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Dedicated to Professor Jan Stochel
on the occasion of his 70th birthday

SHIFTED MODEL SPACES AND THEIR ORTHOGONAL DECOMPOSITIONS

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Abstract. We generalize certain well known orthogonal decompositions of model spaces and obtain similar decompositions for the wider class of shifted model spaces, allowing us to establish conditions for near invariance of the latter with respect to certain operators which include, as a particular case, the backward shift S^*. In doing so, we illustrate the usefulness of obtaining appropriate decompositions and, in connection with this, we prove some results on model spaces which are of independent interest. We show moreover how the invariance properties of the kernel of an operator T, with respect to another operator, follow from certain commutation relations between the two operators involved.

Keywords: model space, Toeplitz operator, Toeplitz kernel, truncated Toeplitz operator, nearly invariant, shift invariant.

Mathematics Subject Classification: 47B32, 47B35, 30H10.

1. INTRODUCTION

Let θ be an inner function, i.e., $\theta \in H^\infty(\mathbb{D})$ with $|\theta| = 1$ a.e. on $\mathbb{T}$, and let H^2_+ denote the Hardy space of the unit disk, $H^2_+ := H^2(\mathbb{D})$.

The model space K_θ associated with θ is defined by

$$K_\theta = H^2_+ \ominus \theta H^2_+. \tag{1.1}$$

Model spaces and operators defined on them have attracted enormous attention for their properties and applications (see for example [7]) and the references therein). It is well known that the class of model spaces coincides with the class of all proper invariant subspaces of H^2_+ for

$$S^* = P^+ \bar{z} P^+|_{H^2_+}, \tag{1.2}$$

where P^+ denotes the orthogonal projection from $L^2 := L^2(\mathbb{T})$ onto H^2_+. Another well known property of model spaces is that if α is an inner function dividing θ, which we denote by $\alpha \leqslant \theta$, meaning that $\frac{\theta}{\alpha} \in H^\infty := H^\infty(\mathbb{D})$, then K_θ admits two orthogonal decompositions:

$$K_\theta = K_\alpha \oplus \alpha K_{\frac{\theta}{\alpha}}, \qquad K_\theta = K_{\frac{\theta}{\alpha}} \oplus \frac{\theta}{\alpha} K_\alpha. \tag{1.3}$$

One can look at those orthogonal sums as describing the behaviour of different parts of K_θ, in particular certain invariance properties, with respect to multiplication by $\bar{\alpha}$ and α ($M_{\bar{\alpha}}$ and M_α, respectively). Indeed, regarding the first decomposition in (1.3), and denoting

$$H^2_- = \bar{z}\overline{H^2_+} = L^2 \ominus H^2_+, \tag{1.4}$$

we have that

$$\bar{\alpha} K_\alpha \cap K_\theta = \{0\}, \quad \text{with} \quad \bar{\alpha} K_\alpha \subset H^2_-, \quad \text{and} \quad \bar{\alpha}(\alpha K_{\frac{\theta}{\alpha}}) = K_{\frac{\theta}{\alpha}} \subset K_\theta \tag{1.5}$$

while, regarding the second decomposition in (1.3),

$$\alpha K_{\frac{\theta}{\alpha}} \subset K_\theta \quad \text{and} \quad \alpha(\tfrac{\theta}{\alpha} K_\alpha) = \theta K_\alpha \cap K_\theta = \{0\} \quad \text{with} \quad \alpha(\tfrac{\theta}{\alpha} K_\alpha) \subset \theta H^2_+. \tag{1.6}$$

So we see that one of the terms on the right hand side of each of the decompositions (1.3) “stays” in K_θ after multiplication by $\bar{\alpha}$ or α (depending on the decomposition), while the other terms are mapped into a space which is disjoint from K_θ, and either “goes” to H^2_- or to θH^2_+.

Taking this perspective enables us to generalize (1.3) and obtain similar decompositions when α does not divide θ, by looking at the model space K_θ as a Toeplitz kernel, $K_\theta = \ker T_{\bar{\theta}}$, where

$$T_g = P^+ g P^+|_{H^2}, \quad \text{for } g \in L^\infty, \tag{1.7}$$

(the symbol g will be always identified with M_g, the multiplication operator by g) and asking which elements of this kernel are mapped into the same space by $M_{\bar{\alpha}}$ or M_α.

Thus in Section 2 we generalize the decompositions (1.3) to include the case where α does not divide θ and we study their relation with the usual conjugation on the model space K_θ. In Section 3 we use some of those results to obtain orthogonal decompositions for the wider class of shifted model spaces and we establish conditions for near invariance of the latter with respect to certain operators which include, as a particular case, the backward shift S^*. In doing so, we illustrate the usefulness of obtaining appropriate decompositions and, in connection with this, we prove some results on model spaces which are of independent interest. In Section 4 we consider the particular case of shifted model spaces of the form zK_θ and apply the previous results to study the relations between K_θ and its images by the shift S and the backward shift S^*, to answer the question when is S^*K_θ exactly equal to K_θ and to describe the orthogonal projections from L^2 onto the kernels of a particular type of Toeplitz operators. Finally, in Section 5 we study how the invariance properties of the kernel of an operator T, with respect to another operator, follow from certain commutation relations between the two operators involved.

2. MODEL SPACES AND ORTHOGONAL DECOMPOSITIONS

We will use here the following notation. Let $\mathcal{H}$ be a Hilbert space, H and $\mathcal{M}$ be closed subspaces of $\mathcal{H}$ with $\mathcal{M} \subset H \subset \mathcal{H}$, and let $X \in \mathcal{B}(\mathcal{H})$. Then we define

$$\mathcal{M}_X = \{f \in \mathcal{M} : Xf \in \mathcal{M}\}. \tag{2.1}$$

We use the notation $[f] := \operatorname{span}\{f\}$.

Proposition 2.1. *Let θ and α be inner functions. Then*

$$(K_\theta)_\alpha = (\ker T_{\bar\theta})_\alpha = \ker T_{\bar\theta\alpha}, \quad (K_\theta)_{\bar\alpha} = (\ker T_{\bar\theta})_{\bar\alpha} = \alpha \ker T_{\bar\theta\alpha}. \tag{2.2}$$

Proof. Note that $f, \alpha f \in \ker T_{\bar\theta}$ if and only if $f \in H^2_+$, $\bar\theta f = f_- \in H^2_-$, $\bar\theta\alpha f = h_- \in H^2_-$. This is equivalent to $f \in H^2_+$, $\bar\theta\alpha f = h_- \in H^2_-$, i.e., $f \in \ker T_{\bar\theta\alpha}$.

Since $f \in (\ker T_{\bar\theta})_{\bar\alpha}$ if and only if $\bar\alpha f \in (\ker T_{\bar\theta})_\alpha$, it follows that $(\ker T_{\bar\theta})_{\bar\alpha} = \alpha \ker T_{\bar\theta\alpha}$. □

There is a relation between the spaces in the first and the second sets of equalities in (2.2), given by the usual conjugation C_θ on K_θ, defined by

$$C_\theta f = \theta \bar z \bar f. \tag{2.3}$$

Proposition 2.2. *Let θ and α be inner functions. Then*

$$C_\theta (K_\theta)_\alpha = (K_\theta)_{\bar\alpha} \tag{2.4}$$

Proof. For $f \in K_\theta$, we have that

$$f \in C_\theta (\ker T_{\bar\theta})_\alpha = C_\theta (\ker T_{\bar\theta\alpha})$$

if and only if $\bar\theta\alpha(\theta\bar z\bar f) = \alpha\bar z\bar f = f_- \in H^2_-$. So, if $f \in C_\theta(\ker T_{\bar\theta\alpha})$, then $f = \alpha\bar z\bar f_-$ with $\bar z\bar f_- \in H^2_+$ and $\bar\theta\alpha(\bar z\bar f_-) = \bar\theta\alpha(\bar\alpha f) = \bar\theta f \in H^2_-$. Therefore $\bar z\bar f_- \in \ker T_{\bar\theta\alpha}$ and $f \in \alpha \ker T_{\bar\theta\alpha}$. Conversely, if $f \in (\ker T_{\bar\theta})_{\bar\alpha}$, then $f \in K_\theta$ and $\bar\alpha f \in H^2_+$, so $\alpha\bar z\bar f \in \bar z\overline{H^2_+} = H^2_-$ which is equivalent to $f \in C_\theta(\ker T_{\bar\theta\alpha})$. □

Proposition 2.3. *Let α, θ be inner functions. Then*

$$K_\theta \ominus \alpha \ker T_{\bar\theta\alpha} = P_\theta K_\alpha.$$

Proof. It is clear that $P_\theta K_\alpha \subset K_\theta$ and $P_\theta K_\alpha \subset (\alpha \ker T_{\bar\theta\alpha})^\perp$ because, for any $f_\alpha \in K_\alpha$, $g \in \ker T_{\bar\theta\alpha}$,

$$\langle P_\theta f_\alpha, \alpha g\rangle = \langle f_\alpha, \alpha g\rangle = 0.$$

Conversely, suppose that $f \in K_\theta$ and $f \in (P_\theta K_\alpha)^\perp$. Then, for all $k_\alpha \in K_\alpha$, $0 = \langle f, P_\theta k_\alpha\rangle = \langle f, k_\alpha\rangle$, so $f \in K_\theta \cap \alpha H^2_+ = \alpha \ker T_{\bar\theta\alpha}$ by Lemma 2.4 below. □

Lemma 2.4. *Let α, θ be inner functions. Then*

$$\ker T_{\bar\theta\alpha} = \bar\alpha K_\theta \cap H^2_+ = \bar\alpha K_\theta \cap K_\theta.$$

Proof. Note that $f \in \ker T_{\bar{\theta}\alpha}$ if and only if $f \in H^2_+$, $\bar{\theta}\alpha f = f_- \in H^2_-$. In other words $\alpha f \in K_\theta$, $f \in H^2_+$ which is equivalent to $f \in \bar{\alpha} K_\theta \cap H^2_+ = \bar{\alpha} K_\theta \cap K_\theta$. □

We can now state the following generalization of (1.3).

Theorem 2.5. *Let α, θ be inner functions. Then*

$$K_\theta = P_\theta K_\alpha \oplus \alpha \ker T_{\bar{\theta}\alpha}, \quad K_\theta = P_\theta C_\theta K_\alpha \oplus \ker T_{\bar{\theta}\alpha}.$$

Proof. The first equality is an immediate consequence of Proposition 2.3, while the second equality follows from Proposition 2.1 and 2.2 and the properties of conjugations. □

Remark 2.6. The equalities in Theorem 2.5 can also be expressed in the form

$$K_\theta = P_\theta K_\alpha \oplus (K_\theta)_{\bar{\alpha}}, \quad K_\theta = P_\theta C_\theta K_\alpha \oplus (K_\theta)_\alpha, \tag{2.5}$$

with

$$(P_\theta K_\alpha)\backslash\{0\} \underset{M_{\bar{\alpha}}}{\longrightarrow} L^2\backslash H^2_+, \quad (P_\theta C_\theta K_\alpha)\backslash\{0\} \underset{M_\alpha}{\longrightarrow} H^2_+\backslash K_\theta. \tag{2.6}$$

Thus Theorem 2.5 generalizes (1.3) in the sense that it reduces to (1.3) when $\alpha \leqslant \theta$, but also in the sense that it describes analogously certain invariance properties under multiplication by α or $\bar{\alpha}$.

It may happen, in Theorem 2.5, that $\ker T_{\bar{\theta}\alpha} = \{0\}$, which means that no non-zero element of K_θ is mapped by M_α or $M_{\bar{\alpha}}$ into K_θ. The relations between $\ker T_g$ and $\ker T_{\alpha g}$, where $g \in L^\infty$ and α is an inner function, were studied in [4] where the following was proved.

Theorem 2.7 ([4, Theorem 6.2])**.** *If $g \in L^\infty$ and α is a finite Blaschke product (denoted $\alpha \in FBP$), then*

$$\dim \ker T_g < \infty \quad \textit{if and only if} \quad \dim \ker T_{\alpha g} < \infty \tag{2.7}$$

and, if $\dim \ker T_g < \infty$, *then for any inner α,*

$$\dim \ker T_{\alpha g} = \max\{0, \dim \ker T_g - \dim K_\alpha\}. \tag{2.8}$$

In particular, if $\theta \in FBP$, then

$$\ker T_{\bar{\theta}\alpha} = \{0\} \quad \textit{if and only if} \quad \dim K_\theta \leqslant \dim K_\alpha. \tag{2.9}$$

However, if neither θ nor α belong to FBP, it may be difficult to see whether or not we have $\ker T_{\bar{\theta}\alpha} = \{0\}$. From Theorem 2.5 we now obtain the following necessary and sufficient condition.

Proposition 2.8. *Let α, θ be inner functions. Then*

$$\ker T_{\bar{\theta}\alpha} = \{0\} \quad \textit{if and only if} \quad K_\theta = P_\theta K_\alpha. \tag{2.10}$$

As an example of application of Proposition 2.8, take $\theta = \exp(\frac{t-1}{t+1})$, $\alpha = \exp(\frac{t+1}{t-1})$. We have that $\ker T_{\bar{\theta}\alpha} = \{0\}$ ([4, Example 6.3]) so we conclude that

$$P_{\exp(\frac{t-1}{t+1})} K_{\exp(\frac{t+1}{t-1})} = K_{\exp(\frac{t-1}{t+1})}.$$

3. NEAR INVARIANCE PROPERTIES OF A SHIFTED MODEL SPACE

It is clear from Theorem 2.5 that multiplication by an inner function α and by its conjugate $\bar{\alpha}$, which we will call right and left generalized shift (or simply right and left shift), respectively, act differently on the subspaces of $\mathcal{M}$ which are not mapped into $\mathcal{M}$ by M_α or $M_{\bar{\alpha}}$.

In the case of the first equality in Theorem 2.5, from the first relation in (2.6) we see that K_θ is *nearly $\bar{\alpha}$-invariant* ([2]) for any inner function α, i.e.,

$$\text{for all } f \in K_\theta \text{ if } \bar{\alpha} f \in H^2_+, \text{ then } \bar{\alpha} f \in K_\theta, \tag{3.1}$$

so that no element of K_θ is mapped by a left generalized shift into $H^2_+ \backslash K_\theta$. On the other hand, regarding multiplication by α, we have that K_θ is *H^2_+-stable* ([5]) for right shifts, i.e., $\alpha K_\theta \subset H^2_+$, and from the second relation in (2.6) we see that, if $P_\theta C_\theta K_\alpha$ is finite dimensional (which happens in particular if $\alpha \in FBP$), then

$$\text{for all } f \in K_\theta,\ \alpha f \in K_\theta \oplus \mathcal{F}, \tag{3.2}$$

where $\mathcal{F} \subset H^2_+$ is a finite dimensional space of dimension m. If (3.2) holds, then K_θ is *almost-invariant* for $T_\alpha = P^+ \alpha P^+|_{H^2_+}$ *with defect* m ([1]), i.e., $T_\alpha K_\theta \subset K_\theta \oplus \mathcal{F}$.

More generally, we have the following definition.

Definition 3.1 ([5])**.** Let $\mathcal{M}$ and H be closed subspaces of a Hilbert space $\mathcal{H}$, with $\mathcal{M} \subset H \subset \mathcal{H}$, and let $X \in B(\mathcal{H})$. We say that $\mathcal{M}$ is *nearly X-invariant* with respect to (w.r.t.) H if and only if

$$f \in \mathcal{M},\ Xf \in H \Longrightarrow Xf \in \mathcal{M}; \tag{3.3}$$

$\mathcal{M}$ is *nearly X-invariant w.r.t. H with defect m* if and only if

$$f \in \mathcal{M},\ Xf \in H \Longrightarrow Xf \in \mathcal{M} \oplus \mathcal{F} \tag{3.4}$$

where $\mathcal{F} \subset H$ is finite dimensional with $\dim \mathcal{F} = m$, and we assume that m is the smallest possible dimension of such a space $\mathcal{F}$. We say that $\mathcal{M}$ is *H-stable* for X if $X(\mathcal{M}) \subset H$ and, in that case, $\mathcal{M}$ is *almost-invariant* for $P_H X|_H$ with defect m, if and only if $X\mathcal{M} \subset \mathcal{M} \oplus \mathcal{F}$ where $\mathcal{F}$ is finite dimensional with $\dim \mathcal{F} = m$. If $m = 0$, then M is an invariant subspace for $P_H X|_H$ and for X.

Remark 3.2. Note that if $Xf \in H$, then $Xf = P_H X f$. Thus, if (3.3) holds, we can also say that $\mathcal{M}$ is *nearly $P_H X|_H$–invariant.* For instance, if $X = \bar{z}$ (identifying $\bar{z}$ with $M_{\bar{z}}$) then, since $\bar{z} f \in H^2_+$ is equivalent to having $f(0) = 0$, (3.3) is equivalent to

$$f \in \mathcal{M},\ f(0) = 0 \Longrightarrow S^* f \in \mathcal{M} \tag{3.5}$$

which is the usual definition for a nearly S^*-invariant subspace of H^2_+ ([10,13]) (also called nearly $\bar{z}$-invariant subspace [2]).

Remark 3.3. Clearly, if $\mathcal{M}$ is invariant for $P_H X|_H$, then it is nearly X-invariant (w.r.t. H). Indeed, if $f \in \mathcal{M}$ and $Xf \in H$, then $Xf = P_H Xf \in \mathcal{M}$. One may ask when is the converse true, i.e., when is a nearly X-invariant (w.r.t. H) space $\mathcal{M}$ invariant for $P_H X|_H$. If $\mathcal{M}$ is nearly X-invariant (w.r.t. H) or, equivalently, nearly $P_H X|_H$-invariant, then

$$\mathcal{M} = \mathcal{M}_X \oplus (\mathcal{M} \ominus \mathcal{M}_X) \tag{3.6}$$

where $\mathcal{M} \ominus \mathcal{M}_X \underset{X}{\longrightarrow} \mathcal{H} \backslash H$. Thus $\mathcal{M}$ is invariant for $P_H X|_H$ if and only if

$$P_H X(\mathcal{M} \ominus \mathcal{M}_X) \subset \mathcal{M}.$$

For example, if we take $\mathcal{M} = K_\theta$, which is nearly S^*-invariant, as all Toeplitz kernels, and also S^*-invariant ($S^* = P^+ \bar{z} P^+|_{H^2_+}$), we have

$$K_\theta = (K_\theta)_{\bar{z}} \oplus [k_0^\theta] \tag{3.7}$$

and we see that

$$S^* k_0^\theta = -\overline{\theta(0)} S^* \theta = -\overline{\theta(0)} \tilde{k}_0^\theta \in K_\theta,$$

where $k_0^\theta = 1 - \overline{\theta(0)}\theta$ and $\tilde{k}_0^\theta = \bar{z}(\theta - \theta(0))$.

Model spaces are a very important type of Toeplitz kernels insofar as they are the only ones to be S^*-invariant; furthermore, all Toeplitz kernels take the form gK_θ for same inner function θ and some outer function g satisfying certain additional conditions ([9]). However, model spaces can also be seen as particular cases of kernels of truncated Toeplitz operators (TTO), namely

$$A_G^\theta = P_\theta G P_\theta|_{K_\theta} \tag{3.8}$$

with $G \in \overline{H^\infty}$ ([3,11]). Unlike Toeplitz kernels, however, one can also have kernels of TTO which take the more general form

$$\ker A_G^\theta = \alpha K_\beta \tag{3.9}$$

where α, β are inner functions dividing θ ($\alpha\beta = \theta$). We call αK_β a *shifted model space*. This is the case, in particular, when $G \in H^\infty$.

Let us consider an example (which we will take as a starting point for the results that follow):

$$\ker A_\theta^{\theta z} = zK_\theta. \tag{3.10}$$

Using the first decomposition for K_θ in Theorem 2.5 for $\alpha = z$, we have

$$zK_\theta = z^2 \ker T_{\bar{\theta} z} \oplus [zk_0^\theta] \tag{3.11}$$

where we use the notation $[f] = \operatorname{span}\{f\}$. It is not difficult to see from here that zK_θ is nearly S^*-invariant with defect 1, the defect space ([1]) being $[k_0^\theta] := \operatorname{span}\{k_0^\theta\}$, and almost-invariant with defect 1 for S^*.

More generally, for a shifted model space αK_β we may ask how multiplication by $\bar{\alpha}$ acts on it. Using the first decomposition in Theorem 2.5 or, equivalently, in (2.5), we have that

$$\alpha K_\beta = \alpha (K_\beta)_{\bar{\alpha}} \oplus \alpha P_\beta K_\alpha \tag{3.12}$$

where $\alpha(K_\beta)_{\bar{\alpha}} \xrightarrow[M_{\bar{\alpha}}]{} \alpha K_\beta$ and $\alpha P_\beta K_\alpha \xrightarrow[M_{\bar{\alpha}}]{} L^2 \backslash \alpha H^2_+$.

But, since the space αK_β is defined by two inner functions α and β, we might also ask what are its invariance properties with respect to multiplication by $\bar{\beta}$. In this section we study the nearly $\bar{\beta}$-invariance of a shifted model space αK_β.

We start by showing that no non-zero element of αK_β is mapped into the same space by multiplication by $\bar{\beta}$ (in contrast with what happens when multiplying by $\bar{\alpha}$).

Proposition 3.4. *Let α, β be inner functions. Then*

$$(\alpha K_\beta)_{\bar{\beta}} = \{0\}. \tag{3.13}$$

Proof. We have that $\varphi \in \alpha K_\beta$ if and only if $\bar{\alpha}\varphi \in K_\beta$. In other words, $\bar{\alpha}\varphi \in H^2_+$, $\bar{\beta}\bar{\alpha}\varphi = \varphi_- \in H^2_-$, which implies that $\bar{\beta}\varphi \in \alpha H^2_-$. Thus, if $\bar{\beta}\varphi \in \alpha K_\beta \subset \alpha H^2_+$, then we must have $\bar{\beta}\varphi = 0$, i.e., $\varphi = 0$. □

Now we obtain an orthogonal sum decomposition of αK_β describing two parts of that space on which multiplication by $\bar{\beta}$ acts differently. Note that $\alpha K_\beta \subset K_{\alpha\beta}$.

Theorem 3.5. *Let α, β be inner functions. Then:*

(1) $\alpha K_\beta = (\alpha K_\beta \cap \beta K_\alpha) \oplus P_{\alpha K_\beta} K_\beta = (\alpha K_\beta \cap \beta K_\alpha) \oplus Q_\alpha K_\beta$ *where* $P_{\alpha K_\beta} = \alpha P_\beta \bar{\alpha} I$ *is the orthogonal projection from* L^2 *onto* αK_β *and* $Q_\alpha = I - P_\alpha = P^- + \alpha P^+ \bar{\alpha} I$,
(2) $(\alpha K_\beta \cap \beta K_\alpha) \backslash \{0\}$ *consists of the elements of* αK_β *which are mapped by* $M_{\bar{\beta}}$ *into* $K_{\alpha\beta} \backslash (\alpha K_\beta)$,
(3) $Q_\alpha K_\beta \backslash \{0\}$ *is mapped by* $M_{\bar{\beta}}$ *into* $L^2 \backslash H^2_+$.

Proof. (1) Suppose that $\varphi \in \alpha K_\beta$ and $\varphi \perp P_{\alpha K_\beta} K_\beta$. Then, for all $f_\beta \in K_\beta$,

$$0 = \langle \varphi, P_{\alpha K_\beta} f_\beta \rangle = \langle \varphi, f_\beta \rangle$$

so $\varphi \perp K_\beta$, i.e., $\varphi \in \beta H^2_+$. So there exist $h_\beta \in K_\beta$ such that $\varphi = \alpha h_\beta$ and $f_+ \in H^2_+$ with

$$\alpha h_\beta = \beta f_+ \qquad \text{which is equivalent to} \qquad \bar{\beta} h_\beta = \bar{\alpha} f_+,$$

and, since $\bar{\beta} h_\beta \in H^2_-$, we see that $f_+ \in K_\alpha$. Therefore $\varphi \in \alpha K_\beta \cap \beta K_\alpha$ and it follows that the first equality in Theorem 3.5 holds. The second equality is a consequence of Lemma 3.6 below, which may have an independent interest.

(2) If $f \in \alpha K_\beta \cap \beta K_\alpha$, then $\bar{\beta} f \in \bar{\beta}\alpha K_\beta \cap K_\alpha \subset K_\alpha \subset K_{\alpha\beta}$, so $\bar{\beta} f \in K_{\alpha\beta}$. On the other hand, it is clear that $\bar{\beta} f \in K_\alpha$ implies that $\bar{\beta} f \in K_{\alpha\beta} \backslash \alpha K_\beta$. Conversely, consider the subspace K' of αK_β consisting of all the elements in αK_β which are mapped by $M_{\bar{\beta}}$ into $K_{\alpha\beta}$. We have

$$K' = \{\varphi \in \alpha K_\beta : \bar{\beta}\varphi \in K_{\alpha\beta}\} = \alpha K_\beta \cap (K_{\alpha\beta})_{\bar{\beta}} = \alpha K_\beta \cap \beta K_\alpha \tag{3.14}$$

by Proposition 2.1.

(3) If $\bar{\beta}Q_\alpha f_\beta \in H^2_+$ for any $f_\beta \in K_\beta$, then $\bar{\beta}Q_\alpha f_\beta \in K_{\alpha\beta}$ because $Q_\alpha f_\beta \in K_{\alpha\beta}$ and $K_{\alpha\beta}$ is nearly $\bar{\beta}$-invariant. So, $Q_\alpha f_\beta \in K' = \alpha K_\beta \cap \beta K_\alpha$ by (3.14). On the other hand, $Q_\alpha K_\beta \perp (\alpha K_\beta \cap \beta K_\alpha)$ so the only element of $Q_\alpha K_\beta$ which is mapped by $M_{\bar{\beta}}$ into $K_{\alpha\beta}$ is zero. It follows that $\bar{\beta}Q_\alpha(K_\beta \backslash \{0\}) \subset L^2 \backslash H^2_+$. □

Lemma 3.6. *Let α, β be inner functions. Then, with the same notation as in Theorem 3.5,*

$$P_{\alpha K_\beta} K_\beta = Q_\alpha K_\beta. \tag{3.15}$$

Proof. Let h_β be any element of K_β. Then

$$\begin{aligned} P_{\alpha K_\beta} h_\beta &= \alpha P_\beta \bar{\alpha} h_\beta = \alpha\beta P^- \bar{\beta} P^+ \bar{\alpha} h_\beta = \alpha\beta P^- \bar{\beta}(I - P^-)\bar{\alpha} h_\beta \\ &= h_\beta - \alpha P^- \bar{\alpha} h_\beta = \alpha P^+ \bar{\alpha} h_\beta = Q_\alpha h_\beta. \end{aligned}$$ □

The decompositions in Theorem 3.5 allow us to establish necessary and sufficient conditions for αK_β to be nearly $\bar{\beta}$-invariant (w.r.t. H^2_+, w.r.t. $K_{\alpha\beta}$) or nearly $\bar{\beta}$-invariant with defect. Note that, since model spaces are nearly $\bar{\beta}$-invariant (in H^2_+) for any inner function β and $\alpha K_\beta \subset K_{\alpha\beta}$, saying that $f \in \alpha K_\beta$, $\bar{\beta} f \in H^2_+$ is equivalent to saying that $f \in \alpha K_\beta$, $\bar{\beta} f \in K_{\alpha\beta}$. Thus αK_β is nearly $\bar{\beta}$-invariant w.r.t. H^2_+ if and only if it is $\bar{\beta}$-invariant w.r.t. $K_{\alpha\beta}$.

Firstly, however, we prove some results that will be used later but are of independent interest.

Lemma 3.7. *Let α, β be inner functions. Then $Q_\alpha K_\beta = \{0\}$ if and only if $\beta \leqslant \alpha$.*

Proof. Indeed, $Q_\alpha K_\beta = \{0\}$ if and only if $K_\beta \subset K_\alpha$ which is equivalent to $\beta \leqslant \alpha$. □

Lemma 3.8. *Let δ and β be inner functions. Then*

$$\beta K_\delta \subset \delta K_\beta \quad \textit{if and only if} \quad \delta \leqslant \beta. \tag{3.16}$$

Proof. Let $\delta \leqslant \beta$ and $f \in \beta K_\delta$, i.e., $f = \beta h_\delta$ with $h_\delta \in K_\delta$. Then $\bar{\delta} f = \bar{\delta}\beta h_\delta \in H^2_+$ because $\bar{\delta}\beta \in H^\infty$. On the other hand,

$$\bar{\beta}(\bar{\delta} f) = \bar{\beta}(\bar{\delta}\beta h_\delta) = \bar{\delta} h_\delta \in H^2_-,$$

so $\bar{\delta} f \in K_\beta$ and we have $\bar{\delta}\beta K_\delta \subset K_\beta$. Conversely, suppose that $\beta K_\delta \subset \delta K_\beta$. Then $\beta K_\delta \cap \delta K_\beta = \beta K_\delta$ which, by Theorem 3.5(1), means that $Q_\beta K_\delta = \{0\}$. This is equivalent to $K_\delta \subset K_\beta$ so $\delta \leqslant \beta$. □

Corollary 3.9. *Let α, β be inner functions and let $\delta = g.c.d.(\alpha, \beta)$. Then*

$$\tfrac{\alpha\beta}{\delta} K_\delta \subset \alpha K_\beta \cap \beta K_\alpha.$$

Proof. We have that $\beta K_\delta \subset \delta K_\beta$ by (3.16), so $\alpha \frac{\beta}{\delta} K_\delta \subset \alpha K_\beta$. Analogously, since $\alpha K_\delta \subset \delta K_\alpha$, then $\frac{\beta\alpha}{\delta} K_\delta \subset \beta K_\alpha$. □

Lemma 3.10. *Let α, β be inner functions and let $\delta = g.c.d.(\alpha, \beta)$. We have*

$$\alpha K_\beta \cap \beta K_\alpha = \{0\} \quad \textit{if and only if} \quad \delta \in \mathbb{C}.$$

Proof. From Corollary 3.9 it follows that

$$\text{if } \alpha K_\beta \cap \beta K_\alpha = \{0\}, \text{ then } \delta \in \mathbb{C}.$$

Conversely, suppose that α and β are relatively prime. Then for any $f \in \alpha K_\beta \cap \beta K_\alpha$ there exist $f_\beta \in K_\beta$, $f_\alpha \in K_\alpha$ such that

$$\alpha f_\beta = \beta f_\alpha.$$

Since α and β are relatively prime we must have $f_\alpha \in \alpha H^2_+$, $f_\beta \in \beta H^2_+$, which implies that $f_\beta = f_\alpha = 0$. □

Now recall that αK_β is nearly $\bar{\beta}$-invariant (w.r.t. H^2_+, w.r.t. $K_{\alpha\beta}$) if and only if

$$f \in \alpha K_\beta\,, \; \bar{\beta} f \in K_{\alpha\beta} \Longrightarrow \bar{\beta} f \in \alpha K_\beta, \tag{3.17}$$

which is equivalent to

$$f \in K' \Longrightarrow \bar{\beta} f \in \alpha K_\beta, \tag{3.18}$$

where K' is given by (3.14). If $K' = \{0\}$ the implication is trivially true and we say that αK_β is trivially nearly $\bar{\beta}$-invariant.

Proposition 3.11. *Let α, β be inner functions. Then the following are equivalent:*

(i) αK_β *is nearly $\bar{\beta}$-invariant (w.r.t. H^2_+, w.r.t. $K_{\alpha\beta}$),*
(ii) $\alpha K_\beta \cap \beta K_\alpha = \{0\}$,
(iii) αK_β *is trivially nearly $\bar{\beta}$-invariant (w.r.t. H^2_+, w.r.t. $K_{\alpha\beta}$),*
(iv) α *and* β *are relatively prime, i.e., g.c.d.$(\alpha, \beta) \in \mathbb{C}$.*

Proof. Conditions (i), (ii), (iii) are equivalent because, by Theorem 3.5(2), $\bar{\beta}(K'\backslash\{0\}) \subset K_{\alpha\beta}\backslash\alpha K_\beta$ so, if $\bar{\beta} f \in \alpha K_\beta$ holds for all $f \in K'$, as in (3.18), then we must have $K' = \alpha K_\beta \cap \beta K_\alpha = \{0\}$.

(iii)⇔(iv) by Lemma 3.10.

□

Corollary 3.12. *We have that, on the one hand,*

$$\alpha K_\beta = P_{\alpha K_\beta} K_\beta = Q_\alpha K_\beta \quad \textit{if and only if} \quad g.c.d.(\alpha, \beta) \in \mathbb{C} \tag{3.19}$$

and, in this case,

$$(\alpha K_\beta)\backslash\{0\} \underset{M_{\bar{\beta}}}{\longrightarrow} L^2\backslash H^2_+; \tag{3.20}$$

on the other hand,

$$\alpha K_\beta = \alpha K_\beta \cap \beta K_\alpha \quad \textit{if and only if} \quad \beta \leqslant \alpha \tag{3.21}$$

and, in this case,

$$(\alpha K_\beta)\backslash\{0\} \underset{M_{\bar{\beta}}}{\longrightarrow} K_{\alpha\beta}\backslash(\alpha K_\beta). \tag{3.22}$$

Corollary 3.13. *Let $\alpha, \beta \in H^\infty$ be inner functions. Then:*

(1) αK_β *is $K_{\alpha\beta}$-stable for $M_{\bar{\beta}}$ if and only if $\beta \leqslant \alpha$,*
(2) αK_β *is nearly $\bar{\beta}$-invariant with defect (w.r.t. H^2_+, w.r.t. $K_{\alpha\beta}$) if and only if $\alpha K_\beta \cap \beta K_\alpha$ is finite dimensional.*

4. THE CASE $\alpha = z$

If $\alpha = z$, we can obtain several interesting properties by using the decompositions in Theorem 2.5. In that case we have:

$$K_z = \mathbb{C}, \quad P_\theta K_z = [k_0^\theta], \quad P_\theta C_\theta K_z = [\tilde{k}_0^\theta] \tag{4.1}$$

where

$$k_0^\theta = 1 - \overline{\theta(0)}\theta, \quad \tilde{k}_0^\theta = \bar{z}(\theta - \theta(0)). \tag{4.2}$$

Thus, from Theorem 2.5

$$K_\theta = [k_0^\theta] \oplus z \ker T_{\bar{\theta} z}, \quad K_\theta = [\tilde{k}_0^\theta] \oplus \ker T_{\bar{\theta} z}. \tag{4.3}$$

We start by showing that, just as the decompositions (1.3) are expressed in terms of model spaces, (4.3) can also be expressed in terms of Toeplitz kernels.

Lemma 4.1. *Let θ be an inner function. Then*

$$[k_0^\theta] = \ker T_{\bar{z}\frac{\overline{k_0^\theta}}{k_0^\theta}}.$$

Proof. We have that, for $\varphi_\pm \in H^2_\pm$,

$$\bar{z}\frac{\overline{k_0^\theta}}{k_0^\theta}\varphi_+ = \varphi_- \quad \text{if and only if} \quad (k_0^\theta)^{-1}\varphi_+ = z(\overline{k_0^\theta})^{-1}\varphi_-$$

and, since $k_0^\theta, (k_0^\theta)^{-1} \in H^\infty$, the left hand side of the last identity is in H^2_+, while the right hand side represents a function in $\overline{H^2_+}$, so both sides must be equal to a constant. Therefore $\varphi_+ = \lambda k_0^\theta$, $\lambda \in \mathbb{C}$. □

From (4.3) and Lemma 4.1 we get the following.

Proposition 4.2. *Let θ be an inner function. Then*

$$K_\theta = \ker T_{\bar{\theta}} = \ker T_{\bar{z}\frac{\overline{k_0^\theta}}{k_0^\theta}} \oplus z \ker T_{z\bar{\theta}} \tag{4.4}$$

and

$$K_\theta = \ker T_{\bar{\theta}} = \frac{\tilde{k}_0^\theta}{k_0^\theta} \ker T_{\bar{z}\frac{\overline{k_0^\theta}}{k_0^\theta}} \oplus \ker T_{z\bar{\theta}}. \tag{4.5}$$

Note that, if $\theta(0) = 0$, these equalities become $K_\theta = K_z \oplus z K_{\frac{\theta}{z}}$ and $K_\theta = K_{\frac{\theta}{z}} \oplus \frac{\theta}{z} K_z$, respectively.

We can also use (4.3) to get a better understanding of the relations between K_θ and $SK_\theta = zK_\theta$, on the one hand, and between K_θ and S^*K_θ, on the other.

We have, using the second equality in (4.3),

$$SK_\theta = zK_\theta = [z\tilde{k}_0^\theta] \oplus z \ker T_{\bar{\theta} z}, \tag{4.6}$$

where $z \ker T_{\bar{\theta} z} \subset K_\theta$, but $z\tilde{k}_0^\theta = \theta - \theta(0) \in H^2_+ \setminus K_\theta$.

So we never have $SK_\theta \subset K_\theta$. However, it follows that

$$SK_\theta \subset K_\theta \oplus \mathbb{C}\theta \tag{4.7}$$

because

$$z\tilde{k}_0^\theta = \theta - \theta(0) = \theta - \theta(0)k_0^\theta - \theta(0)\overline{\theta(0)}\theta = -\theta(0)k_0^\theta + (1 - \theta(0)\overline{\theta(0)})\theta$$

where the first term on the right hand side belongs to K_θ and the second term belongs to $\mathbb{C}\theta$ (see also [1] where (4.7) was proved differently).

On the other hand, using the first equality in (4.3), we have

$$S^*K_\theta = [S^*k_0^\theta] \oplus \ker T_{\bar{\theta}z} = [\overline{\theta(0)}\,\tilde{k}_0^\theta] \oplus \ker T_{\bar{\theta}z}. \tag{4.8}$$

Since K_θ is invariant for S^*, it is natural to ask when is S^*K_θ exactly equal to K_θ. This has a simple answer, comparing the right hand side of (4.8) with the second decomposition in (4.3). We immediately see the following (see also [6]):

Proposition 4.3. *Let θ be an inner function. Then $S^*(K_\theta) = K_\theta$ if and only if $\theta(0) \neq 0$. If $\theta(0) = 0$, then $S^*(K_\theta) = \ker T_{\bar{\theta}z} \subsetneq K_\theta$.*

Remark 4.4. This result can be generalized to obtain necessary and sufficient conditions for $T_{\bar{\alpha}}K_\theta = K_\theta$ from Theorem 3.5 in a similar way, when α is inner. Necessary and sufficient conditions for $T_{\bar{a}}K_\theta = K_\theta$ when $a \in H^\infty$ is outer, where obtained in [6, Lemma 4.9].

Yet another interesting result that one can get from (4.3) is the answer to the following question: how do we decompose a given function in K_θ according to (4.3)? This is equivalent to asking how to define the orthogonal projections associated with the orthogonal sums in (4.3), in particular $P_{\ker T_{\bar{\theta}z}}$. Note that the orthogonal projection from L^2 onto the kernel of a Toeplitz operator T_G is not easy to define, unless one has a representation of the form $\ker T_G = gK_\theta$, where g is an outer function satisfying certain conditions (g is called the extremal function for $\ker T_G$) and θ is an inner function; in that case $P_{\ker T_G} = P_{gK_\theta} = gP_\theta\bar{g}I$ ([8,9]). However, that representation is not known in general. Using (4.3) for $k \in K_\theta$, we have

$$k = \lambda_1 k_0^\theta + (k - \lambda_1 k_0^\theta) \quad \text{with} \quad \lambda_1 \in \mathbb{C} \tag{4.9}$$

such that

$$k - \lambda_1 k_0^\theta \perp k_0^\theta, \quad (k - \lambda_1 k_0^\theta)(0) = 0. \tag{4.10}$$

It follows that $\lambda_1 = k(0)/k_0^\theta(0)$, so

$$k = \tfrac{k(0)}{k_0^\theta(0)} k_0^\theta + \left(k - \tfrac{k(0)}{k_0^\theta(0)} k_0^\theta\right) \tag{4.11}$$

with $k - \frac{k(0)}{k_0^\theta(0)} k_0^\theta \in z \ker T_{\bar{\theta}z}$. This is equivalent to

$$k = \tfrac{k_0^\theta \otimes k_0^\theta}{k_0^\theta(0)} k + \tfrac{1}{k_0^\theta(0)} D_{k_0^\theta} k, \tag{4.12}$$

where

$$D_{k_0^\theta} k = \begin{vmatrix} k & k(0) \\ k_0^\theta & k_0^\theta(0) \end{vmatrix}, \quad k \in K_\theta, \tag{4.13}$$

and we see that $\frac{1}{k_0^\theta(0)} D_{k_0^\theta} P_\theta$ is the orthogonal projection from L^2 onto $z \ker T_{\bar{\theta} z}$.

Analogously, from the second decomposition in (4.3), for any $k \in K_\theta$ we have

$$k = \lambda_2 \tilde{k}_0^\theta + (k - \lambda_2 \tilde{k}_0^\theta). \tag{4.14}$$

Since

$$C_\theta k = \bar{\lambda}_2 k_0^\theta + (C_\theta k - \bar{\lambda}_2 k_0^\theta) \quad \text{with} \quad \bar{\lambda}_2 = \frac{(C_\theta k)(0)}{k_0^\theta(0)}, \tag{4.15}$$

we have, taking into account that $k_0^\theta(0) \in \mathbb{R}$,

$$k = \frac{\overline{(C_\theta k)(0)}}{k_0^\theta(0)} \tilde{k}_0^\theta + \left(k - \frac{\overline{(C_\theta k)(0)}}{k_0^\theta(0)} \tilde{k}_0^\theta\right) = \frac{\tilde{k}_0^\theta \otimes \tilde{k}_0^\theta}{k_0^\theta(0)} k + \frac{1}{k_0^\theta(0)} D_{\tilde{k}_0^\theta} k \tag{4.16}$$

(the first summand belongs to $[\tilde{k}_0^\theta]$ and the second to $\ker T_{\bar{\theta} z}$), where $D_{\tilde{k}_0^\theta}$ is defined by

$$D_{\tilde{k}_0^\theta} k = \begin{vmatrix} k & \overline{(C_\theta k)(0)} \\ \tilde{k}_0^\theta & k_0^\theta(0) \end{vmatrix} = \begin{vmatrix} k & \overline{(C_\theta k)(0)} \\ C_\theta k_0^\theta & k_0^\theta(0) \end{vmatrix}, \tag{4.17}$$

and we see that $\frac{1}{k_0^\theta(0)} D_{\tilde{k}_0^\theta} P_\theta$ is the orthogonal projection from L^2 onto $\ker T_{\bar{\theta} z}$.

Note that when $\theta(0) = 0$, so that $\ker T_{\bar{\theta} z} = K_{\frac{\theta}{z}}$, we have $k_0^\theta(0) = 1$ and $D_{k_0^\theta} P_\theta$, $D_{\tilde{k}_0^\theta} P_\theta$ coincide with $P_{z K_{\frac{\theta}{z}}} = z P_{\frac{\theta}{z}} \bar{z} I$ and $P_{\frac{\theta}{z}}$, respectively.

We have thus proved the following:

Proposition 4.5. *If θ is an inner function, then*

$$P_{\ker T_{\bar{\theta} z}} = \frac{1}{k_0^\theta(0)} D_{\tilde{k}_0^\theta} P_\theta \quad \text{and} \quad P_{z \ker T_{\bar{\theta} z}} = \frac{1}{k_0^\theta(0)} D_{k_0^\theta} P_\theta,$$

where $D_{\tilde{k}_0^\theta}$ and $D_{k_0^\theta}$ are defined by (4.13) *and* (4.17)*, respectively.*

5. INVARIANCE PROPERTIES AND COMMUTATION RELATIONS

Model spaces are invariant for S^* in H^2_+ and, more generally, for any Toeplitz operator with anti-analytic symbol; Toeplitz kernels are nearly S^*-invariant; kernels of truncated Toeplitz operators are nearly S^*-invariant with defect $m \leqslant 1$.

We now study here how those invariance properties and, in general, the invariance properties of the kernel of an operator T, with respect to another operator, follow from certain commutation relations between the two operators involved.

Let us start with a general situation. Let H be a closed subspace of $\mathcal{H}$. Our first result is a very simple one.

Proposition 5.1. *Let X_H and T_H be bounded operators on H. If $X_H T_H = T_H X_H$, then $\ker T_H$ is invariant for X_H.*

Proof. Let $X_H T_H = T_H X_H$ and let f be any element of $\ker T_H$. Then

$$T_H(X_H f) = X_H(T_H f) = 0,$$

so $X_H f \in \ker T_H$. □

Now let $X \in \mathcal{B}(\mathcal{H})$ and let H be a closed subspace of $\mathcal{H}$. Let $X_H = P_H X|_H$. Consider the operator $T_H \in \mathcal{B}(H)$. Now recall from Definition 3.1 that $\ker T_H$ is nearly X-invariant with respect to H if

$$f \in \ker T_H\,,\ Xf \in H \Longrightarrow Xf = X_H f \in \ker T_H. \tag{5.1}$$

Then we will say that $\ker T_H$ is *nearly X_H-invariant*. Similarly, $\ker T_H$ is nearly X-invariant with respect to H with defect $\mathcal{F} \subset H$ if

$$f \in \ker T_H\,,\ Xf \in H \Longrightarrow Xf = X_H f \in \ker T_H \oplus \mathcal{F}. \tag{5.2}$$

Then we will say that $\ker T_H$ is *nearly X_H-invariant with defect m*. Recall that $\mathcal{F}$ is finite dimensional with $\dim \mathcal{F} = m$, and we assume that m is the smallest possible dimension of such a space $\mathcal{F}$.

With this definition, for any inner θ, saying for instance that $K_\theta = \ker T_{\bar\theta} \subset H^2$ ($T_{\bar\theta}$ is the Toeplitz operator with symbol $\bar\theta$) is nearly S^*-invariant is equivalent to saying that K_θ in nearly $\bar z$-invariant w.r.t. H^2 ([2]).

Proposition 5.2. *Let $X \in \mathcal{B}(\mathcal{H})$ and let $X_H = P_H X|_H$. Let T_H be a bounded operator on H. If $T_H X_H = X_H T_H$ on $H_X = \{f \in H : Xf \in H\}$, then $\ker T_H$ is nearly X_H-invariant.*

Proof. Let $f \in \ker T_H$ and $Xf \in H$. Then

$$T_H(X_H f) = X_H T_H f = 0,$$

so $X_H f \in \ker T_H$. □

As an illustration of this result, consider a Toeplitz operator T_g, $g \in L^\infty$, and let $h_- \in \overline{H^\infty}$. In general, T_g and T_{h_-} do not commute, unless $g \in \overline{H^\infty}$, so we can apply Proposition 5.1 only in the latter case. That is the case of a model space $K_\theta = \ker T_{\bar\theta}$. However, if $f \in (H^2_+)_{h_-} = \{f \in H^2_+ : h_- f \in H^2_+\}$, then

$$T_g T_{h_-} f = T_g h_- f = P^+ g h_- f = P^+ h_- g f = P^+ h_- P^+ g f = T_{h_-} T_g f,$$

so we conclude that $\ker T_g$ is nearly T_{h_-}-invariant (as it is known). In particular, $\ker T_g$ is nearly S^*-invariant.

Proposition 5.3. *Let $X \in \mathcal{B}(\mathcal{H})$ and let $X_H = P_H X|_H$. Let T_H be a bounded operator on H. If $T_H X_H - X_H T_H$, restricted to H_X, is a finite rank operator with rank r, i.e., there is $\tilde{\mathcal{F}} \subset H$ with $\dim \tilde{\mathcal{F}} = r$, $r < \infty$ such that*

$$(T_H X_H - X_H T_H) f \in \tilde{\mathcal{F}}, \quad \textit{for all} \quad f \in H_X.$$

Then $\ker T_H$ is nearly X_H-invariant with defect $m \leqslant r$.

Proof. Let $f \in \ker T_H$, $Xf \in H$. Then

$$T_H X_H f = X_H T_H f + \tilde{f} \quad \text{with} \quad \tilde{f} \in \tilde{\mathcal{F}}$$

if and only if

$$T_H X_H f = \tilde{f} \quad \text{with} \quad \tilde{f} \in \tilde{\mathcal{F}}.$$

To simplify the proof, we will assume that $\dim \tilde{\mathcal{F}} = 1$ (the reasoning is analogous if $\dim \tilde{\mathcal{F}}$ is higher). Let $\tilde{f}_0 \in \tilde{\mathcal{F}} \backslash \{0\}$ and define

$$H_{\tilde{f}_0} = \{f \in \ker T_H \cap X_H : (T_H X_H - X_H T_H) f = \tilde{f}_0\}.$$

Note that $H_{\tilde{f}_0}$ may be empty, in which case the assumptions of Proposition 5.2 are satisfied and $\ker T_H$ is nearly X_H-invariant. If $H_{\tilde{f}_0}$ is nonempty, then choosing any element $f_0 \in H_{\tilde{f}_0}$ we have that, for every $f \in \ker T_H \cap H_X$, there exists $\lambda_f \in \mathbb{C}$ such that $T_H(X_H f - \lambda_f X_H f_0) = 0$, so $X_H f - \lambda_f (X_H f_0) \in \ker T_H$ and therefore

$$X_H f \in \ker T_H + \operatorname{span}\{X_H f_0\}.$$ □

It was shown in [12] that kernels of truncated Toeplitz operators are either nearly S^*-invariant, or nearly S^*-invariant with defect 1. We recover here these results by a different method, applying Proposition 5.3. We start by an auxiliary result.

Lemma 5.4. *Let θ be an inner function. If $h_+ \in H^2_+$, then*

$$P_\theta \bar{z} \theta h_+ = h_+(0) \tilde{k}_0^\theta, \quad \tilde{k}_0^\theta = \bar{z}(\theta - \theta(0)).$$

Proof. Calculate

$$\begin{aligned} P_\theta \bar{z} \theta h_+ &= \theta P^- \bar{\theta} \bar{z} (\theta h_+ - \theta(0) h_+(0)) \\ &= \theta P^- \bar{z} h_+ - \theta(0) h_+(0) \bar{z} \\ &= \theta \bar{z} h_+(0) - \theta(0) h_+(0) \bar{z} = h_+(0) \tilde{k}_0^\theta. \end{aligned}$$ □

Remark 5.5. For an inner function θ we have that K_θ is nearly S^*-invariant, i.e.,

$$f \in K_\theta, \bar{z} f \in H^2_+ \Longrightarrow \bar{z} f = P^+ \bar{z} f = S^* f \in K_\theta, \tag{5.3}$$

and, on the other hand, for all $f \in K_\theta$,

$$S^*_\theta f = P_\theta \bar{z} f = P^+ \bar{z} f = S^* f. \tag{5.4}$$

Therefore, saying that $\mathcal{M} \subset K_\theta$ is nearly S^*_θ-invariant, i.e.,

$$f \in \mathcal{M}, \ \bar{z} f \in K_\theta \Longrightarrow P_\theta \bar{z} f = S^*_\theta f \in \mathcal{M} \tag{5.5}$$

is equivalent to saying that $\mathcal{M} \subset K_\theta$ is nearly S^*-invariant, i.e.,

$$f \in \mathcal{M}, \ \bar{z} f \in H^2_+ \Longrightarrow P^+ \bar{z} f = S^* f \in \mathcal{M}. \tag{5.6}$$

The same is true for near S^*_θ-invariance and near S^*-invariance with defect.

Proposition 5.6. *Let $g \in L^\infty$. Then for all $f \in \ker A_g^\theta \cap (K_\theta)_{\bar{z}}$ we have*

$$(A_g^\theta S_\theta^* - S_\theta^* A_g^\theta)f = \lambda_f \tilde{k}_0^\theta \tag{5.7}$$

with $\lambda_f = (P^+(g\bar{\theta}f))(0)$ and $\tilde{k}_0^\theta(z) = \bar{z}(\theta(z) - \theta(0))$.

Proof. Let us calculate:

$$\begin{aligned} A_g^\theta S_\theta^* f =& A_g^\theta S^* f = P_\theta g P_\theta \bar{z} f \\ =& P_\theta g P^+ \bar{z} f = P_\theta g \bar{z} f = P_\theta \bar{z}(P_\theta + \theta P^+ \bar{\theta}) g f \\ =& P_\theta \bar{z} P_\theta g f + P_\theta \bar{z} \theta (P^+ \bar{\theta} g f) \\ =& S_\theta^* A_g^\theta f + \lambda_f \tilde{k}_0^\theta \quad \text{with} \quad \lambda_f = (P^+(g\bar{\theta}f))(0). \end{aligned}$$

□

Note that

$$f \in \ker A_g^\theta \quad \text{if and only if} \quad gf = f_- + \theta f_+ \quad \text{with} \quad f_- \in H_-^2, f_+ \in H_+^2, \tag{5.8}$$

where

$$f_+ = P^+ g\bar{\theta} f,$$

so

$$f_+(0) = (P^+(g\bar{\theta}f))(0) = 0 \quad \text{if and only if} \quad \bar{z} f_+ \in H^2. \tag{5.9}$$

We can now state the following.

Proposition 5.7. *Let $g \in L^\infty$ and assume that* $\ker A_g^\theta \neq \{0\}$*. Then* $\ker A_g^\theta$ *is nearly S^*-invariant with defect 1 if and only if*

$$f(0) = 0 \quad \textit{for all } f \in \ker A_g^\theta; \tag{5.10}$$

otherwise $\ker A_g^\theta$ *in nearly S^*-invariant.*

Proof. From Propositions 5.6 and 5.3 we have that $\ker A_g^\theta$ is S^*-invariant with defect at most equal to 1; from Proposition 5.2 it follows that $\ker A_g^\theta$ is nearly S^*-invariant if and only if λ_f in (5.7) is equal to 0 for all $f \in \ker A_g^\theta \cap (K_\theta)_{\bar{z}}$, i.e.,

$$(P^+(g\bar{\theta}f))(0) = 0 \quad \text{for all } f \in \ker A_g^\theta \cap (K_\theta)_{\bar{z}}. \tag{5.11}$$

If $f(0) = 0$ for all $f \in \ker A_g^\theta$, then $\ker A_g^\theta$ cannot be a nearly S^*-invariant subspace of H_+^2, and therefore it must have defect 1.

Suppose now that there exists $f_1 \in \ker A_g^\theta$ with $f_1(0) \neq 0$. Then we have, for $f \in \ker A_g^\theta \cap (K_\theta)_{\bar{z}}$,

$$\begin{aligned} gf_1 = f_1^- + \theta f_1^+ \quad &\text{with} \quad f_1^- \in H_-^2, f_1^+ \in H_+^2, \\ gf = f_- + \theta f_+ \quad &\text{with} \quad f_- \in H_-^2, f^+ \in H_+^2 \end{aligned}$$

and, multiplying one equation by f and the other by f_1, we get

$$ff_1^- + \theta f f_1^+ = f_1 f_- + \theta f_1 f_+$$

which is equivalent to

$$(\bar{\theta}f)f_1^-(\bar{\theta}f_1)f_- = f_1 f_+ - f f_1^+.$$

The left hand side of the last equality belongs to $(H^2_-)^2$ and the right hand side to $(H^2_+)^2$ so both sides must be zero, so $f_1 f_+ - f f_1^+ = 0$.

Since $f(0) = 0$ and $f_1(0) \neq 0$, we conclude that we must have $f_+(0) = 0$ and therefore (5.11) holds and $\ker A_g^\theta$ is S^*-invariant. □

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