha_2-\beta}(1-t_2)^{\alpha_2}e^{-\gamma y t_2}(t_1-t_2)^{\beta}\\ &= \int_0^{\infty e^{\sqrt{-1}\theta}} dt_1 \int_0^{1} ds_2\, t_1(1-t_1)^{-\alpha_2-\beta}e^{\gamma x t_1} s_2^{-\alpha_1-\alpha_2-\beta}(1-t_1s_2)^{\alpha_2}e^{-\gamma y t_1 s_2}(1-s_2)^{\beta}.\end{aligned}$$

We apply Lemma 4.1 to

$$h(t_1,s_2)=(1-t_1)^{-\alpha_2-\beta}(1-t_1s_2)^{\alpha_2}(1-s_2)^{\beta},$$

and then obtain

$$\begin{aligned}\Delta_{00}=&\sum_{m=0}^{M-1}(-1)^m\sum_{i=0}^{m}\binom{-\alpha_2-\beta}{m-i}\binom{\alpha_2}{i}I_i^{m*}+\sum_{n=0}^{N-1}(-1)^n\sum_{j=0}^{n}\binom{\alpha_2}{j}\binom{\beta}{n-j}I_j^{*n}\\&-\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\sum_{k=0}^{\min\{m,n\}}(-1)^{m+n-k}\binom{-\alpha_2-\beta}{m-k}\binom{\alpha_2}{k}\binom{\beta}{n-k}I^{mn}+\tilde{R}_{MN},\end{aligned}$$

where

$$\begin{aligned}I_i^{m*}&=\int_0^{\infty e^{\sqrt{-1}\theta}}dt_1\int_0^1 ds_2\,t_1^{m+1}e^{\gamma xt_1}s_2^{-\alpha_1-\alpha_2-\beta+i}(1-s_2)^{\beta}e^{-\gamma yt_1s_2},\\I_j^{*n}&=\int_0^{\infty e^{\sqrt{-1}\theta}}dt_1\int_0^1 ds_2\,t_1^{j+1}(1-t_1)^{-\alpha_2-\beta}e^{\gamma xt_1}s_2^{-\alpha_1-\alpha_2-\beta+n}e^{-\gamma yt_1s_2},\\I^{mn}&=\int_0^{\infty e^{\sqrt{-1}\theta}}dt_1\int_0^1 ds_2\,t_1^{m+1}e^{\gamma xt_1}s_2^{-\alpha_1-\alpha_2-\beta+n}e^{-\gamma yt_1s_2},\\\tilde{R}_{MN}&=\int_0^{\infty e^{\sqrt{-1}\theta}}dt_1\int_0^1 ds_2\int_0^1 d\sigma\int_0^1 d\tau\,t_1^{M+1}e^{\gamma xt_1}s_2^{-\alpha_1-\alpha_2-\beta+N}e^{-\gamma yt_1s_2}\\&\quad\times(1-\sigma)^{M-1}(1-\tau)^{N-1}\frac{h^{(M,N)}(t_1\sigma,s_2\tau)}{(M-1)!(N-1)!}.\end{aligned}$$

We evaluate the integrals I_i^{m*},I_j^{*n},I^{mn}. By the changes $\gamma xt_1=-u_1$ and $zs_2=-u_2$, we have

$$\begin{aligned}I_i^{m*}=&\left(-\frac{1}{\gamma x}\right)^{m+2}\left(-\frac{1}{z}\right)^{-\alpha_1-\alpha_2-\beta+i+1}\\&\times\int_0^{+\infty}du_1\int_0^{-z}du_2\,u_1^{m+1}e^{-u_1}u_2^{-\alpha_1-\alpha_2-\beta+i}\left(1+\frac{u_2}{z}\right)^{\beta}e^{-u_1u_2}.\end{aligned}$$

Here we introduce the function

$$H(a,b,c;z)=\int_0^{+\infty}du_1\int_0^{z}du_2\,u_1^{a}e^{-u_1}u_2^{b}\left(1+\frac{u_2}{z}\right)^{c}e^{-u_1u_2}.$$

Then we have

$$I_i^{m*} = \left(-\frac{1}{\gamma x}\right)^{m+2} \left(-\frac{1}{z}\right)^{-\alpha_1-\alpha_2-\beta+i+1} H(m+1, -\alpha_1-\alpha_2-\beta+i, \beta; z).$$

By the changes of integral variables $\gamma x t_1 = -u_1, z u_1 s_2 = -v_2$, we get

$$I_j^{*n} = \left(-\frac{1}{\gamma x}\right)^{j+2} \left(-\frac{1}{z}\right)^{-\alpha_1-\alpha_2-\beta+n+1} \int_0^{+\infty} du_1 \int_0^{-zu_1} dv_2$$
$$\times u_1^{\alpha_1+\alpha_2+\beta-n+j} \left(1+\frac{u_1}{\gamma x}\right)^{-\alpha_2-\beta} e^{-u_1} v_2^{-\alpha_1-\alpha_2-\beta+n} e^{-v_2}.$$

Here we note

$$\int_0^{-zu_1} v_2^{-\alpha_1-\alpha_2-\beta+n} e^{-v_2} = \gamma(-\alpha_1-\alpha_2-\beta+n+1, -zu_1),$$

where $\gamma(a, x)$ is the incomplete gamma function of the first kind:

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t}\, dt.$$

The asymptotic behavior of the incomplete gamma function is known. The difference of $\Gamma(a)$ from $\gamma(a, x)$ is the incomplete gamma function $\Gamma(a, x)$ of the second kind, and its asymptotic behavior is given by

$$\Gamma(x,a) = \Gamma(a) - \gamma(a,x)$$
$$= x^{a-1} e^{-x} \left[1 + \sum_{n=1}^{\infty} (a-1)(a-2)\cdots(a-n) x^{-n}\right].$$

Therefore, in evaluating asymptotic behaviors, we can replace $\gamma(a, x)$ by $\Gamma(a)$. Thus we obtain

$$I_j^{*n} = \left(-\frac{1}{\gamma x}\right)^{j+2} \left(-\frac{1}{z}\right)^{-\alpha_1-\alpha_2-\beta+n+1}$$
$$\times \left[\int_0^{+\infty} u_1^{\alpha_1+\alpha_2+\beta-n+j} \left(1+\frac{u_1}{\gamma x}\right)^{-\alpha_2-\beta} e^{-u_1}\, du_1 \right.$$
$$\left. \times \Gamma(-\alpha_1-\alpha_2-\beta+n+1) + o(e^{-\delta|z|})\right]$$
$$= \left(-\frac{1}{\gamma x}\right)^{j+2} \left(-\frac{1}{z}\right)^{-\alpha_1-\alpha_2-\beta+n+1}$$
$$\times \left[G(\alpha_1+\alpha_2+\beta-n+j, -\alpha_2-\beta; \gamma x)\Gamma(-\alpha_1-\alpha_2-\beta+n+1) + o(e^{-\delta|z|})\right]$$

with some positive constant δ.

The evaluation of I^{mn} is similar. We get

$$I^{mn} = \left(-\frac{1}{\gamma x}\right)^{m+2}\left(-\frac{1}{z}\right)^{-\alpha_1-\alpha_2-\beta+n+1} \times \left[\Gamma(\alpha_1+\alpha_2+\beta-n+m+1)\Gamma(-\alpha_1-\alpha_2-\beta+n+1)+o(e^{-\delta|z|})\right].$$

In a similar way, we can evaluate $\tilde{R}_{MN}$. Summing up the results, we obtain the following asymptotic behavior. We set

$$\Phi_{00} = \left(-\frac{1}{\gamma x}\right)^{2}\left(-\frac{1}{z}\right)^{-\alpha_1-\alpha_2-\beta+1}.$$

Then we have

$$\begin{aligned}\frac{\Delta_{00}}{\Phi_{00}} &= \sum_{m=0}^{M-1}\sum_{i=0}^{m}\binom{-\alpha_2-\beta}{m-i}\binom{\alpha_2}{i}H(m+1,-\alpha_1-\alpha_2-\beta+i,\beta;z)(-z)^{-i}(-\gamma x)^{-m}\\ &+(-1)^{-\alpha_1-\alpha_2-\beta+1}\sum_{n=0}^{N-1}\Gamma(-\alpha_1-\alpha_2-\beta+n+1)\\ &\quad\times\sum_{j=0}^{n}(-1)^j\binom{\alpha_2}{j}\binom{\beta}{n-j}G(\alpha_1+\alpha_2+\beta-n+j,-\alpha_2-\beta;\gamma x)(-\gamma x)^{-j}(-z)^{-n}\\ &-\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\Gamma(\alpha_1+\alpha_2+\beta-n+m+1)\Gamma(-\alpha_1-\alpha_2-\beta+n+1)\\ &\quad\times\sum_{k=0}^{\min\{m,n\}}(-1)^k\binom{-\alpha_2-\beta}{m-k}\binom{\alpha_2}{k}\binom{\beta}{n-k}(-\gamma x)^{-m}(-z)^{-n}\\ &+(-\gamma x)^{-M}(-z)^{-N}R^{00}_{MN},\end{aligned} \tag{4.8}$$

where R^{00}_{MN} is bounded.

4.4. ASYMPTOTIC BEHAVIORS OF Δ_{10} AND Δ_{11}

In a similar argument, we can evaluate the integrals Δ_{10} and Δ_{11}.

For the integral Δ_{10}, we set

$$\Phi_{10} = -e^{\gamma x}\left(\frac{1}{\gamma x}\right)^{-\alpha_2-\beta+1}\left(\frac{1}{\gamma y}\right)^{-\alpha_1-\alpha_2-\beta+1}.$$

Then we have

$$\begin{aligned}\frac{\Delta_{10}}{\Phi_{10}} &= \sum_{m=0}^{M-1} \Gamma(-\alpha_2 - \beta + m + 1) \\ &\quad \times \sum_{i=0}^{m} \binom{\alpha_1 + \alpha_2}{m - i} \binom{\beta}{i} G(-\alpha_1 - \alpha_2 - \beta, \alpha_2 + \beta - j; -\gamma y)(-\gamma x)^{-m} \\ &\quad + \sum_{n=0}^{N-1} \Gamma(-\alpha_1 - \alpha_2 - \beta + n + 1) \\ &\quad \times \sum_{j=0}^{n} \binom{\alpha_2}{n - j} \binom{\beta}{j} G(-\alpha_2 - \beta, \alpha_1 + \alpha_2 + \beta - j; -\gamma x)(-\gamma y)^{-n} \\ &\quad - \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \Gamma(-\alpha_2 - \beta + m + 1)\Gamma(-\alpha_1 - \alpha_2 - \beta + n + 1)(-\gamma x)^{-m}(-\gamma y)^{-n} \\ &\quad + (-\gamma x)^{-M}(-\gamma y)^{-N} R_{MN}^{10},\end{aligned} \tag{4.9}$$

where R_{MN}^{10} is bounded.

For the integral Δ_{11}, we set

$$\Phi_{11} = (-1)^{\beta} e^{\gamma x} e^{-\gamma y} \left(\frac{1}{\gamma x}\right)^2 z^{-\alpha_2 - 1}.$$

Then we have

$$\begin{aligned}\frac{\Delta_{11}}{\Phi_{11}} &= \sum_{m=0}^{M-1} (-1)^m \sum_{i=0}^{m} \binom{\alpha_1 + \alpha_2}{m - i} \binom{-\alpha_1 - \alpha_2 - \beta}{i} H(m + 1, \alpha_2 + i, \beta; -z) z^{-i} (\gamma x)^{-m} \\ &\quad + \sum_{n=0}^{N-1} (-1)^n \Gamma(\alpha_2 + n_1) \\ &\quad \times \sum_{j=0}^{n} \binom{-\alpha_1 - \alpha_2 - \beta}{j} \binom{\beta}{n - j} G(-\alpha_2 - n + j, \alpha_1 + \alpha_2; -\gamma x)(\gamma x)^{-j}(-z)^{-n} \\ &\quad - \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \Gamma(-\alpha_2 + m - n + 1)\Gamma(\alpha_2 + n + 1) \\ &\quad \times \sum_{k=0}^{\min\{m,n\}} (-1)^{m^k} \binom{\alpha_1 + \alpha_2}{m - k} \binom{\beta}{n - k} \binom{-\alpha_1 - \alpha_2 - \beta}{k} (\gamma x)^{-m}(-z)^{-n} \\ &= (\gamma x)^{-M} (-z)^{N} R_{MN}^{11},\end{aligned} \tag{4.10}$$

where R_{MN}^{11} is bounded.

4.5. RESULT OF THE LOCAL ANALYSIS IN S

To summarize the above analysis, we have the following result.

Proposition 4.2. *For each $i, j = 0, 1$, the function Δ_{ij}/Φ_{ij} is asymptotically developable on S. The explicit evaluations are given by* (4.7), (4.8), (4.9) *and* (4.10). *The integrals $\Delta_{01}, \Delta_{00}, \Delta_{10}, \Delta_{11}$ correspond to the asymptotic behaviors*

$$e^{-\gamma y} x^{-\alpha_1-\alpha_2} y^{-\alpha_2}, \quad (y/x)^{\alpha_1+\alpha_2+\beta}, \quad e^{\gamma x} x^{\alpha_2+\beta} y^{\alpha_1+\alpha_2+\beta}, \quad e^{\gamma x} e^{-\gamma y} (y/x)^{-\alpha_2},$$

respectively, obtained in Proposition 3.2.

Remark 4.3. According to the definition in [3, Definition 3], the functions Δ_{ij}/Φ_{ij} are $(1, 1)$-Gevrey strongly asymptotically developable in S.

4.6. ASYMPTOTIC BEHAVIORS IN THE SECTORIAL DOMAIN S'

Next we consider the asymptotic behaviors in the sectorial domain S' which is defined by (4.3). We fix $(x, y) \in S'$. Then we can take θ' satisfying

$$\begin{aligned} \frac{\pi}{2} &< \arg\gamma + \arg x + \theta' < \frac{3}{2}, \\ -\frac{\pi}{2} &< \arg\gamma + \arg y + \theta' < \frac{\pi}{2}, \\ -\pi &< \theta' < 0. \end{aligned} \tag{4.11}$$

We define 2-chains Δ'_{ij} $(i, j = 0, 1)$ by

$$\begin{aligned} \Delta'_{01} &= (t_1 : 0 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 1 \to \infty e^{\sqrt{-1}\theta'}), \\ \Delta'_{00} &= (t_1 : 0 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 0 \to t_1), \\ \Delta'_{10} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 0 \to \infty e^{\sqrt{-1}\theta'}), \\ \Delta'_{11} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 1 \to t_1) \end{aligned}$$

with the branches

$$\Delta'_{01} : \begin{cases} \arg t_1 = \theta', \\ \arg(1 - t_1) = 0 & \text{at } t_1 = 0, \\ \arg t_2 = 0 & \text{at } t_2 = 1, \\ \arg(1 - t_2) = \theta' + \pi, \\ \arg(t_1 - t_2) = \pi & \text{at } (t_1, t_2) = (0, 1), \end{cases}$$

$$\Delta'_{00} : \begin{cases} \arg t_1 = \theta', \\ \arg(1 - t_1) = 0 & \text{at } t_1 = 0, \\ \arg t_2 = \theta', \\ \arg(1 - t_2) = 0 & \text{at } t_2 = 0, \\ \arg(t_1 - t_2) = \theta', \end{cases}$$

$$\Delta'_{10} : \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1, \\ \arg(1 - t_1) = \theta' + \pi, \\ \arg t_2 = \theta', \\ \arg(1 - t_2) = 0 & \text{at } t_2 = 0, \\ \arg(t_1 - t_2) = 0 & \text{at } (t_1, t_2) = (1, 0), \end{cases}$$

$$\Delta'_{11} : \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1, \\ \arg(1 - t_1) = \theta' + \pi, \\ \arg t_2 = 0 & \text{at } t_2 = 1, \\ \arg(1 - t_2) = \theta' + \pi, \\ \arg(t_1 - t_2) = \theta'. \end{cases}$$

It is shown in a similar manner that each Δ'_{ij} has the same asymptotic behavior as Δ_{ij}. Namely the following holds.

Proposition 4.4. *For each $i, j = 0, 1$, the function Δ'_{ij}/Φ_{ij} is asymptotically developable on S'. The explicit evaluations are the same as those for Δ_{ij}. The integrals $\Delta'_{01}, \Delta'_{00}, \Delta'_{10}, \Delta'_{11}$ correspond to the asymptotic behaviors*

$$e^{-\gamma y} x^{-\alpha_1 - \alpha_2} y^{-\alpha_2},\ (y/x)^{\alpha_1 + \alpha_2 + \beta},\ e^{\gamma x} x^{\alpha_2 + \beta} y^{\alpha_1 + \alpha_2 + \beta},\ e^{\gamma x} e^{-\gamma y} (y/x)^{-\alpha_2},$$

respectively, obtained in Proposition 3.2.

5. GLOBAL ANALYSIS AND STOKES PHENOMENON

Lastly in this section, we study the Stokes phenomenon. Since we have obtained the fundamental systems of solutions $\{\Delta_{ij}\}$ and $\{\Delta'_{ij}\}$ which have the same asymptotic behaviors in S and S', respectively, the linear relations among these two fundamental systems give the Stokes multipliers. We shall obtain the linear relations by deforming the twisted cycles.

We first take (x, y) in S, and then move it continuously to S'. We denote this continuous movement of (x, y) by σ. Then we should move θ continuously to θ', which makes the deformation of the cycles Δ_{ij}. We denote the results of the deformation by $\sigma_* \Delta_{ij}$.

First we consider the analytic continuation $\sigma_* \Delta_{01}$. Since the direction θ to ∞ is decreased to θ', 1-chains $(t_1 : 0 \to \infty e^{\sqrt{-1}\theta})$ and $(t_2 : 1 \to \infty e^{\sqrt{-1}\theta})$ rotate. The 1-chain $(t_2 : 1 \to \infty e^{\sqrt{-1}\theta})$ simply moves to $(t_2 : 1 \to \infty e^{\sqrt{-1}\theta'}) =: T_2$, while the 1-chain $(t_1 : 0 \to \infty e^{\sqrt{-1}\theta})$ should avoid the point $t_1 = 1$, and then takes the form as in Figure 4. Then we deform the latter 1-chain as in Figure 5, and divide it into three 1-chains

$$\begin{aligned} A &= (t_1 : 0 \to \infty e^{\sqrt{-1}\theta'}), \\ B &= (t_1 : \infty e^{\sqrt{-1}\theta'} \to 1), \\ C &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \end{aligned}$$

as described also in Figure 5.

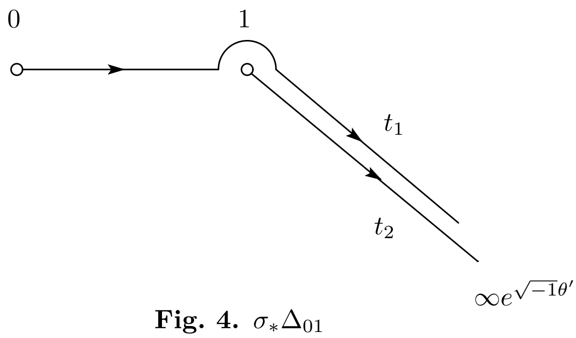

Fig. 4. $\sigma_*\Delta_{01}$

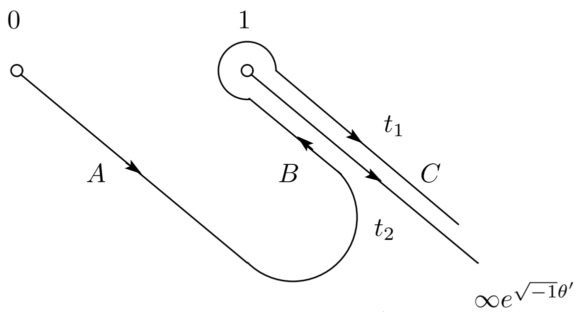

Fig. 5. Deformation of $\sigma_*\Delta_{01}$

Thus, at this moment, we have

$$\sigma_*\Delta_{01} = A \times T_2 + B \times T_2 + C \times T_2.$$

On the chains $B \times T_2$ and $C \times T_2$, there is the point $t_1 = t_2$. Then we divide B and C into two parts as

$$B = (t_1 : \infty e^{\sqrt{-1}\theta'} \to t_2) + (t_1 : t_2 \to 1),$$
$$C = (t_1 : 1 \to t_2) + (t_1 : t_2 \to \infty e^{\sqrt{-1}\theta'}).$$

Therefore we obtain

$$\sigma_*\Delta_{01} = \mathrm{I}_{01} + \mathrm{II}_{01} + \mathrm{III}_{01} + \mathrm{IV}_{01} + \mathrm{V}_{01},$$

where

$$\mathrm{I}_{01} = (t_2 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_1 : 0 \to \infty e^{\sqrt{-1}\theta'}),$$
$$\mathrm{II}_{01} = (t_2 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_1 : \infty e^{\sqrt{-1}\theta'} \to t_2),$$
$$\mathrm{III}_{01} = (t_2 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_1 : t_2 \to 1),$$
$$\mathrm{IV}_{01} = (t_2 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_1 : 1 \to t_2),$$
$$\mathrm{V}_{01} = (t_2 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_1 : t_2 \to \infty e^{\sqrt{-1}\theta'}).$$

We can get the branch on each component by the continuation from the branch on Δ_{01}. The results are the following:

$$\mathrm{I}_{01}:\begin{cases}\arg t_1=\theta', & \\ \arg(1-t_1)=0 & \text{at } t_1=0,\\ \arg t_2=0 & \text{at } t_2=1,\\ \arg(1-t_2)=\theta'+\pi, & \\ \arg(t_1-t_2)=\pi & \text{at } (t_1,t_2)=(0,1),\end{cases}$$

$$\mathrm{II}_{01}:\begin{cases}\arg t_1=0 & \text{at } t_1=1,\\ \arg(1-t_1)=\theta'+\pi, & \\ \arg t_2=0 & \text{at } t_2=1,\\ \arg(1-t_2)=\theta'+\pi, & \\ \arg(t_1-t_2)=\theta'+2\pi, & \end{cases}$$

$$\mathrm{III}_{01}:\begin{cases}\arg t_1=0 & \text{at } t_1=1,\\ \arg(1-t_1)=\theta'+\pi, & \\ \arg t_2=0 & \text{at } t_2=1,\\ \arg(1-t_2)=\theta'+\pi, & \\ \arg(t_1-t_2)=\theta'+\pi, & \end{cases}$$

$$\mathrm{IV}_{01}:\begin{cases}\arg t_1=0 & \text{at } t_1=1,\\ \arg(1-t_1)=\theta'-\pi, & \\ \arg t_2=0 & \text{at } t_2=1,\\ \arg(1-t_2)=\theta'+\pi, & \\ \arg(t_1-t_2)=\theta'+\pi, & \end{cases}$$

$$\mathrm{V}_{01}:\begin{cases}\arg t_1=0 & \text{at } t_1=1,\\ \arg(1-t_1)=\theta'-\pi, & \\ \arg t_2=0 & \text{at } t_2=1,\\ \arg(1-t_2)=\theta'+\pi, & \\ \arg(t_1-t_2)=\theta'. & \end{cases}$$

In a similar way, we can describe the analytic continuations $\sigma_*\Delta_{00}, \sigma_*\Delta_{10}$ and $\sigma_*\Delta_{11}$. For $\sigma_*\Delta_{00}$, we deform $\sigma_*\Delta_{00}$ as in Figure 6, and divide the result into seven chains:

$$\sigma_*\Delta_{00}=\mathrm{I}_{00}+\mathrm{II}_{00}+\mathrm{III}_{00}+\mathrm{IV}_{00}+\mathrm{V}_{00}+\mathrm{VI}_{00}+\mathrm{VII}_{00},$$

where

$$\begin{aligned}
\mathrm{I}_{00} &= (t_1 : 0 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 0 \to t_1),\\
\mathrm{II}_{00} &= (t_1 : \infty e^{\sqrt{-1}\theta'} \to 1) \times (t_2 : 0 \to \infty e^{\sqrt{-1}\theta'}),\\
\mathrm{III}_{00} &= (t_1 : \infty e^{\sqrt{-1}\theta'} \to 1) \times (t_2 : \infty e^{\sqrt{-1}\theta'} \to t_1),\\
\mathrm{IV}_{00} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 0 \to \infty e^{\sqrt{-1}\theta'}),\\
\mathrm{V}_{00} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : \infty e^{\sqrt{-1}\theta'} \to t_1),\\
\mathrm{VI}_{00} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : t_1 \to 1),\\
\mathrm{VII}_{00} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 1 \to t_1).
\end{aligned}$$

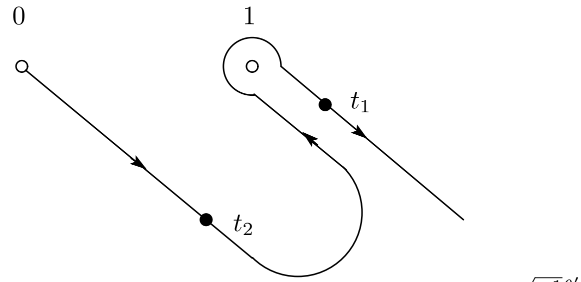

Fig. 6. Deformation of $\sigma_* \Delta_{00}$

The branches of these chains are given by

$$\mathrm{I}_{00} : \begin{cases} \arg t_1 = \theta', & \\ \arg(1-t_1) = 0 & \text{at } t_1 = 0, \\ \arg t_2 = \theta', & \\ \arg(1-t_2) = 0 & \text{at } t_2 = 0, \\ \arg(t_1 - t_2) = \theta', & \end{cases}$$

$$\mathrm{II}_{00} : \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1, \\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = \theta', & \\ \arg(1-t_2) = 0 & \text{at } t_2 = 0, \\ \arg(t_1 - t_2) = 0 & \text{at } (t_1, t_2) = (1, 0), \end{cases}$$

$$\mathrm{III}_{00} : \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1, \\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1, \\ \arg(1-t_2) = \theta' + \pi, & \\ \arg(t_1 - t_2) = \theta' + \pi, & \end{cases}$$

$$\mathrm{IV}_{00}:\begin{cases}\arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta' - \pi, & \\ \arg t_2 = \theta', & \\ \arg(1-t_2) = 0 & \text{at } t_2 = 0,\\ \arg(t_1 - t_2) = 0 & \text{at } (t_1,t_2) = (1,0),\end{cases}$$

$$\mathrm{V}_{00}:\begin{cases}\arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta' - \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1,\\ \arg(1-t_2) = \theta' + \pi, & \\ \arg(t_1 - t_2) = \theta' + \pi, & \end{cases}$$

$$\mathrm{VI}_{00}:\begin{cases}\arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta' - \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1,\\ \arg(1-t_2) = \theta' + \pi, & \\ \arg(t_1 - t_2) = \theta', & \end{cases}$$

$$\mathrm{VII}_{00}:\begin{cases}\arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta' - \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1,\\ \arg(1-t_2) = \theta' - \pi, & \\ \arg(t_1 - t_2) = \theta'. & \end{cases}$$

These chains are illustrated in Figure 7.

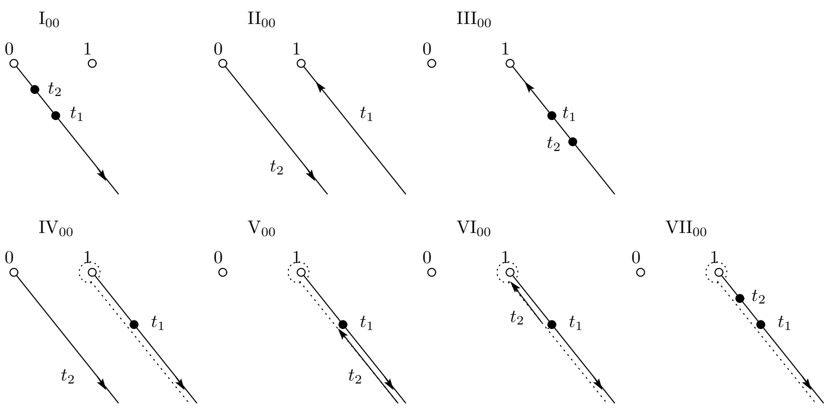

Fig. 7. 2-chains I_{00}–VII_{00}

The analytic continuation $\sigma_*\Delta_{10}$ is very similar to $\sigma_*\Delta_{01}$. Then we omit the figures, and give only the result:

$$\sigma_*\Delta_{10} = \mathrm{I}_{10} + \mathrm{II}_{10} + \mathrm{III}_{10} + \mathrm{IV}_{10} + \mathrm{V}_{10},$$

where

$$\begin{aligned}
\mathrm{I}_{10} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 0 \to \infty e^{\sqrt{-1}\theta'}),\\
\mathrm{II}_{10} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : \infty e^{\sqrt{-1}\theta'} \to t_1),\\
\mathrm{III}_{10} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : t_1 \to 1),\\
\mathrm{IV}_{10} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 1 \to t_1),\\
\mathrm{V}_{10} &= (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : t_1 \to \infty e^{\sqrt{-1}\theta'})
\end{aligned}$$

with the branches given by

$$\mathrm{I}_{10} : \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = \theta', & \\ \arg(1-t_2) = 0 & \text{at } t_2 = 0,\\ \arg(t_1 - t_2) = 0 & \text{at } (t_1, t_2) = (1, 0), \end{cases}$$

$$\mathrm{II}_{10} : \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1,\\ \arg(1-t_2) = \theta' + \pi, & \\ \arg(t_1 - t_2) = \theta' + \pi, & \end{cases}$$

$$\mathrm{III}_{10} : \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1,\\ \arg(1-t_2) = \theta' + \pi, & \\ \arg(t_1 - t_2) = \theta', & \end{cases}$$

$$\mathrm{IV}_{10} : \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1,\\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1,\\ \arg(1-t_2) = \theta' - \pi, & \\ \arg(t_1 - t_2) = \theta', & \end{cases}$$

$$\mathrm{V}_{10}: \begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1, \\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1, \\ \arg(1-t_2) = \theta' - \pi, & \\ \arg(t_1 - t_2) = \theta' - \pi. & \end{cases}$$

The analytic continuation $\sigma_* \Delta_{11}$ is simple. We have only to rotate the direction from θ to θ'. Thus the result is given by

$$\sigma_* \Delta_{11} = (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : 1 \to t_1)$$

with the branch given by

$$\begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1, \\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1, \\ \arg(1-t_2) = \theta' + \pi, & \\ \arg(t_1 - t_2) = \theta'. & \end{cases}$$

The results $\sigma_* \Delta_{ij}$ of the analytic continuation will be described by the cycles Δ'_{ij} defined in Section 4.6, however at this moment, we need one extra cycle $\tilde{\Delta}'$ defined by

$$\tilde{\Delta}' = (t_1 : 1 \to \infty e^{\sqrt{-1}\theta'}) \times (t_2 : t_1 \to \infty e^{\sqrt{-1}\theta'})$$

with

$$\begin{cases} \arg t_1 = 0 & \text{at } t_1 = 1, \\ \arg(1-t_1) = \theta' + \pi, & \\ \arg t_2 = 0 & \text{at } t_2 = 1, \\ \arg(1-t_2) = \theta' + \pi, & \\ \arg(t_1 - t_2) = \theta' + \pi. & \end{cases}$$

By using this together with Δ'_{ij}, we can describe the result. We set

$$e_1 = e^{2\pi\sqrt{-1}(\alpha_2+\beta)}, \ e_2 = e^{-2\pi\sqrt{-1}\alpha_2}.$$

Then we have

$$\begin{aligned} \sigma_* \Delta_{01} &= \Delta'_{01} + e_1(1-e_2)\Delta'_{11} + (e_1 - 1)\tilde{\Delta}', \\ \sigma_* \Delta_{00} &= \Delta'_{00} + (e_1 - 1)\Delta'_{10} + (1-e_1)\tilde{\Delta}' + e_1(e_2 - 1)\Delta'_{11}, \\ \sigma_* \Delta_{10} &= \Delta'_{10} + (e_1^{-1} - 1)\tilde{\Delta}' + (e_2 - 1)\Delta'_{11}, \\ \sigma_* \Delta_{11} &= \Delta'_{11}. \end{aligned} \tag{5.1}$$

On the other hand, we can show the following.

Proposition 5.1. *We have*

$$\tilde{\Delta}' = -e^{2\pi\sqrt{-1}\beta}\Delta'_{11}.$$

Proof. We construct a twisted 3-chain in the complex (t_1, t_2)-space. First we construct a 2-chain in the complex t_1-plane. Let L be the half line in the t_1-plane from 1 to $\infty e^{\sqrt{-1}\theta'}$. Draw a simple loop C with base point $\infty e^{\sqrt{-1}\theta'}$ surrounding L in the positive direction. Take any $0 < \delta \leq 1$. For each $s \in C$, let $\ell_s \in L$ be the point on L such that the distance $d(s, L)$ is equal to the distance $d(s, \ell_s)$. Let s_δ be the point on the segment $\overline{s\ell_s}$ such that $d(s_\delta, \ell_s) = \delta d(s, \ell_s)$. Then the points s_δ $(s \in C)$ make a loop C_δ surrounding L and located inside C. Then the union

$$\bigcup_{0<\delta\leq 1} C_\delta$$

becomes a locally finite 2-chain. We illustrate this 2-chain in Figure 8.

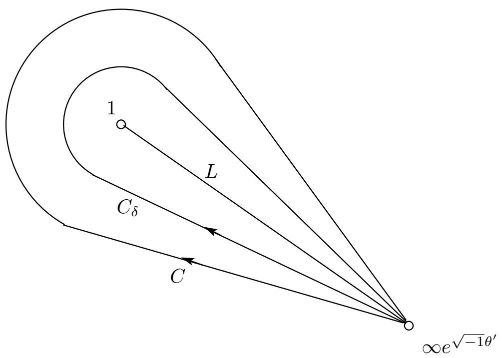

Fig. 8. L, C, C_δ in t_1-plane

Next, for each $t_1 \in C_\delta$, we draw a path B_{t_1} from $\infty e^{\sqrt{-1}\theta'}$ to 1 in the complex t_2-plane in the following way. We copy L and C_δ to the t_2-plane, and denote them by L' and C'_δ, respectively. We regard as $t_1 \in C'_\delta$, and let C'_{δ,t_1} be the part of C'_δ from $\infty e^{\sqrt{-1}\theta'}$ to t_1. Deform L' in a way to avoid C'_{δ,t_1} so that C'_{δ,t_1} is contained in the domain surrounded by L and the result of the deformation. We denote by B_{t_1} the result of the deformation. The path B_{t_1} is illustrated in Figure 9.

In this way, we obtain the 3-chain

$$A = \left\{ (t_1, t_2) \middle| \, t_1 \in \bigcup_{0<\delta\leq 1} C_\delta, t_2 \in B_{t_1} \right\}$$

in (t_1, t_2)-space. The 3-chain A becomes a twisted 3-chain by the following reason. Take $t'_1, t''_1 \in C_\delta$ close each other on the opposite sides of L such that t'_1 is on the part of C_δ before encircling 1. Also take $t'_2 \in B_{t'_1}$ (resp. $t''_2 \in B_{t''_1}$) closer to 1 than the copy of t'_1 (resp. t''_1) in the t_2-plane.

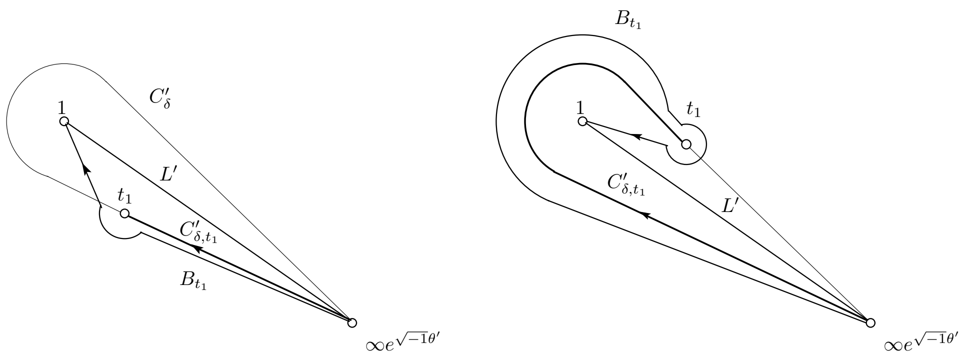

Fig. 9. B_{t_1} in t_2-plane

We assume that t_2' and t_2'' are close each other. See Figure 10. Then it follows

$$\begin{aligned}\arg(1-t_1'') &\approx \arg(1-t_1') - 2\pi,\\ \arg(1-t_2'') &\approx \arg(1-t_2') - 2\pi,\\ \arg(t_1''-t_2'') &\approx \arg(t_1'-t_2') - 2\pi.\end{aligned}$$

Therefore, thank to the resonance of the exponents, the branches of

$$(1-t_1)^{-\alpha_2-\beta}(1-t_2)^{\alpha_2}(t_1-t_2)^{\beta}$$

at (t_1', t_2') and (t_1'', t_2'') coincide. This implies that the branches of the both sides of L coincide, which means that the 3-chain A becomes a twisted 3-chain. Moreover, by assuming $\Re(\beta) > 0$, we may fill up the gap $\{(t_1, t_2) \in A \mid t_1 \in L, t_2 \in L'\}$ in A.

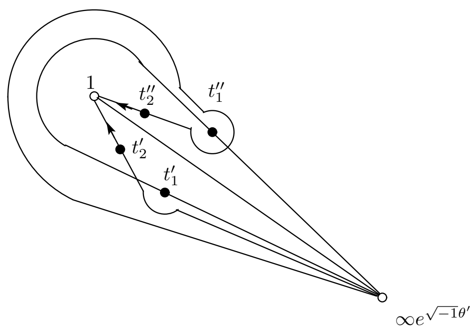

Fig. 10. t_1', t_2', t_1'', t_2''

Hence the boundary ∂A as a twisted chain is homologous to 0. Note that

$$\partial A = \{(t_1, t_2) \in A \mid t_1 \in C\}.$$

We illustrate ∂A in Figure 11; Figure 11(a) shows the state where t_1 is on the part of C before encircling 1, and Figure 11(b) shows the state after encircling 1. We define twisted 2-chains $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ as in Figure 11. Topologically, the chains Π_3 and Π_4 coincide with $\pm\Pi_2$. Then we look at the branches on these chains. On Π_3, $\arg(1-t_1)$ and $\arg(1-t_2)$ are decreased by 2π from those on Π_2, and then the branch on Π_3 is obtained from that on Π_2 by multiplying

$$e^{-2\pi\sqrt{-1}(-\alpha_2-\beta)}e^{-2\pi\sqrt{-1}\alpha_2} = e^{2\pi\sqrt{-1}\beta}.$$

The orientations of t_1 and t_2 are both inverse, and then the signature of Π_3 coincides with Π_2. Therefore we obtain

$$\Pi_3 = e^{2\pi\sqrt{-1}\beta}\Pi_2.$$

On Π_4, the arguments of $(1-t_1)$ and $(1-t_2)$ are the same as on Π_3, and $\arg(t_1-t_2)$ is decreased by 2π. Then, taking the orientation into account, we have

$$\Pi_4 = -e^{-2\pi\sqrt{-1}\beta}e^{2\pi\sqrt{-1}\beta}\Pi_2 = -\Pi_2.$$

Therefore we obtain

$$\partial A = \Pi_1 + \Pi_2 + e^{2\pi\sqrt{-1}\beta}\Pi_2 - \Pi_2 = \Pi_1 + e^{2\pi\sqrt{-1}\beta}\Pi_2.$$

Since ∂A is homologous to 0, we get

$$\Pi_1 = -e^{2\pi\sqrt{-1}\beta}\Pi_2.$$

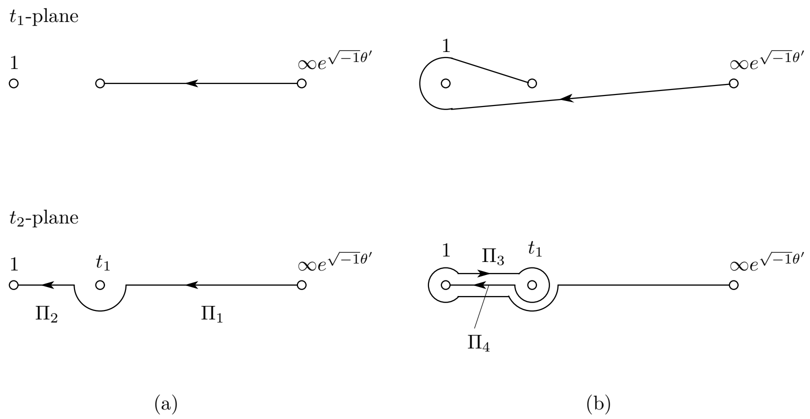

Fig. 11. 2-chains Π_1–Π_4

Comparing with the definitions of Δ'_{11} and $\tilde{\Delta}'$, we have

$$\Delta'_{11} = \Pi_2, \ \tilde{\Delta}' = \Pi_1.$$

Thus we arrive at the assertion of the proposition. □

Applying Proposition 5.1 to (5.1), we get the final result.

Theorem 5.2. *The analytic continuations* $\sigma_*\Delta_{ij}$ $(i,j=0,1)$ *are written in terms of* Δ'_{ij} $(i,j=0,1)$ *as*

$$\begin{aligned}
\sigma_*\Delta_{01} &= \Delta'_{01} + e^{2\pi\sqrt{-1}(\alpha_2+\beta)}(1-e^{2\pi\sqrt{-1}\beta})\Delta'_{11},\\
\sigma_*\Delta_{00} &= \Delta'_{00} + (e^{2\pi\sqrt{-1}(\alpha_2+\beta)}-1)\Delta'_{10} + e^{2\pi\sqrt{-1}(\alpha_2+\beta)}(e^{2\pi\sqrt{-1}\beta}-1)\Delta'_{11},\\
\sigma_*\Delta_{10} &= \Delta'_{10} + (e^{2\pi\sqrt{-1}\beta}-1)\Delta'_{11},\\
\sigma_*\Delta_{11} &= \Delta'_{11}.
\end{aligned} \tag{5.2}$$

The coefficients in the right hand side of (5.2) that are different from 1 are the Stokes multipliers. The existence of the Stokes multipliers tells that the formal solutions of (2.6) actually diverge, which is not derived from Gérard–Sibuya theorem [2, Theorem 2.1].

It is worth noting that the asymptotic behaviors are described for solutions in sectorial domains S and S', which are not polysectors. Gérard–Sibuya [2], Majima [9] and several other people (including the author [3]) considered asymptotic analysis on a *polysector*. However, even in our simple case, we should take not a polysector but a sectorial domain. This will bring a new issue to the asymptotic analysis in several variables.

As mentioned in Introduction, existence of resonances makes the global analysis difficult in general. Actually in the above argument for our Pfaffian system, we come to a difficulty in describing the result of analytic continuations of Δ_{ij}; we have to introduce an extra cycle $\tilde{\Delta}'$. Usually in non-resonant case, relations among twisted cycles can be obtained from Cauchy's theorem for 1-chains ([1]). This is the method *from below*. We used this method in deforming $\sigma_*\Delta_{01}$ as in Figures 5.2 and 5.3. However, in our resonant case, Cauchy's theorem does not supply a sufficient number of relations. Then we use the idea to make a 3-chain and to derive a relation among 2-chains as the boundary of the 3-chain. This is the method *from above*. This method is possible in our case owing to the resonances. The method from above may become a useful tool in the global analysis in general resonant case.

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